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LIYANG YANG

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AN EXPLICIT CM TYPE NORM FORMULA AND EFFECTIVE NONVANISHING OF CLASS GROUP L-FUNCTIONS FOR CM FIELDS

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We show that the central value of class group L-functions of general CM fields can be expressed in terms of derivatives of real-analytic Hilbert Eisenstein series at CM points. Using this in conjunction with an explicit CM type norm formula established in Section 3, following an idea of Iwaniec and Kowalski (2004), we obtain a conditional explicit lower bound for class numbers of CM fields under the assumption $\zeta_K\left(\frac{1}{2}\right) \ll_F \log D_{K/F}$ (note that GRH implies $\zeta_K\left(\frac{1}{2}\right) \leq 0$). Some results in the proof lead to an *effective* nonvanishing result for class group L-functions of general CM fields, generalizing the only known ineffective results. Moreover, combining the CM type norm formula with Barquero-Sanchez and Masri's work (2016), we shall deduce an explicit Chowla–Selberg formula for *all* abelian CM fields.

1. Introduction

1.1. ***A lower bound for the class number of CM fields.*** For imaginary quadratic fields $K = \mathbb{Q}(\sqrt{-D})$, Gauss' class number problem has for a long time inspired the study of lower bounds of $h(-D)$, the class number of K. Also, the magnitude of $h(-D)$ is closely related to the exceptional characters, i.e., those characters χ such that $L(s, \chi)$ has a real zero near $s = 1$ [Landau 1918; Goldfeld 1975; Goldfeld and Schinzel 1975; Granville and Stark 2000]. A repelling property of the exceptional zero gives the result $h(-D) \to \infty$ as $D \to \infty$ [Deuring 1933; Heilbronn 1934]. Landau [1935] then obtained the lower bound $h(-D) \gg_\epsilon D^{1/8-\epsilon}$ by a quantitative analysis of the repelling effects. Siegel [1935] got a stronger result: $h(-D) \gg_\epsilon D^{1/2-\epsilon}$. See [Iwaniec 2006] for a more concrete historical introduction. However, all these results suffer from the serious defect of being ineffective. Hence one can not use them to determine the fields of class number one. Also, there are many other situations requiring an effective lower bound for $h(-D)$, for example, by genus theory, the Euler idoneal number problem calls for an effective lower bound

$$h(-D) \gg D^{c'/\log\log D}, \quad \text{with } c' > \log 2.$$

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Generally one hopes to show that $h_K \geq C D_K^c$ for some positive (absolute) constants c and C, where h_K is the class number and D_K is the absolute discriminant.

Unconditionally, Stark [1974, Theorem 2] gave an effective lower bound for the class number of a CM field K of the shape $h_K \gg_{n,\epsilon} D_{K/F}^{1/2-1/2n} D_F^{\epsilon}$, where $\epsilon > 0$, F is a totally real subfield of K with $n = [F : \mathbb{Q}]$, and D_F is the absolute discriminant of F and $D_{K/F} = D_K \cdot D_F^{-2}$. When $[K : \mathbb{Q}] \geq 20$, Hoffstein [1979] generalized Stark's result by removing the D_F^{ϵ}-term and computing the implied constant explicitly. Also, for any CM field K, Odlyzko [1975] gives an effective lower bound of h_K in terms of the degree of K. For $n = 1$ the known unconditional lower bound is $\prod_{p \mid D_K} (1 - 2\sqrt{p}/(p+1))^{-1} \log D_K$ due to [Goldfeld 1985; Gross and Zagier 1986]. On the other hand, if we assume the grand Riemann hypothesis, the exponent c can be taken to be $\frac{1}{2} - \epsilon$ for any $\epsilon > 0$. Moreover, assuming the Dedekind zeta function $\zeta_K(s)$ has a Siegel zero, Louboutin [1994] obtained effective lower bounds for h_K, with $c = \frac{1}{4}$.

It is well known that the class group L-functions of an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-D})$ can be expressed in terms of values of the real-analytic Eisenstein series for $SL_2(\mathbb{Z})$ at Heegner points [Duke et al. 1995]. Based on this fact, Iwaniec and Kowalski [2004] obtained an effective lower bound for the class number $h_K \gg D^{1/4} \log D$ assuming that $L\big(\frac{1}{2}, \chi_{K/\mathbb{Q}}\big) \geq 0$, where $\chi_{K/\mathbb{Q}}$ is the quadratic character corresponding the extension $K/\mathbb{Q}$; see Section 2. In this paper we generalize Iwaniec and Kowalski's result to arbitrary CM fields and obtain an expression for class group L-functions in terms of derivatives of real-analytic Hilbert Eisenstein series at CM points. Due to the estimates on the Fourier expansions we show that for any CM field K, if $\zeta_K(\frac{1}{2}) \leq 0$, then $c = \frac{1}{4}$ is admissible, and the implied constant is effective.

A precise statement is our Theorem A. To achieve it, we shall introduce some analytic objects with respect to F. We let γ_F^* be the normalized Euler–Kronecker constant, i.e.,

$$\gamma_F^* = \lim_{s \to 1} \left(\rho_F^{-1} \zeta_F(s) - \frac{1}{s-1} \right). \tag{1}$$

For any CM extension K/F, we will always denote by D_F (resp. D_K) the absolute discriminant of F (resp. K). Let h_F (resp. h_K) be the class number of F (resp. K). Then we have:

Theorem A. *Let $F/\mathbb{Q}$ be a totally real field of degree n. Let K/F be a CM extension. Assume that*

$$\zeta_K\big(\tfrac{1}{2}\big) \leq \frac{\rho_F}{4[\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot h_F} \log \frac{\sqrt{D_K}}{D_F}. \tag{2}$$

Then we have

$$h_K \geq {\Psi_F}^{-1} \cdot D_K^{1/4} \log \frac{\sqrt{D_K}}{D_F}, \tag{3}$$

where

$$\Psi_F := \pi^{-n} \rho_F^{-1} h_F D_F^2 + e^{2\gamma_F^* + 2n + 2} \Upsilon_F^* D_F^{3/2},$$

and $\Upsilon_F^* = 4\gamma_F^* + 3 \log D_F + 4n + \sqrt{7n} + 8$.

Remark. Noting that $\zeta_K(s)$ is continuous on $\left(\frac{1}{2}, 1\right)$ and $\lim_{s\to 1^-} \zeta_K(s) = -\infty$, the grand Riemann hypothesis then gives $\zeta_K\left(\frac{1}{2}\right) \leq 0$, which is stronger than the assumption (2). Unconditionally, it is only known that $\zeta_{\mathbb{Q}}\left(\frac{1}{2}\right) < 0$ currently. Also, we can keep the normalized Euler–Kronecker constant γ_F^* here as an invariant of F. We refer to Theorem 7 in [Murty and Van Order 2007] for an elementary upper bound for γ_F^* and to Theorem 1 in [Ihara 2006] for a conditional upper bound.

Remark. Note that Stark's bound is sharper than (3) except $[K : \mathbb{Q}] \leq 4$. However, the proof of Theorem A, when invoked with Stark's result, leads to effective results on nonvanishing of class group L-functions (and their derivatives) for CM fields. Note that previous results are all ineffective. See Section 1.2 for more details.

Also, we point out that, with a little bit more work, the inequality (3) can be naturally generalized to a conditional lower bound for $h_{\mathfrak{O}}$, where $\mathfrak{O}$ is an order of K and $h_{\mathfrak{O}}$ denotes the number of proper $\mathfrak{O}$-ideal classes of $\mathfrak{O}$, since everything in this paper has a counterpart in the order case.

The outline of the proof to Theorem A is described in Section 2.2 below, with the new ingredients involved. See the rest of the sections for lengthy details.

Also, we have the following corollary to go beyond [Iwaniec and Kowalski 2004] by plugging the upper bound for γ_F^* given in Lemma 20 into (3). It is of course weaker than the conditional result in [Ihara 2006] under GRH; however, it is simple enough compared to the elementary bound in [Murty and Van Order 2007].

Corollary 2. *Let notation be as before. Then there exist absolute constants* $c_1, c_2 > 0$ *such that if* $\zeta_K\left(\frac{1}{2}\right)$ *satisfies* (2), *then we have*

$$h_K^- \geq c_1 D_F^{-c_2} \cdot D_K^{1/4} \log D_{K/F}, \tag{4}$$

where h_K^- *is the relative class number of* K/F *and* $D_{K/F} = D_F^{-2} D_K$ *is the relative discriminant of* K/F.

Remark. Combining the analytic class number formula with a result of Louboutin [1994, Proposition A], one actually sees that if we assume $\zeta_K\left(\frac{1}{2}\right) \leq 0$, then essentially $h_K^- \gg D_K^{1/4}$, where the implied constant is effective. According to the conditional estimate $\gamma_F^* \ll \log\log D_F$ [Ihara 2006], it is expected that one can take $c_2 = 3 + \epsilon$ in (4) for any $\epsilon > 0$. So (4) is an improvement of Louboutin's result if we fix F and let D_K vary. Also, by Hermite's theorem [1857] one sees that almost every h_K^- is a positive power of $D_{K/F}$.

1.2. ***Nonvanishing of class group L-functions.*** It is an important problem in number theory to determine whether an interesting L-function is nonvanishing at the central value $s = \frac{1}{2}$. However, it is usually pretty difficult to check individually. A common strategy is to consider instead the average of L-functions over a family. One fruitful way for dealing with such averages combines periods relations of Waldspurger type with the equidistribution of special points on varieties (see, e.g., [Michel and Venkatesh 2007; Masri 2010]). However, in this paper, we will use purely analytic methods to obtain asymptotic expressions for some averages of weighted class group L-functions. This has several advantages. For instance, one can obtain effective nonvanishing results, namely, avoiding using Siegel's bound.

Although one can obtain the nonvanishing results on the critical line $\mathrm{Re}(s) = \frac{1}{2}$ by the same method, here we just focus on the central value $s = \frac{1}{2}$ for simplicity. Precisely, we will use byproducts from the proof of Theorem A to obtain a lower bound for the number of nonvanishing class group L-functions as (5). This bound, as can be seen, is almost as good as the conjectural magnitude. Moreover, a significant difference between our approach and that in [Masri 2010] is that subconvexity bounds are not essential for us. Therefore, one might be able to handle higher derivative cases by general convex bounds for Dirichlet series via our methods here.

Theorem B. *Let $F/\mathbb{Q}$ be a totally real field and K/F is a CM extension. Denote by $L_K^{(0)}(\chi, \frac{1}{2}) = L_K(\chi, \frac{1}{2})$ and $L_K^{(1)}(\chi, \frac{1}{2}) = L_K'(\chi, \frac{1}{2})$ the derivative of $L_K(\chi, s)$ at $s = \frac{1}{2}$. Then for any $\epsilon > 0$ we have*

$$\#\{\chi \in \widehat{Cl(K)} : L_K^{(k)}(\chi, \tfrac{1}{2}) \neq 0\} \gg_F \frac{h_K}{\log D_K}, \quad k = 0, 1, \tag{5}$$

where the implied constant in (5) *is computable.*

Remark. When $F = \mathbb{Q}$, i.e., K is imaginary quadratic, Blomer [2004] followed [Duke et al. 1995] to prove a better lower bound

$$\#\{\chi \in \widehat{Cl(K)} : L_K(\chi, \tfrac{1}{2}) \neq 0\} \geq c \cdot h_K \prod_{p \mid D_K} \left(1 - \frac{1}{p}\right)$$

for some constant $c > 0$ and all sufficiently large D_K. Once D_K is chosen, the constant c can be taken explicitly. However, one does not know how large D_K must be chosen for this lower bound to be valid because of an application of Siegel's lower bound for $L(1, \chi_{K/F})$. For a general CM extension K/F, when F has trivial class group, Masri [2010] proved a lower bound as

$$\#\{\chi \in \widehat{Cl(K)} : L_K(\chi, \tfrac{1}{2}) \neq 0\} \gg_{F,\epsilon} D_K^{1/100-\epsilon}. \tag{6}$$

Again, this bound is ineffective since Siegel's bound is used here. The exponent $\frac{1}{100}$ comes from an application of subconvexity bound for GL(2).

Together with Stark's effective lower bound, Theorem B implies that:

Corollary 4. *Let $F/\mathbb{Q}$ be a totally real field of degree n and K/F a CM extension. Then we have*

$$\#\{\chi \in \widehat{Cl(K)} : L_K^{(k)}(\chi, \tfrac{1}{2}) \neq 0\} \gg_{F,\epsilon} \frac{D_K^{1/2-1/(2n)}}{\log D_K}, \quad k = 0, 1.$$

Moreover, the implied constants are computable.

Remark. By (5) and Theorem A, if we assume (2), in particular, if we assume $\zeta_K(\frac{1}{2}) \leq 0$, then we obtain

$$\#\{\chi \in \widehat{Cl(K)} : L_K^{(k)}(\chi, \tfrac{1}{2}) \neq 0\} \gg_{F,\epsilon} D_K^{1/4}, \quad k = 0, 1,$$

where the implied constant is computable. Moreover, Siegel's theorem gives that

$$\#\{\chi \in \widehat{Cl(K)} : L_K^{(k)}(\chi, \tfrac{1}{2}) \neq 0\} \gg_{F,\epsilon} \frac{D_K^{1/2}}{\log D_K}, \quad k = 0, 1,$$

where the implied constant is ineffective. This is a significant improvement of (6).

1.3. ***An explicit Chowla–Selberg formula for general abelian CM fields.*** The celebrated Chowla–Selberg formula was first proved for imaginary quadratic fields in [Selberg and Chowla 1967] by analytic methods. A geometric interpretation was given by Gross [1978]. Yoshida [2003] obtained such a formula for arbitrary CM fields. For any abelian CM field which contains a totally real subfield with trivial narrow class group, Barquero-Sanchez and Masri [2016], combining Lerch's identity [1897] and the results in [Deninger 1984], were able to obtain an explicit Chowla–Selberg formula in terms of (generalized) gamma functions, paralleling the original Chowla–Selberg formula.

In this section, we point out that Barquero-Sanchez and Masri's idea works for *all* CM fields with our Proposition 15 being the new input. Hence we shall avoid repeating the proof of [Barquero-Sanchez and Masri 2016] and just state the generalized formula below.

Let K be an abelian CM field and F is its maximal totally real subfield. Then there is some $N \in \mathbb{N}_{\geq 1}$ such that $K \subset \mathbb{Q}(\zeta_N)$ with $\zeta_N = e^{2\pi i/N}$. Let H_K (resp. H_F) be the subgroup of $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q})$ which fixes K (resp. F). Fix an isomorphism $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \simeq (\mathbb{Z}/N\mathbb{Z})^\times$. Set $X_K = \{\chi \in \widehat{(\mathbb{Z}/N\mathbb{Z})}^\times : \chi|_{X_K} = 1\}$. Let X_F be defined similarly. For any $\chi \in X_K$, it can be written as $\chi = \chi^*\chi_0$, where χ^* is primitive and χ_0 is trivial. Write c_χ the conductor of χ^*. Since χ^* is uniquely determined by χ, then c_χ is well-defined. Define the Gauss sum associated with $\chi \in X_E$ to be

$$\tau(\chi) = \sum_{k=1}^{c_\chi} \chi(k) e^{2\pi k i / c_\chi}.$$

Define the function $\Gamma_2(w) = e^{R(w)}$, $\mathrm{Re}(w) > 0$, where

$$R(x) = \lim_{n\to\infty}\left(-\zeta_{\mathbb{Q}}''(0) + x\log^2 n - \log^2 x - \sum_{k=1}^{n-1}\bigl(\log^2(x+k) - \log^2 k\bigr)\right).$$

This function $\Gamma_2(w)$ is defined in [Deninger 1984] and it is analogous to $\Gamma(s)/\sqrt{2\pi}$. Now we state a general version of the Chowla–Selberg formula for abelian CM fields.

Theorem C. *Let $F/\mathbb{Q}$ be a totally real field of degree n and K/F a CM extension with $K/\mathbb{Q}$ abelian. Let Φ be the fixed CM type for K as before. For any fractional ideal $\mathfrak{a}$ of K, denote by $\mathfrak{f}_{\mathfrak{a}}$ a fixed integral ideal in the Steinitz class of $\mathfrak{a}$ with minimal absolute norm, and write $z_{\mathfrak{a}}$ for the corresponding CM point of type (K, Φ). Then*

$$\prod_{[\mathfrak{a}]\in Cl(K)} H(z_{\mathfrak{a}}, \mathfrak{f}_{\mathfrak{a}}) = C_F^K \prod_{\chi\in X_K\setminus X_F}\prod_{k=1}^{c_\chi}\Gamma\left(\frac{k}{c_\chi}\right)^{\frac{h_K\chi(k)}{2L(0,\chi)}} \prod_{\substack{\chi\in X_F\\ \chi\neq 1}}\prod_{k=1}^{c_\chi}\Gamma_2\left(\frac{k}{c_\chi}\right)^{\frac{h_K\tau(\chi)\bar{\chi}(k)}{2c_\chi L(1,\chi)}},$$

where for any fractional ideal $\mathfrak{b}$ of F,

$$H(z;\mathfrak{b}) = [N_\Phi(\mathrm{Im}(z))N_{F/\mathbb{Q}}(\mathfrak{b})^{-1}]^{1/h_F}\phi(z;\mathfrak{b}),$$

and $\phi(z;\mathfrak{b})$ is defined via

$$\log\phi(z;\mathfrak{b}) = -\frac{2\pi^n D_F^{1/2} y^{\sigma}}{\rho_F N_{F/\mathbb{Q}}(\mathfrak{b})}\cdot\zeta_F(2,[\mathcal{O}_K]) - \frac{2\rho_F^{-1}}{D_F N_{F/\mathbb{Q}}(\mathfrak{b})}\sum_{b\in F^\times}|N(b)|^{-1}\lambda(b,0)\mathbf{e}(bz),$$

and

$$\lambda(b,s) = \sum\sum_{\substack{(a,c)\in\mathfrak{b}^{-1}\mathfrak{d}^{-1}\times\mathcal{O}_F^\times/\mathcal{O}_F^{\times,+}\\ ac=b}} |c|^{(1-2s)\sigma}.$$

Also, the constant C_F^K equals $(2^{-n-1}\pi^{-1}{D_K}^{-1/2}D_F)^{h_K/2}$.

