

UNIQUENESS OF STATIONARY STATES FOR SINGULAR KELLER-SEGEL TYPE MODELS

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ABSTRACT. We consider a generalised Keller-Segel model with non-linear porous medium type diffusion and non-local attractive power law interaction, focusing on potentials that are more singular than Newtonian interaction. We show uniqueness of stationary states (if they exist) in any dimension both in the diffusion-dominated regime and in the fair-competition regime when attraction and repulsion are in balance. As stationary states are radially symmetric decreasing, the question of uniqueness reduces to the radial setting. Our key result is a sharp generalised Hardy-Littlewood-Sobolev type functional inequality in the radial setting.

Keywords: uniqueness, Hardy-Littlewood-Sobolev inequality, aggregation-diffusion equation, Keller-Segel model.

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1. INTRODUCTION

We consider a family of partial differential equations modelling self-attracting diffusive particles at the macroscopic scale,

$$(1.1) \quad \begin{cases} \partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla S), & t > 0, \quad x \in \mathbb{R}^N, \\ \rho(t = 0, x) = \rho_0(x), \end{cases}$$

where the diffusion exponent $m > 1$ is of porous medium type [27]. Since equation (1.1) is positivity preserving, conserves mass, and is invariant by translation, we impose

$$\rho_0(x) \geq 0, \quad \int_{\mathbb{R}^N} \rho_0(x) dx = M, \quad \int_{\mathbb{R}^N} x \rho_0(x) dx = 0$$

for some fixed mass $M > 0$, and it follows that the same holds true for the solution $\rho(t, x)$. The mean-field potential $S(x) := W(x) * \rho(x)$ depends non-locally on the solution $\rho(t, x)$ through convolution with the interaction potential $W(x)$. Depending on the context and the application, different choices of repulsive or attractive potentials are used to model pair-wise interactions between particles, see for instance [3, 18, 12, 14] and the references therein. Here, we focus on attractive singular power-law potentials $W(x) = W_k(x)$,

$$W_k(x) := \frac{|x|^k}{k}, \quad k < 0.$$

For $W_k \in L^1_{loc}(\mathbb{R}^N)$, we require $k > -N$. Whilst for $k > 1 - N$, the gradient $\nabla S_k := \nabla (W_k * \rho)$ is well defined, it becomes a singular integral in the range $-N < k \leq 1 - N$, and we thus define it via a Cauchy principal value. Hence, the mean-field potential gradient in equation (1.1) is given by

$$(1.2) \quad \nabla S_k(x) := \begin{cases} \nabla W_k * \rho, & \text{if } k > 1 - N, \\ \int_{\mathbb{R}^N} \nabla W_k(x - y) (\rho(y) - \rho(x)) dy, & \text{if } -N < k \leq 1 - N. \end{cases}$$

Writing $k = 2s - N$ with $s \in (0, \frac{N}{2})$, the convolution term S_k is governed by a fractional diffusion process,

$$c_{N,s}(-\Delta)^s S_k = \rho, \quad c_{N,s} = (2s - N) \frac{\Gamma(\frac{N}{2} - s)}{\pi^{N/2} 4^s \Gamma(s)} = \frac{k\Gamma(-k/2)}{\pi^{N/2} 2^{k+N} \Gamma(\frac{k+N}{2})},$$

and so the system (1.1) can be interpreted as

$$\begin{cases} \partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla (-\Delta)^{-s} \rho), & t > 0, \quad x \in \mathbb{R}^N, \\ \rho(t=0, x) = \rho_0(x). \end{cases}$$

Models with this type of non-local interaction have been considered in [8, 7] in the repulsive case. Since the non-linear diffusion acts as a repulsive force between particles, one expects competing effects between the diffusion term and the non-local attractive forces, which motivates the study of equilibria of the system. For certain choices of parameters m and k , diffusion may overcome attraction, and no stationary states for (1.1) exist. In this case, we seek self-similar profiles instead as they are the natural candidates characterizing the long-time behaviour of the system. Self-similar profiles of equation (1.1) are stationary states of a suitably rescaled aggregation-diffusion equation with an additional confining potential. Combining both the original and rescaled system, we write

$$(1.3) \quad \partial_t \rho = \Delta \rho^m + \nabla \cdot (\rho \nabla S_k) + \mu_{\text{resc}} \nabla \cdot (x\rho), \quad t > 0, \quad x \in \mathbb{R}^N,$$

with $\mu_{\text{resc}} = 0$ for original variables, and $\mu_{\text{resc}} = 1$ for rescaled variables. For details on the change of variables transforming (1.1) into (1.3), see [11].

The competing effects of attractive and repulsive forces can also be observed on the level of the energy functional corresponding to equation (1.3):

$$\mathcal{F}[\rho] = \frac{1}{m-1} \int \rho^m + \frac{1}{2} \iint W_k(x-y) \rho(x) \rho(y) dx dy + \frac{\mu_{\text{resc}}}{2} \int |x|^2 \rho(x) dx.$$

More precisely, we can write equation (1.3) as

$$\partial_t \rho = \nabla \cdot \left(\rho \nabla \frac{\delta \mathcal{F}}{\delta \rho} \right)$$

where the first variation of \mathcal{F} is given by

$$\frac{\delta \mathcal{F}}{\delta \rho}[\rho](x) = \frac{m}{m-1} \rho^{m-1} + W_k * \rho + \mu_{\text{resc}} \frac{|x|^2}{2},$$

and so solutions to (1.3) are gradient flows in the 2-Wasserstein metric with respect to the energy \mathcal{F} , see [2, 28]. One simple way to observe the competition between the diffusion and aggregation term in original variables $\mu_{\text{resc}} = 0$ is to consider mass-preserving dilations

$$\rho_\lambda(x) := \lambda^N \rho(\lambda x).$$

Substituting ρ_λ into \mathcal{F} with $\mu_{\text{resc}} = 0$, we see that the two contributions to the energy are homogeneous with different powers,

$$\mathcal{F}[\rho_\lambda] = \frac{\lambda^{N(m-1)}}{m-1} \int \rho^m + \frac{\lambda^{-k}}{2} \iint W_k(x-y) \rho(x) \rho(y) dx dy,$$

and one observes different types of behaviour depending on the relation between the parameters N , m and k . The energy functional is homogeneous if attraction and repulsion are in balance, so that the two terms of the energy scale with the same power, that is, if $m = m_c$ for

$$m_c := 1 - \frac{k}{N}.$$

This motivates the definition of three different regimes: the *diffusion-dominated regime* $m > m_c$, the *fair-competition regime* $m = m_c$, and the *attraction-dominated regime* $0 < m < m_c$.

We will here concentrate on the diffusion-dominated and fair-competition regimes, $m \geq m_c$ in the more singular range $-N < k \leq 2 - N$. For a detailed overview of the different regimes and recent results, see [11].

Let us start by summarising the properties of system (1.3) that are known in the literature. We begin by making precise our notion of stationary states.

Definition 1. *Given $\bar{\rho} \in L^1_+(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ we call it a stationary state for the evolution problem (1.3) if $\bar{\rho}^m \in H^1_{loc}(\mathbb{R}^N)$, $\nabla \bar{S}_k \in L^1_{loc}(\mathbb{R}^N)$ is as in (1.2) for $\bar{S}_k := W_k * \bar{\rho}$, and it satisfies*

$$\nabla \bar{\rho}^m = -\bar{\rho} \nabla \bar{S}_k - \mu_{\text{resc}} x \bar{\rho}.$$

In the case of the fair-competition regime $m = m_c$ in original variables $\mu_{\text{resc}} = 0$, a similar critical mass phenomenon occurs as for the classical Keller-Segel model [6, 4, 10] with logarithmic interaction and linear diffusion. More precisely, it was shown in [5] that there exists a critical mass M_c in the case of Newtonian interaction $k = 2 - N$ for which infinitely many stationary states exist. For sub-critical masses $0 < M < M_c$, no stationary states exist as diffusion overcomes attraction, but solutions exist globally in time and decay in a self-similar fashion. For super-critical masses $M > M_c$, attraction overcomes diffusion, and solutions cease to exist in finite time. As shown in [11, 12], this dichotomy holds in fact in the full range $-N < k < 0$ in the fair competition regime $m = m_c$.

In the case of the fair-competition regime $m = m_c$ in rescaled variables $\mu_{\text{resc}} = 1$ and for subcritical masses $0 < M < M_c$, we have existence of a stationary solution $\bar{\rho}_M$ with mass $\int \bar{\rho}_M = M$ by [11, Theorem 2.9], which corresponds to a self-similar profile for equation (1.3) in original variables with $\mu_{\text{resc}} = 0$. Uniqueness of this stationary state is known in one-dimension, as well as convergence in 2-Wasserstein distance of solutions under certain assumptions on the transport map between the solution and the stationary state, see [12]. In higher dimensions, uniqueness and convergence results were shown in [29] for the special case of the Newtonian interaction kernel $k = 2 - N$, but the questions of convergence and uniqueness have previously not been answered for more general k .

As soon as $m > m_c$, we expect regularising effects from the dominating diffusive term. For the diffusion-dominated regime $m > m_c$ in original variables $\mu_{\text{resc}} = 0$, uniform L^∞ -bounds were obtained in [26, 9] for any initial mass $M > 0$ in the case of Newtonian interactions $k = 2 - N$. Recently, further results on the existence, boundedness and regularity of solutions have been obtained in [30]. Moreover, the existence of global minimisers for the energy functional \mathcal{F} for any mass $M > 0$ was shown in [13, 17] for Newtonian interactions $k = 2 - N$ and for more general interaction kernels $-N < k < 0$ in [16]. Minimisers of the energy functional \mathcal{F} are stationary states of equation (1.3) thanks to the gradient flow structure, as long as they are regular enough. As a direct consequence, we obtain existence of stationary states in the above cases. The uniqueness of the stationary state for $m > m_c$ was shown in one dimension in [16] using optimal transport techniques, and in the Newtonian case $k = 2 - N$ in any dimension $N \geq 3$ in [22] by a dynamical argument. The general case however has not been answered yet up to now.

