Modelling consumer behaviour in the presence of network effects

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Except where otherwise indicated, this thesis is my own original work.

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Declaration of Authorship

I, Felipe Maldonado, hereby declare that this thesis titled: *Modelling consumer behaviour in the presence of network effects* and the work presented in it are my own. I confirm that:

- This work was done wholly while in candidature for a research degree at this University.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

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Abstract

Consumer choice models are a key component in fields such as Revenue Management and Transport Logistics, where the demands for certain products or services are assumed to follow a particular form, and sellers or market-makers use that information to adjust their strategies accordingly, choosing for example which products to display (assortment problem) or their prices (pricing problem).

In the last couple of decades, online markets have gained relevance, providing a setting where consumers can easily compare different products, before deciding to buy them. More information is now available, and the purchasing decisions not only depend on the quality, prices and availability of the products, but also on what previous consumers think about them (a phenomenon commonly known as Network Effects). Hence, in order to create a suitable model for this kind of market, it is relevant to understand how the collective decisions affect the market evolution.

In this thesis we consider a particular subset of those online markets, cultural markets, where the products are for example songs, video games or e-books. This kind of market has the special feature that its products have unlimited supply (since they are just a digital copy), and therefore we can exploit this in our models, to justify assumptions of the asymptotic behaviour of the market.

We study some variations of the traditional Multinomial Logit (MNL) model, characterising the behaviour of consumers, where their purchasing decisions are affected by the quality and prices (initially fixed) of the available products, as well as their visibilities in the market interface and the consumption patterns of previous users. We focus particularly on the parameters associated to the network effects, where depending on the strength of the network effects, it is possible to explain: herd behaviours, where an alternative overpowers the rest; as well as more well-distributed settings, where all the alternatives receive enough attention. Which we interpret as a notion of fairness, since higher quality products get a larger market share.

Finally, using the model where market shares are distributed according to the quality of the products, we study pricing strategies, where sellers can either collaborate or compete. We analyse the effect of both type of strategies into the choice model.

Preface

The contributions made in this thesis are listed in the following publications of peer-reviewed or in preparation papers. Each chapter is associated to one or more publications

Chapter 3:

- Abeliuk, A.; Berbeglia, G.; **Maldonado, F.**; and Van Hentenryck, P., 2016. Asymptotic optimality of myopic optimization in trial-offer markets with social influence. *In the 25th International Joint Conference on Artificial Intelligence* (IJCAI- 16).
- Van Hentenryck, P.; Abeliuk, A.; Berbeglia, F.; Berbeglia, G.; and **Maldonado, F.**, 2016. Aligning popularity and quality in online cultural markets. *In the Proceedings of the International AAAI Conference on Web and Social Media* (ICWSM 2016)

Chapter 4: **Maldonado, F.**; Van Hentenryck, P.; Berbeglia, G.; and Berbeglia, F., 2018. Popularity signals in trial-offer markets with social influence and position bias. *European Journal of Operational Research,* 266(2), pp. 775-793.

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Chapter 1

Introduction

1.1 Thesis Statement

The main goal of this thesis is to understand how consumer behaviour is affected by network effects, given by the consumption history, as well as position biases, where only selected products are shown in the first positions. For chapters 3 and 4 we consider a two steps model based on a Multinomial Logit model, where the consumers can try a product and then decide whether to buy it or not. In the last chapter, we use a variation of the Multinomial Logit model with network effects, where we incorporate the prices into the purchasing decisions. Using these models we provide to the firm running the market and the sellers, the best strategies that they can follow based on the available information, so they can maximise their revenues.

Although many concepts used in this thesis can have a variety of meanings, for the purpose of this thesis we consider them defined as follows:

- **(un)Predictability:** We say that the outcome of a market is *predictable*, if given a fixed set of products, the long-run distribution of the market is defined only by the intrinsic characteristic of the products (such as their quality) and the market structure (such as the chosen ranking). If on the other hand, the initial appeals and early dynamics heavily affect the market outcomes, then we say that they are *unpredictable*.

- **Inequality:** We say that a market outcome is *unequal*, if it suffers from "Rich-gets-Richer" effects, where popular products take over the whole market.

- **Inefficiency:** We say that a market outcome is *inefficient* if it leads to suboptimal results, where for example "bad" products are very popular and "good" products take the last positions in the rankings.

Our primary objective is to maximise the market efficiency, represented by the expected number of purchases. Note also that the higher this objective is, the lower the probability that consumers try a product but then decide not to purchase it. Hence, if we interpret this last action as an inefficiency, maximising the expected efficiency of the market also minimises unproductive trials.

1.2 Introduction

The widespread use of internet, has created many new type of markets that are reshaping the global economy, for example, people now watch movies on Netflix instead of renting a DVD at Blockbuster.

These Internet-based markets do not necessary follow the same rules as traditional markets (which have been well studied for decades), since their structure can be fairly different, where for example products can have unlimited supply (e.g., digital goods like songs), and millions of users from all across the world can access to them instantaneously. All these new type of markets open research opportunities in many disciplines such as Economics, Operations Research and Computer Science, where researchers could tackle problems like novel pricing schemes, subscription-based fees, recommendations systems and many more.

A very interesting feature of these markets is the effect of consumption history, reflected in a *social signal* (e.g., 5 stars rating, number of views, etc.), over the purchasing decisions of upcoming customers, phenomenon that in the literature is referred as **social influence** or **network effects**. Consumers make their purchasing decisions (choose one product over the others, or do not purchase anything) not only based on the quality and prices of the available alternatives, but also based on market-specific features such as rating systems that keep track of past consumption and opinions. These network effects become even more relevant when the prior information about the products is scarce, so the willingness to try/pay is heavily influenced by the opinion of the rest (*Wisdom-of-the-Crowd* e.g., Wang et al. [2014]).

In this thesis we will pay special attention to a sub-class of internet-based markets, called Trial-Offer markets, which are a setting where consumers can try products before deciding to buy them or not, an example of these markets are the Freemium phone apps, where you get for free the basic version, but you can pay to obtain the complete one (or an ad-free version). Many authors (e.g.,[Salganik et al., 2006; Tucker and Zhang, 2011; Viglia et al., 2014]) have explored the impact of network effects on consumer behaviour in these markets, where consumer can experience different types of social signals: A market place may report the number of past purchases of a product, its consumer ratings, and/or its consumer recommendations. Recent studies [Engstrom and Forsell, 2014; Viglia et al., 2014] however came to the conclusion that the popularity signal (based on the number of past purchases or the market share) has a much stronger impact on consumer behaviour than the average consumer rating signal. These two experimental studies were conducted in very different settings, using the Android application platform in one case and hotel selection in the other. Consumer preferences are also influenced by product visibilities, a phenomenon that has been widely observed in internet advertisement (e.g., [Craswell et al., 2008]), in online stores such as Expedia, Amazon, and iTunes, as well as physical retail stores (see, e.g., [Lim et al., 2004]).

Despite the ubiquitousness of social signals in internet-based markets (including for songs, albums, movies, hotels, and cell phones to name only a few), there is considerable debate in the scientific community about the benefits of network effects.

Many researchers have pointed out the potential negative implications of network effects. The seminal work of Salganik et al. [Salganik et al., 2006] on the MusicLab experimental market demonstrated that network effects can introduce significant unpredictability, inequality, and inefficiency in Trial-Offer markets. To investigate this hypothesis experimentally, they created an artificial music market called the MusicLab. Participants in the MusicLab were presented a list of unknown songs from unknown bands, each song being described by its name and band. The participants were divided into two groups exposed to two different experimental conditions: the *independent* condition and the *social influence* condition. In the first group (independent condition), participants were provided with no additional information about the songs. Each participant would decide which song to listen to from a random list. After listening to a song, the participant had the opportunity to download it. In the second group (social influence condition), each participant was provided with an additional information: The number of times the song was downloaded by earlier participants. Moreover, these participants were presented with a list ordered by the number of downloads. Additionally, to investigate the impact of social influence, participants in the second group were distributed in eight "worlds" evolving completely independently. In particular, participants in one world had no visibility about the downloads and the rankings in the other worlds. The MusicLab is a Trial-Offer market that provides an experimental test-bed for measuring the unpredictability of cultural markets. By observing the evolution of different worlds given the same initial conditions, the MusicLab provides unique insights on the impact of social influence and the resulting unpredictability.

These results were reproduced by many researchers (e.g., Salganik and Watts [2009, 2008]; Muchnik et al. [2013]; van de Rijt et al. [2014]). More recently, [Hu et al., 2015] studied a newsvendor problem with two substitutable products with the same quality in which consumer preferences are affected by past purchases. The authors showed that the market is unpredictable but can become less so if one of the products has an initial advantage. [Altszyler et al., 2017] has recently studied the impact of product appeal (how attractive they are) and product quality (how good they are) in a trial-offer market model with network effects under a finite time horizon. The authors showed that there exists a logarithmic trade-off between the two: the final product market share remains constant if a decrease in product quality is followed by an exponential increase in the product appeal. Other researchers have focused on understanding when these undesirable side-effects arise and where they come from. [Ceyhan et al., 2011] studied a market specified by a Logit model where a constant J captures the strength of the social signal. They showed that the market behaviour (e.g., whether it is predictable) depends on the strength of the social signal. Their results did not consider product visibilities, which is another important aspect of Trial-Offer markets. Indeed, various researchers (e.g., Lerman and Hogg [2014]; Abeliuk et al. [2015, 2017]) indicated that unpredictability and inefficiencies often depend on how products are displayed in the market. In particular, Abeliuk et al. [2015] shows that social influence/network effects are always beneficial[1] in expectation when the products are ordered by the *performance ranking* that maximises the purchases greedily at each step. This result was obtained using the generalised Multinomial Logit model proposed by [Krumme et al., 2012] to reproduce the MusicLab experiments. This thesis seeks to expand our understanding of network effects in consumer choice, and explores the role of the social signal in conjunction with product visibilities and prices. The starting point is the generalised Multinomial Logit model of [Krumme et al., 2012], which we extend to vary the strength of the social signal. More precisely, we start considering a Trial-Offer market where the probability of purchasing product i at time t is given by

$$\pi_i(\phi^t) = \frac{v_{\sigma_i} q_i f(\phi_i^t)}{\sum_{j=1}^{n} v_{\sigma_j} q_j f(\phi_j^t)} \tag{1.1}$$

where σ is a bijection from n products to n positions (representing a ranking), $v_k \in \mathbb{R}$ is the visibility of position k, $q_i \in \mathbb{R}$ is the inherent quality of product i, ϕ_i^t is the market share of product i at time t, and $f : \mathbb{R} \to \mathbb{R}$ is a positive continuous increasing function that represents the strength of the network effects. We notice that if in Equation (1.1) we consider $f(x) = K$, with K a constant, then we recover a version of standard Multinomial Logit model (e.g., see McFadden et al. [1973]), since in that case, π_i does not change over time. In this thesis we will explore different settings where we first fix the function f, as $f(x) = x$, and we vary the rankings induced by σ, making more emphasis on the effects of position biases into a consumer choice model. We then fix σ, and modify f, to be $f(x) = x^r$, $r > 0, r \neq 1$ providing a family of functions indexed by r, where we will study the effect on the consumption behaviour related to different values of r, what we will call the strength of the social signal. Finally in a related model, deduced from a variation of a multinomial logit model where network effects are incorporated, the parameters of visibilities are absorbed, and $f(x) = x^r$, $0 < r < 1$, we include the prices of each product into the model, so we can study how the sellers strategies change depending on the consumer choice model, and the strength of the social signals.

The theoretical results will be complemented with some agent-based simulations, where we will explore the performance of the suggested market designs. The numerical examples also will help to clarify when it is beneficial to include different ranking policies and social signals.

1.3 Related work

1.3.1 The Wisdom-of-the-Crowd and the Rich-Get-Richer Effect

In the absence of prior information about the alternatives, people tend to rely in what previous consumers have said or done, observed through an aggregate information given for example by past consumption or a rating/review system. As Lorenz et al.

[1] increase the expected number of purchases

[2011] estates, the Wisdom of the Crowd effect is not a socio-psychological effect but a statistical phenomenon, based on the aggregation of estimates over individuals, where the opinion of the majority has an impact on the decisions performed by the individuals. The effect of the Wisdom-of-the-Crowd can be so strong that people leaning towards a particular option, seem to change their mind due to what the majority says. This effect has been studied in many fields, such as psychology, economics, marketing, etc., where new users seem to believe that old users had a strong reason or better information for choosing the options they chose and decide to copy them. This phenomenon can be observed for instance in stock markets (Surowiecki [2004]), cryptocurrencies (Gusev [2018]), adoption of software (Fan et al. [2015]), and many others.

However, this imitation effect, can lead to irrational herding effects, where everyone copies what the rest is doing, not necessarily choosing the best options along the way. In order to investigate the effects of social influence is that many researchers have tested how reliable those systems can be, particularly about the unpredictability generated by network effects. The experimental evidence from MusicLab suggests that the relationship between quality and popularity can be significantly distorted by social influence and position biases (where most popular products were allocated in the first positions of a ranking). These relationships have also been observed experimentally by Stoddard [2015], who studied the relationship between the intrinsic article quality and its popularity in the social news sites `Reddit and Hacker` news. The author proposed a Poisson regression model to estimate the demand for an article based on its quality, past views, and age among others. The results obtained after an estimation of each intrinsic article from these social news site showed that the most popular articles are typically the articles with the highest quality. Another study of social influence was carried out by Tucker and Zhang [2011]. The authors conducted a field experiment which showed that popularity information may benefit products with narrow appeal significantly more than those with a broad appeal. Along these lines, Sipos et al. [2014] analysed the voting behaviour of users from `Amazon` product reviews when answering the question "Was this review helpful to you?" and how these votes relate to quality. The results showed that votes not only depend on the inherent quality of reviews, but also on the position where the review was presented in the ranked list. The authors also concluded that the ranking process converges and that the relative ordering of reviews stabilises during the 4 months data was collected.