Remark. It is clear that by definition $H(z_{\mathfrak{a}}, \mathfrak{f}_{\mathfrak{a}})$ is independent of the choice of $\mathfrak{f}_{\mathfrak{a}}$. If $h_F^- = 1$, we may take $\mathfrak{f}_{\mathfrak{a}} = \mathcal{O}_F$ to recover Theorem 1.1 in [Barquero-Sanchez and Masri 2016]. Combined with Colmez's theorem [1993], the formula above can be used to compute the average of Faltings heights of certain CM abelian varieties.

2. Proof of Theorem A in imaginary quadratic case

2.1. ***Review of the imaginary quadratic case.*** We start by reviewing the case of Theorem A in the imaginary quadratic case. This is Iwaniec and Kowalski's original idea. For the sake of illustration, we give a brief proof following [Iwaniec and Kowalski 2004].

Let $K=\mathbb{Q}(\sqrt{-D})$ be a imaginary quadratic field. Since $\mathbb{Q}$ has class number 1, we can often factor a nonzero integral ideal uniquely as $(l)\mathfrak{a}$ where $l\in\mathbb{Z}_{>0}$ and $\mathfrak{a}$ is a primitive ideal, i.e., $\mathfrak{a}$ has no rational integer factors other than ± 1.

If $\mathfrak{a}$ is primitive, then it is generated by

$$\mathfrak{a}=\left[a,\ \frac{b+i\sqrt{D}}{2}\right],$$

where $a=N\mathfrak{a}$ and b solves the congruence $b^2+D\equiv 0\pmod{4a}$, and is determined modulo $2a$.

Conversely, given such a and b we get a primitive ideal $\mathfrak{a}=\left[a,\ \frac{b+i\sqrt{D}}{2}\right]$. Thus there exists a one-to-one correspondence between the primitive ideals and the points

$$z_{\mathfrak{a}}:=\frac{b+i\sqrt{D}}{2a}\in\mathbb{H}\quad\text{determined by modulo 1.}$$

These will be called the Heegner points. Moreover, we have $\mathfrak{a}^{-1}=[1,\bar{z}_{\mathfrak{a}}]$. Then according to [Duke et al. 1995] one has the following formula:

$$\frac{1}{h}\sum_{\chi\in\widehat{Cl}(K)}\chi(\mathfrak{a})L_K(s,\chi)=w^{-1}\left(\frac{\sqrt{D}}{2}\right)^{-s}\zeta(2s)E(z_{\mathfrak{a}},s),\tag{7}$$

where $h=h_K$ is the class number, w is the root of unity of K, $\mathfrak{a}$ is any primitive ideal, $z_{\mathfrak{a}}$ is the Heegner point, and $E(z,s)$ is the real analytic Eisenstein series of weight 0 for the modular group. The Eisenstein series $E(z,s)$ admits the Fourier expansion:

$$\Theta(s)E(z,s)=\Theta(s)y^s+\Theta(1-s)y^{1-s}+4y^{1/2}\sum_{k=1}^{\infty}\sum_{mn=k}\left(\frac{m}{n}\right)^{it}K_{it}(2\pi ky)\cos(2\pi kx),$$

where $\Theta(s):=\pi^{-s}\Gamma(s)\zeta(2s)$. Applying Fourier inversion we get from (7) that

$$L_K(s,\chi)=w^{-1}\left(\frac{\sqrt{D}}{2}\right)^{-s}\zeta(2s)\sum_{z_{\mathfrak{a}}\in\Lambda_D}\bar{\chi}(\mathfrak{a})E(z_{\mathfrak{a}},s),\tag{8}$$

where $\Lambda_D:=\{z_{\mathfrak{a}}\in\mathcal{F}:\ \mathfrak{a}\text{ primitive}\}$ and $\mathcal{F}$ is the fundamental domain for $SL_2(\mathbb{Z})$.

Clearly from the Fourier expansion we have $E\left(z,\frac{1}{2}\right)\equiv 0$, since $\zeta(2s)\sim\frac{1}{2s-1}$ when $s\to\frac{1}{2}$ and the right-hand side is well defined.

Thus take the derivative of (8) at $s=\frac{1}{2}$ and note $K_0(y)\ll y^{-1/2}e^{-y}$ to get

$$\begin{aligned}L_K\big(\tfrac{1}{2},\chi\big)&=\frac{\sqrt{2}}{w}|D|^{-1/4}\sum_{\mathfrak{a}}\bar{\chi}(\mathfrak{a})E'\big(z_{\mathfrak{a}},\tfrac{1}{2}\big)\\&=\frac{1}{w}\sum_{\mathfrak{a}}\frac{\bar{\chi}(\mathfrak{a})}{\sqrt{a}}\bigg\{\log\frac{\sqrt{|D|}}{2a}+4\sum_{n=1}^{\infty}\tau(n)K_0\bigg(\frac{\pi n\sqrt{|D|}}{a}\bigg)\cos\bigg(\frac{\pi nb}{a}\bigg)\bigg\}\\&=\frac{1}{2}\sum_{\mathfrak{a}}\frac{\bar{\chi}(a)}{\sqrt{a}}\log\frac{\sqrt{|D|}}{2a}+O\big(h(D)|D|^{-1/4}\big),\end{aligned}$$

since $\#\Lambda_D=h(D)$.

Assuming $L\big(\frac{1}{2},\chi_D\big)\geq 0$, i.e., $L_K\big(\frac{1}{2},\chi_0\big)=\zeta\big(\frac{1}{2}\big)L\big(\frac{1}{2},\chi_D\big)\leq 0$, we derive

$$h(D)|D|^{-1/4}\gg\sum_{\mathfrak{a}}\frac{1}{\sqrt{a}}\log\frac{\sqrt{|D|}}{2a}=\sum_{\mathfrak{a}}\frac{1}{\sqrt{a}}\log\frac{\sqrt{|D|}}{a}+O\big(h(D)|D|^{-1/4}\big).$$

Thus we have $h(D)|D|^{-1/4}\gg\sum_{\mathfrak{a}}\frac{1}{\sqrt{a}}\log\frac{\sqrt{|D|}}{a}\gg\log|D|$, which implies:

Theorem 6 [Iwaniec and Kowalski 2004]. *Let notation be as before. Assume that $L\big(\frac{1}{2},\chi_D\big)\geq 0$. Then we have*

$$h(D)\gg|D|^{1/4}\log|D|. \tag{9}$$

Remark. The implied constant in (9) is absolute. Actually, by estimating everything explicitly one can get an explicit lower bound:

Theorem 7 [Dittmer et al. 2015]. *Let notation be as before. Assume $L\big(\frac{1}{2},\chi_D\big)\geq 0$, then for any $\epsilon\in\big(0,\frac{1}{2}\big)$ we have*

$$h(D)\geq 0.1265\epsilon|D|^{1/4}\log|D|\quad\textit{for all } D\geq 200^{1/(1-2\epsilon)}.$$

2.2. ***Sketch of proof of Theorem A and Theorem B in the general CM case.*** To generalize Iwaniec and Kowalski's results to the CM fields case, one has to establish a general form of the Eisenstein period (7) (see below). The analogue of (7) in the CM case is easy to build if F, the totally real subfield, has trivial narrow class group (see, e.g., [Masri 2010]). In general, one needs to compute CM points associated to each Steinitz class. However, the situation is quite different from the imaginary quadratic case since generally, integral representatives in $Cl(F)$ bounded by Minkowski bounds may not necessarily be primitive. In addition, the CM type norm of the imaginary parts of CM points should be computed explicitly in order to compute the constant terms of Fourier coefficients of Eisenstein series. These problems are solved by the crucial Proposition 13 in Section 3.2 below. Roughly speaking, fix a CM type Φ on K, then a fractional ideal $\mathfrak{a}\subset K$ corresponds to a CM point $z_{\mathfrak{a}}$ in a Hilbert modular variety (see Section 3); we calculate the CM type norm $N_\Phi(y_\mathfrak{a})$ explicitly, where $y_\mathfrak{a}$ the imaginary part of $z_\mathfrak{a}$. Also, unlike the cases in [Iwaniec and Kowalski 2004] or [Masri 2010], generally a single Hilbert Eisenstein series does not have a functional equation, but it turns out that it does vanish at the central point $s = \frac{1}{2}$. Then we prove Proposition 15, which is an expression for $L_K\left(\frac{1}{2}, \chi\right)$ in terms of derivatives of Eisenstein series similar to (7). Then following Iwaniec and Kowalski's idea, an explicit lower bound for $L_K\left(\frac{1}{2}, \chi_0\right)$ is given in Proposition 18, Section 4.1. Furthermore, several effective estimates such as bounds for $L_K(1, \chi)$ and the normalized Euler–Kronecker constant γ_F^* are established in Section 4. With all these preparations, we eventually complete the proof of Theorem A in Section 4.2.

The above mentioned results such as Proposition 15 and Fourier expansion of derivatives of Hilbert Eisenstein series (i.e., Lemma 17) will lead to a lower bound for the first moment of class group L-functions and an upper bound for the second moment for class group L-functions; see (52) and (53). Then a standard technique using the Cauchy inequality will imply the $k = 0$ case of Theorem B. The $k = 1$ case simply follows from the $k = 0$ case and functional equation.

3. Generalization to the CM case

3.1. *Hilbert Modular varieties and CM zero-cycles.*

3.1.1. *The basic correspondence.* Let $F/\mathbb{Q}$ be a totally real extension of degree n. For any $S \subset F$, let S^+ be the subset of S consisting of totally positive elements. Given a fractional ideal $\mathfrak{f} \subset F$, define

$$\Gamma(\mathfrak{f}) := \left\{ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, F) : \quad a, d \in \mathcal{O}_F,\ b \in \mathfrak{f},\ c \in \mathfrak{f}^{-1} \right\}.$$

Let $\mathbb{H}$ be the upper half plane. Then $\Gamma(\mathfrak{f})$ acts on $\mathbb{H}^n$ via

$$\gamma \cdot z = (\sigma_1(\gamma) z_1, \ldots, \sigma_n(\gamma) z_n) \quad \text{for all } z = (z_1, \ldots, z_n) \in \mathbb{H}^n.$$

Recall that the quotient

$$X(\mathfrak{f}) := \Gamma(\mathfrak{f}) \backslash \mathbb{H}^n$$

is the open Hilbert modular variety associated to $\mathfrak{f}$. It's known [Goren 2002, Theorem 2.17] that $X(\mathfrak{f})$ parameterizes isomorphism classes of triples (A, i, m) where (A, i) is an abelian variety with real multiplication $i : \mathcal{O}_F \hookrightarrow \mathrm{End}(A)$ and

$$m : (\mathfrak{M}_A, \mathfrak{M}_A^+) \to \left((\mathfrak{o}_F \mathfrak{f})^{-1}, (\mathfrak{o}_F \mathfrak{f})^{-1,+}\right)$$

is an $\mathcal{O}_F$-isomorphism $\mathfrak{M}_A \xrightarrow{\sim} (\mathfrak{o}_F \mathfrak{f})^{-1}$ which maps $\mathfrak{M}_A^+$ to $(\mathfrak{o}_F \mathfrak{f})^{-1,+}$, where

$$\mathfrak{M}_A := \{\lambda : A \to A^\vee \mid \lambda \text{ is a symmetric } \mathcal{O}_F\text{-linear homomorphism}\}$$

is the polarization module of A and $\mathfrak{M}_A^+$ is its positive cone.

Let K/F be a CM extension and let $\Phi = (\sigma_1, \dots, \sigma_n)$ be a CM type of K. Then $z = (A, i, m) \in X(\mathfrak{f})$ is a CM point of type (K, Φ) if one of the following equivalent definitions holds:

(a) Because $z \in \mathbb{H}^n$, there is a point $\tau \in K$ such that

$$\Phi(\tau) = (\sigma_1(\tau), \dots, \sigma_n(\tau)) = z$$

and $\Lambda_\tau = \mathfrak{f} + \mathcal{O}_F \tau$ is a fractional ideal of K.

(b) (A, i') is a CM abelian variety of type (K, Φ) with complex multiplication $i : \mathcal{O}_K \hookrightarrow \mathrm{End}(A)$ such that $i = i' |_{\mathcal{O}_F}$.

To relate CM points with ideals of K, recall that we have fixed $\epsilon_0 \in K^\times$ such that $\overline{\epsilon_0} = -\epsilon_0$ and $\Phi(\epsilon_0) \in \mathbb{H}^n$. Let $\mathfrak{a}$ be a fractional ideal of K and $\mathfrak{f}^{\mathfrak{a}} := \epsilon_0 \mathfrak{o}_{K/F} \mathfrak{a}\bar{\mathfrak{a}} \cap F$.

Then by Lemma 3.1 in [Bruinier and Yang 2006], the ideal class of $\mathfrak{f}^{\mathfrak{a}}$ is the Steinitz class of $\mathfrak{a} \subset K$ as a projective $\mathcal{O}_F$-module. Then it is clear that the CM abelian variety $(A_{\mathfrak{a}} = \mathbb{C}^n/\Phi(\mathfrak{a}), i)$ has the polarization module

$$(\mathfrak{M}_A, \mathfrak{M}_A^+) \to \big((\mathfrak{o}_F \mathfrak{f}^{\mathfrak{a}})^{-1}, (\mathfrak{o}_F \mathfrak{f}^{\mathfrak{a}})^{-1,+}\big).$$

To give an $\mathcal{O}_F$-isomorphism between the above pair and $((\mathfrak{o}_F \mathfrak{f})^{-1}, (\mathfrak{o}_F \mathfrak{f})^{-1,+})$ amounts to giving some $r \in F^+$ such that $\mathfrak{f}^{\mathfrak{a}} = r\mathfrak{f}$. Therefore, to give a CM point $(A, i, m) \in X(\mathfrak{f})$ is the same as to give a pair $(\mathfrak{a}, r)$, where $\mathfrak{a}$ is a fractional ideal of K and $\mathfrak{f}^{\mathfrak{a}} = r\mathfrak{f}$ for some $r \in F^+$. Two such pairs $(\mathfrak{a}_1, r_1)$ and $(\mathfrak{a}_2, r_2)$ are equivalent if there exists an $\gamma \in K^\times$ such that $\mathfrak{a}_2 = \gamma\mathfrak{a}_1$ and $r_2 = r_1 \gamma\bar{\gamma}$. We write $[\mathfrak{a}, r]$ for the class of pair $(\mathfrak{a}, r)$ and identify it with its associated CM point $(A_{\mathfrak{a}}, i, m) \in X(\mathfrak{f})$.

Note that for any fractional ideal $\mathfrak{f} \subset K$ and any $r \in F^+$ we have the natural isomorphism of varieties:

$$\tau : X(\mathfrak{f}) \xrightarrow{\sim} X(r\mathfrak{f}), \quad z = (z_i) \mapsto rz = (\sigma_i(r) z_i).$$

3.1.2. *Upper bounds of Minkowski type and Steinitz class.* Let $L/\mathbb{Q}$ be a number field of signature (r_1, r_2). Let n be the degree of $L/\mathbb{Q}$, then $n = r_1 + 2r_2$. Minkowski showed that there is a constant $M(r_1, r_2)$ only depending on the signature such that for any $\mathcal{C} \in Cl(L)$, there exists an integral ideal $\mathfrak{a}_{\mathcal{C}} \in \mathcal{C}$ satisfying $N_{L/\mathbb{Q}}(\mathfrak{a}_{\mathcal{C}}) \le M(r_1, r_2)\sqrt{D_L}$, where D_L is the absolute discriminant of L.

We can make the corollary in [Zimmert 1981, p. 374] more explicit. In fact, an elementary estimate gives that

$$\log \frac{D_L}{N_{L/\mathbb{Q}}(\mathfrak{a}_{\mathcal{C}})} \ge (2\log 2 + \gamma) r_1 + (\log 2\pi + 2\gamma) r_2 - \sqrt{7n}. \tag{10}$$

Then for $n \ge 6$, the right-hand side is always positive since $\sqrt{n} \le \sqrt{r_1} + \sqrt{2r_2}$. Let $M(n) := (4^{r_2} n!)/(\pi^{r_2} n^n)$. Then $M(n) \cdot \sqrt{D_L}$ gives the Minkowski constant of $L/\mathbb{Q}$.

Combining $M(n)$ with (10) we can take

$$M(r_1, r_2) := \min\{e^{-(2\log 2+\gamma)r_1-(\log 2\pi+2\gamma)r_2+\sqrt{7n}}\cdot 1_{n\geq 7}+M(6)\cdot 1_{n\leq 6},\ M(n)\}. \tag{11}$$

In particular, when n is large, we have $M(r_1, r_2) \leq 50.7^{-r_1/2} \cdot 19.9^{-r_2}$. From now on, we shall fix $M(r_1, r_2)$ in (11). For totally real extension $F/\mathbb{Q}$, each ideal class $\mathcal{C} \in Cl(F)$ contains an integral ideal $\mathfrak{f}_{\mathcal{C}}$ satisfying $N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathcal{C}}) \leq M(n, 0)\sqrt{D_F}$.