Finally, we point out that existence and uniqueness of stationary solutions have also been obtained for $m = 2$ under suitable assumptions in the case of integrable attractive interaction potentials in [21].

Our goal here is to extend the results on the uniqueness of stationary states of system (1.3) to more singular k , higher dimensions N and any $m \geq m_c$ by building on the techniques employed in [12].

An important point to make is that due to the results in [15, 16], any stationary solution in the sense of Definition 1 in all the cases for m , W and μ_{resc} discussed in the previous paragraphs

are radially symmetric decreasing about their centre of mass and compactly supported. This means that the question of uniqueness for stationary states is reduced to the radial setting. To this end, we rewrite (1.3) in radial variables,

$$(1.4) \quad \partial_t (r^{N-1}\rho) = \partial_r (r^{N-1}\partial_r \rho^m) + \partial_r (r^{N-1}\rho \partial_r (W_k * \rho(r))) + \mu_{\text{resc}} \partial_r (r^N \rho),$$

and work completely in the radial setting from now on. Let

$$\mathcal{Y}_M := \left\{ \rho \in L^1_+(\mathbb{R}^N) \cap L^m(\mathbb{R}^N) : \|\rho\|_1 = M, \int x \rho(x) dx = 0, \mu_{\text{resc}} \int |x|^2 \rho(x) dx < \infty \right\}.$$

and its radial subset

$$\mathcal{Y}_M^* := \{ \rho \in \mathcal{Y}_M : \rho^* = \rho \},$$

where ρ^* denotes the symmetric decreasing rearrangement of ρ .

Theorem 2 (Sharp Functional Inequality). *Let $N \geq 2$, $k \in (-N, 2 - N]$ and $m \geq m_c$. If (1.4) admits a radial stationary density $\bar{\rho}$ in \mathcal{Y}_M^* , then*

$$\mathcal{F}[\rho] \geq \mathcal{F}[\bar{\rho}], \quad \forall \rho \in \mathcal{Y}_M^*,$$

with the equality cases given by $\bar{\rho}$, and by its dilations if $m = m_c$ and $\mu_{\text{resc}} = 0$.

From the above, we can deduce the following uniqueness result.

Corollary 3 (Uniqueness). *Let $N \geq 3$ and $-N < k \leq 2 - N$.*

- (i) *If $m > m_c$ and $\mu_{\text{resc}} = 0$, then there is at most one stationary state of (1.3) for any mass $M > 0$ and any centre of mass. Moreover, it coincides with the global minimiser for \mathcal{F} in \mathcal{Y}_M^* for any $M > 0$ (up to translations), as long as $m_c < m < m_*$, where*

$$m^* := \begin{cases} \frac{2-k-N}{1-k-N}, & \text{if } -N < k < 1 - N, \\ +\infty & \text{if } 1 - N \leq k \leq 2 - N. \end{cases}$$

- (ii) *If $m = m_c$ and $\mu_{\text{resc}} = 1$, then there exists at most one stationary state to (1.3) for any $0 < M < M_c$ and with zero centre of mass. Moreover, it coincides with the global minimiser for \mathcal{F} in \mathcal{Y}_M^* .*
- (iii) *If $m = m_c$ and $\mu_{\text{resc}} = 0$, then there exists at most one stationary state (up to dilations and translations) to (1.3) for the critical mass $M = M_c$. Moreover, it coincides with the global minimiser for \mathcal{F} in $\mathcal{Y}_{M_c}^*$.*

The strategy of the proof for our main result Corollary 3 is to show that all radially symmetric stationary states are in fact global minimisers of \mathcal{F} as stated in Theorem 2. The existence of the global minimiser in the above ranges has been proven in [11, 16].

In Section 2, we set up the necessary notation and take advantage of the radial symmetry to derive an explicit formula for the mean-field interaction potential in terms of hypergeometric functions. Relevant results about hypergeometric functions are summarised in Appendix A. In Section 3, we prove a characterisation of radially symmetric stationary states that allows us to show the functional inequality in Theorem 2 using optimal transport tools. The key ingredient for the proof is a convexity estimate on the radial interaction potential, see Lemma 7. The proof of the convexity estimate is more involved, and postponed to Appendix B for the convenience of the reader. This crucial lemma is the reason we are restricted to the upper bound $2 - N$ in k with this approach of reasoning. We conclude with the proof of Corollary 3, which follows directly from the statement of Theorem 2.

2. POTENTIALS OF RADIAL FUNCTIONS

For any radial function $\rho : \mathbb{R}^N \rightarrow \mathbb{R}$, still denoting by ρ the radial profile of ρ ,

$$\int_{\mathbb{R}^N} \rho(x) dx = \int_0^\infty \int_0^{2\pi} \rho(r) r^{N-1} \sin^{N-2}(\theta) d\theta dr = \sigma_N \int_0^\infty \rho(r) r^{N-1} dr,$$

where $\sigma_N = 2\pi^{(N/2)}/\Gamma(N/2)$ denotes the surface area of the N -dimensional unit ball. Further, if $\delta \geq 1$, then $(\cdot)^\delta$ is convex on the interval $[a, b]$, and by Jensen's inequality

$$(2.1) \quad \left(\int_a^b \rho(r) \frac{Nr^{N-1} dr}{b^N - a^N} \right)^\delta \leq \int_a^b \rho(r)^\delta \frac{Nr^{N-1} dr}{b^N - a^N}$$

with equality if and only if ρ is constant on $[a, b]$.

In what follows, we rewrite the interaction term using polar coordinates.

Lemma 4. For $N \geq 2$ and a given radial function $\rho : \mathbb{R}^N \rightarrow \mathbb{R}$, we have for $|x| = r$

$$|x|^k * \rho(x) = 2^{N-2} \sigma_{N-1} \int_0^\infty (r + \eta)^k H\left(a, b; c; \frac{4r\eta}{(r + \eta)^2}\right) \rho(\eta) d\eta$$

where $H(a, b; c; \cdot)$ is given in terms of hypergeometric functions as defined in (A.2) with

$$a = -\frac{k}{2}, \quad b = \frac{N-1}{2}, \quad c = N-1.$$

Proof. We compute as in [25, Theorem 5], see also [3], [19] or [20, §1.3],

$$|x|^k * \rho(x) = \sigma_{N-1} \int_0^\infty \left(\int_0^\pi (|x|^2 + \eta^2 - 2|x|\eta \cos \theta)^{k/2} \sin^{N-2} \theta d\theta \right) \rho(\eta) \eta^{N-1} d\eta.$$

Let us define

$$\Theta_{k,N}(r, \eta) := \sigma_{N-1} \int_0^\pi (r^2 + \eta^2 - 2r\eta \cos(\theta))^{k/2} \sin^{N-2}(\theta) d\theta = \begin{cases} r^k \vartheta_{k,N}(\eta/r), & \eta < r, \\ \eta^k \vartheta_{k,N}(r/\eta), & r < \eta, \end{cases}$$

where, for $u \in [0, 1)$,

$$\begin{aligned} \vartheta_{k,N}(u) &:= \sigma_{N-1} \int_0^\pi (1 + u^2 - 2u \cos(\theta))^{k/2} \sin^{N-2}(\theta) d\theta \\ &= \sigma_{N-1} (1 + u)^k \int_0^\pi \left(1 - 4 \frac{u}{(1+u)^2} \cos^2\left(\frac{\theta}{2}\right) \right)^{k/2} \sin^{N-2}(\theta) d\theta. \end{aligned}$$

Using the change of variables $t = \cos^2\left(\frac{\theta}{2}\right)$, we get from the integral formulation of hypergeometric functions (A.2),

$$\begin{aligned} \vartheta_{k,N}(u) &= 2^{N-2} \sigma_{N-1} (1 + u)^k \int_0^1 \left(1 - \frac{4u}{(1+u)^2} t \right)^{k/2} t^{\frac{N-3}{2}} (1-t)^{\frac{N-3}{2}} dt \\ &= 2^{N-2} \sigma_{N-1} (1 + u)^k H(a, b; c; z) \end{aligned}$$

with $z = 4u/(1+u)^2$. □

Our goal is to extend the techniques in [12] to higher dimensions in the case of more singular interaction kernels $-N < k \leq 2 - N$. For this purpose we rewrite the interaction functional in radial coordinates. We will make use of the formulation in terms of hypergeometric functions as introduced in Lemma 4.