One of the most common phenomena observed in online markets is what some researchers call *Matthew effect* or *rich-get-richer effect*, where the probability of choosing a product is proportional to its current level of popularity, hence, the most popular products get reinforced its popularity with every market interaction. These effects has been observed in many experimental settings (e.g., Salganik et al. [2006]; Salganik and Watts [2009, 2008]; Lorenz et al. [2011]; Muchnik et al. [2013]; van de Rijt et al. [2014]). Where for example in Lorenz et al. [2011] the authors study how the subjects of an experiment of factual questions, tend to change their mind after being presented with the average answers of their peers, concluding that social influence can undermine the wisdom of the crowd effect. Along the same lines Muchnik et al. [2013] designed an experiment on a social news aggregation Web site, where they analysed effects such as, the bias created over individual rating behaviour due to the ratings given by past users; and how positive and negative social influence (whether an article was considered good or bad, respectively) created herding effects, for instance, an article that displays a considerable number of down-votes (up-votes) may lead to upcoming users to also down-vote (up-vote) it, reinforcing the bias towards that article.

But not everything is negative about social influence, as Duncan Watts (Watts [2012]) estates that, when social influence is affecting the market, prediction of popularity can be a nearly impossible problem. However a more hands-on approach can be taken, what he calls **measure and react**, explained as in this kind of market, instead of trying to predict how the users will behave or defining rules to control the customers' behaviour, it is preferable to measure directly the market interactions between users and products and react accordingly. Following this idea some research (e.g., Abeliuk et al. [2015, 2017]; Van Hentenryck et al. [2016]) has focused on the design of more appropriate ranking systems (different than the popularity ranking) that can control the disparities created by social influence, claiming that unpredictability is not an inherent property of this type of markets, where consumers can observe what the rest is doing, but its related to how the information is transferred to the users, and in some ways this can be controlled through a proper design of the market.

Abeliuk et al. [2017] shows that the use of the *quality ranking* that ranks products in decreasing order of quality, reduces the unpredictability associated to social influence (in comparison with the popularity ranking). The authors report the results of an experimental study, where participants were shown a list of ten science stories displayed in a column and asked them to read one story, and later recommend it if they found it interesting. The participants were assigned (uniformly at random) into one of four different experimental conditions that vary depending on how the stories are ordered and whether social signals are displayed or not. If no social signals were present (independent condition), then participants saw only story titles and short abstracts. When social signals were displayed, each participant was provided with additional information in the form of the number of recommendations that each story received from prior participants in that experiment, displayed as "popularity bars".

As it will be presented in Chapters 3 and 4, we will consider a similar idea in our model, presenting a ranking method that mitigates the disparities between popularity and quality that emerge from social influence and position bias. A key feature of Trial-Offer markets is its decomposition into two stages, a sampling stage where participants decide which product to try followed by a buying stage where participants decide whether to buy or not the product sampled at the previous stage. Our results rely on the natural assumption that social influence and position biases have a greater effect on the decisions taken in the sampling stage than on the buying stage. Thus, popularity as proxy of quality is distorted by the noise of the first stage. Our ranking policy uses a proxy for quality based only on the second stage, which can be interpreted as the posterior probability of buying an item given that it was sampled.

1.3.2 Online Advertising and Negative Externalities

Markets like Google Ads, where the firms compete for positions in the search results when certain keywords are typed, are an example of online sponsored search market, where the market-manager selects the set of ads and their order of appearance that will maximise the expected revenue. A very popular method, used both as a research model and with industry applications is the separated *click-through-rates (CTR)* model (see for example Aggarwal et al. [2008]; Lahaie et al. [2007]; Edelman et al. [2007]), where the firms that own the ads will pay a fixed amount of money every time someone clicks on their ad, where certain ads can pay very little but their popularity compensates, providing a considerable expected revenue (Gomes et al. [2009]). The separability assumption estates that the CTR is just determinate as the product of a position factor, and a quality factor for each one of the ads, however, this does not consider the fact that due to a limited attention-span of the audience, at each stage the chances of one ad being clicked depends upon the performance of the other ads. In particular, highly attractive ads can undermine the performance of ads with higher quality, in the literature, this is called *negative externalities* (Gomes et al. [2009]; Jeziorski and Segal [2012]).

Ad auction is a well-studied scenario with negative externalities where the allocation of slots is assigned to ad bidders by an auctioneer (Kempe and Mahdian [2008]; Aggarwal et al. [2008]; Ghosh and Mahdian [2008]; Cavallo and Wilkens [2014]; Hummel and McAfee [2014]). In Ghosh and Mahdian [2008] the authors introduce the problem of modelling externalities in online advertising, and study the winner determination problem under these models, which in the most general cases turns out to be computationally hard (even to approximate). They pay special attention to the lead generation business where the goal is to sell credible leads (such as personal information of a potential customer) to advertising companies, interested in such leads. The advertisers then contact the potential customers offering them quotes about their services. This model of advertising is commonly used for insurances companies, or telecommunication companies. Where the objective of the firms is, that the leads are sent to fewer competing entities (being as exclusive as possible). A similar phenomenon is explained in a famous article from 1986 (Katz and Shapiro [1986]), where the authors discuss about the case of the competing technologies for video recording, Beta and VHS (mutually incompatible formats), where Beta was considered of superior quality (since it had better resolution and more stable images), but the recorders were more expensive, leading to opting for VHS instead, becoming the most popular option.

A general assumption in many models related to online advertising is that once an user selects one of the options, he only can consume it or leave the market. Although, more general approaches has been taken, where some researchers has focussed their work into the creation of interpretative models. Representing in this way, more complex behaviours. An example of these models is known as a *cascade model* (e.g., Kempe and Mahdian [2008]), where users can select an option and have the extra alternative to move to another one with certain probabilities, and then repeat the same process. In a recent paper, Gao et al. [2017] studies a monopolistic pricing optimisation, under a general cascade model, where users can select one product, and then they can decide to buy it, go to the next one, or abandon the market with certain transition probabilities. They first assume a complete information setting (all the transition probabilities are known) and deduce an optimal price for the products. In the second half of the paper, the authors study a more realistic setting, where the transition probabilities are unknown but they can be learnt, proposing two different approximations algorithms to do that, providing as well some theoretical performance guarantees for them.

With the appearance of online social networks, recent work in optimal auction for a single good has also considered positive network externalities, where the utility of an individual consumer for the good increases with the number of network neighbours using the same good (Hartline et al. [2008]; Haghpanah et al. [2013]; Munagala and Xu [2014]). Similar cases can be observed in online games (Liu et al. [2015]), software and operative systems (Tellis et al. [2009]), where products are more attractive when more people use them.

1.3.3 Revenue Management and Discrete Choice Models

Capturing the way people make decisions has been a problem of interest across different disciplines for many decades, having on one hand classic models from Economic Theory, and on the other hand data-driven approaches from Machine Learning, two different perspectives that aim to the same: understand consumer behaviour, and eventually predict with certain accuracy future outcomes. Many features have been considered into these models (e.g., type of users, willingness to pay, etc.), trying to establish what is more relevant to the consumers, leading to better structured markets.

From the Economic Theory perspective, predicting the sales quantities is a key element in the field of Revenue Management, where sellers have to decide what products to sell and their best prices that maximise their revenues (among other decisions). In order to do that, it is required to have at least an estimation of the consumers' demands for each one of the products, such a problem has been widely studied in Economics, where classic models assume that each user obtains certain utility for buying a particular product (given by a real number), so among all the available discrete options, consumers try to maximise their utilities. It is important to notice that this can be as general as possible, where for instance, among the available options we can consider bundles of products as a single one, or include the no-purchase option as a fictional product that captures the consumers that do not buy anything.

An important subclass of discrete choice models are special cases of the Random

Utility Models (RUM) (Block et al. [1959]). Among the most common Random Utility Models, we can find the Multinomial Logit (MNL) model, originally introduced by Luce [1959], widely used in fields such as Psychology, Marketing, Economics, and Computer Science, where it is often used for operational and managerial decisions problems such as assortment optimisation (Wang and Wang [2016]), pricing (Besbes and Sauré [2016]), scheduling (Feldman et al. [2014]), and top K-rankings (Chen et al. [2018]).

The MNL model has many advantages due to the simplicity on how it is defined, leading to desirable results like being computational tractable (Assortment optimisation can be computed in polynomial time [Talluri and Van Ryzin [2004]]), however it exhibits the property known as independence of irrelevant alternatives (IIA), which states that the ratio of the probabilities of being chosen between two alternatives, is independent of the rest of alternatives. In practice, this property is often violated, particularly when there are more than two similar alternatives. To overcome this limitation, several extensions have been proposed, among them we can find the Nested Multinomial Logit (NMNL) model (Williams [1977]), where the alternatives are grouped in nests, choosing each nest follows a MNL model, and choosing each alternative within each nest, is also chosen accordingly a MNL. The Mixed Multinomial Logit (MMNl) model (Daly and Zachary [1978]) that considers random utilities and integrates the original MNL model over the distribution of utilities. Some of the downsides of these more general choice models is the computational complexity associated to them, while problems like assortment (choosing the subset of products that maximise the expected revenue) under the MNL model admits a polynomial-time algorithm (Talluri and Van Ryzin [2004]; Rusmevichientong et al. [2010a]), in the case when consumers follow either a NMNL or MMNL model, the optimal assortment problem is NP-hard (Davis et al. [2014]; Rusmevichientong et al. [2010b] respectively]).

In a different class of models we can find the Markov Chain model (Blanchet et al. [2016]) in which states are products, and the individuals choose product i with a probability p_i or decides to move to product j with a transition probability p_{ij}. The authors prove that the assortment problem under this model can be solved efficiently in polynomial time. Berbeglia [2016] shows that the Markov Chain Model and more generally, the discrete choice models based on random walks, are a special case of the Random Utility Models.

As Berbeglia [2018] states, RUM's fail to explain several choice phenomena, such as the *decoy effect*, where the inclusion of a similar but inferior product into the option set, can increase the probabilities of being chosen for some of the original products (a typical example is to include a medium size popcorn with a price close to the large size option). Hence, more complex consumer behaviour has led to the inclusion of more general choice models that are not RUM's such as the Perception-Adjusted Luce model (PALM) (Echenique et al. [2018]), the General Attraction Model (GAM) (Gallego et al. [2014]), the General Luce Model (Echenique and Saito [2015]), and the General Stochastic Preferences Berbeglia [2018]). PALM for example considers a perception effect, where the individuals check sequentially their alternatives according a perception priority order, and the probability of choosing an alternative is affected by the probability of not choosing alternatives with higher priority.

A big part of the problems studied with the use of discrete choice models are either assortment or pricing problems (or a combination of both), where researchers weight the trade-off between having a more general model and its computational complexity. The different types of problem studied under these models has lead to many extensions. In Besbes and Sauré [2016] for example, the authors present a model where the demands follow a MNL model, they analyse equilibrium outcomes when different firms compete and face a display constraint (assortment), each retailer needs to choose strategically which products to show and what prices in order to maximise their revenues. The analysis is separated in two parts, the first part is when the prices are fixed by an external agent and the firms only compete in assortments, and the second is when they compete on both assortment and pricing, for this last one the prices are chosen according an assortment maximisation strategy. Li and Huh [2011] on the other hand, study a case where a Nested Logit model (including MNL as a special case, when there is only one nest) represents the demands of consumers, defining what the authors call the *market share* of the products, the authors find an optimal price, that maximises the revenue for a monopolist selling multiple products. A price and quantity competition are also studied under simpler conditions for the case of an oligopoly.

Some recent research have also incorporated the effect of past purchases (network effects) into a MNL model for consumer choice, for example in Wang and Wang [2016] and Du et al. [2016], the authors propose a model that focuses in a monopolistic environment (studying assortment, and pricing optimisation respectively), defining a consumer utility function affected linearly by network effects. Their models has led to many related research and extensions (e.g., Cui and Zhu [2016]; Chen and Chen [2017]). It is worth mentioning that the models presented by Du et al. [2016] and Cui and Zhu [2016] have among their results, that for the homogeneous case (identical products), if the network effects are strong enough then the optimal price assigns the same price to all the products except for one (arbitrary) product, which gets a lower price. This result differs from the classical MNL model without network effects, where in such a case, all the products have the same price. In our model on the other hand, the presence of network effects does not affect that outcome, obtaining the same price for all products. That price depends on a network parameter r, and when $r \to 0$ we recover the prices from the MNL without network effects.

Similarly, Abeliuk et al. [2016] studies an assortment optimisation problem under a MNL model that presents both, social influence and position biases, proposing a greedy policy that finds the optimal assortment and positioning in polynomial time, which holds true even in the case where capacity constrains that limit the number of visible products, are present.

Most of the previous research that include network effects into their models have focused on monopolistic markets. Among the exceptions we can find Li and Huh [2011] and Chen and Chen [2017], where the latter analyses a duopoly where the firms compete using the market share as a decision variable (instead of the price), finding multiple Nash Equilibria depending on the strength of the network effects and the quality of the products. In contrast to them, as we will see in Chapter 5, when we study a competition between sellers, our model leads to a unique Nash Equilibrium.

Also in the competition research literature, we can find a recent paper, Feng and Hu [2017] where the authors provide a game theoretical approach to a market where the strategic sellers decide to enter if their expected revenues are positive, their managerial decision is the investment in the quality of their products. After the quality game is played, sequential customers enters to the market and base their purchasing decisions on the qualities and the current sales volume. Unlike the model presented in Chapter 5, they do not consider a no purchase option, since they assume the prices are the same for every product and fixed beforehand, their main focus is on the quality competition game.