Now we fix a set of fractional ideals

$$\mathcal{I}_F^+ := \{\mathfrak{f} :\ \mathfrak{f} \subset \mathcal{O}_F \text{ and } N(\mathfrak{f}) \leq M(n, 0)\sqrt{D_F}\}, \tag{12}$$

such that

$$Cl(F)^+ = \{[\mathfrak{f}] :\ \mathfrak{f} \in \mathcal{I}_F^+\},$$

where $Cl(F)^+$ denotes the narrow ideal class group of F. For simplicity, we assume $\mathcal{O}_F \in \mathcal{I}_F^+$. Then for any fractional ideal $\mathfrak{a} \subset K$, there exists a unique $\mathfrak{f} \in \mathcal{I}_F^+$ such that

$$\mathfrak{f}^{\mathfrak{a}} := \epsilon_0 \mathfrak{o}_{K/F}\mathfrak{a}\bar{\mathfrak{a}} \cap F \in [\mathfrak{f}],$$

i.e., we can find some $r \in F^+$ such that $\mathfrak{f}^{\mathfrak{a}} = r\mathfrak{f}$. Then by the above discussion, $[\mathfrak{a}, r]$ gives a CM point in $X(\mathfrak{f})$. Actually we can construct the CM point more explicitly. To achieve this, let's recall the standard result:

Proposition 8 [Yoshida 2003, Proposition 2.1, p. 179]. *Let F be an arbitrary algebraic number field and K/F be an algebraic extension of degree n. Let $\mathfrak{a} \subset K$ be a fractional ideal. Then there exist $\alpha_1, \dots, \alpha_n \in F$ and a fractional ideal $\mathfrak{f} \subset F$ such that*

$$\mathfrak{a} = \mathcal{O}_F\alpha_1 \oplus \cdots \oplus \mathcal{O}_F\alpha_{n-1} \oplus \mathfrak{f}\alpha_n.$$

Moreover, we have

$$[\mathfrak{f}] = [\mathfrak{c} \cdot N_{K/F}(\mathfrak{a})] \in Cl(K),$$

where $\mathfrak{c}$ is a fractional ideal of F, independent of $\mathfrak{a}$, such that

$$[\mathfrak{c}^2] = [D_{K/F}], \quad \textit{where } D_{K/F} := N_{K/F}(\mathfrak{o}_{K/F}).$$

Proof. The first part of the assertion comes from the structure theorem for a finitely generated torsion free module over a Dedekind domain. Then we can write

$$\mathcal{O}_K = \mathcal{O}_F\alpha_1 \oplus \cdots \oplus \mathcal{O}_F\alpha_{n-1} \oplus \mathfrak{c}\alpha_n;$$
$$\mathfrak{a} = \mathcal{O}_F\beta_1 \oplus \cdots \oplus \mathcal{O}_F\beta_{n-1} \oplus \mathfrak{f}\beta_n,$$

where $\{\alpha_1, \dots, \alpha_n\}$ and $\{\beta_1, \dots, \beta_n\}$ are basis of K over F and $\mathfrak{f}$ and $\mathfrak{b}$ are fractional ideals of F. Then there exists some $\gamma \in \mathrm{GL}(n, F)$ such that $\gamma\alpha_i = \beta_i$, $1 \leq i \leq n$. Take $x \in \mathbb{A}_F^\times$ such that $\mathrm{div}(x) = \mathfrak{c}^{-1}\mathfrak{f}$ and set $y = \mathrm{diag}[x, 1, \dots, 1]$. Then clearly

$y\gamma\mathcal{O}_K=\mathfrak{a}$. On the other hand we have $\mathfrak{a}=a\mathcal{O}_K$, where $a\in\mathbb{A}_K^\times$ such that $\mathrm{div}(a)=\mathfrak{a}$. Therefore we have

$$a^{-1}y\gamma\mathcal{O}_K=\mathcal{O}_K,$$

which gives that $N_{K/F}(\mathfrak{a}^{-1}\mathfrak{c}^{-1}\mathfrak{f})=\det(\gamma)^{-1}N_{K/F}(a^{-1}y\gamma)\in\mathcal{O}_F^\times$, i.e.,

$$[N_{K/F}(\mathfrak{a})]=[N_{K/F}(\mathfrak{c}^{-1}\mathfrak{f})]\in Cl(F).$$

Therefore the last assertion is reduced to the case $\mathfrak{a}=\mathcal{O}_K$.

Let $\{\alpha_1',\dots,\alpha_n'\}$ be the dual basis of K/F with respect to the relative trace $Tr_{K/F}$. Then we have

$$\mathfrak{d}_{K/F}^{-1}=\mathcal{O}_F\alpha_1'\oplus\cdots\oplus\mathcal{O}_F\alpha_{n-1}'\oplus\mathfrak{c}^{-1}\alpha_n',$$

where $\mathfrak{d}_{K/F}$ is the relative different with respect to K/F. Then by the above discussion (i.e., taking $\mathfrak{a}=\mathfrak{d}_{K/F}^{-1}$) we have

$$[N_{K/F}(\mathfrak{d}_{K/F}^{-1})]=[\mathfrak{c}^{-1}\cdot\mathfrak{c}^{-1}]\in Cl(K).$$

Hence we have $[D_{K/F}]=[N_{K/F}(\mathfrak{d}_{K/F}^{-1})]=[\mathfrak{c}^2]\in Cl(K)$. □

Let $\mathfrak{a}$ be any fractional ideal of K, let $[\mathfrak{f}^{\mathfrak{a}}]$ be the Steinitz class of $\mathfrak{a}$. Denote by

$$St_{\mathfrak{a}}=\Big\{\mathfrak{f}_{\mathfrak{a}}:\ N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})=\min_{\mathfrak{f}\in[\mathfrak{f}^{\mathfrak{a}}]}N_{F/\mathbb{Q}}(\mathfrak{f})\Big\}. \tag{13}$$

Given a fractional ideal $\mathfrak{a}\subset K$, take an $\mathfrak{f}_{\mathfrak{a}}\in St_{\mathfrak{a}}$. Without loss of generality, we may assume that $\mathfrak{f}_{\mathfrak{a}}\in\mathcal{I}_F^+$. We then fix this choice for any fractional ideal $\mathfrak{a}\in K$ once and for all. Then by Proposition 8 there is a decomposition

$$\mathfrak{a}=\mathcal{O}_F\alpha\oplus\mathfrak{f}_{\mathfrak{a}}\beta. \tag{14}$$

By the above proposition and the definition of $\mathfrak{f}^{\mathfrak{a}}$ we can take a appropriate β such that there exists some $r\in F^+$ such that $\mathfrak{f}^{\mathfrak{a}}=r\mathfrak{f}_{\mathfrak{a}}$.

Define $z_{\mathfrak{a}}:=\frac{\alpha}{\beta}$. Then we have as in the proof of Lemma 3.2 of [Bruinier and Yang 2006] that

$$(\bar\alpha\beta-\alpha\bar\beta)\mathfrak{f}\mathcal{O}_K=\mathfrak{d}_{K/F}\mathfrak{a}\bar{\mathfrak{a}}.$$

Then we have

$$\epsilon_0(\bar\alpha\beta-\alpha\bar\beta)=r\epsilon\quad\text{for some }\epsilon\in\mathcal{O}_F^\times.$$

Replacing β by $\epsilon^{-1}\beta$ if necessary, we can assume $\epsilon=1$. This implies that

$$\epsilon_0(\bar z-z)=\frac{r}{\beta\bar\beta}\in F^\times,$$

and thus $z_{\mathfrak{a}}\in K^\times\cap\mathbb{H}^n=\{z\in K^\times:\Phi(z)\in\mathbb{H}^n\}$. Moreover, z represents the CM point $[\mathfrak{a},r]\in X(\mathfrak{f}_{\mathfrak{a}})$.

Let $\mathcal{CM}(K, \Phi, \mathfrak{f})$ be the set of CM points $[\mathfrak{a}, r] \in X(\mathfrak{f})$ which we regard as a CM 0-cycle in $X(\mathfrak{f})$. Let

$$\mathcal{CM}(K, \Phi) := \sum_{[\mathfrak{f}] \in Cl(F)^+} \mathcal{CM}(K, \Phi, \mathfrak{f}).$$

We have the natural surjective map

$$\mathcal{CM}(K, \Phi) \twoheadrightarrow Cl(K), \quad [\mathfrak{a}, r] \mapsto [\mathfrak{a}].$$

The fiber is indexed by $\epsilon \in \mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^{\times}$, since every element in the fiber of $\mathfrak{a}$ is of the form $[\mathfrak{a}, r\epsilon]$ with r fixed and $\epsilon \in \mathcal{O}_F^{\times,+}$ a totally positive unit. Note that $\sharp(\mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^{\times}) \leq 2$.

3.2. *Representation of ideals.* Let $z_\mathfrak{a}$ be the CM point corresponding to the fractional ideal $\mathfrak{a}$. Write $x_\mathfrak{a}$ (resp. $y_\mathfrak{a}$) to be the real part (resp. imaginary part) of $z_\mathfrak{a}$. To prove our main results, we need to compute $y_\mathfrak{a}$ explicitly. We start with recalling some definition.

Definition 9 (primitive ideals). Let $\mathfrak{a}$ be a fractional ideal of $\mathcal{O}_K$. We say that $\mathfrak{a}$ is primitive if $\mathfrak{a}$ is an integral ideal of $\mathcal{O}_K$ and if for any nontrivial integral ideal $\mathfrak{n}$ of $\mathcal{O}_F$, $\mathfrak{n}^{-1}\mathfrak{a}$ is not an integral ideal.

Fact 10. *For any fractional ideal $\mathfrak{a}$ of $\mathcal{O}_K$, there exists a unique fractional ideal $\mathfrak{n}$ of F such that $\mathfrak{n}^{-1}\mathfrak{a}$ is a primitive ideal. The ideal $\mathfrak{n}$ will be called the content of the ideal $\mathfrak{a}$.*

Let K/F be a CM extension. There exists some $D \in F^\times/(F^2 \cap F^\times)$ such that $K = F(\sqrt{D})$. We may assume $D \in \mathcal{O}_F$ and fix this choice once and for all. Let $\mathfrak{q}$ be the index-ideal $\left[\mathcal{O}_K : \mathcal{O}_F[\sqrt{D}]\right]$. Set $\tilde{\mathfrak{q}} = \mathfrak{q}\mathcal{O}_K$.

Proposition 11 [Cohen 2000, Section 2.6]. *Let $\mathfrak{a}$ be a fractional ideal of K. There exist unique ideals $\mathfrak{n}$ and $\mathfrak{m}$ and an element $b \in \mathcal{O}_F$ such that*

$$\mathfrak{a} = \mathfrak{n}(\mathfrak{m} \oplus \mathfrak{q}^{-1}(-b + \sqrt{D})), \tag{15}$$

where $\mathfrak{q}$ is the index-ideal $\left[\mathcal{O}_K : \mathcal{O}_F[\sqrt{D}]\right]$. In addition, we have the following:

1. $\mathfrak{n}$ *is the content of* $\mathfrak{a}$.
2. $\mathfrak{a}$ *is an integral ideal of* $\mathcal{O}_K$ *if and only if* $\mathfrak{n}$ *is an integral ideal of* $\mathcal{O}_F$.
3. $\mathfrak{a}$ *is primitive in* K/F *if and only if* $\mathfrak{n} = \mathcal{O}_F$.
4. $\mathfrak{m}$ *is an integral ideal and* $\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{m}\mathfrak{n}^2$.

Remark. The element b is determined by the modulo relation

$$\begin{cases} \delta - b \in \mathfrak{q}, \\ b^2 + D \in \mathfrak{m}\mathfrak{q}^2, \end{cases}$$

where $\delta \in \mathcal{O}_F$ comes from the corresponding pseudomatrix on the basis $(1, \sqrt{D})$ [Cohen 2000, Corollary 2.2.9].

The equations (14) and (15) give us two decompositions of a fractional ideal $\mathfrak{a}$ of K. However, the main obstacle comes from the factor $\mathfrak{n}$ in (15). We may not easily get rid of $\mathfrak{n}$ unless the ideal class group $Cl(F)$ is trivial. Noting that $\mathfrak{n}$ is a content, one natural way is to use the decompositions to construct a group of primitive representatives of the ideal class group $Cl(K)$ such that the CM norms of the imaginary part of the corresponding CM points can be computed explicitly. In fact, It can be seen from the definition that an integral ideal $\mathfrak{a}$ of $\mathcal{O}_K$ is primitive if and only if its primary decomposition is of the following form:

$$\mathfrak{a} = \prod_j \mathfrak{P}'_j \cdot \prod_i \mathfrak{P}_i^{\alpha_i} \overline{\mathfrak{P}}_i^{\beta_i},$$

where $\mathfrak{P}'_j$ are ramified primes and $\mathfrak{P}_i$ are splitting primes with $\alpha_i \cdot \beta_i = 0$. In particular, every split prime ideal of $\mathcal{O}_K$ is primitive. On the other hand, by the Chebotarev density theorem, there exist a group of representatives of $Cl(K)$ consisting of split prime ideals. This gives us a set of primitive representatives of $Cl(K)$. However, since we have to bound these representatives uniformly (as can be seen in the last section) and it is not easy to give such a bound for splitting ideals in each ideal class, we move on in another way.

It's well known that, for any fractional $\mathcal{O}_F$-ideals $\mathfrak{a}$ and $\mathfrak{b}$, we have the isomorphism $\mathfrak{a} \oplus \mathfrak{b} \simeq \mathcal{O}_F \oplus \mathfrak{ab}$. But this is not enough to make (15) into the form of (14), we need to make the isomorphism into an identity.

Lemma 12. *Suppose K/F is a finite extension of number fields. Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of $\mathcal{O}_F$. Let α, β be two elements in $K^\times$. Assume that $a \in \mathfrak{a}$, $b \in \mathfrak{b}$, $c \in \mathfrak{b}^{-1}$ and $d \in \mathfrak{a}^{-1}$ such that $ad - bc = 1 \in F$. Set*

$$(\alpha', \beta') := (\alpha, \beta) \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

then we have

$$\mathfrak{a}\alpha + \mathfrak{b}\beta = \mathcal{O}_F\alpha' + \mathfrak{ab}\beta'.$$

Proof. We have $\alpha' = a\alpha + b\beta$ and $\beta' = c\alpha + d\beta$. Hence

$$\mathcal{O}_F\alpha' + \mathfrak{ab}\beta' \subset (\mathcal{O}_F \cdot a + \mathfrak{ab} \cdot c)\alpha + (\mathcal{O}_F \cdot b\mathfrak{ab} \cdot d)\beta \subset \mathfrak{a}\alpha + \mathfrak{b}\beta.$$

Conversely, we have $\alpha = d\alpha' - b\beta'$ and $\beta = -c\alpha' + a\beta'$. Hence

$$\mathfrak{a}\alpha + \mathfrak{b}\beta \subset \mathcal{O}_F\alpha' + \mathfrak{ab}\beta'. \qquad \square$$

Let $z_\mathfrak{a}$ be the associate CM point to $\mathfrak{a}$. Define the CM type norm of $y_\mathfrak{a}$ as $N_\Phi(y_\mathfrak{a}) := \prod_{\sigma\in\Phi} \sigma(y_\mathfrak{a})$. In Section 3.3 we will see that CM type norms show up naturally in Fourier coefficients of Hilbert Eisenstein series and their derivatives at the central value. According the period formula (27) CM type norms of imaginary parts of CM points also connect with central values of class group L-functions. By the above preparation we can prove an explicit expression of $N_\Phi(y_{\mathfrak{a}})$ as follows:

Proposition 13. *Let notation be as before. Then we have*

$$N_\Phi(y_{\mathfrak{a}}) = \frac{N_{K/\mathbb{Q}}(c_{\mathfrak{a}})N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})N_{F/\mathbb{Q}}(\mathfrak{q})^2}{2^n N_{K/\mathbb{Q}}(\mathfrak{a})} \cdot \frac{\sqrt{D_K}}{D_F}, \tag{16}$$

where $\mathfrak{f}_{\mathfrak{a}} \in St_{\mathfrak{a}}$, $c_{\mathfrak{a}}$ is an element in the content of $\mathfrak{a}\tilde{\mathfrak{q}}^{-1}$ that is of the minimal absolute norm, and D_K (resp. D_F) is the absolute discriminant of $K/\mathbb{Q}$ (resp. $F/\mathbb{Q}$).

Proof. By the argument in the above remark, we may assume $\mathfrak{a}$ is an integral ideal of $\mathcal{O}_K$ such that $\mathfrak{a}\tilde{\mathfrak{q}}^{-1}$ is integral. Let $\mathfrak{n}$ be the content of $\mathfrak{a}\tilde{\mathfrak{q}}^{-1}$, then $\mathfrak{n}$ is integral. Noting that $\mathfrak{q} \subset \mathcal{O}_F$, hence by (15) we have the decomposition

$$\mathfrak{a} = \mathfrak{n}\cdot(-b+\sqrt{D}) \oplus \mathfrak{n}^{-1}\bar{\mathfrak{q}}^{-1}\mathfrak{a}\bar{\mathfrak{a}} = \mathfrak{n}\cdot(-b+\sqrt{D}) \oplus \mathfrak{n}^{-1}\mathfrak{q}^{-1}\mathfrak{a}\bar{\mathfrak{a}}, \tag{17}$$

where $b \in \mathcal{O}_F$ and $\mathfrak{q}$ is the index-ideal $\left[\mathcal{O}_K : \mathcal{O}_F[\sqrt{D}]\right]$.