Lemma 5. *Let $N \geq 2$ and $k > -N$. The attractive mean-field potential rewrites as follows under radial symmetry for $|x| = r$:*

$$(2.2) \quad |x|^k * \rho(x) = r^k \int_{\eta=0}^r \vartheta_{k,N} \left(\frac{\eta}{r} \right) \rho(\eta) \eta^{N-1} d\eta + \int_{\eta=r}^{\infty} \eta^k \vartheta_{k,N} \left(\frac{r}{\eta} \right) \rho(\eta) \eta^{N-1} d\eta,$$

where

$$(2.3) \quad \vartheta_{k,N}(s) = d_N F \left(-\frac{k}{2}, 1 - \frac{k+N}{2}; \frac{N}{2}; s^2 \right), \quad d_N := 2^{N-2} \sigma_{N-1} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}.$$

Proof. As in the proof of Lemma 4,

$$\begin{aligned} |x|^k * \rho(x) &= \int_0^{\infty} \Theta_{k,N}(r, \eta) \rho(\eta) \eta^{N-1} d\eta \\ &= \int_0^r r^k \vartheta_{k,N} \left(\frac{\eta}{r} \right) \rho(\eta) \eta^{N-1} d\eta + \int_r^{\infty} \eta^k \vartheta_{k,N} \left(\frac{r}{\eta} \right) \rho(\eta) \eta^{N-1} d\eta, \end{aligned}$$

and by (A.2), $\vartheta_{k,N}$ can be written as

$$\vartheta_{k,N}(s) = d_N (1+s)^k F(a, b; c; z), \quad d_N := 2^{N-2} \sigma_{N-1} \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)},$$

with

$$a = -\frac{k}{2}, \quad b = \frac{N-1}{2}, \quad c = N-1, \quad z = \frac{4s}{(1+s)^2}.$$

Using the quadratic transformation (A.4), we have for $s \in (0, 1)$

$$F \left(-\frac{k}{2}, \frac{N-1}{2}; N-1; \frac{4s}{(1+s)^2} \right) = (1+s)^{-k} F \left(-\frac{k}{2}, 1 - \frac{k+N}{2}; \frac{N}{2}; s^2 \right),$$

and so (2.2)-(2.3) follow. \square

As a consequence, equation (1.4) rewrites as

$$(2.4) \quad \begin{aligned} \partial_t (r^{N-1} \rho) &= \partial_r (r^{N-1} \partial_r \rho^m) \\ &+ \partial_r \left(r^{N-1} \rho \partial_r \left(\frac{r^k}{k} \int_{s=0}^r \vartheta_{k,N} \left(\frac{s}{r} \right) \rho(s) s^{N-1} ds + \int_{s=r}^{\infty} \frac{s^k}{k} \vartheta_{k,N} \left(\frac{r}{s} \right) \rho(s) s^{N-1} ds \right) \right) \\ &+ \mu_{\text{resc}} \partial_r (r^N \rho). \end{aligned}$$

Similarly, using (2.2), the free energy becomes

$$(2.5) \quad \begin{aligned} \mathcal{F}[\rho] &= \frac{\sigma_N}{m-1} \int_{r=0}^{\infty} \rho(r)^m r^{N-1} dr + \frac{\sigma_N}{2} \int_{r=0}^{\infty} \rho(r) (W_k * \rho)(r) r^{N-1} dr \\ &+ \mu_{\text{resc}} \frac{\sigma_N}{2} \int_{r=0}^{\infty} r^2 \rho(r) r^{N-1} dr \\ &= \frac{\sigma_N}{m-1} \int_{r=0}^{\infty} \rho(r)^m r^{N-1} dr + \frac{\sigma_N}{2} \int_{r=0}^{\infty} \int_{s=0}^r \frac{r^k}{k} \vartheta_{k,N} \left(\frac{s}{r} \right) \rho(r) \rho(s) r^{N-1} s^{N-1} ds dr \\ &+ \frac{\sigma_N}{2} \int_{r=0}^{\infty} \int_{s=r}^{\infty} \frac{s^k}{k} \vartheta_{k,N} \left(\frac{r}{s} \right) \rho(r) \rho(s) r^{N-1} s^{N-1} ds dr \\ &+ \mu_{\text{resc}} \frac{\sigma_N}{2} \int_{r=0}^{\infty} r^2 \rho(r) r^{N-1} dr \\ &= \frac{\sigma_N}{m-1} \int_{r=0}^{\infty} \rho(r)^m r^{N-1} dr + \sigma_N \int_{r=0}^{\infty} \int_{s=0}^r \frac{r^k}{k} \vartheta_{k,N} \left(\frac{s}{r} \right) \rho(r) \rho(s) r^{N-1} s^{N-1} ds dr \\ &+ \mu_{\text{resc}} \frac{\sigma_N}{2} \int_{r=0}^{\infty} r^2 \rho(r) r^{N-1} dr, \end{aligned}$$

where we swapped the order of integration in the second last line.

In the Newtonian case $k = 2 - N$, we can simplify further. We denote by $M_\rho(\cdot)$ the cumulative mass of ρ inside balls,

$$M_\rho(r) = \sigma_N \int_{s=0}^r \rho(s) s^{N-1} ds.$$

Recall that by Newton's Shell theorem [24],

$$\partial_r \left(\frac{r^{2-N}}{2-N} * \rho \right) (r) = r^{1-N} M_\rho(r).$$

Then for the harmonic case $k = 2 - N$, equation (1.1) writes in radial coordinates as

$$\partial_t (r^{N-1} \rho) = \partial_r (r^{N-1} \partial_r \rho^m) + \partial_r (\rho M_\rho) + \mu_{\text{resc}} \partial_r (r^N \rho).$$

We have (cf. the Newton's Theorem, [24, Theorem 9.7]):

$$\sigma_{N-1} \int_{\theta=0}^{\pi} |r^2 + \eta^2 - 2r\eta \cos(\theta)|^{(2-N)/2} \sin(\theta)^{N-2} d\theta = \sigma_N (r \wedge \eta)^{2-N}, \quad r \wedge \eta = \max(r, \eta).$$

Therefore, the interaction term of the energy simplifies and we obtain

$$\begin{aligned} \iint_{\mathbb{R}^N \times \mathbb{R}^N} |x-y|^{2-N} \rho(x) \rho(y) dx dy &= \sigma_N \int_{r=0}^{\infty} \left(\int_{\eta=0}^{\infty} \sigma_N (r \wedge \eta)^{2-N} \rho(\eta) \eta^{N-1} d\eta \right) \rho(r) r^{N-1} dr \\ &= \sigma_N \int_{r=0}^{\infty} \left(\sigma_N r^{2-N} \int_{\eta=0}^r \rho(\eta) \eta^{N-1} d\eta + \sigma_N \int_{\eta=r}^{\infty} \rho(\eta) \eta d\eta \right) \rho(r) r^{N-1} dr \\ &= 2\sigma_N \int_{r=0}^{\infty} \rho(r) M_\rho(r) r dr \end{aligned}$$

by changing the domain of integration in the second term. As a consequence the expression for the energy functional \mathcal{F} in radial coordinates in the case $k = 2 - N$ is

$$(2.6) \quad \mathcal{F}[\rho] = \frac{\sigma_N}{m-1} \int_{r=0}^{\infty} \rho(r)^m r^{N-1} dr + \frac{\sigma_N}{2-N} \int_{r=0}^{\infty} \rho(r) M_\rho(r) r dr + \mu_{\text{resc}} \frac{\sigma_N}{2} \int_{r=0}^{\infty} r^2 \rho(r) r^{N-1} dr.$$

3. FUNCTIONAL INEQUALITY

In the sequel we drop the indices in the notation $\vartheta_{k,N}$ for simplicity, and we write

$$d\bar{r} := \bar{\rho}(r) r^{N-1} dr.$$

Lemma 6 (Characterisation of steady states). *Let $N \geq 2$ and $k > -N$. Then any stationary state $\bar{\rho}$ of the radial system (2.4) can be written in the form*

$$(3.1) \quad \begin{aligned} \bar{\rho}(r)^m &= \int_{s=r}^{\infty} \int_{t=0}^s \left(s^{k-N} \vartheta \left(\frac{t}{s} \right) - \frac{s^{k-1-N}}{k} t \vartheta' \left(\frac{t}{s} \right) \right) d\bar{t} d\bar{s} \\ &\quad + \int_{s=r}^{\infty} \int_{t=s}^{\infty} s^{1-N} \frac{t^{k-1}}{k} \vartheta' \left(\frac{s}{t} \right) d\bar{t} d\bar{s} + \mu_{\text{resc}} \int_{s=r}^{\infty} s^{2-N} d\bar{s}. \end{aligned}$$

In the case $k = 2 - N$, the above simplifies to

$$(3.2) \quad \bar{\rho}^m(r) = \int_{s=r}^{\infty} M_{\bar{\rho}}(s) s^{2-2N} d\bar{s} + \mu_{\text{resc}} \int_{s=r}^{\infty} s^{2-N} d\bar{s}.$$

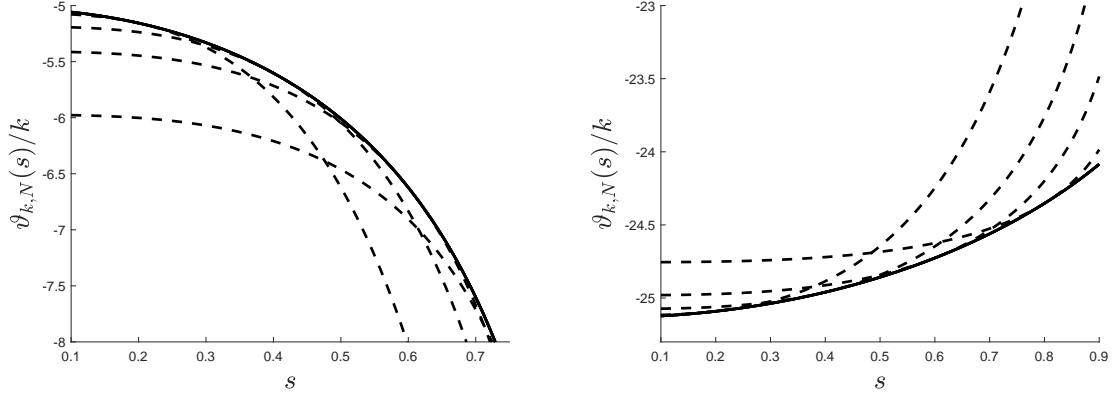


FIGURE 1. Numerical illustration of Lemma 7 for $N = 3$, and (left) $k = -2.5$ or (right) $k = -0.5$ for tangents $c = 0.2, 0.4, 0.6, 0.8$ (dotted lines). For $k < 2 - N$ where convexity holds all tangents lie below the curve $\vartheta_{k,N}(s)/k$ (black line) as shown in Lemma 7, which does not hold for $k > 2 - N$.