In a different stream of literature some researches have focussed on social networks and pricing decisions over the services provided (e.g., Sääskilahti [2015]; Chen et al. [2011]; Candogan et al. [2010]; Crapis et al. [2016]), due to the nature of this type of network, most of the research in this area only analyses monopolistic pricing, however some extra complexities have also been included into their models , such as incomplete information. For example Crapis et al. [2016] considers a model where the qualities of the products have a random distortion, and the preferences for each product follow a known distribution, the author study the monopolist's pricing problem where sequential customers arrive and face the decision of buying or taking an outside option. Under some conditions based on social interactions, the products' qualities eventually can be learnt, and under this setting two pricing policies are proposed (static price, and single change price).

1.3.4 Asymptotic Analyses

As early dynamics in a market tend to be very noisy, some research has focused in the study of asymptotic performance of markets, trying in the first place to detect some patterns in the consumer behaviour in the long run, and then make strategic decisions about the design of the market to control inefficiencies.

The Trial-Offer market studied in this thesis generalises the model proposed by Krumme et al. [2012], exploring various strengths for the social signal as indicated in Equation 1.1. The case of a linear social signal ($r = 1$), will be analysed first, since this has been given significant attention (e.g., Salganik et al. [2006]) . Abeliuk et al. [2015] proposed the *performance ranking* which orders the products optimally at each time t given the appeals, qualities, visibilities, and market shares. They show that, when the *performance ranking* (a greedy algorithm that chooses the permutation of products, that maximises the expected number of purchases) is used, the market always benefits from social influence in expectation. In Chapter 3 we will study the quality and performance rankings and we will show that the market converges almost surely to a monopoly for the highest-quality product, indicating that the quality and performance rankings are both optimal and predictable asymptotically. These results extend well-known theorems on Polya urns and their generalisations (e.g., Mahmound [2008]; Chung et al. [2003]; Renlund [2010]).

[Ceyhan et al., 2011] study a choice probability $C_i^J(\phi^t)$, where J represents the strength of the social signal, and prove some general convergence results under some assumptions. In particular, they use the ODE method Ljung [1977] and a Lyapunov function (e.g., Kushner and Yin [2003]) to prove that the market converges to an equilibrium when the Jacobian of C_i^J is symmetric (which is not the case when product visibilities are present). They also study in detail the case where the market follows a Logit model of the form

$$C_i^J(\phi) = \frac{e^{J\phi_i + q_i}}{\sum_j e^{J\phi_j + q_j}}$$

where J is a constant capturing the strength of the social influence signal. They show that there exists a parameter J^* such that the market converges toward a unique equilibrium when $J < J^*$ and to a monopoly when $J \geq J^*$. No analytical characterisation of the equilibrium when $J < J^*$ is presented.

It is interesting to contrast these and our results. Observe first that the proof technique used in Ceyhan et al. [2011] relies on the fact that the Jacobian of C_i^J is symmetric, which is not the case for Trial-Offer markets with product visibilities.

It is also useful to mention that different, theoretical and experimental, approaches to the use of social influence are present in the literature. For instance, [Yuan and Hwarng, 2016] describe a demand-based pricing model under social influence and capture its behaviour with a dynamical system that evolve to some stable or chaotic equilibria depending on the strength of the social signal. [Stummer et al., 2015] introduces an agent-based model for repeated purchase decisions addressing different types of innovation diffusion and their perceived attributes; They also used this methodology to an application concerned with second-generation biofuel.

1.4 Thesis Outline

The rest of this thesis is structured as follows. In Chapter 2 we explain the background of techniques and results that will be used throughout the thesis, making special emphasis in *stochastic approximation algorithms* and *ordinary differential equations.*

Chapter 3 starts defining a general model for the Trial-Offer market, as well as some special properties related to Chapter 2, where we show that the market share can be represented as a Stochastic Approximation Algorithm. We then develop on a basis of this general model of consumer choice, exploring one of the key managerial decisions, *the rankings*. We fixed the network effect function to be $f(x) = x$, giving a linear growth to the way previous purchases affect the future consumption patterns, this is one of the most common social signals used in both literature and industry applications (MusicLab uses this, for example). We study the long term behaviour of the market under two regimes: the *quality ranking* where products are ordered decreasingly in terms of their intrinsic quality, and the *performance ranking* which greedily optimise the positions of the products for any new consumer. We show that under certain conditions, in both cases the market converges to a monopoly of the highest quality product, making the market completely predictable and asymptotically optimal. We also develop some computational experiments to complement the theoretical findings, and to give some insights about the general behaviour of the market.

Chapter 4 on the other hand, considers only static rankings (like the quality ranking) but we add complexity, including a family of social signal functions indexed by a parameter $r > 0, r \neq 1$(known as the social signal or network effect parameter), where the functions are given by $f(x) = x^r$. The main objective is to study whether we can represent a market evolution to something different than a monopoly for a particular product, having on the other hand, a better distributed and fairer market. We prove that with probability 1, when $r < 1$, the vector of market shares converges to a distribution, where all the products have a strictly positive chance of being purchased. We analyse the properties of this equilibrium, such as stability and monotony in terms of the parameters. Many computational experiments are developed, to show the behaviour of the market equilibrium and its properties in terms of the parameter r. We also study the case when $r > 1$ showing that the equilibria of the Trial-Offer market are given by monopolies for each product and other type of equilibria (e.g, a market share consisting on a distribution $60\%, 40\%, 0\%$ for a market with 3 products). We prove that, when $r > 1$, the equilibria that are not monopolies are unstable (under certain conditions). As a result, the market will likely converge to a monopoly for some product, highly affected by initial conditions and early dynamics.

In Chapter 5 we consider a slightly different model, to study the problem from the sellers perspective, where we start from a variation of a Multinomial Logit model with network effects for consumer choice, incorporating the prices of the products into the model. Naturally we also consider a no purchase option, which represents the cases where consumers decide not to purchase anything. We analyse two different settings that strategic sellers can follows, they can either cooperate or compete with each other. In the first case the sellers adopt a monopolistic strategy trying to maximise the overall revenue. We find in this case an optimal static price strategy. If on the other hand, they decide to compete (so they can maximise their own revenues), a price competition game is induced, we prove that this game has a unique pure Nash Equilibrium, providing a greedy algorithm to compute it. We also analyse the behaviour of the prices and revenues in terms of the network effect parameter, as well as how the expected consumer utilities are compared under both strategies.

Finally Chapter 6 explains the main contributions of this work and establish the open questions that can lead to some future work.

Chapter 2

Background

This chapter has the purpose of introducing the necessary background that will be used throughout the thesis. Section 2.1 explains some key results about Ordinary Differential Equations and Stability Analysis. In Section 2.2 we review Stochastic Approximation Algorithms, where we are particularly interested in Robbins-Monro Algorithms. Section 2.3 presents some of the key results from Benaïm [1999] that will be used in this thesis.

2.1 Differential Equations and Stability

This section starts with a brief introduction of Ordinary Differential Equations (ODE) and some stability criteria (e.g., see Hirsch et al. [2012]; Jordan and Smith [1999]). Consider the following differential equation

$$\frac{dy}{dt} = F(y) \tag{2.1}$$

where F is some vector field. The concept of equilibrium is central in the study of asymptotic behaviour for this type of equation.

Definition 2.1. *A vector $y^* \in \mathbb{R}^n$ is an equilibrium for differential equation* (2.1) *if $F(y^*) = 0$.*

We are interested in equilibria that satisfy (at least) certain stability criteria.

Definition 2.2. *An equilibrium y^* is said to be stable for Equation* (2.1) *if, given $\epsilon > 0$, there exists $\delta > 0$ such that $\|y(t) - y^*\| < \epsilon$ for all $t > 0$ and for all y such that $\|y - y^*\| < \delta$. We say that y^* is asymptotically stable if it also satisfies*

$$\lim_{t\to\infty} y(t) = y^*.$$

Remark 2.1. When an equilibrium y^* is not stable, we say that y^* is *unstable*.

The asymptotic stability of an equilibrium y^* can be characterised in terms of the Jacobian matrix $JF(y^*) = (\frac{\partial F_i(y^*)}{\partial y_j})_{i,j}$ $(i, j \in \{1, \ldots, n\})$ as estated by the following Theorem (see, for instance, Jordan and Smith [1999] p. 440). The following two theorems will be used in Chapter 4 to show, that under certain circumstances, some market equilibria are unstable.

Theorem 2.1. *Let y^* be an equilibrium for the differential equation* (2.1)*. If the eigenvalues of the Jacobian matrix $JF(y^*)$ all have negative real part, then y^* is asymptotically stable. If, on the other hand, $JF(y^*)$ has at least one eigenvalue with a positive real part, then y^* is unstable.*

A well-known result from linear algebra (see, for example, Robinson [2006] p. 296) establishes the connection between the trace of a square matrix and its eigenvalues. In the following, we use $tr(A) = \sum_i^n a_{ii}$ to denote the trace of matrix A where a_{ii} are the diagonal entries of matrix A.

Theorem 2.2. *Let A be a $n \times n$ matrix, and $\lambda_1, ..., \lambda_n$ its eigenvalues. Then $tr(A) = \sum_{i=1}^n \lambda_i$.*

2.1.1 Convergence of sequences

In Chapter 5 we will propose a greedy algorithm to compute some equilibrium price, the correctness of that algorithm it is based on a classic result about convergence of subsequences, known as The Bolzano-Weierstrass Theorem, which can be stated as follows

Theorem 2.3. *Every bounded sequence in $\mathbb{R}^n$ has a convergent subsequence.*

Its proof can be found for example in Burk [2011] (Theorem 2.6).

2.2 Stochastic Approximation Algorithms

The main results of this thesis rely on Stochastic Approximation Algorithms, and the ODE method. We will use the definitions and results from this section to characterise the long term behaviour of our market model. We consider now one of the most well studied Stochastic Approximation Algorithms, a Robbins-Monro Algorithm (RMA) [Kushner and Yin, 2003; Duflo and Wilson, 1997] which can be defined as follows.

Definition 2.3 (Robbins-Monro Algorithm)**.** *A Robbins-Monro Algorithm (RMA) is a discrete time stochastic process $\{x^k\}_{k\geq 0}$ whose general structure is specified by*

$$x^{k+1} - x^k = \gamma^{k+1}[F(x^k) + U^{k+1}], \tag{2.2}$$

where

- *x^k takes its values in some Euclidean space (e.g., $\mathbb{R}^n$);*
- *γ^k is deterministic and satisfies $\gamma^k > 0$, $\sum_{k\geq 1} \gamma^k = \infty$, and $\lim_{k\to\infty} \gamma^k = 0$;*
- *$F : \mathbb{R}^n \to \mathbb{R}^n$ is a deterministic continuous vector field;*

- $\mathbb{E}[U^{k+1}|\mathcal{F}^k] = 0$, *where $\mathcal{F}^k$ is the natural filtration of the entire process.*[1].

A RMA $\{x^k\}_{k\geq 0}$ where x^k has n coordinates is said to be n-dimensional.
Robbins-Monro algorithms are particularly interesting because, under certain conditions on x^k, γ^k, and U^{k+1}, their asymptotic behaviour, i.e., the values of x^k when $k \to \infty$, is closely related to the asymptotic behaviour of the following continuous dynamic process:

$$\frac{dx^t}{dt} = F(x^t). \tag{2.3}$$

This idea, called *the ODE Method*, was introduced by Ljung [1977] and has been extensively studied (e.g., [Borkar and Meyn, 2000; Duflo and Wilson, 1997; Kushner and Yin, 2003]). Consider again the RMA $\{x^k\}_{k\geq 0}$ defined in (2.2) and the following hypotheses:

H1: $\sup_k \mathbb{E}[\|U^{k+1}\|^2] < \infty$;

H2: $\sum_k (\gamma^k)^2 < \infty$;

H3: $\sup_k \|x^k\| < \infty$.

We will now present a theorem establishing the connection between the discrete stochastic process (2.2) and the continuous process defined by (2.3). This connection requires the concept of Internally Chain Transitivity (ICT) sets. These ICT sets include equilibria, periodic orbits of (2.3), and possibly more complicated sets.

To define ICT sets formally for the purpose of this thesis, we use Proposition 5.3 in Benaïm [1999] that proves that the concepts of internally chain recurrent and internally chain transitive set are equivalent when the set over which F is defined is connected, which will be the case here.

Definition 2.4 ((ϵ, T)-Chains Conley [1978])**.** *Consider $\epsilon > 0$, $T > 0$, a set $A \subset \mathbb{R}^n$, and two points $x, y \in A$. There is an (ϵ, T)-chain of length k in A between x and y if there exist k solutions $\{y_1, ..., y_k\}$ of* (2.3) *and their associated times $\{t_1, ..., t_k\}$ with $t_i > T$ such that*

1. *$y_i^t \in A$ for all $t \in [0, t_i]$ and for all $i \in \{1, ..., k\}$;*
2. *$\|y_i^{t_i} - y_{i+1}^0\| < \epsilon$ for all $i \in \{1, ..., k-1\}$;*
3. *$\|y_1^0 - x\| < \epsilon$ and $\|y_k^{t_k} - y\| < \epsilon$.*

We are now in a position to define ICT sets, which is derived from the definition of Internally Chain Recurrent sets introduced by Conley [1978].

Definition 2.5 (ICT Sets)**.** *A closed set A is said Internally Chain Transitive (ICT) for the dynamics* (2.3) *if it is compact, connected, and for all $\epsilon > 0, T > 0$ and $x, y \in A$, there exists an (ϵ, T)-chain in A between x and y.*

[1] $\mathcal{F}^k$, the natural filtration, is the σ-field generated by the history $\{x^l : l \leq k\}$

The following theorem, due to Benaïm [1999] and whose proof is in Section 2.3, links the behaviour of the limit set $L\{x^k\}_{k\geq 0}$ of any sample path $\{x^k\}_{k\geq 0}$ for Equation (2.2) and the limit sets of the solution to Equation (2.3).