Let $c_{\mathfrak{a}} \in \mathfrak{n}$ be an element of the minimal absolute norm, and we fix one such choice for each $\mathfrak{a}$ once and for all. Then by Lemma 12 we have

$$\mathfrak{a} = \mathfrak{n}\cdot(-b+\sqrt{D}) \oplus \mathfrak{n}^{-1}\mathfrak{q}^{-1}\mathfrak{a}\bar{\mathfrak{a}} = \mathcal{O}_F\cdot c_{\mathfrak{a}}\cdot(-b+\sqrt{D}) \oplus \mathfrak{q}^{-1}\mathfrak{a}\bar{\mathfrak{a}}\cdot c_{\mathfrak{a}}^{-1}. \tag{18}$$

The direct sum in the right-hand side of the above identity can be verified easily from the proof of Lemma 12. Also noting that by the definition of $\mathfrak{q}$ we have $\mathfrak{d}_{K/F} = 4D\mathfrak{q}^{-2}$, where $\mathfrak{d}_{K/F}$ is the relative ideal-discriminant, then $[\mathfrak{q}^{-1}\mathfrak{a}\bar{\mathfrak{a}}]$ is the Steinitz class of $\mathfrak{a}$.

Combining the decomposition (17) with (14), i.e., $\mathfrak{a} = \mathcal{O}_F\alpha \oplus \mathfrak{f}_{\mathfrak{a}}\beta$, we have, by the uniqueness of Steinitz class, that

$$\alpha = c_{\mathfrak{a}}\cdot(-b+\sqrt{D})\epsilon \quad \text{and} \quad \mathfrak{f}_{\mathfrak{a}}\beta = \mathfrak{q}^{-1}\mathfrak{a}\bar{\mathfrak{a}}\cdot c_{\mathfrak{a}}^{-1},$$

for some unit $\epsilon \in \mathcal{O}_F^\times$. So we have

$$y_{\mathfrak{a}} = \mathfrak{Im}(z_{\mathfrak{a}}) = \frac{c_{\mathfrak{a}}\cdot\mathfrak{Im}(\sqrt{D})}{\beta}.$$

Noting that $\mathfrak{d}_{K/F} = N_{K/F}(\mathfrak{d}_{K/F})$ and $D_K = D_F^2 N_{F/\mathbb{Q}}(\mathfrak{d}_{K/F})$, we thus obtain

$$N_\Phi(y_{\mathfrak{a}}) = \prod_{\sigma\in\Phi}\left(\frac{c_{\mathfrak{a}}\cdot\sqrt{D}}{\beta}\right) = \frac{N_{K/\mathbb{Q}}(c_{\mathfrak{a}})N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})N_{F/\mathbb{Q}}(\mathfrak{q})^2}{2^n N_{K/\mathbb{Q}}(\mathfrak{a})} \cdot \frac{\sqrt{D_K}}{D_F}. \qquad \square$$

Remark. From the above expression, it is clear that $N_\Phi(y_{\mathfrak{a}})$ is independent of a particular choice of $\mathfrak{f}_{\mathfrak{a}} \in St_{\mathfrak{a}}$. Also, the term $N_{K/\mathbb{Q}}(c_{\mathfrak{a}})$ in the right-hand side of (16) does not depend on a particular choice of $c_{\mathfrak{a}}$. In fact (16) shows that $N_\Phi(y_{\mathfrak{a}})$ is independent of the choice of a particular representative of the class $[\mathfrak{a}]$. This is because the factors $N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})$ and $N_{K/\mathbb{Q}}(c_{\mathfrak{a}}\mathfrak{a}^{-1})$ are both invariant under scalar multiplication by $K^\times$.

We will always fix the CM type Φ in this paper. For the sake of simplicity, we will write $y_{\mathfrak{a}}^{\sigma}$ for the CM type norm $N_{\Phi}(y_{\mathfrak{a}})$ in computations in the following parts.

3.3. ***Hilbert Eisenstein series.*** Let notation be as before. Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of F. Take φ to be the characteristic function of the closure of $\mathfrak{ab}\oplus\mathfrak{b}$. Let $\varphi_{\mathfrak{ab}}$ be the characteristic function of the closure of $\mathfrak{ab}$, and $\varphi_{\mathfrak{b}}$ be the characteristic function of the closure of $\mathfrak{b}$. Then we define

$$G_{\mathbf{k}}(z,s;\varphi)=y^{-\mathbf{k}/2+s\sigma}\sum_{(c,d)\in F^{2,\times}/\mathcal{O}_F^{\times}}\varphi(c,d)(cz+d)^{-\mathbf{k}}|cz+d|^{\mathbf{k}-2s\sigma}.$$

Define

$$\Gamma_{\mathfrak{a}}:=\left\{\gamma\in\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}(2,F):\ a,d\in\mathcal{O}_F,\ b\in\mathfrak{a}^{-1},\ c\in\mathfrak{a},\ \det\gamma\in\mathcal{O}_F^{+}\right\}.$$

Then clearly $\varphi(x\gamma)=\varphi(x)$ for $x\in F\oplus F$, $\gamma\in\Gamma_{\mathfrak{a}}$. From now on, we assume $\mathbf{k}=0$. One can check that

$$G_{\mathbf{k}}(\gamma z,s;\mathfrak{a},\mathfrak{b})=G_{\mathbf{k}}(z,s;\mathfrak{a},\mathfrak{b})\quad\text{for all }\gamma\in\Gamma_{\mathfrak{a}}.$$

Let $G(z,s;\mathfrak{a},\mathfrak{b}):=G_0(z,s;\mathfrak{a},\mathfrak{b})$ and define the regularized Eisenstein series as

$$E(z,s;\mathfrak{a},\mathfrak{b}):=\zeta_F(2s)^{-1}G(z,s;\mathfrak{a},\mathfrak{b}),\ \mathrm{Re}(s)>1.$$

Then based on the Fourier expansion of $G(z,s;\mathfrak{a},\mathfrak{b})$ [Yoshida 2003, Chapter V] we have the explicit Fourier expansion:

$$\begin{aligned}E(z,s;\mathfrak{a},\mathfrak{b})&=N(\mathfrak{b})^{-2s}y^{s\sigma}\frac{\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F(2s)}\\&\quad+\left(\frac{\sqrt{\pi}\,\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}\right)^{n}D_F^{-1/2}N(\mathfrak{b})^{-1}y^{(1-s)\sigma}N(\mathfrak{ab})^{1-2s}\frac{\zeta_F(2s-1,[\mathfrak{ab}]^{-1})}{\zeta_F(2s)}\\&\quad+\left(\frac{2\pi^{s}}{\Gamma(s)}\right)^{n}\frac{y^{\sigma/2}}{D_F^{1/2}N(\mathfrak{b})\zeta_F(2s)}\sum_{b\in F^{\times}}|N(b)|^{s-1/2}\lambda(b,s)\,\mathbf{e}(bx)\\&\quad\times\prod_{v\in\mathbf{J}_{\infty}}K_{s-\frac{1}{2}}(2\pi y_v|b_v|).\end{aligned}$$

We have the following Laurent expansion of (partial) Dedekind zeta function around $s=1$:

$$\zeta_F(s,\mathcal{C})=\frac{h_F^{-1}\rho_F}{s-1}+\gamma_{F,[\mathcal{C}]}+O(s-1),\tag{19}$$

where $\mathcal{C}$ is an ideal class in $Cl(F)$, and $\rho_F = 2^n h_F R_F w_F^{-1} D_F{}^{-1/2}$ is the residue of $\zeta_F(s)$ at $s = 1$. In particular, around $s = 1$ we have

$$\zeta_F(s) = \frac{\rho_F}{s-1} + \gamma_F + O(s-1), \tag{20}$$

where

$$\gamma_{F,\mathcal{C}} := \lim_{s\mapsto 1}\left\{\zeta_F(s,\mathcal{C}) - \frac{h_F^{-1}\rho_F}{s-1}\right\};$$

and by $\zeta_F(s) = \sum_{[\mathcal{C}]\in Cl(F)} \zeta_F(s,\mathcal{C})$ we have

$$\gamma_F = \sum_{\mathcal{C}\in Cl(F)} \gamma_{F,\mathcal{C}} = \rho_F \gamma_F^*,$$

where γ_F^* is defined in (1). Constants γ_F and $\gamma_{F,\mathcal{C}}$ are called unnormalized Euler–Kronecker constants with respect to $F/\mathbb{Q}$, which we will deal with later.

From the Fourier expansion above we see that $E(z, s; \mathfrak{a}, \mathfrak{b})$ has a meromorphic continuation to $\mathbb{C}$ with a simple pole at $s = 1$ with residue

$$\mathrm{Res}_{s=1} E(z, s; \mathfrak{a}, \mathfrak{b}) = \frac{2^{n-1}\pi^n R_F}{w_F D_F N(\mathfrak{b}) N(\mathfrak{a}\mathfrak{b}) \zeta_F(2)}.$$

We have the Taylor expansion around $s = 0$ such that $\Gamma(s) = s^{-1} + O(1)$ and

$$\zeta_F(s) = -\frac{h_F R_F}{w_F} s^{n-1} + O(s^n), \tag{21}$$

where R_F is the regulator and w_F is the number of roots of unity. Then $E(z, s; \mathfrak{a}, \mathfrak{b})$ is holomorphic at $s = \frac{1}{2}$. Moreover, we can show actually $E(z, s; \mathfrak{a}, \mathfrak{b})$ vanishes at $s = \frac{1}{2}$ for all $z \in \mathbb{C}$.

Lemma 14. *Let notation be as above. Then we have*

$$E\big(z, \tfrac{1}{2}; \mathfrak{a}, \mathfrak{b}\big) \equiv 0 \quad \text{for all } z \in \mathbb{H}^n.$$

Proof. Let $\mathfrak{f} \subset F$ be a fractional ideal and $\tilde{\mathfrak{f}}$ be its dual, i.e., $[\mathfrak{f}] \cdot [\tilde{\mathfrak{f}}] = [\mathfrak{D}]$, where $\mathfrak{D}$ is the different of $F/\mathbb{Q}$. Let

$$\mathcal{Z}_F(s, [\mathfrak{f}]) := \mathcal{Z}_{F,\infty}(s)\zeta_F(s, [\mathfrak{f}]) \quad \text{and} \quad \mathcal{Z}_F(s) := \mathcal{Z}_{F,\infty}(s)\zeta_F(s),$$

where $\mathcal{Z}_{F,\infty}(s) := D_F{}^{s/2}\pi^{-ns/2}\Gamma(s/2)^n$. It is well known that we have the following functional equation for the partial completed zeta function:

$$\mathcal{Z}_F(s, [\mathfrak{f}]) = \mathcal{Z}_F(1-s, [\tilde{\mathfrak{f}}]). \tag{22}$$

An immediate consequence of this is the functional equation for the completed Dedekind zeta function (obtained adding the partial ones), which has exactly the same form. Also, all the partial zeta functions have a simple pole at $s = 1$ with the same residue $2^n R_F w_F^{-1} D_F^{-1/2}$.

Let's introduce some functions to simplify the notations. Define

$$\begin{aligned} M_1(s) &:= \mathcal{Z}_F(2s) N(\mathfrak{b})^{-2s} y^{s\sigma} \frac{\zeta_F(2s, [\mathfrak{b}]^{-1})}{\zeta_F(2s)} \\ &= \mathcal{Z}_{F,\infty}(2s) N(\mathfrak{b})^{-2s} y^{s\sigma} \zeta_F(2s, [\mathfrak{b}]^{-1}); \end{aligned}$$

$$M_2^*(s) := \left(\frac{\sqrt{\pi}\,\Gamma\big(s-\frac{1}{2}\big)}{\Gamma(s)} \right)^n D_F^{-1/2} N(\mathfrak{b})^{-1} y^{(1-s)\sigma} N(\mathfrak{a}\mathfrak{b})^{1-2s} \frac{\zeta_F(2s-1, [\mathfrak{a}\mathfrak{b}]^{-1})}{\zeta_F(2s)};$$

$$\begin{aligned} M_2(s) &:= \mathcal{Z}_F(2s) M_2^*(s) \\ &= \mathcal{Z}_{F,\infty}(2s-1) N(\mathfrak{b})^{-1} y^{(1-s)\sigma} N(\mathfrak{a}\mathfrak{b})^{1-2s} \zeta_F(2s-1, [\mathfrak{a}\mathfrak{b}]^{-1}). \end{aligned}$$

Then by (22) (taking $\mathfrak{f} = \mathfrak{a}\mathfrak{b}$) we see that $M_2(1-s) = H(s) M_1(s)$, where

$$H(s) := N(\mathfrak{b})^{2s-1} N(\mathfrak{a}\mathfrak{b})^{2s-1} \cdot \frac{\zeta_F(2s, [\widetilde{\mathfrak{a}\mathfrak{b}}]^{-1})}{\zeta_F(2s, [\mathfrak{b}]^{-1})}.$$

Since the Dirichlet series $\zeta_F(2s, [\mathfrak{b}]^{-1})$ absolutely converges when $\mathfrak{Re}(s) > \frac{1}{2}$, $H(s)$ is holomorphic when $\mathfrak{Re}(s) > \frac{1}{2}$, and can be continued to a meromorphic function on $\mathbb{C}$. Since all the partial zeta functions have a simple pole at $s = 1$ with the same residue, we see $H(s)$ is holomorphic at $s = \frac{1}{2}$ and $H\big(\frac{1}{2}\big) = 1$.

Let $L(s)^{-1} := H(s) H(1-s)$, then we have

$$L(s)^{-1} = \frac{\zeta_F(2s, [\widetilde{\mathfrak{a}\mathfrak{b}}]^{-1}) \zeta_F(2-2s, [\widetilde{\mathfrak{a}\mathfrak{b}}]^{-1})}{\zeta_F(2s, [\mathfrak{b}]^{-1}) \zeta_F(2-2s, [\mathfrak{b}]^{-1})}.$$

Likewise, $L(s)$ is a meromorphic function on $\mathbb{C}$ and is analytic at $s = \frac{1}{2}$, with $L\big(\frac{1}{2}\big) = 1$. So we have $M(1-s) = H(s) M_1(S) + H(s) L(s) M_2(s)$. Now let

$$E_1(s) := \mathcal{Z}_F(2s) \left(\frac{2\pi^s}{\Gamma(s)} \right)^n \frac{y^{\sigma/2}}{D_F^{1/2} N(\mathfrak{b}) \zeta_F(2s)} = 2^n D_F^s \frac{y^{\sigma/2}}{D_F^{1/2} N(\mathfrak{b})};$$

$$\begin{aligned} E_2(s) &:= \sum_{b \in F^\times} |N(b)|^{s-1/2} \lambda(b, s) \mathbf{e}(bx) \prod_{v \in \mathbf{J}_\infty} K_{s-\frac{1}{2}}(2\pi y_v |b_v|) \\ &= \sum_{b \in F^\times} |N(b)|^{s-1/2} \sum\sum_{\substack{(a,c) \in \mathfrak{b}^{-1}\mathfrak{o}^{-1} \times \mathcal{C} \\ ac = b}} |N(c)|^{1-2s} \mathbf{e}(bx) \prod_{v \in \mathbf{J}_\infty} K_{s-\frac{1}{2}}(2\pi y_v |b_v|) \\ &= \sum_{b \in F^\times} \sum\sum_{\substack{(a,c) \in \mathfrak{b}^{-1}\mathfrak{o}^{-1} \times \mathcal{C} \\ ac = b}} \left(\frac{|N(a)|}{|N(c)|} \right)^{s-\frac{1}{2}} \mathbf{e}(bx) \prod_{v \in \mathbf{J}_\infty} K_{s-\frac{1}{2}}(2\pi y_v |b_v|). \end{aligned}$$

Also set $E(s) = E_1(s)E_2(s)$. In fact both $E_1(s)$ and $E_2(s)$ are entire functions. Let's briefly explain why $E_2(s)$ is entire: by Turan's inequality for Bessel functions, $\log K_\nu(x)$ is convex. Also note the fact that

$$\lim_{x\mapsto+\infty} \frac{\log K_\nu(x)}{x} = -1,$$

hence we have $K_\nu(x) \leq (eK_\nu(1))e^{-x}$. Namely, the Bessel K-function has exponential decay, which forces the sum in $E_2(s)$ to converge absolutely. Also one sees easily that $E_1(s) = D_F^{2s-1}E_1(1-s)$. So we have

$$E(1-s) = D_F^{1-2s}E_1(s)E_2(1-s).$$

Let $\mathcal{E}(z, s; \mathfrak{a}, \mathfrak{b}) := \mathcal{Z}_F(2s)E(z, s; \mathfrak{a}, \mathfrak{b})$ be completed Eisenstein series, then by the Fourier expansion of $E(z, s; \mathfrak{a}, \mathfrak{b})$ we have

$$\mathcal{E}(z, s; \mathfrak{a}, \mathfrak{b}) = M(s) + E(s) = M_1(s) + M_2(s) + E_1(s)E_2(s).$$

By the above computation we have

$$\mathcal{E}(z, 1-s; \mathfrak{a}, \mathfrak{b}) = H(s)M_1(s) + L(s)H(s)M_2(s) + D_F^{1-2s}E_1(s)E_2(1-s).$$

Therefore, we have (noting that $K_\nu(x) = K_{-\nu}(x)$)

$$\begin{aligned} E\big(z, \tfrac{1}{2}; \mathfrak{a}, \mathfrak{b}\big) &= \lim_{s\mapsto\frac{1}{2}} \frac{\mathcal{E}(z, s; \mathfrak{a}, \mathfrak{b})}{\mathcal{Z}_{F,\infty}(2s)\zeta_F(2s)} = \lim_{s\mapsto\frac{1}{2}} \frac{\mathcal{E}(z, 1-s; \mathfrak{a}, \mathfrak{b})}{\mathcal{Z}_{F,\infty}(2-2s)\zeta_F(2-2s)} \\ &= -\lim_{s\mapsto\frac{1}{2}} \frac{H(s)M_1(s) + L(s)H(s)M_2(s) + D_F^{1-2s}E_1(s)E_2(1-s)}{\mathcal{Z}_{F,\infty}(2-2s)\zeta_F(2s)} \\ &= -\lim_{s\mapsto\frac{1}{2}} \frac{H\big(\frac{1}{2}\big)M_1(s) + L\big(\frac{1}{2}\big)H\big(\frac{1}{2}\big)M_2(s) + E_1\big(\frac{1}{2}\big)E_2\big(\frac{1}{2}\big)}{\mathcal{Z}_{F,\infty}(2s)\zeta_F(2s)} \\ &= -\lim_{s\mapsto\frac{1}{2}} \frac{\mathcal{E}(z, s; \mathfrak{a}, \mathfrak{b})}{\mathcal{Z}_{F,\infty}(2s)\zeta_F(2s)} = -E\big(z, \tfrac{1}{2}; \mathfrak{a}, \mathfrak{b}\big). \end{aligned}$$

Thus we have $E\big(z, \frac{1}{2}; \mathfrak{a}, \mathfrak{b}\big) = 0$. □

Remark. Note that $E(z, s; \mathfrak{a}, \mathfrak{b})$ may not have a functional equation, since the Hilbert modular variety may have several cusps. The Eisenstein matrix will always have a functional equation. However, when $\mathfrak{a} = \mathfrak{b} = \mathcal{O}_F$, there is only one cusp. In fact, we see that in this situation $H(s) = L(s) = 1$ and $E(s) = E(1-s)$, then we have the functional equation

$$\mathcal{E}(z, s; \mathfrak{a}, \mathfrak{b}) = \mathcal{E}(z, 1-s; \mathfrak{a}, \mathfrak{b}),$$

which gives immediately that $E\big(z, \frac{1}{2}; \mathfrak{a}, \mathfrak{b}\big) = 0$. This is the case in [Iwaniec and Kowalski 2004].