Proof. By Definition 1, any stationary state $\bar{\rho}$ satisfies

$$-r^{N-1} \frac{d}{dr} \bar{\rho}(r)^m = \bar{\rho}(r) r^{N-1} \frac{d}{dr} (W_k * \bar{\rho})(r) + \mu_{\text{resc}} r^N \bar{\rho}(r).$$

We deduce that

$$\bar{\rho}(r)^m = \int_{s=r}^{\infty} \bar{\rho}(s) \frac{d}{ds} (W_k * \bar{\rho})(s) ds + \mu_{\text{resc}} \int_{s=r}^{\infty} s^{2-N} \bar{\rho}(s) s^{N-1} ds.$$

It remains to examine the term $(d/ds)(W_k * \bar{\rho})(s)$. We differentiate the expression (2.2) to obtain

$$\frac{d}{ds} (W_k * \bar{\rho})(s) = \int_{t=0}^s \left(s^{k-1} \vartheta \left(\frac{t}{s} \right) - \frac{s^{k-2}}{k} t \vartheta' \left(\frac{t}{s} \right) \right) \bar{\rho}(t) t^{N-1} dt + \int_{t=s}^{\infty} \frac{t^{k-1}}{k} \vartheta' \left(\frac{s}{t} \right) \bar{\rho}(t) t^{N-1} dt.$$

This yields the claimed characterisation. For $k = 2 - N$, the result follows directly from Newton's Shell Theorem, see the end of Section 2. \square

It follows from the above characterisation that for any function $g : [0, \infty) \rightarrow \mathbb{R}$, a stationary state of (1.4) satisfies

$$\begin{aligned} \int_{a=0}^{\infty} g(a) \bar{\rho}^m(a) a^{N-1} da &= \int_{a=0}^{\infty} \int_{b=a}^{\infty} \int_{s=0}^b g(a) a^{N-1} \left[b^{k-N} \vartheta \left(\frac{s}{b} \right) - \frac{b^{k-1-N}}{k} s \vartheta' \left(\frac{s}{b} \right) \right] d\bar{s} d\bar{b} da \\ (3.3) \quad &+ \int_{a=0}^{\infty} \int_{b=a}^{\infty} \int_{s=b}^{\infty} g(a) a^{N-1} b^{1-N} \frac{s^{k-1}}{k} \vartheta' \left(\frac{b}{s} \right) d\bar{s} d\bar{b} da \\ &+ \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{b=a}^{\infty} g(a) a^{N-1} b^{2-N} d\bar{b} da. \end{aligned}$$

This expression will be useful for proving the functional inequality in Theorem 2. Moreover, in order to prove Theorem 2, we seek an inequality of the following type:

$$\frac{\vartheta_{k,N}(s)}{k} \geq \alpha + \beta (1 - s^N)^{k/N}.$$

The constants α and β are chosen so that the above inequality is an equality at zero and first order for a convenient choice of s (to be chosen later). This writes into the following lemma:

Lemma 7. *Assume $N \geq 2$ and $k \in (-N, 2 - N)$. The following inequality holds true for any $(s, c) \in (0, 1)^2$:*

$$(3.4) \quad \frac{\vartheta_{k,N}(s)}{k} \geq \alpha(c) + \beta(c) (1 - s^N)^{k/N} .$$

with the two factors given by

$$\alpha(c) := \frac{\vartheta_{k,N}(c)}{k} + \frac{1}{k^2} c^{1-N} (1 - c^N) \vartheta'_{k,N}(c) \leq 0 ,$$

and

$$\beta(c) := -\frac{1}{k^2} c^{1-N} (1 - c^N)^{1-k/N} \vartheta'_{k,N}(c) \leq 0 .$$

Note that this crucial lemma is the reason we are restricted to the upper bound $2 - N$ in k , see Figure 1. The proof is postponed to the Appendix B due to technicality.

Proof of Theorem 2. We begin with the more complicated case $k \in (-N, 2 - N)$ as the harmonic case $k = 2 - N$ will follow in a similar manner. The energy of the stationary state $\bar{\rho}$ is given by

$$\frac{1}{N\sigma_N} \mathcal{F}[\bar{\rho}] = \frac{1}{N(m-1)} \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} da + \frac{1}{Nk} \int_{a=0}^{\infty} \int_{b=0}^a a^k \vartheta \left(\frac{b}{a} \right) d\bar{b} d\bar{a} + \frac{\mu_{\text{resc}}}{2N} \int_{a=0}^{\infty} a^2 d\bar{a} .$$

Choosing $g = id$ in (3.3) and rewriting the domain of integration, we obtain

$$\begin{aligned} & \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} da \\ &= \int_{a=0}^{\infty} \int_{b=a}^{\infty} \int_{s=0}^b \left[b^{k-N} \vartheta \left(\frac{s}{b} \right) - \frac{b^{k-1-N}}{k} s \vartheta' \left(\frac{s}{b} \right) \right] a^{N-1} d\bar{s} d\bar{b} da \\ &+ \frac{1}{k} \int_{a=0}^{\infty} \int_{b=a}^{\infty} \int_{s=b}^{\infty} b^{1-N} s^{k-1} \vartheta' \left(\frac{b}{s} \right) a^{N-1} d\bar{s} d\bar{b} da + \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{b=a}^{\infty} a^{N-1} b^{2-N} d\bar{b} da \\ &= \int_{b=0}^{\infty} \int_{s=0}^b \left(\int_{a=0}^b a^{N-1} da \right) \left[b^{k-N} \vartheta \left(\frac{s}{b} \right) - \frac{b^{k-1-N}}{k} s \vartheta' \left(\frac{s}{b} \right) \right] d\bar{s} d\bar{b} \\ &+ \frac{1}{k} \int_{b=0}^{\infty} \int_{s=b}^{\infty} \left(\int_{a=0}^b a^{N-1} da \right) b^{1-N} s^{k-1} \vartheta' \left(\frac{b}{s} \right) d\bar{s} d\bar{b} + \mu_{\text{resc}} \int_{b=0}^{\infty} \left(\int_{a=0}^b a^{N-1} da \right) b^{2-N} d\bar{b} \\ &= \frac{1}{N} \left(\int_{b=0}^{\infty} \int_{s=0}^b b^k \vartheta \left(\frac{s}{b} \right) d\bar{s} d\bar{b} + \mu_{\text{resc}} \int_{b=0}^{\infty} b^2 d\bar{b} \right) . \end{aligned}$$

Hence,

$$\frac{1}{Nk} \int_{a=0}^{\infty} \int_{b=0}^a a^k \vartheta \left(\frac{b}{a} \right) d\bar{b} d\bar{a} = \frac{1}{k} \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} da - \frac{\mu_{\text{resc}}}{kN} \int_{a=0}^{\infty} a^2 d\bar{a} ,$$

and so we conclude

$$(3.5) \quad \frac{1}{N\sigma_N} \mathcal{F}[\bar{\rho}] = \int_{a=0}^{\infty} \left(\frac{1}{N(m-1)} + \frac{1}{k} \right) \bar{\rho}(a)^m a^{N-1} da + \frac{\mu_{\text{resc}}}{N} \left(\frac{1}{2} - \frac{1}{k} \right) \int_{a=0}^{\infty} a^2 d\bar{a} .$$

Next, we write the energy $\mathcal{F}[\rho]$ in terms of $\bar{\rho}$ and our goal is to find suitable estimates from below. For a given stationary state $\bar{\rho} \in \mathcal{Y}_M^*$ and solution $\rho \in \mathcal{Y}_M^*$ of (2.4), we denote by ψ the radial profile of the convex function whose gradient pushes forward the measure $\bar{\rho}(a) a^{N-1} da$ onto $\rho(r) r^{N-1} dr$:

$$\psi' \# (\bar{\rho}(a) a^{N-1} da) = \rho(r) r^{N-1} dr .$$

Changing variables $r = \psi'(a)$, we have

$$\rho(r) r^{N-1} dr = \bar{\rho}(a) a^{N-1} da =: d\bar{a}$$

and

$$\rho(r) = \frac{a^{N-1}\bar{\rho}(a)}{\psi'(a)^{N-1}\psi''(a)} = \frac{\bar{\rho}(a)}{\varphi(a)}, \quad \text{for } \varphi(a) := \frac{1}{Na^{N-1}} \frac{d}{da} (\psi')^N(a).$$

Then the repulsive term of the functional $\mathcal{F}[\rho]$ rewrites

$$\frac{\sigma_N}{m-1} \int_{r=0}^{\infty} \rho(r)^m r^{N-1} dr = \frac{\sigma_N}{m-1} \int_{a=0}^{\infty} \varphi(a)^{1-m} \bar{\rho}(a)^m a^{N-1} da,$$

and following (2.5), the interaction term becomes

$$\sigma_N \int_{r=0}^{\infty} \int_{s=0}^r \frac{r^k}{k} \vartheta\left(\frac{s}{r}\right) \rho(r)\rho(s)r^{N-1}s^{N-1} ds dr = \frac{\sigma_N}{k} \int_{a=0}^{\infty} \int_{b=0}^a (\psi'(a))^k \vartheta\left(\frac{\psi'(b)}{\psi'(a)}\right) d\bar{b}d\bar{a}.$$