Theorem 2.4 (Benaïm [1999]). *Let $\{x^k\}_{k\geq 0}$ be a Robbins-Monro algorithm* (2.2) *satisfying hypotheses $H1 - H3$ where F is a bounded locally Lipschitz vector field (e.g., a bounded $\mathcal{C}^1$ function). Then, with probability 1, the limit set $L\{x^k\}_{k\geq 0}$ is internally chain transitive for Equation* (2.3).

2.3 Key Results from Benaïm [1999]

Consider the set of solutions of the differential equation (2.3), we say that $Y = (Y_t)_{t\in\mathbb{R}}$ is the *flow induced* by the vector field F, where Y_t are the local unique solutions of (2.3) with $x^0 = x_0 \in \Delta^{n-1}$. Benaïm defines the following useful concept: A continuous function $X : \mathbb{R}_+ \to \mathbb{R}^n$ is an *Asymptotic pseudo-trajectory* for Y if for any $T > 0$

$$\lim_{t\to\infty} \sup_{0\leq h\leq T} dist(X(t+h), Y_h(X(t))) = 0.$$

Recall now that the Robbins-Monro Algorithm (2.2) is defined by

$$x^{k+1} - x^k = \gamma^{k+1}[F(x^k) + U^{k+1}],$$

Let $\tau_k = \sum_{i=1}^{k} \gamma^i, \tau_0 = 0$ and define the affine interpolated process $Z(t)$:

$$Z(t) = x^k + [t - \tau_k]\frac{x^{k+1} - x^k}{\gamma^{k+1}}, \quad \tau_k \geq t \geq \tau_{k+1}. \tag{2.4}$$

Consider also the map $m : \mathbb{R}_+ \to \mathbb{N}$ defined by $m(t) = \sup\{k \geq 0 : t \geq \tau_k\}$.

Proposition 2.1 (Proposition 4.1 in Benaïm [1999]). *Let F be a bounded locally Lipschitz vector field. Assume that*

A1.1 For all $T > 0$,

$$\lim_{l\to\infty} \sup\{\|\sum_{i=n}^{k-1} \gamma^{i+1}U^{i+1}\| : k = n+1, ..., m(\tau_l + T)\} = 0.$$

A1.2 $\sup_k \|x^k\| < \infty$.

Then the interpolated process $Z(t)$ is an asymptotic pseudotrajectory of the flow induced by F.

Proposition 2.2 (Proposition 4.2 in Benaïm [1999]). *Let ϕ^k be the Robbins-Monro Algorithm* (3.7). *Suppose that, for some $b \geq 2$,*

$$\sup_k \mathbb{E}(\|U^{k+1}\|^b) < \infty,$$

and

$$\sum_k [\gamma^k]^{1+b/2} < \infty.$$

Then assumption $A1.1$ of Proposition 2.1 holds with probability 1.

Let $X : \mathbb{R}_+ \to M$ be an asymptotic pseudotrajectory of an induced flow Φ, with M some metric space. The limit set $L(X)$ of X is the set of limits of convergent sequences $X(t_k)$, $t_k \to \infty$.

Theorem 2.5 (Theorem 5.7 *i.* in Benaïm [1999])**.** *Let X be a precompact asymptotic pseudotrajectory of Φ. Then $L(X)$ is Internally Chain Transitive.*

The following Theorem is an adaptation of Corollary 10.3 in Chichilnisky [1998] (p.23), which is itself, a particular case of Theorem 2.4.

Theorem 2.6. *Let $Eq = \{e \in \mathbb{R}^n : F(e) = 0\}$ be the set of Equilibria for Equation* (2.3)*. If Eq is composed by asymptotically stable equilibria and every trajectory of* (2.3) *converges to one of the equilibrium $e \in Eq$. Then with probability one, a sample path of the process* (3.7) *converges to one of these equilibria.*

Theorem 2.6 will link the asymptotic behaviour of the RMA that defines our model (see Chapter 3), with the equilibria of a related continuous dynamic (described by a ODE). To show that the only possible ICT sets are equilibria, we will solve the dynamical systems (ODE + initial condition), and we will show that given any initial condition, the solutions of the ODE must converge to some $e \in Eq$.

2.3.1 Proof of Theorem 2.4

Proof of Theorem 2.4. According to hypotheses $H1 - H2$, Proposition 2.2 holds for $b = 2$. As a result, we can apply Proposition 2.1 and $Z(t)$ from Equation (2.4) is almost surely an asymptotic pseudo-trajectory for the flow induced by F. Using $H3$, then $Z(t)$ is precompact. Finally, using Theorem 2.5, the limit set $L\{x^t\}_{t\geq 0}$ is an ICT for Equation (2.3). □

Throughout the thesis we will use Theorems 2.4 and 2.6 to show that the market share described in with our model converges with probability 1 to the equilibria of a related continuous dynamic.

Chapter 3

Position bias over a model of consumer choice with social influence

This chapter is reproduced with changes from

i) Abeliuk, A.; Berbeglia, G.; **Maldonado, F.**; and Van Hentenryck, P., 2016. Asymptotic optimality of myopic optimization in trial-offer markets with social influence. *In the 25th International Joint Conference on Artificial Intelligence* (IJCAI-16).

ii) Van Hentenryck, P.; Abeliuk, A.; Berbeglia, F.; Berbeglia, G.; and **Maldonado, F.**, 2016. Aligning popularity and quality in online cultural markets. *In the Proceedings of the International AAAI Conference on Web and Social Media* (ICWSM 2016)

And using techniques developed in **Maldonado, F.**; Van Hentenryck, P.; Berbeglia, G.; and Berbeglia, F., 2018. Popularity signals in trial-offer markets with social influence and position bias. *European Journal of Operational Research, 266(2), pp. 775-793.* to define a general version of the Trial-Offer market model. Where my main contributions for the papers i) and ii) are in the theoretical section of them, providing a representation of the market as a Robbins-Monro Algorithm, characterising its possible equilibria, and finally proving how the market converges (or not) to them.

Motivation In this chapter, we first introduce formally a model for a Trial-Offer Market, we deduce some general results valid for any ranking policy and social signal, and then we establish some of the special interesting cases: to study the effect of position biases in this kind of market, we consider two different ranking policies, one static and the other one dynamic, most of the theoretical results are directly applicable for both type of ranking (e.g., Theorem 3.1), however in others, we need to separate the analysis for each policy (e.g., Theorem 3.3). We will be interested into understanding the long term behaviour of the consumption decisions under these policies, and to do that we will use Robbins-Monro Algorithms to model the expected market shares. We finally complement the theoretical results with some computational experiments, where we compare the different behaviour of the market depending on the type of ranking that was used.

3.1 Introduction

Social influence is ubiquitous in cultural markets. From book recommendations in `Amazon`, to song popularities in `iTunes`, and article rankings in the online version of the `New York Times` or the `Reddit and Hacker` site, social influence has become a critical aspect of the customer experience. Social influence may appear through different social signals such as the number of past purchases; consumer ratings; and/or consumer recommendations, depending on the market and/or platform.

Social influence is often also reinforced by position bias (e.g., Lerman and Hogg [2014]), as consumer preferences are also affected considerably by the visibility of the choices. In digital markets, the impact of visibility on consumer behaviour has been widely observed in internet advertisement where sophisticated mathematical models have been developed to determine the relative importance of the different ads positions, in online stores such as `Amazon` and `iTunes`, and in online travel agents such as `Expedia` and `Orbitz` among others.

In this chapter, we start defining a model for Trial-Offer markets, where we establish some general results (that will be used in the upcoming chapters as well). We then consider some specific settings where the the network effects grow linearly, and the rankings are of two different types. The first one known as the *performance ranking*, introduced in Abeliuk et al. [2015]. This ranking is a myopic policy that dynamically maximises the efficiency of the market for each incoming participant, taking into account the inherent quality of products, position bias, and network effects. The second ranking is referred to as *quality ranking*, proposed in Van Hentenryck et al. [2015], in which product qualities are first recovered (using sampling and/or reinforcement learning) and then used to display products in decreasing order of quality: this policy reinforces the appeal of quality products with position bias.

We investigate the quality and performance rankings both computationally, using the generative model of the MusicLab proposed in Krumme et al. [2012], and theoretically by modelling the Trial-Offer market as a discrete choice model based on a multinomial logit (Luce [1965]) with network effects. We study the asymptotic convergence of the market shares under those two policies. Showing that these myopic policies are optimal and predictable asymptotically, in addition to being optimal at each step. From a technical standpoint, our analysis is the first to provide theoretical guarantees over a dynamic policy in cultural markets. Moreover, computational results show that the rate of convergence for the performance ranking is considerably faster than the quality ranking.

Our work is a step toward the understanding and development of expressive computational models for long-term effect of social influence (including unpredictability), an open question raised by Kleinberg (2008). Our main contributions for this chapter can be summarised as follows:

1. The theoretical results show that the Trial-Offer market is optimal asymptotically when the quality/performance rankings are used and converges almost surely to a monopoly for the highest quality product.

2. The computational results show that both, the quality and performance rankings under social influence, significantly improves market efficiency, decreases unpredictability, and identifies "blockbusters". It provides significant improvements over the popularity ranking.

Our results provide an interesting contrast with the conclusions of Salganik et al. [2006]. The quality and performance rankings align quality and popularity, making the market efficient and predictable. In other words, *it is not social influence per se that makes markets unpredictable: It is the way it is used that may lead to unpredictability and inefficiency.*

The rest of this chapter is organised as follows. Section 3.2 introduces Trial-Offer markets and Section 3.3 surveys the ranking policies used in this chapter. Section 3.5 describes the computational experiments, which motivate the theoretical study presented in Section 3.4, including the benefits of position bias.

3.2 Trial-Offer Markets

The chapter builds on the work by Krumme et al. [2012] who propose a framework in which consumer choices are captured by a Multinomial Logit model whose product utilities depend on the product appeal, position bias, and a social influence signal representing past purchases. A marketplace consists of a set of n items that we call [n]. Each item $i \in [n]$ is characterised by two values:

1. its *appeal* $a_i > 0$ which represents the inherent preference of trying item i;

2. its *quality* $q_i > 0$ which represents the conditional probability of purchasing item i given that it was tried.

This thesis assumes that the appeals and the qualities are known. Abeliuk et al. [2015] have shown that these values can be recovered accurately and quickly, either before or during the market execution using the approximation suggested by Krumme et al.:

$$a_i \sim \frac{s_i}{\sum_j s_j},$$

and

$$q_i \sim \frac{d_i}{s_i},$$

where s_i and d_i are the samplings and purchases of product i at some point in time, respectively.

The objective of the firm running this market is to maximise the total expected number of purchases. To achieve this, one of the key managerial decision of the firm is what is known as the ranking policy [Abeliuk et al., 2015], which consists in deciding how to display the products in the market (e.g., where to display a product on a web page). Here we assume that, at the beginning of the market, the firm decides upon a ranking for the items, i.e., an assignment of items to positions in the marketplace. Each position j has a visibility v_j which represents the inherent probability of trying an item in position j. A ranking σ is a permutation of the items and $\sigma_i = j$ means that item i is placed in position j ($j \in [n]$). When a customer enters the market, she observes all the items and their *social signals* as a function of the values of the previous purchases $d^t = (d^t_1, \ldots, d^t_n)$.

As it was explained in 1.1 our main goal is to maximise the market efficiency, represented by the expected number of purchases. Note also that the higher this objective is, the lower the probability that consumers try a product but then decide not to purchase it. Hence, if we interpret this last action as an inefficiency, maximising the expected efficiency of the market also minimises unproductive trials. We also examine a number of questions about the market including (1) What is the best way to allocate the products to positions? (2) Is it beneficial to display a social signal, e.g., the number of past purchases, to customers? (3) Is the market predictable?

Since we are interested in the long-term effects of social influence, we consider a multi-period, dynamic market where consumers arrive sequentially, one per time period. Upon arrival, a consumer is able to observe the aggregate purchase decisions of her predecessors. Denote by $d^t = \left(d^t_1, \ldots, d^t_n\right)$ the total number of consumers who purchased some product i until the beginning of period t. If item i is purchased at time t, then the purchase vector becomes

$$d_j^{t+1} = \begin{cases} d_j^t + 1 & \text{if } j = i; \\ d_j^t & \text{otherwise.} \end{cases}$$

The probability that the consumer arriving at period t will try product i if items are displayed using position assignment σ is given by

$$\mathcal{P}_i(\sigma, d^t) = \frac{v_{\sigma_i} f(a_i + d^t_i)}{\sum_{j=1}^n v_{\sigma_j} f(a_j + d^t_j)}.$$

Where $f : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous positive function (e.g., $f(x) = x^r$, with $r > 0$ or $f(x) = \log(x+1)$). Observe that consumer choice preferences for trying the products are essentially modelled as a discrete choice model based on a Multinomial Logit (Luce [1965]) in which product utilities are affected by their position. The market uses the number of purchases d^t_i of product i at time t, as input for the social signal function f. However, other social signals such as the market share, used in online site such as `iTunes` (with the right choice of function f), can be shown to be equivalent, as it can be seen in Remmark 3.1 . Let $\phi^t = \left(\phi^t_1, \ldots, \phi^t_n\right)$ denote the market shares at time t in terms of the total number of purchases d^t, i.e.,

$$\phi_i^t = \frac{d_i^t}{\sum_{j=1}^n d_j^t},$$

where ϕ^t lives in the n dimensional simplex, this is,

$$\phi^t \in \Delta^n = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1 \text{ and } \sum_{i=1}^n x_i = 1\}.$$

The probability of trying product i can be rewritten as a function of ϕ^t, yielding,

$$\mathcal{P}_i(\sigma, \phi^t) = \frac{v_{\sigma_i} f(\phi_i^t)}{\sum_{j=1}^n v_{\sigma_j} f(\phi_j^t)}, \tag{3.1}$$

where, for simplicity, the vector d^0 is initialised with the products appeals, i.e., $d_t^0 = a_i$ (see Appendix A [Market share versus purchases], for a general version). Both notations are convenient to stress different results; we use market shares when analysing the asymptotic behaviour of the market and the number of purchases for analysing the transient behaviour of the market. Since q_i is the conditional probability of purchasing product i given that it was tried and hence, the conditional expected number of purchases at time t is given by $\sum_{i=1}^n \mathcal{P}_i(\sigma, d^t)\; q_i$.