3.4. ***Periods of Eisenstein series.*** In this section we combine the discussion in last two subsections to show the class group L-function $L_K(\chi, s)$ can be expressed as a weighted period of the Eisenstein series $E(z, s; \mathfrak{f})$ with respect to the CM 0-cycles $\mathcal{CM}(K, \Phi, \mathfrak{f})$, where $[\mathfrak{f}] \in Cl(F)^+$.

Recall that we have the natural surjective map

$$\mathcal{CM}(K, \Phi) \twoheadrightarrow Cl(K), \quad [\mathfrak{a}, r] \mapsto [\mathfrak{a}].$$

And the fiber is indexed by $\epsilon \in \mathcal{O}_F^{\times,+}/N_{K/F}\mathcal{O}_K^{\times}$ with order at most 2.

Recall that by Proposition 13, since K/F is a CM extension of number fields of degree $2n$, we have that (see [Zimmert 1981]) for each ideal class $\mathcal{C} \in Cl(K)$, there exists an integral ideal $\mathfrak{a}_{\mathcal{C}} \in \mathcal{C}$ such that

$$N_{K/\mathbb{Q}}(\mathfrak{a}_{\mathcal{C}}) \leq M(0, n)\sqrt{D_K}.$$

Clearly, we may assume $\mathfrak{a}_{[\mathcal{O}_K]} = \mathcal{O}_K$. Thus we can define a set of representatives of $Cl(K)$ as

$$\mathcal{I}_K := \{\mathfrak{a} : \ \mathfrak{a} = \mathfrak{a}_{\mathcal{C}}\mathfrak{q} \ \text{ for all } \mathcal{C} \in Cl(K)\}, \tag{23}$$

where $\mathfrak{q}$ is the index ideal in (15).

For convenience, let us fix $\mathcal{I}_K$ once and for all. Clearly we have

$$Cl(K) = \{[\mathfrak{a}] : \ \mathfrak{a} \in \mathcal{I}_K\}.$$

For any $\mathfrak{a} \in \mathcal{I}_K$, let $y_{\mathfrak{a}}$ be the imaginary part of $z_{\mathfrak{a}}$, the associated CM point. Then by (16) we have

$$y_{\mathfrak{a}}^{\sigma} = N_{\Phi}(y_{\mathfrak{a}}) \geq \frac{M(0, n)^{-1} N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})}{2^n D_F} \quad \text{ for all } \mathfrak{a} \in \mathcal{I}_K. \tag{24}$$

Also, for any fractional $\mathfrak{a}$ of K, there exists a unique $\mathfrak{f}_{\mathfrak{a}} \in \mathcal{I}_F^+$ and a CM point $[\mathfrak{a}, r] \in X(\mathfrak{f}_{\mathfrak{a}}) := \Gamma(\mathfrak{f}_{\mathfrak{a}}) \backslash \mathbb{H}^n$ mapping to $[\mathfrak{a}]$. Note that there are at most 2 preimages of $[\mathfrak{a}] \in Cl(K)$. From now on, we fix one of them $[\mathfrak{a}, r] \in \mathcal{CM}(K, \Phi, \mathfrak{f}_{\mathfrak{a}})$ for all $\mathfrak{a}$.

Then we have a decomposition (14), i.e.,

$$\mathfrak{a} = \mathcal{O}_F \alpha_{\mathfrak{a}} + \mathfrak{f}_{\mathfrak{a}} \beta_{\mathfrak{a}}$$

with $z_{\mathfrak{a}} := \alpha_{\mathfrak{a}}/\beta_{\mathfrak{a}} \in K^{\times} \cap \mathbb{H}^n = \{z \in K^{\times} : \Phi(z) \in \mathbb{H}^n\}$. Moreover, $z_{\mathfrak{a}}$ represents the CM point $[\mathfrak{a}, r]$.

Proposition 15. *Let K be a CM extension of a totally real number field F of degree n, and Φ be a CM type of K. Then we have*

$$L_K(\chi, s) = \frac{(2^n D_F)^s}{D_K^{s/2}[\mathcal{O}_K^{\times} : \mathcal{O}_F^{\times}]} \sum_{[\mathfrak{a}^{-1}] \in Cl(K)} \overline{\chi}([\mathfrak{a}]) N(\mathfrak{f}_{\mathfrak{a}})^s \zeta_F(2s) E(z_{\mathfrak{a}}, s; \mathfrak{f}_{\mathfrak{a}}^{-1}, \mathfrak{f}_{\mathfrak{a}}), \tag{25}$$

where $\mathfrak{f}_{\mathfrak{a}} \in \mathcal{I}_F^+$ is defined as above and $z_{\mathfrak{a}}$ is the corresponding CM point of $\mathfrak{a}$ via the map $\mathcal{CM}(K, \Phi) \twoheadrightarrow Cl(K)$.

Proof. Let $C \in Cl(K)$ be an ideal class. Then there exist a unique primitive ideal $\mathfrak{a} \in \mathcal{I}_K$ such that $[\mathfrak{a}] = C^{-1}$. Hence as $\mathfrak{b}$ runs over integral ideals in C, $\mathfrak{a}\mathfrak{b} = (w)$ runs over principal ideals (w) with $w \in \mathfrak{a}/\mathcal{O}_K^\times$. Let $\sum'$ denote that the summation is taken over nonzero integral variables (e.g., $\sum'_{\mathfrak{a}}$ means the summation is taken over all nonzero integral ideals $\mathfrak{a} \subset K$), then the partial Dedekind zeta function can be written

$$\begin{aligned}\zeta_K(s, C) &= \sum_{\mathfrak{b}\in C}{}' N_{K/\mathbb{Q}}(\mathfrak{b})^{-s} = N_{K/\mathbb{Q}}(\mathfrak{a})^s \sum_{w\in\mathfrak{a}/\mathcal{O}_K^\times}{}' N_{K/\mathbb{Q}}((w))^{-s} \\ &= \frac{N_{K/\mathbb{Q}}(\mathfrak{a})^s}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{(c,d)\in\mathcal{O}_F\oplus\mathfrak{f}_{\mathfrak{a}}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((c\alpha_{\mathfrak{a}} + d\beta_{\mathfrak{a}}))^{-s} \\ &= \frac{N_{K/\mathbb{Q}}(\mathfrak{a})^s N_{K/\mathbb{Q}}((\beta_{\mathfrak{a}}))^{-s}}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{(c,d)\in\mathcal{O}_F\oplus\mathfrak{f}_{\mathfrak{a}}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((cz_{\mathfrak{a}} + d))^{-s} \\ &= \frac{N_{K/\mathbb{Q}}(\mathcal{O}_F z_{\mathfrak{a}} + \mathfrak{f}_{\mathfrak{a}})^s}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{(c,d)\in\mathcal{O}_F\oplus\mathfrak{f}_{\mathfrak{a}}/\mathcal{O}_F^\times}{}' N_{K/\mathbb{Q}}((cz_{\mathfrak{a}} + d))^{-s}.\end{aligned}$$

Write $z_{\mathfrak{a}} = x_{\mathfrak{a}} + iy_{\mathfrak{a}}$, then a calculation with determinants yields

$$N_{K/\mathbb{Q}}(\mathcal{O}_F z_{\mathfrak{a}} + \mathfrak{f}_{\mathfrak{a}}) = y_{\mathfrak{a}}^\sigma N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}}) \cdot \frac{2^n D_F}{\sqrt{D_K}}. \tag{26}$$

By a calculation with the CM type Φ we have $N_{K/\mathbb{Q}}((cz_{\mathfrak{a}} + d)) = |cz_{\mathfrak{a}} + d|^{2\sigma}$, where we have identified $z_{\mathfrak{a}}$ with $\Phi(z_{\mathfrak{a}}) \in \mathbb{H}^n$. Thus by combining the preceding computations we obtain

$$\begin{aligned}\zeta_K(s, C) &= \frac{(2^n D_F N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}}))^s}{D_K^{s/2}[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{(c,d)\in\mathcal{O}_F\oplus\mathfrak{f}_{\mathfrak{a}}/\mathcal{O}_F^\times}{}' y_{\mathfrak{a}}^{s\sigma} |cz_{\mathfrak{a}} + d|^{-2s\sigma} \\ &= \frac{(2^n D_F N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}}))^s}{D_K^{s/2}[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} G(z_{\mathfrak{a}}, s; \mathfrak{f}_{\mathfrak{a}}^{-1}, \mathfrak{f}_{\mathfrak{a}}).\end{aligned}$$

Finally, using that $L_K(\chi, s) = \sum_{C\in Cl(K)} \chi(C)\zeta_K(s, C)$ we obtain (25). □

In particular, it comes from Lemma 14 and Proposition 15 that:

Proposition 16. *Let notations be as above. Then we have*

$$L_K\big(\chi, \tfrac{1}{2}\big) = \frac{2^{n/2-1}\rho_F\sqrt{D_F}}{D_K^{1/4}[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{[\mathfrak{a}^{-1}]\in Cl(K)} \bar{\chi}([\mathfrak{a}])\sqrt{N(\mathfrak{f}_{\mathfrak{a}})} E'\big(z_{\mathfrak{a}}, \tfrac{1}{2}; \mathfrak{f}_{\mathfrak{a}}^{-1}, \mathfrak{f}_{\mathfrak{a}}\big). \tag{27}$$

Remark. Note that $\sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})} E'\big(z_{\mathfrak{a}}, \tfrac{1}{2}; \mathfrak{f}_{\mathfrak{a}}^{-1}, \mathfrak{f}_{\mathfrak{a}}\big)$ is independent of the choice of $\mathfrak{f}_{\mathfrak{a}}$ for any fractional ideal $\mathfrak{a}$ in K. Formula (27) is known when the narrow class group of F is trivial [Masri 2010], in which case one can take $\mathfrak{f}_{\mathfrak{a}}$ to be $\mathcal{O}_K$.

We will use this Eisenstein period formula (27) in conjunction with the CM type norm formula (16), following an idea of Iwaniec and Kowalski [2004], to obtain Theorem A in Section 4.

3.5. *Derivatives of Eisenstein series at the central point.* In this subsection the derivative of the Eisenstein series and its Fourier expansion will be investigated. Further estimates will be provided in the next section. We start from the vanishing property of $E'\big(z,\frac{1}{2};\mathfrak{a},\mathfrak{b}\big)$ as follows.

Lemma 17. *Let $\mathfrak{a}$ and $\mathfrak{b}$ be fractional ideals of F as before. Then we have*

$$\begin{aligned}E'\big(z,\tfrac{1}{2};\mathfrak{a},\mathfrak{b}\big)&\\ =\frac{y^{\sigma/2}}{N_{F/\mathbb{Q}}(\mathfrak{b})}\Bigg\{&2h_F^{-1}\log y^{\sigma}+\frac{4(\gamma_{F,[\mathfrak{b}]^{-1}}-h_F^{-1}\gamma_F)}{h_F\rho_F}\\ &+h_F^{-1}\big[\log N_{F/\mathbb{Q}}(\mathfrak{a}\mathfrak{b}^{-1})-n(\gamma+2\log 2)\big]\\ &+\frac{2^{n+1}}{\rho_F\sqrt{D_F}}\sum_{b\in F^{\times}}\sum_{\substack{(a,c)\in\mathfrak{b}^{-1}\mathfrak{o}^{-1}\times\mathcal{C}\\ ac=b}}\sum \mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_0(2\pi y_v|b_v|)\Bigg\},\end{aligned}$$

where the Euler–Kronecker constants $\gamma_{F,[\mathfrak{b}]^{-1}}$ and γ_F are defined in (19) *and* (20) *respectively*; $\gamma=0.57721\ldots$ *is the Euler–Mascheroni constant and*

$$\mathcal{C}:=(\mathfrak{a}\mathfrak{b}\cap F^{\times})/\mathcal{O}_F^{\times,+}.$$

Proof. To simplify the computation, let's introduce some notation. Set

$$M_1(s):=N_{F/\mathbb{Q}}(\mathfrak{b})^{-2s}y^{s\sigma}\frac{\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F(2s)};$$

$$M_2(s):=\left(\frac{\sqrt{\pi}\,\Gamma\big(s-\frac{1}{2}\big)}{\Gamma(s)}\right)^{n}D_F^{-1/2}N(\mathfrak{b})^{-1}y^{(1-s)\sigma}N(\mathfrak{a}\mathfrak{b})^{1-2s}\frac{\zeta_F(2s-1,[\mathfrak{a}\mathfrak{b}]^{-1})}{\zeta_F(2s)};$$

$$E(s):=\frac{2^n\pi^{ns}y^{\sigma/2}}{\sqrt{D_F}\,\Gamma(s)^nN(\mathfrak{b})\zeta_F(2s)}\sum_{b\in F^{\times}}|N(b)|^{s-\frac{1}{2}}\lambda(b,s)\mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_{s-\frac{1}{2}}(2\pi y_v|b_v|).$$

Note that for convenience, we shorten the notation of the norm $N_{F/\mathbb{Q}}$ to N occasionally. Then clearly $E'\big(z,\frac{1}{2};\mathfrak{a},\mathfrak{b}\big)=M_1'\big(\frac{1}{2}\big)+M_2'\big(\frac{1}{2}\big)+E'\big(\frac{1}{2}\big)$. Also,

$$M_1'(s)=\big[-2\log N(\mathfrak{b})+\log y^{\sigma}\big]\cdot M_1(s)+N(\mathfrak{b})^{-2s}y^{s\sigma}\left(\frac{\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F(2s)}\right)'.$$

By (19) and (20) we have

$$\left(\frac{\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F(2s)}\right)'\bigg|_{s=\frac{1}{2}} = \lim_{s\mapsto\frac{1}{2}} \frac{2\zeta_F'(2s,[\mathfrak{b}]^{-1})\zeta_F(2s)-2\zeta_F'(2s)\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F^2(2s)}$$
$$= \lim_{s\mapsto\frac{1}{2}} \frac{\frac{4\rho_F}{(2s-1)^2}\left(\frac{h_F^{-1}\rho_F}{2s-1}+\gamma_{F,[\mathfrak{b}]^{-1}}\right)-\frac{4h_F^{-1}\rho_F}{(2s-1)^2}\left(\frac{\rho_F}{2s-1}+\gamma_F\right)}{\frac{\rho_F^2}{(2s-1)^2}}$$
$$= \frac{4(\gamma_{F,[\mathfrak{b}]^{-1}}-h_F^{-1}\gamma_F)}{\rho_F}.$$

Note that by (19) and (20) we have

$$M_1\left(\tfrac{1}{2}\right) = \frac{y^{\sigma/2}}{N(\mathfrak{b})}\lim_{s\mapsto\frac{1}{2}}\frac{\zeta_F(2s,[\mathfrak{b}]^{-1})}{\zeta_F(2s)} = \frac{y^{\sigma/2}}{h_F N(\mathfrak{b})}.$$

Thus we have

$$M_1'\left(\tfrac{1}{2}\right) = \frac{y^{\sigma/2}}{h_F N(\mathfrak{b})}\left\{\log\frac{y^\sigma}{N(\mathfrak{b})^2}+\frac{4(\gamma_{F,[\mathfrak{b}]^{-1}}-h_F^{-1}\gamma_F)}{\rho_F}\right\}. \tag{28}$$