We therefore have

$$(3.6) \quad \frac{1}{N\sigma_N} \mathcal{F}[\rho] = \frac{1}{N(m-1)} \int_{a=0}^{\infty} \varphi(a)^{1-m} \bar{\rho}(a)^m a^{N-1} da \\ + \frac{1}{Nk} \int_{a=0}^{\infty} \int_{b=0}^a (\psi'(a))^k \vartheta\left(\frac{\psi'(b)}{\psi'(a)}\right) d\bar{b}d\bar{a} + \frac{\mu_{\text{resc}}}{2N} \int_{a=0}^{\infty} (\psi'(a))^2 d\bar{a}.$$

By Jensen's inequality (2.1), we estimate

$$(3.7) \quad (\psi'(a))^k = a^k \left(\frac{\psi'(a)^N}{a^N} \right)^{k/N} = a^k \left(\int_{s=0}^a \varphi(s) \frac{Ns^{N-1}}{a^N} ds \right)^{k/N} \\ \leq Na^{k-N} \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} ds,$$

$$(3.8) \quad (\psi'(a)^N - \psi'(b)^N)^{k/N} = \left(\frac{\psi'(a)^N - \psi'(b)^N}{a^N - b^N} \right)^{k/N} (a^N - b^N)^{k/N} \\ = (a^N - b^N)^{k/N} \left(\int_{s=b}^a \varphi(s) \frac{Ns^{N-1}}{(a^N - b^N)} ds \right)^{k/N} \\ \leq N(a^N - b^N)^{k/N-1} \int_{s=b}^a \varphi(s)^{k/N} s^{N-1} ds.$$

Using first the comparison equation (3.4) in Lemma 7 with $c = b/a$ and then (3.7)–(3.8), we obtain the estimate

$$\frac{1}{Nk} \int_{a=0}^{\infty} \int_{b=0}^a (\psi'(a))^k \vartheta\left(\frac{\psi'(b)}{\psi'(a)}\right) d\bar{b}d\bar{a} \\ \geq \frac{1}{N} \int_{a=0}^{\infty} \int_{b=0}^a (\psi'(a))^k \alpha\left(\frac{b}{a}\right) d\bar{b}d\bar{a} + \frac{1}{N} \int_{a=0}^{\infty} \int_{b=0}^a \beta\left(\frac{b}{a}\right) ((\psi'(a))^N - (\psi'(b))^N)^{k/N} d\bar{b}d\bar{a} \\ \geq \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \alpha\left(\frac{b}{a}\right) ds d\bar{b}d\bar{a} \\ + \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=b}^a (a^N - b^N)^{k/N-1} \varphi(s)^{k/N} s^{N-1} \beta\left(\frac{b}{a}\right) ds d\bar{b}d\bar{a} \\ = \frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \left(\vartheta\left(\frac{b}{a}\right) + \frac{1}{k} \left(\frac{b}{a}\right)^{1-N} \left(1 - \left(\frac{b}{a}\right)^N\right) \vartheta'\left(\frac{b}{a}\right) \right) ds d\bar{b}d\bar{a} \\ - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=b}^a (a^N - b^N)^{k/N-1} \varphi(s)^{k/N} s^{N-1} \left(\frac{b}{a}\right)^{1-N} \left(1 - \left(\frac{b}{a}\right)^N\right)^{1-k/N} \vartheta'\left(\frac{b}{a}\right) ds d\bar{b}d\bar{a}$$

$$\begin{aligned}
&= \frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \vartheta \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad + \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \left(\frac{b}{a} \right)^{1-N} \left(1 - \left(\frac{b}{a} \right)^N \right) \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=b}^a (a^N - b^N)^{k/N-1} \varphi(s)^{k/N} s^{N-1} \left(\frac{b}{a} \right)^{1-N} \left(1 - \left(\frac{b}{a} \right)^N \right)^{1-k/N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a}.
\end{aligned}$$

Note that the signs in (3.7)–(3.8) flip since $\alpha \left(\frac{b}{a} \right)$ and $\beta \left(\frac{b}{a} \right)$ are non-positive.

The above simplifies to

$$\begin{aligned}
&\frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \vartheta \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad + \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{k-N-1} b^{1-N} (a^N - b^N) \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=b}^a \varphi(s)^{k/N} s^{N-1} b^{1-N} a^{k-1} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&= \frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \vartheta \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad + \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{k-1} b^{1-N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{k-N-1} b \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=b}^a \varphi(s)^{k/N} s^{N-1} a^{k-1} b^{1-N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&= \frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a a^{k-N} \varphi(s)^{k/N} s^{N-1} \vartheta \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad + \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^b \varphi(s)^{k/N} s^{N-1} a^{k-1} b^{1-N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&\quad - \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{k-N-1} b \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&= \frac{1}{k} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} \left[a^{k-N} \vartheta \left(\frac{b}{a} \right) - \frac{a^{k-N-1}}{k} b \vartheta' \left(\frac{b}{a} \right) \right] ds d\bar{b} d\bar{a} \\
&\quad + \frac{1}{k^2} \int_{a=0}^{\infty} \int_{b=0}^a \int_{s=0}^b \varphi(s)^{k/N} s^{N-1} a^{k-1} b^{1-N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a}.
\end{aligned}$$

Next, we make use of the characterisation (3.3) of stationary states for $g(a) = \varphi(a)^{k/N}$, then writing $\int_{a=0}^{\infty} \int_{s=a}^{\infty} = \int_{s=0}^{\infty} \int_{a=0}^s$ and exchanging a and s , we have

$$\begin{aligned}
&\int_{a=0}^{\infty} \varphi(a)^{1-m_c} \bar{\rho}(a)^m a^{N-1} da \\
&= \int_{a=0}^{\infty} \int_{s=a}^{\infty} \int_{b=0}^s \varphi(a)^{k/N} a^{N-1} \left[s^{k-N} \vartheta \left(\frac{b}{s} \right) - \frac{s^{k-N-1}}{k} b \vartheta' \left(\frac{b}{s} \right) \right] da d\bar{b} d\bar{s} \\
&\quad + \int_{a=0}^{\infty} \int_{s=a}^{\infty} \int_{b=s}^{\infty} \varphi(a)^{k/N} a^{N-1} \frac{b^{k-1}}{k} s^{1-N} \vartheta' \left(\frac{s}{b} \right) da d\bar{b} d\bar{s} + \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{s=a}^{\infty} \varphi(a)^{k/N} a^{N-1} s^{2-N} da d\bar{s}
\end{aligned}$$

$$\begin{aligned}
&= \int_{a=0}^{\infty} \int_{s=0}^a \int_{b=0}^a \varphi(s)^{k/N} s^{N-1} \left[a^{k-N} \vartheta \left(\frac{b}{a} \right) - \frac{a^{k-N-1}}{k} b \vartheta' \left(\frac{b}{a} \right) \right] ds d\bar{b} d\bar{a} \\
&\quad + \int_{a=0}^{\infty} \int_{s=0}^a \int_{b=a}^{\infty} \varphi(s)^{k/N} s^{N-1} \frac{b^{k-1}}{k} a^{1-N} \vartheta' \left(\frac{a}{b} \right) ds d\bar{b} d\bar{a} + \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{2-N} ds d\bar{a}.
\end{aligned}$$

Writing $\int_{a=0}^{\infty} \int_{b=a}^{\infty} = \int_{b=0}^{\infty} \int_{a=0}^b$ in the second term only and exchanging a and b , we conclude

$$\begin{aligned}
&\int_{a=0}^{\infty} \int_{s=0}^a \int_{b=0}^a \varphi(s)^{k/N} s^{N-1} \left[a^{k-N} \vartheta \left(\frac{b}{a} \right) - \frac{a^{k-N-1}}{k} b \vartheta' \left(\frac{b}{a} \right) \right] ds d\bar{b} d\bar{a} \\
&\quad + \int_{a=0}^{\infty} \int_{s=0}^a \int_{b=0}^a \varphi(s)^{k/N} s^{N-1} \frac{a^{k-1}}{k} b^{1-N} \vartheta' \left(\frac{b}{a} \right) ds d\bar{b} d\bar{a} \\
&= \int_{a=0}^{\infty} \varphi(a)^{1-m_c} \bar{\rho}(a)^m a^{N-1} da - \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{2-N} ds d\bar{a}.
\end{aligned}$$

Combining with our above estimate, we obtain the following lower bound on the interaction term:

$$\begin{aligned}
&\frac{1}{Nk} \int_{a=0}^{\infty} \int_{b=0}^a (\psi'(a))^k \vartheta \left(\frac{\psi'(b)}{\psi'(a)} \right) d\bar{b} d\bar{a} \\
&\geq \frac{1}{k} \int_{a=0}^{\infty} \varphi(a)^{1-m_c} \bar{\rho}(a)^m a^{N-1} da - \frac{\mu_{\text{resc}}}{k} \int_{a=0}^{\infty} \int_{s=0}^a \varphi(s)^{k/N} s^{N-1} a^{2-N} ds d\bar{a}.
\end{aligned}$$

Next, we estimate the confinement term using Jensen's inequality (2.1) as in (3.7),

$$\begin{aligned}
(\psi'(a))^2 &= a^2 \left(\frac{\psi'(a)^N}{a^N} \right)^{2/N} = a^2 \left(\int_{s=0}^a \varphi(s) \frac{N s^{N-1}}{a^N} ds \right)^{2/N} \\
(3.9) \quad &\geq N a^{2-N} \int_{s=0}^a \varphi(s)^{2/N} s^{N-1} ds.
\end{aligned}$$