Equation (3.1) generalises the multinomial logit model of Krumme et al. [2012] who defines two sets of probabilities, $p_{i,t}^{SI}$ and p_i^I, that capture the probability of trying product i with and without social influence. These probabilities are defined as:

$$p_{i,t}^{SI} = \frac{v_{\sigma(i)}(\alpha a_i + d_i^t)}{\sum_{j=1}^n v_{\sigma(j)}(\alpha a_j + d_j^t)}, \qquad p_i^I = \frac{v_{\sigma(i)} a_i}{\sum_{j=1}^n v_{\sigma(j)} a_j}, \tag{3.2}$$

where α is a parameter to calibrate the strength of the social signal (e.g., $\alpha = 200$ for the MusicLab experiments). Equation (3.1) allows us to recover the formulae (3.2) via some linear transformation of the identity function: $f(\phi_i) = \beta\phi_i + \alpha a_i$, with $\beta = \sum_j d_j$ or $\beta = 0$ for each case.

After having tried product i, a customer decides whether to buy the sampled item and the probability that she purchases item i is given by q_i.

Our goal is to study how the market shares evolve over time when social influence is present. Observe that the probability of trying a product depends on its position in the list, its appeal, and its number of purchases at time t. Note also that in a dynamic market when no social signals are displayed, the purchase history plays no role and hence, the market behaves as a static market. Following Salganik and Watts [2008], we refer to this setup as the independent condition.

3.3 Rankings Policies

This section presents the ranking policies studied in this chapter. Without loss of generality, we assume that the qualities and visibilities are non-increasing, i.e., $q_1 \geq q_2 \geq \cdots \geq q_n$ and $v_1 \geq v_2 \geq \cdots \geq v_n$. We also assume that the qualities and visibilities are known. In practical situations, the product qualities are obviously unknown but Abeliuk et al. (2015) have shown that they can be recovered accurately and quickly.

The *performance ranking* (P-rank), was proposed by Abeliuk et al. (2015) to show the benefits of social influence in cultural markets. It maximises the expected number of purchases at each iteration, exploiting all the available information globally, i.e., the appeal, the visibility, the purchases, and the quality of the products. More precisely, the performance ranking at step t produces a ranking σ_t^* defined as

$$\sigma_t^* = \underset{\sigma \in S_n}{\text{arg-max}} \sum_{i=1}^{n} \mathcal{P}_i(\sigma, d^t) \cdot q_i. \tag{3.3}$$

The performance ranking can be computed in strongly polynomial time and the resulting policy scales to large markets (Abeliuk et al. [2015]).

The *quality ranking* (Q-rank), which simply orders the products by quality, assigning the product of highest quality to the most visible position and so on. With the above assumptions of non-increasing qualities and visibilities, the quality ranking σ satisfies $\sigma_i = i \ \ (1 \leq i \leq n)$ at any given time t.

Our results contrast with the *popularity ranking* (D-rank), used in Salganik et al. [2006] to show the unpredictability caused by social influence in cultural markets. At iteration t, the popularity ranking orders the products by the number of purchases d_i^t, but these purchases do not necessarily reflect the inherent quality of the products, since they depend on how many times the products were tried, which in turn depends on the position and social signal of the product.

We will also annotate the policies with SI or IN to denote whether they are used under the social influence or the independent condition. For instance, P-rank(SI) denotes the policy that uses the performance ranking under the social influence condition, while P-rank(IN) denotes the policy using the performance ranking under the independent condition. We also use rand-rank to denote the policy that simply presents a random order at each period. Under the independent condition, the optimisation problem is the same at each iteration as mentioned earlier. Since the performance ranking maximises the expected purchases at each iteration, it dominates all other policies in this setting (Abeliuk et al. [2015]).

3.4 Theoretical Analysis

In this section some general theoretical results are presented, those results will be also used in the next chapters. We will also analyse in detail the simpler case where the social signal function is the identity, this is $f(a_i + d_i^t) = a_i + d_i^t$, and for that case we will study the different rankings previously introduced.

Asymptotic Behaviour of the Market Share We first characterise a general lemma that defines the probability that the next purchase is product i, which is valid for any ranking σ (static or dynamic) and social signal function f. Using that characterisation, we then prove that the market share ϕ^t can be defined as a suitable Robbins-Monro Algorithm (see Definition 2.3), and we characterise for the studied rankings and social signal $f(x) = x$, the long term behaviour of the market, showing that in each case (Q-RANK and P-RANK) the market converges almost surely to a monopoly of the highest quality product.

Lemma 3.1. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is a positive function, then the probability $\pi_i(\phi)$ that the next purchase is the product i given the market share vector ϕ is given by*

$$\pi_i(\phi) \; = \frac{v_{\sigma_I} q_i f(\phi_i)}{\sum_{j=1}^{n} v_{\sigma_j} q_j f(\phi_j)}. \tag{3.4}$$

Proof. The probability that item i is purchased in the first step is given by

$$\pi_i^{1st}(\phi) = \frac{v_{\sigma_i} f(\phi_i)}{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)} q_i.$$

The probability that item i is purchased in the second step and no item was purchased in the first step is given by

$$\pi_i^{2nd}(\phi) = \left(\frac{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)(1 - q_j)}{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)} \right) \frac{v_{\sigma_i} f(\phi_i)}{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)} q_i.$$

More generally, the probability that item i is purchased in step m while no item was purchased in earlier steps is given by

$$\pi_i^{mth}(\phi) = \left(\frac{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)(1 - q_j)}{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)} \right)^{m-1} \frac{v_{\sigma_i} f(\phi_i)}{\sum\limits_{j=1}^{n} v_j f(\phi_j)} q_i. \tag{3.5}$$

Let $a = (\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j) q_j) / (\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j))$. Observe that, if $q_{max} = \max_{i \in \{1,\ldots,n\}} q_i$, then $0 < a \leq q_{max} \leq 1$. Equation (3.5) becomes

$$\pi_i^{mth}(\phi) = \Big(1 - a\Big)^{m-1} \frac{v_{\sigma_i} f(\phi_i)}{\sum\limits_{j=1}^{n} v_{\sigma_j} f(\phi_j)} q_i.$$

Hence the probability that the next purchase is item i is given by

$$\pi_i(\phi) = \sum_{m=0}^{\infty} \left(1-a\right)^m \frac{v_{\sigma_i} f(\phi_i)}{\sum_{j=1}^{n} v_{\sigma_j} f(\phi_j)} q_i.$$

Since $|1-a| < 1$, we use the geometric series

$$\sum_{m=0}^{\infty} \left(1-a\right)^m = \frac{1}{a},$$

and then, the probability that the next purchase is item i is given by

$$\pi_i(\phi) = \frac{v_{\sigma_i} q_i f(\phi_i)}{\sum_{j=1}^{n} v_{\sigma_j} q_j f(\phi_j)}.$$

□

Remark 3.1. We notice an important equivalence which is valid when $f(x) = x^r$, $r > 0$: Equation (3.4) remains the same when expressed in terms of the number of purchases d_i instead of the market shares. Indeed,

$$\pi_i(\phi) = \frac{v_{\sigma_i} q_i (\phi_i)^r}{\sum_j v_{\sigma_j} q_j (\phi_j)^r} = \frac{v_{\sigma_i} q_i (\frac{d_i}{\sum_k d_k})^r}{\sum_j v_{\sigma_j} q_j (\frac{d_j}{\sum_k d_k})^r} = \frac{v_{\sigma_i} q_i (d_i)^r}{\sum_j v_{\sigma_j} q_j (d_j)^r}. \tag{3.6}$$

Hence, when the social signal function is $f(x) = x^r$, $r > 0$, this model can be interpreted either using the concept of market share or simply the number of purchases.

The following theorem, establishes that the vector of market shares ϕ^k can be expressed as a Robbins-Monro Algorithm.

Theorem 3.1. *The discrete stochastic dynamic process $\{\phi^k\}_{k\geq 0}$ can be modelled as a Robbins-Monro algorithm, given by*

$$\phi^{k+1} = \phi^k + \gamma^{k+1}(\pi(\phi) - \phi + U^{k+1}),$$

where $\gamma^{k+1} = \frac{1}{D^k+1}$, and U^{k+1} is a martingale difference noise.

Proof. First we notice that ϕ^k is not modified if no purchase is made, therefore for the purpose of studying its evolution we only consider a step k when a customer buys something. Denote by e^k the random unit vector whose j^{th} entry is 1 if item j is the next purchase and 0 otherwise. Since $\phi^k = \dfrac{D^k \phi^k}{D^k}$, the market share at time $k+1$ is given by

$$\phi^{k+1} = \frac{D^k \phi^k}{D^k + 1} + \frac{e^k}{D^k + 1},$$

where $D^k = \sum_{t=0}^{k}\sum_{i=1}^{n} d_i^t = k + k_0$, and $k_0 = \sum_{i=1}^{n} a_i$. It follows that

$$\begin{aligned}
\phi^{k+1} &= \frac{(D^k+1)\phi^k}{D^k+1} - \frac{\phi^k}{D^k+1} + \frac{e^k}{D^k+1} \\
&= \phi^k + \frac{1}{D^k+1}(e^k - \phi^k) \\
&= \phi^k + \frac{1}{D^k+1}(\mathbb{E}[e^k|\mathcal{F}^k] - \phi^k + e^k - \mathbb{E}[e^k|\mathcal{F}^k]) \\
&= \phi^k + \frac{1}{D^k+1}(\pi(\phi^k) - \phi^k + e^k - \mathbb{E}[e^k|\mathcal{F}^k]).
\end{aligned}$$

This last equality can be reformulated as

$$\phi^{k+1} = \phi^k + \gamma^{k+1}(F(\phi^k) + U^{k+1}), \tag{3.7}$$

where $\gamma^{k+1} = \frac{1}{D^k+1}$, $F(\phi) = \pi(\phi) - \phi$, and $U^{k+1} = e^k - \mathbb{E}[e^k|\mathcal{F}^k]$.

The above derivation showed that $\{\phi^k\}_{k\geq 0}$ can be expressed through Equation (3.7), where it is easy to see that $\gamma^k > 0$, $\sum_{k\geq 1}\gamma^k = \infty$, $\lim_{k\to\infty}\gamma^k = 0$, and that $\mathbb{E}[U^{k+1}|\mathcal{F}^k]$ is equal to zero. $\square$

Remark 3.2. Note that the function F captures the difference between the probabilities of purchasing the items (given the market shares) and the market shares at each time step. Recall that $\phi^k \in \Delta^n$ for all $k \geq 0$, which is a compact, convex subset of $\mathbb{R}^n$ (and hence connected). We also notice that the only thing we needed to deduce the previous Theorem was having a probability distribution π^k that can be related to the random vector e^k, this can be further generalised for more complex type of distributions (even when their functional form is unknown), more on this can be seen in Appendix A [Generalisations].

3.4.1 Linear network effects, $f(x) = x$.

Now since clearly for the case where $f(\cdot)$ is the identity function, the hypotheses for Theorem 2.4 hold, then in order to understand the asymptotic behaviour of the discrete process that describes the market share ϕ^k, we have to study the asymptotic behaviour of the solutions for the system of ODEs given by

$$\dot{\phi}^t = \pi(\phi^t) - \phi^t, \quad \phi^t \in \Delta^n,$$

to do so, we first study the solutions for the equation $\pi(\phi^t) - \phi^t = 0$ (fixed points for the probability distribution). We will prove that in the case where $f(x) = x$ and $q_1 > q_2 > \cdots > q_n$, the only solutions are the canonical vector of $\mathbb{R}^n$, $\mathfrak{e}_i$, $i \in \{1, \ldots, n\}$, where the coordinates $(\mathfrak{e}_i)_j = 0$ if $j \neq i$, and 1 otherwise. And finally we will prove that with probability 1, the market share converges to one of these solutions. We define $\hat{q}_i := q_i v_{\sigma_i}$, and since the case where all the $\hat{q}_i$ are the same is trivial (if $\hat{q}_i = \hat{q}$ for all i, then $\pi(\phi) = \phi$ at any time), we consider that there exists $m \leq n$ intervals given as follows

$$\hat{q}_1 = \cdots = \hat{q}_{n_1} > \hat{q}_{n_1+1} = \cdots = \hat{q}_{n_2} > \cdots > \hat{q}_{n_{m-1}+1} = \cdots = \hat{q}_{n_m},$$

where we define $n_0 = 1$ and we notice that $n_m = n$. The previous case represents markets where a big list is presented (100 songs for example), and then the last 10 positions have basically the same visibility. Finally for the rest of this chapter, we consider that the social signal is given by $f(x) = x$.