For the $M_2'(\frac{1}{2})$-term, by definition we have

$$\log M_2(s) = C + n\log\Gamma\left(s-\tfrac{1}{2}\right) - n\log\Gamma(s) + (1-s)\log y^\sigma$$
$$+ (1-2s)\log N(\mathfrak{ab}) + \log\zeta_F(2s-1,[\mathfrak{ab}]^{-1}) - \log\zeta_F(2s),$$

where $C := n\log\sqrt{\pi} - \frac{1}{2}\log D_F - \log N(\mathfrak{b})$. From this identity we obtain

$$M_2'(s) = M_2(s)\cdot\left\{\frac{n\Gamma'\left(s-\frac{1}{2}\right)}{\Gamma\left(s-\frac{1}{2}\right)} - \frac{n\Gamma'(s)}{\Gamma(s)} - \log y^\sigma - 2s\log N(\mathfrak{ab})\right.$$
$$\left. + \frac{2\zeta_F'(2s-1,[\mathfrak{ab}]^{-1})}{\zeta_F(2s-1,[\mathfrak{ab}]^{-1})} - \frac{2\zeta_F'(2s-1)}{\zeta_F(2s-1)}\right\}.$$

Since $\Gamma\left(s-\frac{1}{2}\right)\sim\left(s-\frac{1}{2}\right)^{-1}$ around $s=\frac{1}{2}$ and noting (20) and (21), we obtain

$$M_2'\left(\tfrac{1}{2}\right) = M_2\left(\tfrac{1}{2}\right)\lim_{s\mapsto\frac{1}{2}}\left\{-\frac{n}{s-\frac{1}{2}} - \frac{n\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} - \log y^\sigma - 2s\log N(\mathfrak{ab}) + \frac{n-1}{s-\frac{1}{2}} + \frac{1}{s-\frac{1}{2}}\right\}$$
$$= -M_2\left(\tfrac{1}{2}\right)\cdot\left\{\frac{n\Gamma'\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} + \log y^\sigma + \log N(\mathfrak{ab})\right\}.$$

By (20) and (21) and the functional equation (22) one can easily deduce that $M_2\left(\frac{1}{2}\right) = -h_F^{-1}N(\mathfrak{b})^{-1}y^{\sigma/2}$. Hence we have

$$M_2'\left(\tfrac{1}{2}\right) = h_F^{-1}N(\mathfrak{b})^{-1}y^{\sigma/2}\cdot\left\{\frac{n\Gamma'\left(\frac{1}{2}\right)}{\sqrt{\pi}} + \log y^{\sigma} + \log N(\mathfrak{a}\mathfrak{b})\right\}. \tag{29}$$

Now let's compute $\Gamma'\left(\frac{1}{2}\right)$: Differentiating the Hadamard decomposition of $\Gamma(s)^{-1}$ logarithmically at $s = 1$ we see $\Gamma'(1) = -\gamma$ by the definition of γ. Since $\Gamma(s+1) = s\Gamma(s)$, we have $\Gamma'(2) = 1-\gamma$. Now consider the duplication formula

$$\Gamma(2s) = \pi^{-1/2}2^{2s-1}\Gamma(s)\Gamma\left(s+\tfrac{1}{2}\right).$$

Differentiating it at $s = \frac{1}{2}$ we thus obtain $\Gamma'\left(\frac{1}{2}\right) = -\sqrt{\pi}(\gamma + 2\log 2)$. Plug this into (29) to obtain

$$M_2'\left(\tfrac{1}{2}\right) = h_F^{-1}N(\mathfrak{b})^{-1}y^{\sigma/2}\cdot\{\log y^{\sigma} + \log N(\mathfrak{a}\mathfrak{b}) - n(\gamma + 2\log 2)\}. \tag{30}$$

Finally we deal with $E'\left(\frac{1}{2}\right)$-term. Noting that $\lim_{s\mapsto 1/2}\zeta_F^{-1}(2s) = 0$, we have

$$\begin{aligned}
E'\left(\tfrac{1}{2}\right) &= \frac{-2^n y^{\sigma/2}}{\sqrt{D_F}N(\mathfrak{b})}\lim_{s\mapsto\frac{1}{2}}\frac{2\zeta_F'(2s)}{\zeta_F^2(2s)} \\
&\qquad\times\sum_{b\in F^\times}|N(b)|^{s-1/2}\lambda(b,s)\mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_{s-\frac{1}{2}}(2\pi y_v|b_v|) \\
&= \frac{2^{n+1}y^{\sigma/2}}{D_F^{1/2}N(\mathfrak{b})\rho_F}\sum_{b\in F^\times}\lambda\left(b,\tfrac{1}{2}\right)\mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_{s-\frac{1}{2}}(2\pi y_v|b_v|) \\
&= \frac{2^{n+1}y^{\sigma/2}}{D_F^{1/2}N(\mathfrak{b})\rho_F}\sum_{b\in F^\times}\sum_{\substack{(a,c)\in\mathfrak{b}^{-1}\mathfrak{o}^{-1}\times\mathcal{C}\\ ac=b}}\sum\mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_0(2\pi y_v|b_v|).
\end{aligned}$$

Combining this formula with (28) and (30) we thus obtain the conclusion. □

4. Proof of the Main Theorems

4.1. ***Estimates related to L-functions.*** Let F be a totally real number field of degree n. Let K/F be a CM extension and $\widehat{Cl(K)}$ be the dual group of the ideal class group $Cl(K)$. Note that $L(\chi, 1)$ is finite for any nontrivial $\chi\in\widehat{Cl(K)}$, so we can define

$$\mathcal{L}_F := \max_{\chi\in\widehat{Cl(F)}\setminus\{\chi_0\}}|L_F(\chi,1)|. \tag{31}$$

Also, in this paper we will always use $M(r_1, r_2)$ as the generalized Minkowski function defined in (11).

Proposition 18. *Let notation be as above, and χ_0 be the trivial character in $\widehat{Cl(K)}$. Then we have*

$$L_K\big(\chi_0, \tfrac{1}{2}\big) \geq \frac{\rho_F}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]\cdot h_F}\left(\frac{1}{2}\log\frac{\sqrt{D_K}}{D_F} - \Phi_F^0 \cdot h_K D_K^{-1/4}\right). \tag{32}$$

where

$$\Phi_F^0 := \frac{2^{5n/2}M(0,n)D_F^{7/4}h_F^2}{\pi^n \rho_F h_F^+} + e^{2\rho_F^{-1}\mathcal{L}_F + n}\mathcal{L}_F^*\sqrt{D_F},$$

and $\mathcal{L}_F^* = 4\rho_F^{-1}\mathcal{L}_F + \log D_F + (3\log 2 - \log\pi)n + \sqrt{7n} + 4$.

Proof. By Lemma 17 we have $E'\big(z, \frac{1}{2}; \mathfrak{f}_{\mathfrak{a}}^{-1}, \mathfrak{f}_{\mathfrak{a}}\big) = I_M(z;\mathfrak{f}_{\mathfrak{a}}) + I_E(z;\mathfrak{f}_{\mathfrak{a}})$, where

$$I_M(z;\mathfrak{f}_{\mathfrak{a}}) := \frac{y^{\sigma/2}}{N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})h_F}\left\{2\log y^\sigma + \frac{4\Upsilon_{F,[\mathfrak{f}_{\mathfrak{a}}]^{-1}}}{\rho_F} - 2\log N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}}) - (\gamma + 2\log 2)n\right\},$$

$$I_E(z;\mathfrak{f}_{\mathfrak{a}}) := \frac{y^{\sigma/2}}{N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})}\cdot\frac{2^{n+1}}{\rho_F\sqrt{D_F}}\sum_{b\in F^\times}\sum\sum_{\substack{(a,c)\in\mathfrak{f}_{\mathfrak{a}}^{-1}\mathfrak{o}^{-1}\times\mathcal{C}\\ ac=b}}\mathbf{e}(bx)\prod_{v\in\mathbf{J}_\infty}K_0(2\pi y_v|b_v|),$$

where $\mathcal{C} := \mathcal{O}_F^\times/\mathcal{O}_F^{\times,+}$, and

$$\mathcal{L}_{F,[\mathfrak{f}_{\mathfrak{a}}]^{-1}} := \gamma_{F,[\mathfrak{f}_{\mathfrak{a}}]^{-1}} - h_F^{-1}\gamma_F = \frac{1}{h_F}\sum_{\substack{\chi\in\widehat{Cl(F)}\\ \chi\neq\chi_0}}\chi([\mathfrak{f}_{\mathfrak{a}}])L_F(\chi,1).$$

By Proposition 15 we can write $L_K\big(\chi, \frac{1}{2}\big) = L_{M,\chi} + L_{E,\chi}$, where

$$L_{M,\chi} = \frac{2^{n/2}\sqrt{D_F}}{2D_K^{1/4}}\cdot\frac{\rho_F}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]}\sum_{[\mathfrak{a}^{-1}]\in Cl(K)}\bar{\chi}([\mathfrak{a}])\sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})}\cdot I_M(z;\mathfrak{f}_{\mathfrak{a}}),$$

$$L_{E,\chi} = \frac{2^{n/2}\sqrt{D_F}}{2D_K^{1/4}}\cdot\frac{\rho_F}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]}\sum_{[\mathfrak{a}^{-1}]\in Cl(K)}\bar{\chi}([\mathfrak{a}])\sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_{\mathfrak{a}})}\cdot I_E(z;\mathfrak{f}_{\mathfrak{a}}).$$

We will start with bounding $I_E(z;\mathfrak{f}_{\mathfrak{a}})$ and further estimating $L_{E,\chi}$:

$$|I_E(z;\mathfrak{f}_{\mathfrak{a}})| \leq \frac{y^{\sigma/2}}{N(\mathfrak{f}_{\mathfrak{a}})}\cdot\frac{2^{n+1}}{\rho_F\sqrt{D_F}}\sum_{b\in F^\times}\sum\sum_{\substack{(a,c)\in\mathfrak{f}_{\mathfrak{a}}^{-1}\mathfrak{o}^{-1}\times\mathcal{C}\\ ac=b}}\prod_{v\in\mathbf{J}_\infty}|K_0(2\pi y_v|b_v|)|$$

$$= \frac{y^{\sigma/2}}{N(\mathfrak{f}_{\mathfrak{a}})}\cdot\frac{2^{n+1}[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}]}{\rho_F\sqrt{D_F}}\sum_{b\in\mathfrak{f}_{\mathfrak{a}}^{-1}\mathfrak{o}^{-1}\cap F^\times}\prod_{v\in\mathbf{J}_\infty}|K_0(2\pi y_v|b_v|)|$$

To compute $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}]$, let's fix an ordering $(\phi_1, \dots, \phi_n)$ of $\mathrm{Hom}(F, \overline{\mathbb{Q}})$ and consider the homomorphism

$$\tau : \mathcal{O}_F^\times \longrightarrow \{\pm 1\}^n, \quad x \mapsto (\phi_1(x)/|\phi_1(x)|, \dots, \phi_n(x)/|\phi_n(x)|).$$

Then clearly, $\ker(\tau) = \mathcal{O}_F^{\times,+}$. Hence $\mathrm{Im}(\tau) \simeq \mathcal{O}_F^\times/\mathcal{O}_F^{\times,+}$. In fact, by Lemma 11.2 of [Conner and Hurrelbrink 1988] we have

$$\mathrm{coker}(\tau) \simeq \mathrm{Gal}(H_F^+/H_F) \simeq \ker(Cl(F)^+ \to Cl(F)),$$

where H_F is the Hilbert class field of F and H_F^+ is the narrow Hilbert class field of F. By the above isomorphism, we have $[\mathcal{O}_F^\times : \mathcal{O}_F^{\times,+}] = 2^n h_F {h_F^+}^{-1}$. Noting that F is totally real, consider the canonical embedding

$$j : F \longrightarrow F_{\mathbb{R}} := \prod_{v \in \mathbf{J}_\infty} F_v \simeq \mathbb{R}^n.$$

Since $\mathfrak{f}_\mathfrak{a}^{-1}\mathfrak{o}^{-1} \neq 0$, the lattice $\Gamma := j(\mathfrak{f}_\mathfrak{a}^{-1}\mathfrak{o}^{-1})$ is complete in $F_{\mathbb{R}}$. Let $\alpha_1, \dots, \alpha_n$ be a $\mathbb{Z}$-basis of $\mathfrak{f}_\mathfrak{a}^{-1}\mathfrak{o}^{-1}$. Let $\beta_v = j(\alpha_v)$, then we may assume that $\beta_v > 0$, for all $1 \leq v \leq n$. Since $\mathbb{Z}$ is a PID, we have $\Gamma = \mathbb{Z}\beta_1 \oplus \cdots \oplus \mathbb{Z}\beta_n$. Then a computation with determinants gives

$$\prod_{v \in \mathbb{J}_\infty} \beta_v^{-1} = \mathrm{vol}(\Gamma)^{-1} = \frac{1}{\sqrt{D_F}} \cdot N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a}\mathfrak{o}) = N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})\sqrt{D_F}.$$

For any $b \in \mathfrak{f}_\mathfrak{a}^{-1}\mathfrak{o}^{-1} \cap F^\times$, we may write $j(b) = (m_1\beta_1, \dots, m_n\beta_n)$, where $m_i \neq 0$, for all $1 \leq v \leq n$. Otherwise, we may assume $m_1 = 0$. Then there exists some $v \in \mathbf{J}_\infty$ such that $b_v = 0$. Then the minimal polynomial of b has 0 as its root. This is impossible unless $b = 0$.

On the other hand, note that $K_0(x) < K_{1/2}(x) = \sqrt{\frac{\pi}{2x}} e^{-x}$ for all $x > 0$. Combining these results we have

$$\begin{aligned}
|I_E(z; \mathfrak{f}_\mathfrak{a})| &\leq \frac{y^{\sigma/2}}{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})} \cdot \frac{2^{2n+1} h_F}{\rho_F h_F^+ \sqrt{D_F}} \prod_{v \in \mathbf{J}_\infty} \left(\sum_{m=1}^{\infty} \sqrt{\frac{1}{y_v \beta_v m}}\, e^{-2\pi y_v \beta_v m} \right) \\
&= \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})} \cdot \frac{2^{2n+1} h_F}{\rho_F h_F^+ \sqrt{D_F}} \prod_{v \in \mathbf{J}_\infty} \beta_v^{-1/2} \prod_{v \in \mathbf{J}_\infty} \left(\sum_{m=1}^{\infty} \sqrt{\frac{1}{m}}\, e^{-2\pi y_v \beta_v m} \right) \\
&\leq \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})} \cdot \frac{2^{2n+1} h_F}{\rho_F h_F^+ \sqrt{D_F}} \prod_{v \in \mathbf{J}_\infty} \beta_v^{-1/2} \prod_{v \in \mathbf{J}_\infty} \frac{1}{e^{2\pi y_v \beta_v} - 1} \\
&\leq \frac{1}{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})} \cdot \frac{2^{2n+1} h_F}{\rho_F h_F^+ \sqrt{D_F}} \prod_{v \in \mathbf{J}_\infty} \beta_v^{-1/2} \prod_{v \in \mathbf{J}_\infty} \frac{1}{2\pi y_v \beta_v} \\
&\leq \sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})} \cdot \frac{2^{n+1} h_F D_F^{1/4}}{\pi^n \rho_F h_F^+} y^{-\sigma}.
\end{aligned}$$

Therefore, by Proposition 13 and the definition of $\mathcal{I}_K$, we have

$$|I_E(z;\mathfrak{f}_\mathfrak{a})| \le \sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})}\cdot\frac{2^{n+1}h_F D_F^{1/4}}{\pi^n\rho_F h_F^+}\cdot\frac{2^n D_F N_{K/\mathbb{Q}}(\mathfrak{a})}{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2\sqrt{D_K}}$$
$$\le \frac{2^{2n+1}M(0,n)h_F D_F^{5/4}}{\pi^n\rho_F h_F^+\sqrt{N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})}} \qquad \text{for all } \mathfrak{a}\in\mathcal{I}_K.$$

Note that by definition, each $\mathfrak{f}_\mathfrak{a}\in\mathcal{I}_F^+$ is defined in (12). Thus we have

$$|L_{E,\chi}| \le \frac{2^{n/2}\sqrt{D_F}}{2D_K^{1/4}}\cdot\frac{\rho_F}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]}\sum_{\mathfrak{a}\in\mathcal{I}_K}\frac{2^{2n+1}M(0,n)h_F D_F^{5/4}}{\pi^n\rho_F h_F^+}$$
$$\le \frac{1}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]}\cdot\frac{2^{5n/2}M(0,n)D_F^{7/4}h_F}{\pi^n h_F^+}\cdot h_K D_K^{-1/4},$$

where $\mathcal{I}_K$ is defined in (23), so that (24) is available here.