Substituting these estimates into (3.6), we obtain

$$\begin{aligned}
\frac{1}{N\sigma_N} \mathcal{F}[\rho] &\geq \frac{1}{N} \int_{a=0}^{\infty} \left(\frac{\varphi(a)^{1-m}}{m-1} - \frac{\varphi(a)^{1-m_c}}{m_c-1} \right) \bar{\rho}(a)^m a^{N-1} da \\
&\quad + \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{s=0}^a \left(\frac{\varphi(s)^{2/N}}{2} - \frac{\varphi(s)^{k/N}}{k} \right) s^{N-1} a^{2-N} ds d\bar{a}.
\end{aligned}$$

Finally, we make use of the inequalities

$$(3.10) \quad \frac{z^{1-m}}{m-1} - \frac{z^{1-m_c}}{m_c-1} \geq \frac{1}{m-1} - \frac{1}{m_c-1} \quad z \geq 0, \quad m \geq m_c,$$

$$(3.11) \quad \frac{z^{2/N}}{2} - \frac{z^{k/N}}{k} \geq \frac{1}{2} - \frac{1}{k} \quad z \geq 0, \quad k < 0.$$

From the formulation (3.5), we conclude

$$\frac{1}{N\sigma_N} \mathcal{F}[\rho] \geq \frac{1}{N\sigma_N} \mathcal{F}[\bar{\rho}].$$

Equality in Jensen's inequality (2.1) arises if and only if the derivative of the transport map, ψ'' , is a constant function, i.e. when ρ is a dilation of $\bar{\rho}$. In agreement with this, equality in (3.10)–(3.11) is realised if and only if $z = 1$, that is, $\rho = \bar{\rho}$. We conclude that equality in the functional inequality in Theorem 2 is realised if and only if $\rho = \bar{\rho}$, unless $m = m_c$ and $\mu_{\text{resc}} = 0$, in which case the equality cases correspond to dilations of $\bar{\rho}$.

For the harmonic case $k = 2 - N$, the above estimates corresponding to the interaction term of the energy simplify. First of all, the interaction energy of the stationary state can be rewritten as follows using the characterisation (3.2) provided in Lemma 6:

$$\begin{aligned}
& \frac{\sigma_N}{2-N} \int_{a=0}^{\infty} \bar{\rho}(a) M_{\bar{\rho}}(a) a \, da \\
&= \frac{N\sigma_N}{2-N} \int_{a=0}^{\infty} \left(\int_{b=0}^a b^{N-1} \, db \right) a^{1-N} \bar{\rho}(a) M_{\bar{\rho}}(a) \, da \\
&= \frac{N\sigma_N}{2-N} \int_{b=0}^{\infty} \left(\int_{a=b}^{\infty} a^{1-N} \bar{\rho}(a) M_{\bar{\rho}}(a) \, da \right) b^{N-1} \, db \\
&= \frac{N\sigma_N}{2-N} \int_{b=0}^{\infty} \bar{\rho}(b)^m b^{N-1} \, db - \mu_{\text{resc}} \frac{N\sigma_N}{2-N} \int_{b=0}^{\infty} \left(\int_{a=b}^{\infty} a^{2-N} \, d\bar{a} \right) b^{N-1} \, db \\
&= \frac{N\sigma_N}{2-N} \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} \, da - \mu_{\text{resc}} \frac{N\sigma_N}{2-N} \int_{a=0}^{\infty} \left(\int_{b=0}^a b^{N-1} \, db \right) a^{2-N} \, d\bar{a} \\
&= \frac{N\sigma_N}{2-N} \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} \, da - \mu_{\text{resc}} \frac{\sigma_N}{2-N} \int_{a=0}^{\infty} a^2 \, d\bar{a}.
\end{aligned}$$

Substituting into (2.6), we obtain the following expression for the free energy of the stationary states:

$$\frac{1}{N\sigma_N} \mathcal{F}[\bar{\rho}] = \left(\frac{1}{N(m-1)} + \frac{1}{2-N} \right) \int_{a=0}^{\infty} \bar{\rho}(a)^m a^{N-1} \, da + \frac{\mu_{\text{resc}}}{N} \left(\frac{1}{2} - \frac{1}{2-N} \right) \int_{a=0}^{\infty} a^2 \, d\bar{a}.$$

Next, we estimate the interaction term of the free energy for ρ using (3.7) for $k = 2 - N$,

$$\begin{aligned}
\int_{r=0}^{\infty} \rho(r) M_{\rho}(r) r \, dr &= \int_{a=0}^{\infty} M_{\bar{\rho}}(a) (\psi'(a))^{2-N} \, d\bar{a} \\
&\leq N \int_{a=0}^{\infty} \int_{b=0}^a M_{\bar{\rho}}(a) \varphi(b)^{(2-N)/N} b^{N-1} a^{2-2N} \, db d\bar{a} \\
&= N \int_{b=0}^{\infty} \varphi(b)^{(2-N)/N} \left\{ \int_{a=b}^{\infty} M_{\bar{\rho}}(a) a^{2-2N} \, d\bar{a} \right\} b^{N-1} \, db.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\int_{r=0}^{\infty} \rho(r) M_{\rho}(r) r \, dr &= N \int_{b=0}^{\infty} \varphi(b)^{(2-N)/N} \bar{\rho}(b)^m b^{N-1} \, db \\
&\quad - N \mu_{\text{resc}} \int_{b=0}^{\infty} \varphi(b)^{(2-N)/N} \left\{ \int_{a=b}^{\infty} a^{2-N} \, d\bar{a} \right\} b^{N-1} \, db
\end{aligned}$$

since $M_{\rho}(r) = M_{\bar{\rho}}(a)$ and where we used the characterisation (3.2) of stationary states provided in Lemma 6. Estimating the confinement term as in (3.9), we obtain the following estimate on the free energy as given in (2.6):

$$\begin{aligned}
\frac{1}{N\sigma_N} \mathcal{F}[\rho] &\geq \int_{a=0}^{\infty} \left(\frac{\varphi(a)^{1-m}}{N(m-1)} + \frac{\varphi(a)^{(2-N)/N}}{2-N} \right) \bar{\rho}(a)^m a^{N-1} \, da \\
&\quad + \mu_{\text{resc}} \int_{a=0}^{\infty} \int_{b=0}^a \left(\frac{\varphi(b)^{2/N}}{2} - \frac{\varphi(b)^{(2-N)/N}}{2-N} \right) a^{2-N} b^{N-1} \, db d\bar{a}.
\end{aligned}$$

We conclude as before using (3.10)–(3.11). \square

Theorem 2 directly implies Corollary 3:

Proof of Corollary 3. Assume there are two radial stationary states to equation (1.4) with the same mass M : $\bar{\rho}_1, \bar{\rho}_2 \in \mathcal{Y}_M^*$. Then Theorem 2 implies that $\mathcal{F}[\bar{\rho}_1] = \mathcal{F}[\bar{\rho}_2]$, and so $\bar{\rho}_1 = \bar{\rho}_2$ (up to dilations if $m = m_c$, $\mu_{\text{resc}} = 0$ and $M = M_c$). \square

APPENDIX A. PROPERTIES OF HYPERGEOMETRIC FUNCTIONS

In this work, we are making frequent use of the fact that the Riesz potential of a radial function can be expressed in terms of the Gauss Hypergeometric Function,

$$(A.1) \quad F(a, b; c; z) := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

which we define for $z \in (-1, 1)$, with parameters a, b, c being positive. Here $(q)_n$ denotes the Pochhammer symbol,

$$(q)_n := q(q+1) \cdots (q+n-1) \text{ if } n > 0 \quad \text{and} \quad (q)_n = 1 \text{ if } n = 0.$$

In our context, the following analytical continuation allows to establish the link with the Riesz potential,

$$F(a, b; c; z) := \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 (1-zt)^{-a} (1-t)^{c-b-1} t^{b-1} dt,$$

Notice that $F(a, b, c, 0) = 1$ and F is increasing with respect to $z \in (-1, 1)$. Moreover, if $c > 1$, $b > 1$ and $c > a + b$, the limit as $z \uparrow 1$ is finite and it takes the value

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)},$$

see [23, §9.3]. To simplify notation, let us define

$$(A.2) \quad H(a, b; c; z) := \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} F(a, b; c; z) = \int_0^1 (1-zt)^{-a} (1-t)^{c-b-1} t^{b-1} dt.$$

We will also make use of some elementary relations. Firstly, the derivative of F in z is given by [1, 15.2.1]

$$(A.3) \quad \frac{d}{dz} F(a, b; c; z) = \frac{ab}{c} F(a+1, b+1, c+1, z).$$

Further, the following quadratic transformation holds true for hypergeometric functions [1, Formula 15.3.17]:

$$(A.4) \quad F(a, b; 2b; z) = \left(\frac{1}{2} + \frac{1}{2} \sqrt{1-z} \right)^{-2a} F \left(a, a-b + \frac{1}{2}; b + \frac{1}{2}; \left(\frac{1 - \sqrt{1-z}}{1 + \sqrt{1-z}} \right)^2 \right).$$

Finally, we will make use of the following two identities [1, 15.2.18 and 15.2.17],

$$(A.5) \quad (c-a-b)F(a, b; c; z) - (c-a)F(a-1, b; c; z) + b(1-z)F(a, b+1; c; z) = 0,$$

and

$$(A.6) \quad (c-a-1)F(a, b; c; z) + aF(a+1, b; c; z) - (c-1)F(a, b; c-1; z) = 0.$$

APPENDIX B. PROOF OF LEMMA 7

In this appendix we give a complete proof of Lemma 7. The case $N = 2$ will be treated separately, and we present here two different proofs. The first one is in the same spirit as for higher dimensions and uses the integral representation (A.2) to motivate inequality (3.4), a simple consequence of Jensen's inequality. The second proof is much shorter and follows from a simple convexity argument, however, it cannot be generalised to higher dimensions up to our knowledge.