Theorem 3.2. *The solutions for the Equation $\pi(\phi) - \phi = 0$ have the following structure: there exists $j : 0 \le j < m$ such that for all $i : n_j + 1 \le i \le n_{j+1}$, $\phi_i = \dfrac{1}{n_{j+1} - n_j}$, and $\phi_i = 0$ otherwise.*

Proof.

$$\begin{aligned}
\pi(\phi) - \phi &= 0 \\
\pi_i(\phi) - \phi_i &= 0, \forall i \in \{1, \ldots, n\} \\
\frac{\hat{q}_i \phi_i}{\sum_{j=1}^n \hat{q}_j \phi_j} - \phi_i &= 0, \forall i \in \{1, \ldots, n\} \\
\frac{\phi_i}{\sum_{j=1}^n \hat{q}_j \phi_j} \left(\hat{q}_i - \sum_{j=1}^n \hat{q}_j \phi_j \right) &= 0, \forall i \in \{1, \ldots, n\}.
\end{aligned} \tag{3.8}$$

Let $Q := \{i \in [n] : \phi_i \neq 0\}$, then for every $i \in Q$, by Equation (3.8) necessarily

$$\hat{q}_i = \sum_{k=1}^n \hat{q}_k \phi_k = \sum_{k \in Q} \hat{q}_k \phi_k$$

thus, for all $i, k \in Q$, $\hat{q}_i = \hat{q}_k$. Therefore $Q = \{i \in [n] : \hat{q}_i = \hat{q}\}$, hence there must exists $j : 0 \le j < m$ such that $Q = \{n_j + 1, n_j + 2, \ldots, n_{j+1}\}$. Furthermore if $i \notin Q$, then $\phi_i = 0$. If $i \in Q$, since all the qualities are the same, by symmetry $\phi_i = C$ a constant value, then $\sum_{k \in Q} \phi_k = 1$, implies that $|n_{j+1} - (n_j + 1) + 1| C = 1$, where $\phi_i = \dfrac{1}{n_{j+1} - n_j}$.

□

Corollary 3.1. *If $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_n$, then the only solutions for the Equation $\pi(\phi) - \phi = 0$ are the canonical vectors , $\mathfrak{e}_i$, $i \in \{1, \ldots, n\}$.*

Proof. If $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_n$, then $m = n$ and for all $0 \le j < m$, $n_{j+1} - n_j = 1$. Therefore the only possible equilibria are the canonical vectors $\mathfrak{e}_i$, $i \in \{1, \ldots, n\}$.

□

In what follows, we will prove some properties for the performance ranking, P-RANK, that will be useful to apply the ODE method for this dynamic case.

Lemma 3.2. *Let l^* be the optimal list induced by the performance ranking for the static problem (Equation (3.3)), given market share ϕ and λ^* be the expected number of purchases* *given l^* and ϕ (i.e,$\lambda^* = \sum_{i=1}^{n} \mathcal{P}_i(l^*, \phi^t) \cdot q_{l_i^*}$). Then*

$$l^* = \underset{l}{\text{arg-max}} \sum_{i=1}^{n} v_i \phi_{l_i} \left(q_{l_i} - \lambda^* \right).$$

Proof. First observe that

$$\lambda^* = \frac{\sum_{i=1}^{n} v_i \phi_{l_i^*} q_{l_i^*}}{\sum_{i=1}^{n} v_i \phi_{l_i^*}} \Leftrightarrow 0 = \sum_{i=1}^{n} v_i \phi_{l_i^*} \left(q_{l_i^*} - \lambda^* \right). \tag{3.9}$$

Now assume that there exists $\hat{l}$ such that

$$\sum_{i=1}^{n} v_i \phi_{\hat{l}_i} \left(q_{\hat{l}_i} - \lambda^* \right) > \sum_{i=1}^{n} v_i \phi_{l_i^*} \left(q_{l_i^*} - \lambda^* \right) = 0.$$

By reordering the terms, it comes that $\frac{\sum_{i=1}^{n} v_i \phi_{\hat{l}_i} q_{\hat{l}_i}}{\sum_{i=1}^{n} v_i \phi_{\hat{l}_i}} > \lambda^*$, which contradicts the optimality of l^*. □

Lemma 3.2 through the rearrangement inequality provides an important characterization of the optimal ranking at time t.

Corollary 3.2. *Let λ_t^* be the expected number of purchases at time t under the performance ranking. The performance ranking l_t^* satisfies*

$$\phi_{l_{1,t}^*}(q_{l_{1,t}^*} - \lambda_t^*) \geq \ldots \geq \phi_{l_{n,t}^*}(q_{l_{n,t}^*} - \lambda_t^*). \tag{3.10}$$

This corollary indicates that a product with quality greater or equal to λ_t^* is ranked higher than a product with quality smaller than λ_t^*. *This property is independent of the market shares at time t.*

The optimal expected number of purchases (Equation (3.3)) can be written as a function of the market shares:

$$\lambda(\phi) = \frac{\sum_{i=1}^{n} v_i \phi_{l_i^*} q_{l_i^*}}{\sum_{i=1}^{n} v_i \phi_{l_i^*}}.$$

The continuity of $\lambda(\phi)$ is necessary to apply stochastic approximation methods, which are key to the derivation of the asymptotic behaviour of the performance ranking.

Lemma 3.3. *$\lambda(\phi)$ is continuous for all $\phi \in \Delta^n$.*

Proof. $\lambda(\phi)$ is the maximum of continuous functions, and hence continuous as well. □

We now proceed to check the stability of the equilibrium points. In order to use the ODE Method, we study the asymptotic behaviour of the solutions of $\dot{x} = F(x)$.

Theorem 3.3 (Monopoly of Markets). *Consider a Trial-Offer market where $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_n$. Then the market converges almost surely to a monopoly for product 1.*

Proof. We will proceed as follow: first, we study the long term behaviour of the trajectories of the continuous dynamic, proving that they consist only of sample paths converging to some equilibria (in this case a unique equilibrium that is asymptotically stable). Finally, using Theorem 2.6 we conclude that the RMA that describes the discrete dynamic converges almost surely to that equilibrium.

Indeed, we study the asymptotic behaviour of the solutions of $\dot{\phi} = F(\phi)$, or equivalently

$$\dot{\phi}_i^t = F_i(\phi^t) = \phi_i^t\left(\frac{\hat{q}_i}{\sum_{j=1}^n \hat{q}_j \phi_j^t} - 1\right), \quad \forall i \in \{1, ..., n\}.$$

If $\phi_i^t \neq 0, \forall i \in \{1, ..., n\}$, we can rewrite the previous equation as follows:

$$\frac{1}{\hat{q}_i}\left[\frac{\dot{\phi}_i^t}{\phi_i^t} + 1\right] = \sum_{j=1}^n \hat{q}_j \phi_j^t,$$

where the right-hand-side of the equation is the same for every product. Hence,

$$\begin{aligned}
\frac{1}{\hat{q}_i}\left[\frac{\dot{\phi}_i^t}{\phi_i^t} + 1\right] &= \frac{1}{\hat{q}_k}\left[\frac{\dot{\phi}_k^t}{\phi_k} + 1\right], \quad \forall i, k \\
\Leftrightarrow \frac{1}{\hat{q}_i}\frac{d}{dt}\left[\log(\phi_i^t) + t\right] &= \frac{1}{\hat{q}_k}\frac{d}{dt}\left[\log(\phi_k^t) + t\right]. \qquad (3.11)
\end{aligned}$$

Now we separate the analysis depending on the ranking that is being used, we start with the quality ranking, where in particular we have that $\hat{q}_i$ does not change over time, and therefore we have that Equation (3.11) implies the following

$$\begin{aligned}
\frac{1}{\hat{q}_i}\int_0^t \frac{d}{ds}[\log(\phi_i^s) + s]ds &= \frac{1}{\hat{q}_k}\int_0^t \frac{d}{ds}[\log(\phi_k^s) + s]ds \qquad (3.12)\\
\Rightarrow \frac{1}{\hat{q}_i}[\log(\phi_i^t) + t - \log(\phi_i^0)] &= \frac{1}{\hat{q}_k}[\log(\phi_k^t) + t - \log(\phi_k^0)] \\
\Leftrightarrow \frac{1}{\hat{q}_i}\log(\phi_i^t) - \frac{1}{\hat{q}_k}\log(\phi_k^t) &= t\left[\frac{\hat{q}_i - \hat{q}_k}{\hat{q}_k\hat{q}_i}\right] + \frac{1}{\hat{q}_i}\log(\phi_i^0) - \frac{1}{\hat{q}_k}\log(\phi_k^0). \qquad (3.13)
\end{aligned}$$

Now, as the process begins inside of the simplex (i.e., $0 < \phi_i^0 < 1$, for all $i \in \{1, \ldots, n\}$), then $\frac{1}{\hat{q}_i}\log(\phi_i^0) - \frac{1}{\hat{q}_k}\log(\phi_k^0)$ is bounded. In consequence, the behaviour of the solutions, ϕ_i^t, is given by the asymptotic behaviour of $t[\hat{q}_i - \hat{q}_k]$ which depends of the sign of $\hat{q}_i - \hat{q}_k$. Since $\hat{q}_1 > \hat{q}_2 > \cdots > \hat{q}_n$, taking $i = 1$, $k \in \{2, ..., n\}$ in Equation (3.13) yields $t[\hat{q}_1 - \hat{q}_k] \to +\infty$ as $t \to +\infty$. Hence $\frac{1}{\hat{q}_i}\log(\phi_i^t) - \frac{1}{\hat{q}_k}\log(\phi_k^t) \to +\infty$ for all $k > 1$, and consequently $\phi_k^t \to 0$. Since $\sum_{i=1}^n \phi_i^t = 1$, we have that $\phi_1^t \to 1$, i.e., the market converges to a monopoly for the highest-quality product.

On the other hand for the performance ranking we have the following variation of the proof, based on the stability of the equilibria. From any initial condition where the appeals are non-zero, no product will ever reach a market share of exactly one or zero. Hence, we analyse the behaviour of the performance ranking when arbitrarily close to the equilibrium points. Define $\mathfrak{f}^i$ to be a small perturbation from $\mathfrak{e}_i$, the i-th canonical vector, i.e.,

$$\{\mathfrak{f}^i : \mathfrak{f}^i_i = 1 - \epsilon, \mathfrak{f}^i_j > 0, 1 \leq j \leq n, \sum_{k \neq i} \mathfrak{f}^i_k = \epsilon\},$$

where ϵ is an arbitrarily small positive quantity. The expected number of purchases for any $\mathfrak{f}^i$ is $\lambda(\mathfrak{f}^i) \approx q_i$. For $i = 1$, any perturbation of the market shares will slightly decrease the expected number of purchases, i.e., $\lambda(\mathfrak{f}^1) = q_1 - \delta$, $\delta > 0$. Then, $q_1 - \lambda(\mathfrak{f}^1) > 0$ and, for any $k \geq 2$, $q_k - \lambda(\mathfrak{f}^1) \leq 0$. By the condition in Corollary 3.2 and the fact that all $\mathfrak{f}^1_k > 0$, the best quality product q_1 is assigned in the top slot, i.e., $\sigma_1 = 1$. Hence, $v_{\sigma_1} q_1 - v_{\sigma_k} q_k > 0$ for any $k \neq 1$.

Consider the process given by Equation (3.13) with initial condition $x_0 = \mathfrak{f}^1$. The process begins in the simplex, i.e., $0 < \phi^0_i < 1, \forall i$, and hence $\frac{1}{\hat{q}_i} \log(\phi^0_i) - \frac{1}{\hat{q}_k} \log(\phi^0_k)$ is bounded. Since the ranking does not change, the behaviour of the solutions is given by the asymptotic behaviour of $t[v_{\sigma_i} q_i - v_{\sigma_k} q_k]$ which depends on the sign of $v_{\sigma_i} q_i - v_{\sigma_k} q_k$. Taking $i = 1$ and $k \in \{2, ..., n\}$, we have that $v_{\sigma_1} q_1 - v_{\sigma_k} q_k > 0$, and hence $t[v_{\sigma_1} q_1 - v_{\sigma_k} q_k] \to +\infty$ as $t \to +\infty$. Concluding again that ϕ^t converges to $\mathfrak{e}_1$ (a monopoly of the highest quality product), which is a stable equilibrium.

Finally, if for $i > 1$, we consider a perturbation where some product $j : q_j > q_i$ has a small increase in its market such that the expected number of purchases increases very slightly, i.e., $\lambda(\mathfrak{f}^i) = q_i + \delta$, $\delta > 0$. Thus, for small δ, $q_j - \lambda(\mathfrak{f}^i) > 0$ and $q_i - \lambda(\mathfrak{f}^i) = -\delta < 0$, which implies that $\mathfrak{f}^i_j(q_j - \lambda(\mathfrak{f}^i)) > \mathfrak{f}^i_i(q_i - \lambda(\mathfrak{f}^i))$. Therefore, by Corollary 3.2, product j is assigned in a better position than product i, i.e., $v_{\sigma_j} \geq v_{\sigma_i}$ and hence, $v_{\sigma_j} q_j - v_{\sigma_i} q_i > 0$. Consider the process given by Equation (3.13) with initial conditions $\phi^0 = \mathfrak{f}^i$, $i > 1$. At $t = 0$, as described above, it holds that $v_{\sigma_j} q_j - v_{\sigma_i} q_i > 0$ and consequently, at the next period of time $t = 1$, $\frac{1}{\hat{q}_i} \log(\phi^1_i) - \frac{1}{\hat{q}_k} \log(\phi^1_j) > \frac{1}{\hat{q}_i} \log(\phi^0_i) - \frac{1}{\hat{q}_k} \log(\phi^0_j)$. This implies that $\phi^1_j > \phi^0_j$ or $\phi^1_i < \phi^0_i$, which, in either case, indicates that the new state is farther away from the initial state. Hence, $\mathfrak{e}_i$ is an unstable equilibrium.

Summarising, using both ranking, we show that the trajectories of the continuous dynamic converge to a single (asymptotically stable) equilibrium. Therefore, using Theorem 2.6 we conclude that the discrete process (3.7), that describes the market share of the model for the case of linear network effects, converges almost surely to a monopoly of the highest quality product.

□

This result states that, starting from any initial condition where the appeals are non-zero, the market eventually reaches the equilibrium that corresponds to a monopoly for the product of highest quality. This result also implies that the quality and performance rankings are optimal asymptotically, since only the best product is left.