On the other hand, we will give a lower bound for $I_M(z;\mathfrak{f}_\mathfrak{a})$ and $L_{M,\chi}$. Recall the definition of $\mathcal{L}_F$ given in (31); then clearly for any $\mathfrak{a}\in\mathcal{I}_K$, we have

$$\mathcal{L}_F \ge \frac{1}{h_F}\sum_{\chi\in\widehat{Cl(F)}\setminus\{\chi_0\}}|L_F(\chi,1)| \ge |\mathcal{L}_{F,[\mathfrak{f}_\mathfrak{a}]^{-1}}|.$$

Hence by the expression for y^σ given in Proposition 13 we obtain

$$I_M(z;\mathfrak{f}_\mathfrak{a}) \ge \sqrt{\frac{N_{K/\mathbb{Q}}(c_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2}{N_{K/\mathbb{Q}}(\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})}}\left\{2\log\frac{\sqrt{D_K}}{2^n D_F} - C_{F,\mathfrak{a}}\right\}\cdot\frac{D_K^{1/4}}{\sqrt{2^n D_F}h_F}, \tag{33}$$

where the tail $C_{F,\mathfrak{a}}$ is defined as

$$C_{F,\mathfrak{a}} = 2\log\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{N_{K/\mathbb{Q}}(c_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2} + \frac{4\mathcal{L}_F}{\rho_F} + (\gamma+2\log 2)n.$$

Combining (33) with (24) yields

$$L_{M,\chi_0} \ge \frac{\rho_F}{2[\mathcal{O}_K^\times:\mathcal{O}_F^\times]\cdot h_F}\sum_{[\mathfrak{a}^{-1}]\in Cl(K)}\sqrt{\frac{N_{K/\mathbb{Q}}(c_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2}{N_{K/\mathbb{Q}}(\mathfrak{a})}}$$
$$\times\left\{2\log\frac{\sqrt{D_K}}{2^n D_F} - 2\log\frac{N_{K/\mathbb{Q}}(\mathfrak{a})}{N_{K/\mathbb{Q}}(c_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2} - \frac{4\mathcal{L}_F}{\rho_F} - (\gamma+2\log 2)n\right\}$$
$$= \frac{1}{2}\sum_{\mathfrak{a}\in\mathcal{I}_K}\mathcal{N}_\mathfrak{q}(\mathfrak{a})\cdot\left\{2\log\frac{\sqrt{D_K}}{D_F} - 2\log\frac{2^n N_{K/\mathbb{Q}}(\mathfrak{a})}{N_{K/\mathbb{Q}}(c_\mathfrak{a})N_{F/\mathbb{Q}}(\mathfrak{q})^2} - C_F\right\},$$

where

$$\mathcal{N}_\mathfrak{q}(\mathfrak{a}) = \frac{\rho_F\cdot N_{F/\mathbb{Q}}(\mathfrak{q})}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]\cdot h_F}\cdot\sqrt{\frac{N_{K/\mathbb{Q}}(c_\mathfrak{a})}{N_{K/\mathbb{Q}}(\mathfrak{a})}} \quad\text{and}\quad C_F = 4\rho_F^{-1}\mathcal{L}_F + (\gamma+2\log 2)n.$$

Write $\mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* = 2^n N_{K/\mathbb{Q}}(c_{\mathfrak{a}})^{-1} N_{F/\mathbb{Q}}(\mathfrak{q})^{-2} N_{K/\mathbb{Q}}(\mathfrak{a})$, then we can introduce an undetermined parameter T satisfying $0 < T \le M(0,n)\sqrt{D_K}$ such that

$$L_{M,\chi_0} \ge \sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le T}} \mathcal{N}_{\mathfrak{q}}(\mathfrak{a}) \cdot \left(\log \frac{\sqrt{D_K}}{D_F \cdot \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*} - \frac{1}{2} C_F \right) - L^T_{M,\chi_0},$$

where the truncation term L^T_{M,χ_0} is defined as

$$L^T_{M,\chi_0} = \sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \ge T}} \mathcal{N}_{\mathfrak{q}}(\mathfrak{a}) \cdot \left\{ \left| \log \frac{\sqrt{D_K}}{D_F \cdot \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*} \right| + \frac{1}{2} C_F \right\}.$$

We can take $T = T_{K/F} = \min\{(e^{-C_F}\sqrt{D_K})/D_F,\ 2^n M(0,n)\sqrt{D_K}\}$. Then due to (11) and (12) we have $T_{K/F} = e^{-C_F} D_F^{-1}\sqrt{D_K}$ if $n \ge 9$. Note that the choice of $T_{K/F}$ above implies that

$$\log \frac{\sqrt{D_K}}{D_F \cdot \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*} \ge C_F \quad \text{for all } \mathfrak{a} \in \mathcal{I}_K \text{ such that } \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le T_{K/F}.$$

Noting that $\mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le M(0,n)\sqrt{D_K}$, we define

$$\Phi_F = \max_{T_{K/F} \le \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le 2^n M(0,n)\sqrt{D_K}} \left| \log \frac{\sqrt{D_K}}{D_F \cdot \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*} \right| + \frac{1}{2} C_F.$$

Then by monotonicity we obtain

$$\Phi_F = \max\left\{\tfrac{3}{2} C_F,\ \log 2^n M(0,n) D_F + \tfrac{1}{2} C_F\right\} \le \log M(0,n) D_F + 2C_F. \tag{34}$$

When $e^{-C_F} D_F^{-1}\sqrt{D_K} > 2^n M(0,n)\sqrt{D_K}$ (e.g., when $n \ge 9$), this implies that

$$\begin{aligned} L^{T_{K/F}}_{M,\chi_0} &\le \frac{2^{n/2}\rho_F}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot h_F} \sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ T_{K/F} \le \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le 2^n M(0,n)\sqrt{D_K}}} \frac{1}{\sqrt{\mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*}} \cdot \Phi_F \\ &= \frac{\rho_F \Phi_F e^{C_F/2}}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot h_F} \sqrt{D_F} \cdot h_K D_K^{-1/4}. \end{aligned}$$

And when $e^{-C_F} D_F^{-1}\sqrt{D_K} \le 2^n M(0,n)\sqrt{D_K}$ (according to our discussion before, this might happen only when $n \le 8$), we just take $L^{T_{K/F}}_{M,\chi_0} = 0$. Hence, we have

$$\begin{aligned} L_{M,\chi_0} &\ge \frac{1}{2} \sum_{\substack{\mathfrak{a}\in\mathcal{I}_K \\ \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^* \le T_{K/F}}} \mathcal{N}_{\mathfrak{q}}(\mathfrak{a}) \cdot \log \frac{\sqrt{D_K}}{D_F \cdot \mathcal{N}_{\mathfrak{q}}(\mathfrak{a})^*} - L^{T_{K/F}}_{M,\chi_0} \\ &\ge \frac{1}{2} \mathcal{N}_{\mathfrak{q}}(\mathfrak{q}) \cdot \log \frac{\sqrt{D_K}}{D_F} - \frac{\rho_F \Phi_F e^{C_F/2}}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot h_F} \sqrt{D_F} \cdot h_K D_K^{-1/4}. \end{aligned}$$

Since $\mathcal{N}_{\mathfrak{q}}(\mathfrak{q}) = [\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot \rho_F {h_F}^{-1}$, then involving the upper bound of $|L_{E,\chi}|$ developed as before we have

$$L_K\big(\chi_0, \tfrac{1}{2}\big) \geq \frac{\rho_F}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times] \cdot h_F} \left(\frac{1}{2} \log \frac{\sqrt{D_K}}{D_F} - \Phi'_F \cdot h_K D_K^{-1/4} \right). \tag{35}$$

where

$$\Phi'_F := \frac{2^{5n/2} M(0,n) D_F^{7/4} h_F^2}{\pi^n \rho_F h_F^+} + \Phi_F e^{C_F/2} \sqrt{D_F}.$$

By (11) and the usual Minkowski constant $M(n)$ one can obtain an elementary computation of $M(0,n)$, substituting this bound into (34) leads to the inequality

$$\Phi_F \leq 2C_F + \log e^{-(2\gamma + \log 2\pi)n + \sqrt{7n} + 4} D_F,$$

from which one then has

$$\Phi_F \leq 4\rho_F^{-1} \mathcal{L}_F + \log D_F + (3\log 2 - \log \pi)n + \sqrt{7n} + 4. \tag{36}$$

Then the proof follows from the estimate (35) and inequalities (36). □

From Proposition 18 one sees naturally that an upper bound for $|L_F(\chi, 1)|$ is needed to make the inequality (32) more explicit. The desired estimate is provided in the following lemma.

Lemma 19. *Let $F/\mathbb{Q}$ be any field extension of degree $n < +\infty$. Let χ be any nontrivial primitive Grossencharacter of modulus $\mathfrak{m}$. Then we have*

$$|L_F(\chi, 1)| \leq 2 \left[\frac{e}{2n} \log(D_F N_{F/\mathbb{Q}}(\mathfrak{m})) \right]^n. \tag{37}$$

Moreover, if for any $\chi \in \widehat{Cl(F)} \setminus \{\chi_0\}$, we have

$$|L_F(\chi, 1)| \leq \left(1 + \frac{\gamma + 2\log 2 - \log \pi}{2} n + \frac{1}{2} \log D_F + \rho_F^{-1} \gamma_F \right) \cdot \rho_F, \tag{38}$$

where $\rho_F := \mathrm{Res}_{s=1} \zeta_F(s)$ and γ_F is the Euler–Kronecker constant of $F/\mathbb{Q}$.

Proof. Denote by (r_1, r_2) the signature of $F/\mathbb{Q}$. Let b be the number of real places of F dividing the infinite part of the conductor of χ. Let $a := r_1 - b$. Then $a \geq 0$ and $b \geq 0$. Consider the completed Hecke L-function associated with χ:

$$\Lambda_F(\chi, s) := (D_F N_{F/\mathbb{Q}}(\mathfrak{m}))^{s/2} L_{F,\infty}(\chi, s) L_F(\chi, s), \tag{39}$$

where $L_{F,\infty}(\chi, s)$ is the infinite part of $L_F(\chi, s)$, i.e.,

$$L_{F,\infty}(\chi, s) := 2^{-r_2 s} \pi^{-ns/2} \Gamma(s/2)^a \Gamma((s+1)/2)^b \Gamma(s)^{r_2}. \tag{40}$$

Then $\Lambda_F(\chi, s)$ is an entire function satisfying the functional equation:

$$\Lambda_F(\chi, s) = W_\chi \Lambda_F(\overline{\chi}, 1-s), \tag{41}$$

where $W_\chi \in \mathbb{C}$ is the root number with $|W_\chi| = 1$.

Let $\Theta_\chi(x)$ be the inverse Mellin transform of $\Lambda_F(\chi, s)$. Then one can verify easily that (41) yields the functional equation of Θ_χ:

$$\Theta_\chi(x) = x^{-1} W_\chi \Theta_{\overline{\chi}}(x^{-1}). \tag{42}$$

Then by Mellin transform and (42) we have the integral representation of $\Lambda_F(\chi, s)$:

$$\Lambda_F(\chi, s) = \int_0^\infty x^{s-1}\Theta_\chi(x)\,dx = \int_1^\infty x^{s-1}\Theta_\chi(x)\,dx + W_\chi \int_1^\infty x^{-s}\Theta_{\overline{\chi}}(x)\,dx.$$

Since $\Gamma(s)$ is the Mellin transform of the function $f(x) = e^{-x}$, $(x > 0)$, the inverse Mellin transform of $g_{\lambda,\mu}(s) := \Gamma(\lambda s + \mu)$ (for all $\lambda > 0$, $\mu \in \mathbb{R}$) is

$$(\mathcal{M}^{-1} g_{\lambda,\mu})(x) = \lambda^{-1} x^{\mu/\lambda} e^{-x^{1/\lambda}}, \quad \lambda > 0,\ \mu \in \mathbb{R}.$$

Clearly, $\mathcal{M}^{-1} g_{\lambda,\mu}$ is positive. Since the parameters a, b and n are nonnegative integers, we can regard $\Gamma(s/2)^a \Gamma((s+1)/2)^b \Gamma(s)^{r_2}$ as a product of gamma functions of the form $\Gamma(\lambda s)$, $\lambda > 0$. Let $\Theta_{\chi,\infty}$ denote the inverse Mellin transform of $2^{r_2 s}\pi^{ns/2} L_{F,\infty}(\chi, s) = \Gamma(s/2)^a \Gamma((s+1)/2)^b \Gamma(s)^{r_2}$; then we have

$$\Theta_{\chi,\infty}(x) = (\mathcal{M}^{-1,*a} g_{1/2,0} * \mathcal{M}^{-1,*b} g_{1/2,1/2} * \mathcal{M}^{-1,*r_2} g_{1,0})(x) \quad \text{for all } x > 0,$$

where for any $m \in \mathbb{N}_+$, $\mathcal{M}^{-1,*m} g$ denotes the m-fold convolution of $\mathcal{M}^{-1} g$, the inverse Mellin transform of the function g. Hence $\Theta_{\chi,\infty}(x) > 0$ for all $x > 0$.

Note that by definition we have

$$\Theta_\chi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s}\Lambda_F(\chi, s)\,ds = \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a})\Theta_{\chi,\infty}\left(\frac{2^{r_2}\pi^{n/2} N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{D_F N_{F/\mathbb{Q}}(\mathfrak{m})}}\cdot x\right)$$

By the definition of a and b we see $\Theta_{\overline{\chi},\infty} = \Theta_{\chi,\infty}$. Since $\Theta_{\chi,\infty}$ is positive, we have $\Theta_{\overline{\chi}} = \overline{\Theta_\chi}$. Thus for any $s > 1$, we obtain

$$\begin{aligned}
|\Lambda_F(\chi, 1)| &\leq \left|\int_1^\infty \Theta_\chi(x)\,dx\right| + \left|\int_1^\infty x^{-1}\Theta_{\overline{\chi}}(x)\,dx\right| \\
&\leq 2\left|\int_1^\infty \sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_F} \chi(\mathfrak{a})\Theta_{\chi,\infty}\left(\frac{2^{r_2}\pi^{n/2} N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{D_F N_{F/\mathbb{Q}}(\mathfrak{m})}}\cdot x\right)dx\right| \\
&\leq 2\sum_{0 \neq \mathfrak{a} \subset \mathcal{O}_F} \int_0^\infty x^{s-1}\Theta_{\chi,\infty}\left(\frac{2^{r_2}\pi^{n/2} N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{D_F N_{F/\mathbb{Q}}(\mathfrak{m})}}\cdot x\right)dx \\
&= 2(D_F N_{F/\mathbb{Q}}(\mathfrak{m}))^{s/2} L_{F,\infty}(\chi, s)\zeta_F(s).
\end{aligned}$$

From this inequality we obtain the upper bound for $L_F(\chi, 1)$:

$$|L_F(\chi, 1)| \leq 2(D_F N_{F/\mathbb{Q}}(\mathfrak{m}))^{(s-1)/2} \frac{L_{F,\infty}(\chi, s)}{L_{F,\infty}(\chi, 1)} \cdot \zeta(s)^n \quad \text{for all } s > 1. \tag{43}$$

Let

$$H(s) := s^{a+r_2}(s-1)^n \frac{L_{F,\infty}(\chi, s)}{L_{F,\infty}(\chi, 1)} \cdot \zeta(s)^n = \xi(s)^n G(s)^{b+r_2},$$

where

$$\xi(s) := s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s) \quad \text{and} \quad G(s) := \frac{\sqrt{\pi}\,\Gamma((s+1)/2)}{s\Gamma(s/2)}.$$

Then by (43) we have $|L_F(\chi, 1)| \leq 2(D_F N_{F/\mathbb{Q}}(\mathfrak{m}))^{(s_0-1)/2}(s_0-1)^{n-1} \cdot s_0^{-a-r_2} H(s_0)$, where $s_0 := 1 + 2n[\log(D_F N_{F/\mathbb{Q}}(\mathfrak{m}))]^{-1}$. Recall that we have the well known Hadamard decomposition for entire functions

$$\frac{1}{\Gamma(s)} = s e^{\gamma s} \prod_{n=1}^{\infty} \left(1 + \frac{s}{n}\right) e^{-s/n} \quad \text{and} \quad \xi(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho},$$

where $B \in \mathbb{C}$ is a constant and ρ runs through nontrivial zeros of $\zeta(s)$. Then

$$G'(s)/G(s) = (\log G(s))' = \sum_{j=1}^{\infty} (-1)^j (j+s)^{-1} \leq 0 \quad \text{for all } s > 0.$$

This gives that $(\log G)''(s) = \sum_{j=1}^{\infty} (-1)^{j-1}(j+s)^{-2} > 0$ when $s > 0$. Hence $\log G$ is convex when $s > 0$. Now we work out B and thus see for every nontrivial zero $\rho = \sigma + i\tau$, we have $|\tau| \geq 6$. In fact, by definition, $B = \xi(0)'/\xi(0)$. The functional equation $\xi(s) = \xi(1-s)$ gives $\xi(s)'/\xi(s) = -\xi(1-s)'/\xi(1-s)$. Thus $B = -\xi(1)'/\xi(1)$. Therefore,

$$B = \frac{1}{2}\log \pi - \frac{1}{2}\frac{\Gamma'}{\Gamma}\left(\frac{3}{2}\right) - \lim_{s \mapsto 1^+}\left(\frac{\zeta'}{\zeta}(s) + \frac{1}{s-1}\right) = -1 - \frac{\gamma}{2} + \frac{1}{2}\log 4\pi.$$

On the other hand, by $B = -\xi(1)'/\xi(1)$ and symmetry of the nontrivial zeros,

$$B = -\frac{1}{2}\sum_{\rho}\left(\frac{1}{1-\rho} + \frac{1}{\rho}\right) = -\sum_{\rho=\sigma+i\tau} \mathfrak{Re}\frac{1}{\rho} = -\sum_{\rho=\sigma+i\tau} \frac{2\sigma}{\sigma^2+\tau^2}.$$

Then for any $\rho = \sigma + i\tau$ with $\frac{1}{2} \leq \rho \leq 1$, one has $-B \geq 2\sigma \cdot (\sigma^2 + \tau^2)^{-1}$, which gives the lower bound

$$|\tau| \geq \sqrt{\frac{2\sigma}{-B} - \sigma^2} \geq \sqrt{\frac{1}{-B} - \frac{1}{4}} \geq 6.$$

Thus the function

$$h(s) := \sum_{\rho=\sigma+i\tau} \frac{1}{s-\rho} = \sum_{\substack{\rho=\sigma+i\tau \\ \tau\geq 0}} \frac{s-\sigma}{(s-\sigma)^2+\tau^2}, \quad 1 \leq s \leq 6$$

is increasing. So $(\log\xi(s))' = \xi(s)'/\xi(s) = B + h(1) + h(s)$ is increasing when $1 \leq s \leq 6$. Then $\log\xi(s)$ is convex when $1 \leq s \leq 6$. Therefore $\log(s^{-a-r_2}H(s))$ is convex when $1 \leq s \leq 6$.