Proof 1 of Lemma 7 in dimension $N = 2$. We have $d_2 = 2\pi$, and so

$$(B.1) \quad \begin{aligned} \vartheta_{k,2}(t) &= 2\pi F\left(-\frac{k}{2}, -\frac{k}{2}, 1, t^2\right) = \Gamma_k H\left(-\frac{k}{2}, -\frac{k}{2}, 1, t^2\right) \\ &= \Gamma_k \int_{u=0}^1 u^{-k/2-1} (1-u)^{k/2} (1-t^2u)^{k/2} du, \end{aligned}$$

where

$$\Gamma_k := \frac{2\pi}{\Gamma(-k/2)\Gamma(1+k/2)}.$$

We aim at decoupling t and u , which is a crucial step in the forthcoming estimates. By convexity of $(\cdot)^{k/2}$,

$$\begin{aligned} [1-u+(1-t^2)u]^{k/2} &= \left[\alpha \frac{1-u}{\alpha} + (1-\alpha) \frac{(1-t^2)u}{(1-\alpha)} \right]^{k/2} \\ &\leq \alpha \left[\frac{1-u}{\alpha} \right]^{k/2} + (1-\alpha) \left[\frac{(1-t^2)u}{(1-\alpha)} \right]^{k/2}, \end{aligned}$$

where the coefficient α is chosen such that equality arises for $t = c$:

$$\frac{1-u}{\alpha} = \frac{(1-c^2)u}{(1-\alpha)}, \quad \alpha = \frac{1-u}{1-c^2u}, \quad 1-\alpha = \frac{(1-c^2)u}{1-c^2u}.$$

Therefore we have:

$$\begin{aligned} &\int_{u=0}^1 u^{-k/2-1} (1-u)^{k/2} (1-t^2u)^{k/2} du \\ &\leq \int_{u=0}^1 u^{-k/2-1} (1-u)^{k/2} \left\{ (1-u)(1-c^2u)^{k/2-1} + (1-c^2)^{1-k/2} u(1-c^2u)^{k/2-1} (1-t^2)^{k/2} \right\} du \\ &\leq \int_{u=0}^1 u^{-k/2-1} (1-u)^{k/2+1} (1-c^2u)^{k/2-1} du \\ &\quad + (1-c^2)^{1-k/2} (1-t^2)^{k/2} \int_{u=0}^1 u^{-k/2} (1-u)^{k/2} (1-c^2u)^{k/2-1} du \\ &= H\left(-\frac{k}{2}+1, -\frac{k}{2}, 2, c^2\right) + (1-c^2)^{1-k/2} (1-t^2)^{k/2} H\left(-\frac{k}{2}+1, -\frac{k}{2}+1, 2, c^2\right). \end{aligned}$$

To rewrite H in terms of the hypergeometric function F , recall that $\Gamma(z+1)/\Gamma(z) = z$ for any $z \in \mathbb{C}$ that is not an integer less or equal to zero, and so we have $\Gamma(-\frac{k}{2}+1)/\Gamma(-\frac{k}{2}) = -\frac{k}{2}$, and for $k \neq -2n$, $n \in \mathbb{N}_{>0}$,

$$\frac{\Gamma\left(2+\frac{k}{2}\right)}{\Gamma\left(1+\frac{k}{2}\right)} = 1 + \frac{k}{2}.$$

It follows that whenever $k \neq -2n$, $n \in \mathbb{N}_{>0}$,

$$\begin{aligned}
\vartheta_{k,2}(t) &\leq \Gamma_k H\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) + (1 - c^2)^{1-k/2} (1 - t^2)^{k/2} \Gamma_k H\left(-\frac{k}{2} + 1, -\frac{k}{2} + 1, 2, c^2\right) \\
&= 2\pi \left(1 + \frac{k}{2}\right) F\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) \\
\text{(B.2)} \quad &- \pi k (1 - c^2)^{1-k/2} (1 - t^2)^{k/2} F\left(-\frac{k}{2} + 1, -\frac{k}{2} + 1, 2, c^2\right).
\end{aligned}$$

We have on the one hand from (A.3),

$$\text{(B.3)} \quad \vartheta'_{k,2}(c) = \pi c k^2 F\left(-\frac{k}{2} + 1, -\frac{k}{2} + 1, 2, c^2\right).$$

On the other hand, we deduce from (A.5) and (A.6),

$$\begin{aligned}
(1 + k)F\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) &= \frac{k}{2}(1 - c^2)F\left(-\frac{k}{2} + 1, -\frac{k}{2} + 1, 2, c^2\right) \\
&\quad + \left(1 + \frac{k}{2}\right)F\left(-\frac{k}{2}, -\frac{k}{2}, 2, c^2\right), \\
\left(1 + \frac{k}{2}\right)F\left(-\frac{k}{2}, -\frac{k}{2}, 2, c^2\right) &= \frac{k}{2}F\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) + F\left(-\frac{k}{2}, -\frac{k}{2}, 1, c^2\right),
\end{aligned}$$

that

$$\left(1 + \frac{k}{2}\right)F\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) = \frac{k}{2}(1 - c^2)F\left(-\frac{k}{2} + 1, -\frac{k}{2} + 1, 2, c^2\right) + F\left(-\frac{k}{2}, -\frac{k}{2}, 1, c^2\right).$$

Combining the above and (B.1), the last identity rewrites as

$$\text{(B.4)} \quad \left(1 + \frac{k}{2}\right)F\left(-\frac{k}{2} + 1, -\frac{k}{2}, 2, c^2\right) = \frac{(1 - c^2)}{2\pi c k} \vartheta'_{k,2}(c) + \frac{1}{2\pi} \vartheta_{k,2}(c).$$

The two relations (B.3) and (B.4) applied to (B.2) complete the proof of (3.4) in dimension $N = 2$. \square

Alternatively, the special case $N = 2$, $c = b/a$ and $t = \psi'(b)/\psi'(a)$ for $b < a$ can be shown by a simple convexity argument:

Proof 2 of Lemma 7 in dimension $N = 2$. Since $(\cdot)^{k/2}$ is convex, we have directly from the definition of a convex function

$$\begin{aligned}
(\psi'(a)^2 - \psi'(b)^2 u)^{k/2} &= \left((1 - u)a^2 \frac{\psi'(a)^2}{a^2} + u(a^2 - b^2) \left(\frac{\psi'(a)^2 - \psi'(b)^2}{a^2 - b^2} \right) \right)^{k/2} \\
&\leq (1 - u)a^2 (a^2 - b^2 u)^{k/2 - 1} \left(\frac{\psi'(a)^2}{a^2} \right)^{k/2} \\
\text{(B.5)} \quad &+ (a^2 - b^2)u (a^2 - b^2 u)^{k/2 - 1} \left(\frac{\psi'(a)^2 - \psi'(b)^2}{a^2 - b^2} \right)^{k/2}
\end{aligned}$$

Writing $\vartheta_{k,2}(T)$ for $T = \psi'(b)/\psi'(a)$ explicitly in terms of the hypergeometric function F , we have from (B.1) and (B.5):

$$\begin{aligned}
\vartheta_{k,2}(T) &= \Gamma_k \psi'(a)^{-k} \int_0^1 u^{-k/2-1} (1-u)^{k/2} (\psi'(a)^2 - \psi'(b)^2 u)^{k/2} du \\
&\leq \Gamma_k \int_0^1 u^{-k/2-1} (1-u)^{k/2+1} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du \\
&\quad + \Gamma_k \psi'(a)^{-k} \int_0^1 u^{-k/2} (1-u)^{k/2} (a^2 - b^2)^{1-k/2} a^{k-2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} (\psi'(a)^2 - \psi'(b)^2 u)^{k/2} du \\
&= \Gamma_k \int_0^1 u^{-k/2-1} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du \\
&\quad - \Gamma_k \int_0^1 u^{-k/2} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du \\
&\quad + (a^2 - b^2)^{1-k/2} a^{k-2} \left(\Gamma_k \int_0^1 u^{-k/2} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du \right) (1-T^2)^{k/2}.
\end{aligned}$$

Note that

$$\begin{aligned}
\vartheta_{k,2}\left(\frac{b}{a}\right) &= \Gamma_k \int_0^1 u^{-k/2-1} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2} du, \\
\vartheta'_{k,2}\left(\frac{b}{a}\right) &= -\frac{bk}{a} \left(\Gamma_k \int_0^1 u^{-k/2} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du \right), \\
\vartheta_{k,2}\left(\frac{b}{a}\right) - \frac{b}{ak} \vartheta'_{k,2}\left(\frac{b}{a}\right) &= \Gamma_k \int_0^1 u^{-k/2-1} (1-u)^{k/2} \left(1 - \frac{b^2}{a^2} u\right)^{k/2-1} du,
\end{aligned}$$

and so the result is immediate. \square

In higher dimension $N \geq 3$ the proof becomes more involved. As we are not aware of any suitable inequality involving hypergeometric functions, we argue directly from the representation using series (A.1). Further, we will make use of relative convexity properties defined as follows:

Definition 8 (Relative convexity). *Let g and φ be C^2 functions defined on some interval $I \subset \mathbb{R}$. We say that g is convex relatively to φ if and only if the following convexity-like inequality holds true:*

$$\forall (t, c) \in I^2 \quad g(t) \geq \alpha + \beta \varphi(t),$$

where α and β are chosen in order to fulfill zeroth and first-order approximation at $t = c$:

$$\begin{aligned}
\alpha &= g(c) - \frac{g'(c)}{\varphi'(c)} \varphi(c), \\
\beta &= \frac{g'(c)}{\varphi'(c)}.
\end{aligned}$$

In other words, the function $g \circ \varphi^{-1}$ is convex.