Corollary 3.3. *The quality and performance rankings are asymptotically optimal in Trial-Offer markets, this is, the market converges to a monopoly of the highest quality product.*

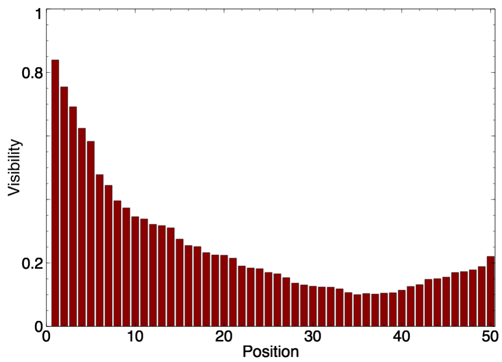

Figure 3.1: The visibility v_p (y-axis) of position p in the song list (x-axis) where $p = 1$ is the top position and $p = 50$ is the bottom position of a single column display.

3.5 Computational Experiments

We now report computational results that illustrate and complement the theoretical analysis presented in the previous section. The computational results use settings that model the MusicLab experiments discussed in Salganik et al. [2006]; Krumme et al. [2012]; Abeliuk et al. [2015]. As mentioned in the introduction, the MusicLab is a Trial-Offer market where participants can try a song and then decide to download it. The generative model of the MusicLab defined in Krumme et al. [2012] is the model of consumer choice with social influence described earlier.

The Experimental Setting The experimental setting uses an agent-based simulation to emulate the MusicLab. Each simulation consists of K iterations and, at each iteration t,

1. the simulator randomly selects a song i according to the probabilities $\pi_i(\sigma, d)$, where σ is the ranking policy under evaluation and d is the social influence signal;

2. the simulator randomly determines, with probability q_i, whether selected song i is downloaded; In the case of a download, the simulator increases the social influence signal for song i, i.e., $d_i^{t+1} = d_i^t + 1$. Otherwise, $d_i^{t+1} = d_i^t$.

Every RR iterations, a new list σ is computed using one of the ranking policies described above. For instance, in the social influence condition of the original MusicLab experiments, the policy ranks the songs by popularity, i.e., the D-rank policy which ranks the songs in decreasing order of download counts. The parameter $RR \geq 1$ is called the refresh rate. The experimental setting, which aims at being close to the MusicLab experiments, considers 50 songs and simulations with 20,000 steps. The songs are displayed in a single column. Figure 3.1 depicts the visibility parameters used in all computational experiments (the exact values can be found in the Appendix A [Dataset]). The visibility profile is based on the analysis in Krumme et al.

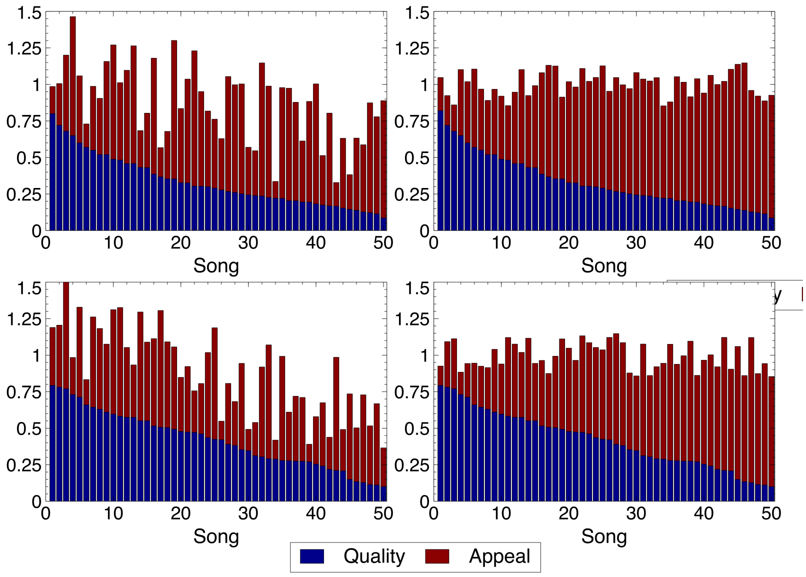

Figure 3.2: The quality q_i (blue) and appeal A_i (red) of song i in the four settings. In the first setting (top left), the qualities and appeals were chosen independently according to a Gaussian distribution. The second setting (top right) explores an extreme case where the appeal is anti-correlated with the quality used in setting 1. In the third setting (bottom left), the qualities and appeals were chosen independently according to a uniform distribution. The fourth setting (bottom right) explores an extreme case where the appeal is anti-correlated with the quality from setting 3.

[2012], indicating that participants are more likely to try songs higher in the list. More precisely, the visibility decreases with the list position, except for a slight increase at the bottom positions (notice that this condition breaks the assumption that the visibilities v_{σ_i} are non-increasing, however in practical terms, this doesn't affect the results, since the only relevant positions are the top of the ranking).

In this section we also use four settings for the quality and appeal of each product, which are depicted in Figure 4.1. In the first setting (top left), the quality and the appeal were chosen independently according to a Gaussian distribution normalised to fit between 0 and 1. The second setting (top right) explores an extreme case where the appeal is negatively correlated with quality. The quality of each product is the same as in the first setting but the appeal is chosen such that the sum of appeal and quality is 1 plus a normally distributed noise. In the third setting (bottom left), the quality and the appeal were chosen independently according to a uniform distribution. The fourth setting (bottom right) also explores an extreme case where the appeal is anti-correlated with quality. The quality of each product is the same as in the third setting but the appeal is chosen such that the sum of appeal and quality is exactly 1. The results were obtained by averaging the results of $W = 400$ simulations.

Recovering the Songs Quality We now show how to recover songs quality in the MusicLab. The key idea is borrowed from Salganik et al. (2006) who stated that *the*

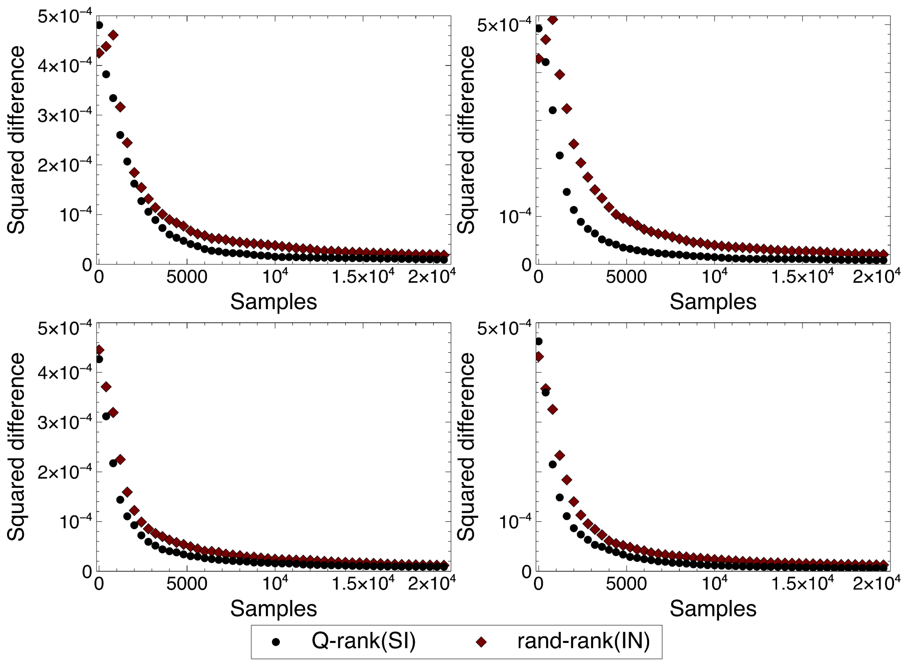

Figure 3.3: Average Squared Difference of Inferred Quality over Time for Different Rankings for the top 10 quality songs. The figure reports the average squared difference $\sum_{i=1}^{n} \frac{(\bar{q}_{i,k}-q_i)^2}{n}$ between the song quality and their predictions for the quality ranking under social influence and the random ranking in the independent condition. The figure shows the four settings in clockwise direction from the top-left plot. The quality of each song was initially approximated with 10 Bernoulli trials.

popularity of a song in the independent condition is a natural measure of its quality and captures both its intrinsic "value" and the preferences of the participants. Expanding on their idea, the popularity of a song in the independent condition and with no position bias is a natural measure of its quality. *However, under social influence, popularity may no longer reflect quality and may be strongly influenced by the visibility and early downloads.*

To approximate the quality of a song, it suffices to sample the participants in an independent world. This can be simulated by using a Bernoulli sampling based on the real quality of the songs. The predicted quality $\bar{q}_i$ of song i is obtained by running m independent Bernoulli trials with probability q_i of success, i.e., $\hat{q}_i = \frac{k}{m}$, where k is the number of successes over the m trials. For a large enough sampling size, $\bar{q}_i$ has a mean of q_i and a variance of $q_i(1-q_i)$. This variance has the desirable property that the quality of a song with a more 'extreme' quality (i.e., a good or a bad song) is recovered faster than those with average quality. In addition, we can merge information about downloads into the prediction as the market with social influence proceeds: At step k, the approximate quality of song i is given by $\bar{q}_{i,k} = \frac{\hat{q}_{i,0} \cdot m + d_{i,k}}{m + s_{i,k}}$, where m is the initial sample size, $d_{i,k}$ and $s_{i,k}$ are the number of downloads and samplings of song i up to step k.

Figure 3.3 presents experimental results about the accuracy of the quality approx-

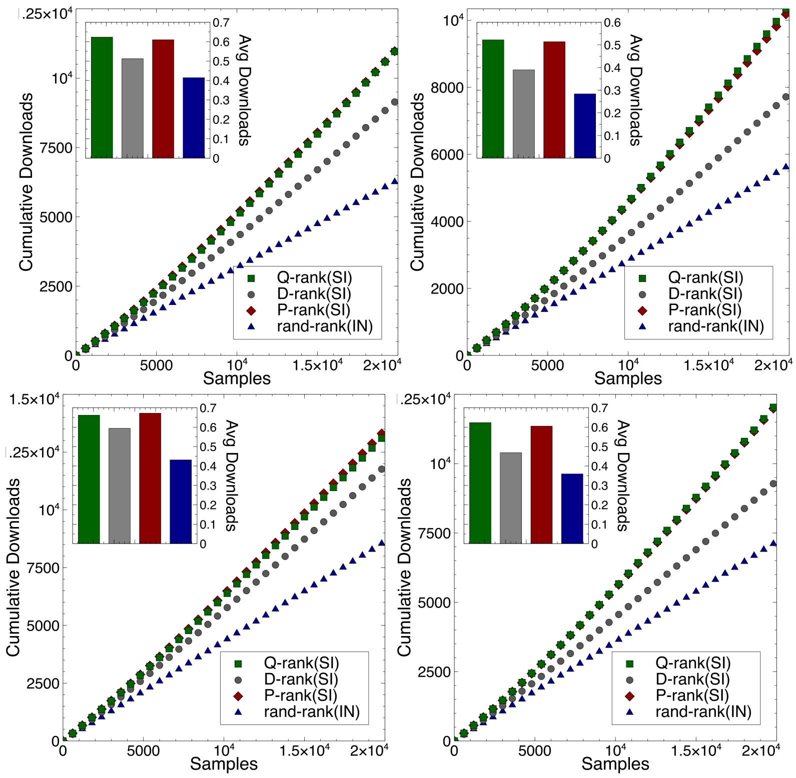

Figure 3.4: The number of downloads over time for the various rankings. The x-axis represents the number of product trials and the y-axis represents the average number of downloads over all experiments. On the upper left corner of each graph, the bar plot depicts the average number of purchases per try for all rankings. The results for the four settings are shown in clockwise direction starting from the top-left figure.

imation for two rankings, assuming an initial independent sampling set of size 10 per song. More precisely, the figure reports the average squared difference between the song qualities and their predictions under the social influence and the independent conditions. In all cases, the results indicate that song qualities are recovered quickly and accurately. Note also that the Q-rank only requires an ordinal ordering of the qualities, not their exact values.

Performance of the Market Figure 3.4 depicts computational results on the expected number of downloads for the various rankings and settings and reveals two findings:

1. The quality ranking exhibits a similar performance to the performance ranking and provides substantial gains in expected downloads compared to the popularity and random rankings. On settings with negative correlations between appeal and quality, the quality ranking performs better than the performance ranking.

2. The benefits of social influence and position bias are complementary and cu-

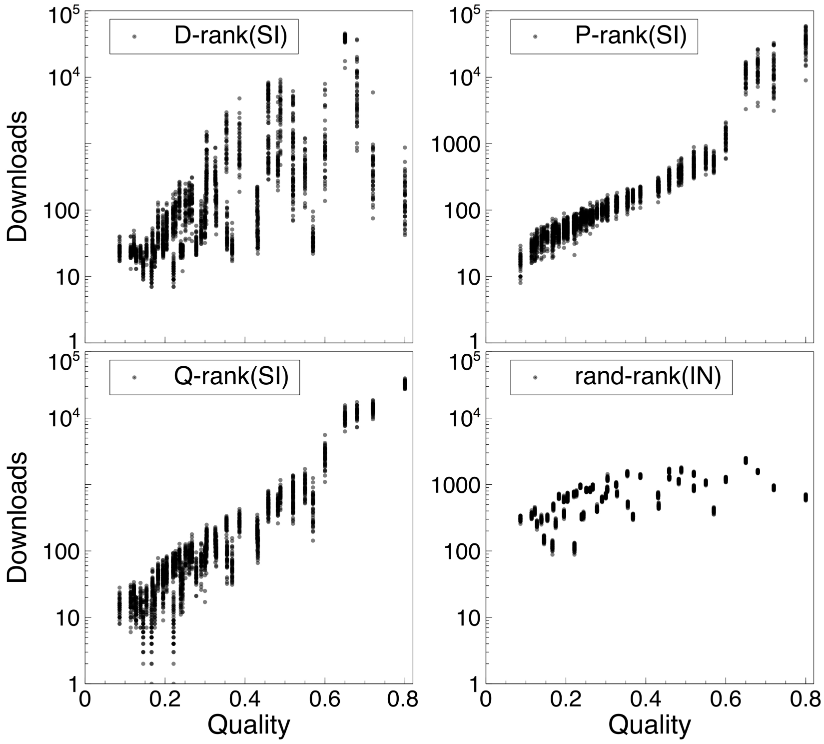

Figure 3.5: The Distribution of Downloads Versus Song Qualities (First Setting). The songs on the x-axis are ranked by increasing quality from left to right. Each dot is the number of download of a product in one of the 400 experiments.

mulative. Both are significant in terms of the expected performance of the market.