By [Zimmert 1981] we know that there exists an integral ideal $\mathfrak{a}$ such that $N_{F/\mathbb{Q}}(\mathfrak{a}) \leq M(r_1, r_2)\sqrt{D_F}$. Then clearly

$$\log(D_F N_{F/\mathbb{Q}}(\mathfrak{m})) \geq \log D_F \geq M(r_1, r_2)^{-1}.$$

Recall that $M(r_1, r_2) \leq M(n) := (4^{r_2}n!)/(\pi^{r_2}n^n)$. Hence

$$s_0 = 1 + 2n[\log(D_F N_{F/\mathbb{Q}}(\mathfrak{m}))]^{-1} \leq 1 + 2nM(r_1, r_2) \leq 1 + 2n \cdot \frac{4^n n!}{\pi^n n^n} \leq 6.$$

By convexity we have $H(s_0) \leq \max\{H(1), H(6)\} = 1$.

Let $\chi \in \widehat{Cl(F)}$ be a nontrivial Hilbert character. We have that

$$Cl(F) \simeq I_F/I_F^{\infty} F^{\times}, \quad \text{where } I_F^{\infty} := \prod_{\mathfrak{p}\mid\infty} F_{\mathfrak{p}}^{\times} \times \prod_{\mathfrak{p}\nmid\infty} \mathcal{O}_{F,\mathfrak{p}}^{\times}.$$

So in this case $b = 0$, $a = r_1$ and $\mathfrak{m} = \mathcal{O}_F$. Then we have the completed L-function (39), where $N_{F/\mathbb{Q}}(\mathfrak{m}) = 1$ and (40) becomes

$$L_{F,\infty}(\chi, s) := 2^{-r_2 s}\pi^{-ns/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}.$$

As before, noting that $\Theta_{\bar{\chi}} = \overline{\Theta_{\chi}}$, Mellin transform and functional equations imply

$$|\Lambda_F(\chi, 1)| \leq \int_1^{\infty} |\Theta_{\chi}(x)|(1+x^{-1})\,dx \leq \int_1^{\infty} |\Theta(x)|(1+x^{-1})\,dx, \tag{44}$$

where Θ_∞ is the inverse Mellin transform of $\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}$ and

$$\Theta(x) := \sum_{0\neq\mathfrak{a}\subset\mathcal{O}_F} \Theta_\infty\biggl(\frac{2^{r_2}\pi^{n/2}N_{F/\mathbb{Q}}(\mathfrak{a})}{\sqrt{D_F}} \cdot x\biggr).$$

Let $c > 10$ and $\Lambda_F(s) := D_F^{s/2}2^{-r_2 s}\pi^{-ns/2}\Gamma(s/2)^{r_1}\Gamma(s)^{r_2}\zeta_F(s)$. Then by (44),

$$\begin{aligned}|\Lambda_F(\chi, 1)| &\leq \int_1^{\infty}\biggl(\frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \Lambda_F(s)x^{-s}\,ds\biggr)(1+x^{-1})\,dx \\ &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty} \widetilde{\Lambda}_F(s)\,ds,\end{aligned}$$

where $\widetilde{\Lambda}_F(s) := \Lambda_F(s)\cdot\bigl(\frac{1}{s}+\frac{1}{s-1}\bigr)$. Denote the right-hand side by I_F. The functional equation $\Lambda_F(s) = \Lambda_F(1-s)$ gives us that $\widetilde{\Lambda}_F(s) = -\widetilde{\Lambda}_F(1-s)$.

Recall that combining the elementary bound and functional equation of $\zeta_F(s)$ and by the Phragmén–Lindelöf theorem we have the fact that

$$\zeta_F(\sigma+it) \ll |t|^{(1-\sigma)/2}\log|t|, \quad 0\le\sigma\le 1,\ |t|\ge 2.$$

By functional calculus we have $\zeta_F(\sigma+it) \ll |t|^{1/2-\sigma}\log|t|$, $\sigma\le 0$, $|t|\ge 2$. Note that in the area $\mathcal{S} := \{s=\sigma+it:\ 1-c\le\sigma\le c, |t|\ge 1\}$ we have uniformly that

$$\Gamma(s) = \sqrt{2\pi}e^{-(\pi/2)|t|}e^{it(\log|t|-1)}e^{(i\pi t/(2|t|))(\sigma-1/2)}(1+O_c(|t|^{-1})).$$

So $\widetilde{\Lambda}_F(s)$ decays exponentially in $\mathcal{S}$ as $|t|\mapsto\infty$. Thus shifting the contour we get

$$\begin{aligned} I_F &= \operatorname{Res}_{s=1}\widetilde{\Lambda}_F(s) + \operatorname{Res}_{s=1}\widetilde{\Lambda}_F(s) + \frac{1}{2\pi i}\int_{1-c-i\infty}^{1-c+i\infty}\widetilde{\Lambda}_F(s)\,ds \\ &= 2\operatorname{Res}_{s=1}\widetilde{\Lambda}_F(s) - \frac{1}{2\pi i}\int_{c-i\infty}^{c+\infty}\widetilde{\Lambda}_F(s)\,ds. \end{aligned}$$

Hence $I_F = \operatorname{Res}_{s=1}\widetilde{\Lambda}_F(s) = \Lambda_{F,1}+\Lambda_{F,2}$, where

$$\Lambda_{F,1} := \lim_{s\to 1}(s-1)\cdot\Lambda_F(s) \quad\text{and}\quad \Lambda_{F,2} := \lim_{s\to 1}(s-1)\cdot\Lambda_F(s)'.$$

By the Laurent expansion of $\zeta_F(s)$ at $s=1$ we have $\lim_{s\to 1}\big(\frac{1}{s-1}+\frac{\zeta_F'}{\zeta_F}(s)\big) = \rho_F^{-1}\gamma_F$. Hence $\Lambda_{F,2}/\Lambda_{F,1} = (\log\Lambda_F(s))'\,|_{s=1}$ is equal to

$$\begin{aligned} &\frac{1}{2}\log D_F - r_2\log 2 - \frac{n}{2}\log\pi + \frac{r_1}{2}\cdot\frac{\Gamma'}{\Gamma}\Big(\frac{1}{2}\Big) + r_2\frac{\Gamma'}{\Gamma}(1) + \rho_F^{-1}\gamma_F \\ &\qquad = \frac{1}{2}\log D_F - \frac{n}{2}\log\pi + \frac{\gamma+2\log 2}{2}\cdot r_1 + (\gamma+\log 2)r_2 + \rho_F^{-1}\gamma_F. \end{aligned}$$

Then by the inequality $|\Lambda_F(\chi,1)|\le I_F$ we obtain (38). □

Now we move on to the Euler–Kronecker constant γ_F. Clearly we need an upper bound for it. The known result on upper bounds for γ_F is essentially $2\log\log\sqrt{D_F}$, which is established under GRH [Ihara 2006]. To prepare for the proof of Theorem A, we give an elementary unconditional effective upper bound for γ_F.

Lemma 20. *Let notation be as before. Then there is an absolute constant $c>0$ such that*

$$-\frac{1}{2}\log D_F - \frac{\gamma+2\log 2-\log\pi}{2}n - 1 \le \gamma_F^* \le c\log D_F. \tag{45}$$

Remark. Note that the main term of this lower bound in (45) is $-\frac{1}{2}\log D_F$, which is slightly better than the general result (i.e., lower bound of main term $-\log D_F$) given in [Ihara 2006]. On the other hand, under GRH, one has $\gamma_F^* \ll \log\log D_F$ according to the main theorems in [Ihara 2006].

Proof. The lower bound for γ_F can be deduced simply from (38). We thus will focus on the upper bound here. For $s=\sigma+it\in\mathbb{C}$ with $\frac{1}{2}\le\sigma\le 1$, we have

$$(s-1)\zeta_F(s) \ll \rho_F|sD_F|^{(1-\sigma)/2}. \tag{46}$$

Note (46) is essentially Theorem 5.31 in [Iwaniec and Kowalski 2004] without $|sD_F|^{\epsilon}$. We drop the ϵ-factor by using a subconvexity bound for ζ_F at $\sigma = \mathrm{Re}(s) = \frac{1}{2}$ as an endpoint rather than using the bound for $\zeta_F(it)$ via the functional equation before applying the Phragmén–Lindelöf theorem. In fact, a subconvexity result (see, e.g., [Venkatesh 2010]) shows that there exists some δ (e.g., one can take $\delta = \frac{1}{200}$) such that $\zeta_F(s) \ll |sD_F|^{1/2-\delta}$, where $s = \frac{1}{2} + it$. Then the Phragmén–Lindelöf theorem implies that

$$(s-1)\zeta_F(s) \ll |sD_F|^{(1/2-\delta)(1-\sigma)}, \quad \text{where } s = \sigma + it,\ \tfrac{1}{2} \le \sigma < 1. \tag{47}$$

Hence (46) comes from (47) and Theorem 5.31 in [Iwaniec and Kowalski 2004].

Let $\mathcal{C}$ be the circle centered at $s = 1$ with radius $r = 1 - (2\log D_F)^{-1}$. Since ζ_F is a meromorphic function with a simple pole at $s = 1$, by Cauchy's theorem we have

$$\gamma_F = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{\zeta_F(s)}{s-1}ds \ll \oint_{\mathcal{C}} \frac{\rho_F}{|s-1|^2}|ds| \ll \rho_F \log D_F.$$

Then the proof follows. □

With these preparations we can now give a proof of our main theorems.

4.2. *Proof of Theorem A.*

Proof. As before, let K/F be a CM extension and $[F:\mathbb{Q}] = n$. Note that for every $\chi \in \widehat{Cl(K)}$, the conductor of χ is $\mathcal{O}_K$. So we have, by Lemma 19, that

$$\rho_F^{-1}\mathcal{L}_F \le 1 + \frac{\gamma + 2\log 2 - \log \pi}{2}n + \frac{1}{2}\log D_F + \gamma_F^*.$$

Then Lemma 20 implies that $\rho_F^{-1}\mathcal{L}_F \le c_1' \log D_F$ for some absolute constant $c_1' > 0$, since a classical lower bound for D_F implies that $\log D_F \gg n$.

Likewise, one has $\mathcal{L}_F^* = 4\rho_F^{-1}\mathcal{L}_F + \log D_F + (4\log 2 - \pi)n + \sqrt{7n} + 4 \le c_2' \log D_F$ for some positive absolute constant c_2'. Then (3) follows from Proposition 18 and thus (4) follows from (3) and elementary computations of $M(n,0)$ and $M(0,n)$. □

4.3. *Proof of Theorem B.* Substituting orthogonality into Proposition 16 we have:

Lemma 21. *Let notations be as above, then we have*

$$L_K\big(\chi, \tfrac{1}{2}\big) = \frac{2^{n/2}\rho_F\sqrt{D_F}}{2D_K^{1/4}[\mathcal{O}_K^\times : \mathcal{O}_F^\times]} \sum_{[\mathfrak{a}^{-1}] \in Cl(K)} \overline{\chi}([\mathfrak{a}])\sqrt{N(\mathfrak{f}_\mathfrak{a})}E'\big(z_\mathfrak{a}, \tfrac{1}{2}; \mathfrak{f}_\mathfrak{a}^{-1}, \mathfrak{f}_\mathfrak{a}\big), \tag{48}$$

and

$$\frac{1}{h_K}\sum_{\chi \in \widehat{Cl(K)}} |L_K(\chi, 1/2)|^2 = \frac{2^{n-2}D_F}{\sqrt{D_K}} \cdot \frac{\rho_F^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2} \times \sum_{[\mathfrak{a}^{-1}] \in Cl(K)} N_{F/\mathbb{Q}}(\mathfrak{f}_\mathfrak{a})\left|E'\big(z_\mathfrak{a}, \tfrac{1}{2}; \mathfrak{f}_\mathfrak{a}^{-1}, \mathfrak{f}_\mathfrak{a}\big)\right|^2.$$

After inserting the Fourier expansion Lemma 17 and the CM type norm formula (16) into (48), we get the following generalization of (7):

Proposition 22. *Let notation be as before. Let* $\mathfrak{a}$ *be a fractional ideal of* K*. Then*

$$\frac{1}{h_K}\sum_{\chi\in\widehat{Cl(K)}}\chi(\mathfrak{a})L_K(\chi,\tfrac{1}{2})=\mathcal{N}_{\mathfrak{q}}(\mathfrak{a})\cdot\big\{\log N_\Phi(y_{\mathfrak{a}})+\mathcal{E}_1(F,K;\mathfrak{a})\big\},\tag{49}$$

where

$$\mathcal{N}_{\mathfrak{q}}(\mathfrak{a})=\frac{\rho_F\cdot N_{F/\mathbb{Q}}(\mathfrak{q})}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]\cdot h_F}\cdot\sqrt{\frac{N_{K/\mathbb{Q}}(c_{\mathfrak{a}})}{N_{K/\mathbb{Q}}(\mathfrak{a})}}.\tag{50}$$

Here $c_{\mathfrak{a}}$ *is given by* (16), *and* $\mathfrak{f}_{\mathfrak{a}}$ *is determined by* $\mathfrak{a}$ *according to* (14). *The error term* $\mathcal{E}_1(F,K;\mathfrak{a})$ *above satisfies that*

$$\mathcal{E}_1(F,K;\mathfrak{a})\ll\log D_F+\frac{h_F^2D_F^{1/4}N(\mathfrak{f}_{\mathfrak{a}})}{\rho_Fh_F^+}N_\Phi(y_{\mathfrak{a}})^{-3/2}.\tag{51}$$

where the implied constant is absolute.

Taking $\mathfrak{a}$ to be trivial in (49), combined with (16), (50) and (51), we have

$$\frac{1}{h_K}\sum_{\chi\in\widehat{Cl(K)}}L_K\big(\chi,\tfrac{1}{2}\big)\gg_F\frac{\log D_K}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]}.\tag{52}$$

On the other hand, substituting Lemma 17 into the second formula in Lemma 21 leads to

$$\begin{aligned}\frac{1}{h_K}\sum_{\chi\in\widehat{Cl(K)}}\big|L_K\big(\chi,\tfrac{1}{2}\big)\big|^2&\ll\frac{2^{n-2}D_F^2}{\sqrt{D_K}}\cdot\frac{\rho_F^2}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]^2}\sum_{[\mathfrak{a}^{-1}]}\big|E'\big(z_{\mathfrak{a}},\tfrac{1}{2};\mathfrak{f}_{\mathfrak{a}}^{-1},\mathfrak{f}_{\mathfrak{a}}\big)\big|^2\\&\ll_F\frac{(\log D_K)^2}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]^2}\sum_{N(\mathfrak{a})\ll\sqrt{D_K}}\frac{1}{N(\mathfrak{a})}.\end{aligned}$$

Then a standard estimate on

$$\sum_{N(\mathfrak{a})\ll\sqrt{D_K}}\frac{1}{N(\mathfrak{a})}$$

implies that

$$\frac{1}{h_K}\sum_{\chi\in\widehat{Cl(K)}}\big|L_K\big(\chi,\tfrac{1}{2}\big)\big|^2\ll_F\frac{(\log D_K)^2}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]^2}\sum_{n\ll\sqrt{D_K}}\frac{d(n)}{n}\ll_F\frac{(\log D_K)^3}{[\mathcal{O}_K^\times:\mathcal{O}_F^\times]^2},\tag{53}$$

where $d(n)$ is the divisor function, and the implied constants are effective.

Proof of Theorem B. By (52), (53) and the Cauchy inequality, we have

$$\begin{aligned}\frac{(h_K \log D_K)^2}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2} &\ll_F \Big| \sum_{\chi \in \widehat{Cl(K)}} L_K\big(\chi, \tfrac{1}{2}\big)\Big|^2 \\ &\ll_F \#\big\{\chi \in \widehat{Cl(K)} : L_K\big(\chi, \tfrac{1}{2}\big) \neq 0\big\} \cdot \sum_{\chi \in \widehat{Cl(K)}} \big|L_K\big(\chi, \tfrac{1}{2}\big)\big|^2 \\ &\ll_{F,\epsilon} \#\big\{\chi \in \widehat{Cl(K)} : L_K\big(\chi, \tfrac{1}{2}\big) \neq 0\big\} \cdot \frac{h_K (\log D_K)^3}{[\mathcal{O}_K^\times : \mathcal{O}_F^\times]^2}.\end{aligned}$$

Now the $k = 0$ case of Theorem B follows. The $k = 1$ case then comes from the $k = 0$ case and logarithmic derivative of the functional equation of $L_K(\chi, s)$. □

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LIYANG YANG
DEPARTMENT OF MATHEMATICS
CALIFORNIA INSTITUTE OF TECHNOLOGY
PASADENA, CA
UNITED STATES

lyyang@caltech.edu

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