A straightforward computation shows that $g \circ \varphi^{-1}$ is convex if and only if the following criterion is valid:

$$\forall t \in (0, 1) \quad g''(t) \geq \frac{\varphi''(t)}{\varphi'(t)} g'(t).$$

Proof of Lemma 7 in dimension $N \geq 3$. By Definition 8, Lemma 7 states that the function $\vartheta(t)/k$ with $\vartheta(\cdot) = \vartheta_{k,N}(\cdot)$ as defined in (2.3) is convex relatively to $(1 - t^N)^{k/N}$. Or alternatively, the function $g(z)$ defined by

$$g(z) := \frac{d_N}{k} F(\bar{a}, \bar{b}; \bar{c}; z) = \frac{d_N}{k} \sum_{n=0}^{\infty} \frac{(\bar{a})_n (\bar{b})_n}{(\bar{c})_n} \frac{z^n}{n!}$$

with

$$\bar{a} := -\frac{k}{2}, \quad \bar{b} := 1 - \frac{k+N}{2}, \quad \bar{c} := \frac{N}{2}$$

is convex relatively to $(1 - z^{N/2})^{k/N}$. This statement is equivalent to the following inequality:

$$(B.6) \quad zg''(z) \geq \left(\frac{k}{2} - 1 + \frac{N-k}{2} \frac{1}{1 - z^{N/2}} \right) g'(z).$$

Note that here $\bar{b} > 0$ since $k < 2 - N$, and so all parameters $\bar{a}, \bar{b}, \bar{c}$ are strictly positive. We now use the two following properties:

- (i) the function g is strictly decreasing when $k \in (-N, 2 - N)$,
- (ii) we have the following sharp inequality for $t \in (0, 1)$:

$$\frac{N}{1 - t^N} \geq \frac{2}{1 - t^2} + \frac{N-2}{2}.$$

The first item (i) is obtained from identity (A.3):

$$g'(z) = \frac{d_N}{k} \frac{\bar{a}\bar{b}}{\bar{c}} F(\bar{a} + 1, \bar{b} + 1, \bar{c} + 1, z) = d_N \left(\frac{k + N - 2}{2N} \right) F(\bar{a} + 1, \bar{b} + 1, \bar{c} + 1, z).$$

To obtain the second item (ii), we need to show that $u(t) \geq 0$ for all $t \in (0, 1)$, where we define

$$u(t) := N(1 - t^2) - 2(1 - t^N) - \left(\frac{N-2}{2} \right) (1 - t^2)(1 - t^N).$$

Note that $u(0) = (N - 2)/2$ and $u(1) = 0$. It is therefore enough to show that $u'(t) \leq 0$ on $(0, 1)$. Differentiating, we have

$$\begin{aligned} u'(t) &= -2Nt + 2Nt^{N-1} + \left(\frac{N-2}{2} \right) [2t(1 - t^N) + Nt^{N-1}(1 - t^2)] \\ &= -(N + 2) \left(t - \frac{N}{2} t^{N-1} + \left(\frac{N-2}{2} \right) t^{N+1} \right). \end{aligned}$$

By Young's inequality,

$$t^{N-1} \leq \frac{2}{N} t^{\theta N/2} + \left(\frac{N-2}{N} \right) t^{\nu N/(N-2)}, \quad \theta = \frac{2}{N}, \quad \nu = \frac{(N-2)(N+1)}{N}.$$

and so $u'(t) \leq 0$ follows. This concludes the prove of (ii).

Hence, in order to show inequality (B.6) it is enough to prove

$$zg''(z) \geq \left(\frac{k}{2} - 1 + \frac{N-k}{N} \frac{1}{1-z} + \frac{(N-k)(N-2)}{4N} \right) g'(z)$$

thanks to identity (ii). This is equivalent to the inequality

$$(1-z)zg''(z) \geq \left(\frac{(N+k)(N-2)}{4N} + \frac{4N - 2Nk - N^2 + (k+2)N - 2k}{4N} z \right) g'(z).$$

Dividing by z and using that $z \leq 1$, it is enough to prove that

$$(1-z)g''(z) \geq \left(\frac{(N+k)(N-2)}{4N} + \frac{4N - 2Nk - N^2 + (k+2)N - 2k}{4N} \right) g'(z),$$

which simplifies to

$$(B.7) \quad (1-z)g''(z) \geq \left(1 - \frac{k}{N} \right) g'(z).$$

We conclude that (B.7) directly implies Lemma 7. We now examine inequality (B.7) term by term using the representation by series. To establish a relation between g' and g'' , we make use of the following identities for the Pochhammer symbol:

$$(q+1)_n = \frac{q+n}{q}(q)_n, \quad (q)_{n+1} = (q+n)(q)_n.$$

We can then write the left hand side of (B.7) as

$$\begin{aligned} & (1-z)g''(z) \\ &= (1-z)d_N \left(\frac{k+N-2}{2N} \right) \frac{(\bar{a}+1)(\bar{b}+1)}{(\bar{c}+1)} \sum_{n=0}^{\infty} \frac{(\bar{a}+2)_n(\bar{b}+2)_n}{(\bar{c}+2)_n} \frac{z^n}{n!} \\ &= d_N \left(\frac{k+N-2}{2N} \right) \frac{(\bar{a}+1)(\bar{b}+1)}{(\bar{c}+1)} \sum_{n=0}^{\infty} \frac{(\bar{a}+2)_n(\bar{b}+2)_n}{(\bar{c}+2)_n} \frac{(z^n - z^{n+1})}{n!} \\ &= d_N \left(\frac{k+N-2}{2N} \right) \frac{(\bar{a}+1)(\bar{b}+1)}{(\bar{c}+1)} \sum_{n=0}^{\infty} \left(\frac{(\bar{a}+1+n)(\bar{b}+1+n) - n(\bar{c}+1+n)}{(\bar{a}+1+n)(\bar{b}+1+n)} \right) \frac{(\bar{a}+2)_n(\bar{b}+2)_n}{(\bar{c}+2)_n} \frac{z^n}{n!} \\ &= d_N \left(\frac{k+N-2}{2N} \right) \sum_{n=0}^{\infty} \left(\frac{(\bar{a}+1+n)(\bar{b}+1+n)}{(\bar{c}+1+n)} - n \right) \left(\frac{(\bar{a}+1)_n(\bar{b}+1)_n}{(\bar{c}+1)_n} \right) \frac{z^n}{n!}. \end{aligned}$$

Comparing this expression term by term to the right hand side of (B.7),

$$\left(1 - \frac{k}{N}\right) g'(z) = \left(1 - \frac{k}{N}\right) d_N \left(\frac{k+N-2}{2N} \right) \sum_{n=0}^{\infty} \left(\frac{(\bar{a}+1)_n(\bar{b}+1)_n}{(\bar{c}+1)_n} \right) \frac{z^n}{n!},$$

we need to show that for all $n \geq 0$,

$$(\bar{a}+1+n)(\bar{b}+1+n) \leq \left(1 - \frac{k}{N} + n\right) (\bar{c}+1+n)$$

(note that the sign has changed due to division by $k+N-2 < 0$). Expanding with respect to n , this is equivalent to

$$\left(\bar{a} + \bar{b} - \bar{c} + \frac{k}{N}\right) n + (\bar{a}+1)(\bar{b}+1) - (\bar{c}+1) \left(1 - \frac{k}{N}\right) \leq 0.$$

We claim that the latter holds true, since we have both

$$\bar{a} + \bar{b} - \bar{c} + \frac{k}{N} = -k + 1 - N + \frac{k}{N} = \frac{(N+k)(1-N)}{N} < 0,$$

and

$$\begin{aligned} (\bar{a}+1)(\bar{b}+1) - (\bar{c}+1) \left(1 - \frac{k}{N}\right) &= \frac{k(N+k)}{4} + 1 - k - N + \frac{k}{N} \\ &= (N+k) \left(\frac{k}{4} + \frac{1}{N} - 1\right) < 0. \end{aligned}$$

Finally, the fact that $\beta(c) \leq 0$ follows directly from $\vartheta'_{k,N}(c) \geq 0$. The sign of $\alpha(c)$ is a consequence of the convexity inequality that we showed above. More precisely, as remarked above, inequality (3.4) is equivalent to

$$g''(c) \geq \frac{\varphi''(c)}{\varphi'(c)} g'(c) \quad \forall c \in (0, 1).$$

Therefore, differentiating $\alpha = g(c) - \frac{g'(c)}{\varphi'(c)}\varphi(c)$ with respect to c , we have

$$\alpha'(c) = \frac{\varphi(c)}{\varphi'(c)^2} (-g''(c)\varphi'(c) + g'(c)\varphi''(c)) \leq 0.$$

Together with

$$\alpha(c) \rightarrow \frac{\vartheta_{k,N}(0)}{k} = \frac{2^{N-2}\sigma_{N-1}}{k} \frac{\Gamma\left(\frac{N-1}{2}\right)^2}{\Gamma(N-1)} < 0 \quad \text{as } c \rightarrow 0^+,$$

we conclude that $\alpha(c) \leq 0$ for all $c \in (0, 1)$. This completes the proof of Lemma 7. \square

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