Predictability of the Market Figures 3.5 and 3.6 depict computational results on the predictability of the market under various ranking policies. The figures plot the number of downloads of each song for the 400 experiments. In the plots, the songs are ranked by increasing quality from left to right on the x-axis. Each dot in the plot shows the number of downloads of a song in one of the 400 experiments. Figures 3.5 and 3.6 present the result for the first and second settings (for appeals and qualities).

The computational results are compelling. Figure 3.5 shows that the best song always receives the most downloads in the quality ranking (with social influence) and that the variance in its number of downloads across the experiments is very small. The performance ranking (with social influence) also performs well although the variance in its downloads is larger. The popularity ranking is highly unpredictable, while the random ranking is highly predictable as one would expect. It is also interesting to note that these observations continue to hold even when the appeal is negatively correlated with quality, as Figure 3.6 indicates. The contrast between the popularity ranking used in Salganik et al. [2006] and the quality ranking is particularly striking.

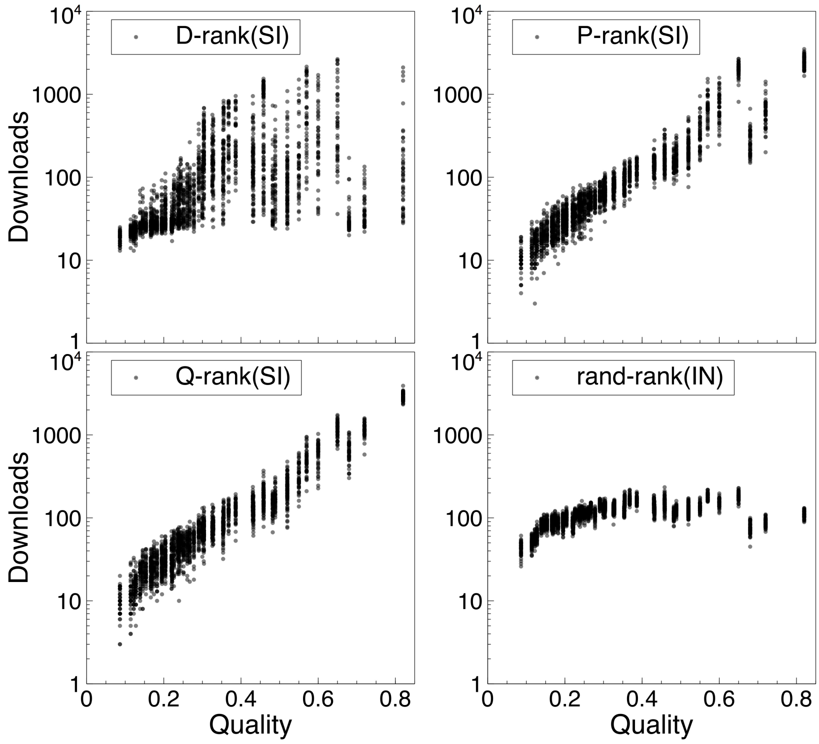

Figure 3.6: The Distribution of Download Versus Song Qualities (Second Setting). The songs on the x-axis are ranked by increasing quality from left to right. Each dot is the number of downloads of a product in one of the 400 experiments.

Figure 3.7 depicts computational results on the distribution of market shares under various ranking policies (in log scale). Each dot is the market share of a product in one of the 100 experiments. Figure 3.7 shows that the best product almost always receives the most purchases in the performance ranking. Quality ranking also performs well although the variance in its market shares is larger. The popularity ranking, while it shows the same overall correlation between quality and market share, exhibits many outliers. In terms of market efficiency, the performance ranking achieves 10% more purchases than the popularity ranking and 8% more than the quality ranking overall. For a single simulation, the performance ranking can achieve up to 23% more purchases than the other rankings.

In conclusion, the theoretical and computational results indicate that the performance ranking has attractive properties for dynamic Trial-Offer markets. It is optimal and predictable asymptotically and it optimises market efficiency at each time point. Computational results also show that it recovers from poor initial conditions much faster than the quality ranking.

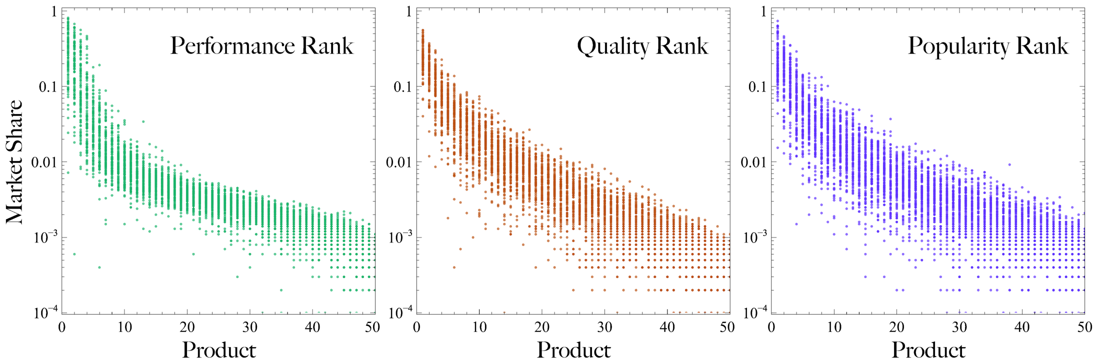

Figure 3.7: The Distribution of Market Shares. The products on the x-axis are ranked by decreasing quality from left to right. Each dot is the market share of a product in one of the 100 experiments. Note that the y-axis is in log scale.

Appendix A

Market share versus purchases

The condition $d_i^0 = a_i$ can be relaxed and the results still hold but the notations become more complicated. Indeed, define the variables $\mu_i^k = \frac{a_i+d_i^k}{\sum_j a_j+d_j^k}$, with $d_i^0 = 0$, consider $\hat{a} = \sum_{j=1}^n a_i$ the cumulative appeal, and $\mathbf{a}, \mathbf{d^k}$ the vectors of appeals and purchases respectively. By definition $\sum_{j=1}^n d_j^k = k$, then we can define the probability function $p(\mu^k)$ by

$$p_i(\mu^k) = \frac{v_i q_i f(\mu_i^k)}{\sum_{j=1}^n v_j q_j f(\mu_j^k)}, \quad i \in \{1, ..., n\}$$

and recover a recurrence for μ as follows:

$$\begin{aligned}
\mu^{k+1} &= \frac{\mathbf{a}+\mathbf{d^k}}{\hat{a}+k+1} + \frac{e^k}{\hat{a}+k+1} \\
&= \frac{\mathbf{a}+\mathbf{d^k}}{\hat{a}+k} \frac{\hat{a}+k}{\hat{a}+k+1} + \frac{e^k}{\hat{a}+k+1} \\
&= \mu^k \frac{\hat{a}+k}{\hat{a}+k+1} + \frac{e^k}{\hat{a}+k+1} \\
&= \mu^k \frac{\hat{a}+k+1}{\hat{a}+k+1} - \frac{\mu^k}{\hat{a}+k+1} + \frac{e^k}{\hat{a}+k+1} \\
&= \mu^k + \frac{1}{\hat{a}+k+1}(p(\mu^k) - \mu^k + e^k - \mathbb{E}[e^k|\mathcal{F}^k]) \\
&= \mu^k + \hat{\gamma}^{k+1}[\hat{F}(\mu^k) + \hat{U}^{k+1}].
\end{aligned}$$

In consequence, all the results from this paper can be translated from the ϕ domain to the μ domain.

Generalisations

We will use the following result, known as Brouwer Fixed Point Theorem.

Theorem 3.4. *Let $\tau : C \to C$ be a continuous function, with $C \subset \mathbb{R}^n$ a convex and compact set, then there exists (at least) one fixed point $x^* \in C$ for f, this is $\tau(x^*) = x^*$.*

Then, we apply this previous theorem to our probability function $p : \Delta^{n-1} \to \Delta^n$, clearly Δ^n is convex and compact, then we can find ϕ^* such that $p(\phi^*) = \phi^*$, or equivalently, ϕ^* is an equilibrium for the continuous dynamic (2.3). As Equilibria Sets are ICT sets for Robbins-Monro Algorithms then we could give a characterisation for the asymptotic behaviour of our discrete dynamic.

Theorem 3.5. *Assuming that for all i, there exists a constant $L_i < 1$ such that $|p_i(\phi^1) - p_i(\phi^2)| \leq L_i \|\phi^1 - \phi^2\|$. Then the continuous dynamic* (2.3) *has only one equilibrium ϕ^*. Furthermore, the set $\{\phi^*\}$ is a global attractor and the discrete process* (3.7) *converge almost surely to ϕ^*.*

Proof. For proving that $\{\phi^*\}$ is a global attractor we find a strict Lyapunov Function, i.e. a function V decreasing through the trajectory ϕ^t, with $V : \Delta^{n-1} \to \mathbb{R}_+$, and $V^{-1}(\{0\}) = \{\phi^*\}$.

Let us define $V(\phi^t) = ||\phi^t - \phi^*||_\infty$, clearly $V(\phi^t) \in \mathbb{R}_+$, and $V(\phi^t) = 0 \Leftrightarrow \phi^t = \phi^*$. Let ϕ^t a solution of (2.3), and $j \in \{1, .., n\}$ the index where the maximum is attained, i.e. $||\phi^t - \phi^*||_\infty = |\phi^t_j - \phi^*_j|$, without loss of generality, we can assume that $\phi^t_j \geq \phi^*_j$ (the other case is analogous), then using that $p_j(\phi^*) - \phi^*_j = 0$ for all j, we have the following:

$$
\begin{aligned}
\frac{dV(\phi^t)}{dt} = \frac{d}{dt}[\phi^t_j - \phi^*_j] = \frac{d\phi^t_j}{dt} \\
&= p_j(\phi^t) - \phi^t_j \\
&= p_j(\phi^t) - p_j(\phi^*) - (\phi^t_j - \phi^*_j) \\
&\leq |p_j(\phi^t) - p_j(\phi^*)| - |\phi^t_j - \phi^*_j| \\
&\leq ||p(\phi^t) - p(\phi^*)||_\infty - ||\phi^t - \phi^*||_\infty \\
&\leq (L-1)||\phi - \phi^*||_\infty \\
&= (L-1)V(\phi^t)
\end{aligned}
$$

Where $L = \max_{i=1,..,n} L_i$. In conclusion V decreases at exponential rate through the trajectories of the solutions in the dynamic (2.3), then $\{\phi^*\}$ is a global attractor (in particular an asymptotic stable equilibrium, since the solutions decay exponentially to it). Using Theorem 2.6 we have that $\{\phi^*\}$ is the only ICT for (2.3), therefore the process (3.7) converge almost surely to $\{\phi^*\}$.

□

Example 3.1. *The probability distribution p given by $p_i(\phi) = \dfrac{\exp(\beta\phi_i + \alpha_i)}{\sum_{j=1}^{n} \exp(\beta\phi_j + \alpha_j)}$ for all $i \in \{1, ..., n\}$, under the condition $\beta < 2$, satisfies the conditions of the previous theorem. Indeed, using the mean value theorem for each p_h we have that*

$$|p_h(\phi^1) - p_h(\phi^2)| \leq [\sup_{x\in\Delta^{n-1};k\in N} \left|\frac{\partial p_h(x)}{\partial x_k}\right|]\|\phi^1 - \phi^2\|$$

On the other hand, for any $h, k \in N$ and $x \in \Delta^n$ we have

$$\begin{aligned}
\frac{\partial p_h(x)}{\partial x_k} &= -\beta\frac{\exp(\beta\phi_h + \alpha_h)}{\sum_{j=1}^{n} \exp(\beta\phi_j + \alpha_j)} - \frac{\beta\exp(\beta\phi_h + \alpha_h)}{(\sum_{j=1}^{n} \exp(\beta\phi_j + \alpha_j))^2}[\exp(\beta\phi_k + \alpha_k) - \sum_{j\neq k} \exp(\beta\phi_j + \alpha_j)] \\
&= \beta p_h(x)[-1 - p_k(x) + \sum_{j\neq k} p_j(x)] \\
&= \beta p_h(x)[-1 - p_k(x) + 1 - p_k(x)] \\
&= -2\beta p_h(x) p_k(x)
\end{aligned}$$

as $p_h(x)p_k(x) \leq \frac{1}{4}$, then $|\dfrac{\partial p_h(x)}{\partial x_k}| \leq 2\beta\frac{1}{4} < 1$.

Dataset

Table 3.1 shows the values of the qualities and appeals for the independent setting (obtained from [Abeliuk et al., 2015]). Table 3.2 shows the values of the visibilities for each position $j \in \{1, \ldots, n\}$.

Product	Quality	Appeal	Product	Quality	Appeal
1	0.8	0.18581654	26	0.278009	0.35136515
2	0.72	0.28594501	27	0.2673	0.78687609
3	0.68	0.52073051	28	0.26083	0.7369193
4	0.65	0.81398644	29	0.2512	0.75227893
5	0.60	0.45868017	30	0.24396	0.32580804
6	0.57	0.15955483	31	0.23941	0.30674759
7	0.55	0.43715743	32	0.23622	0.91103217
8	0.52005	0.38484972	33	0.22629	0.76236248
9	0.52	0.63739211	34	0.2214	0.11459921
10	0.4887	0.78174105	35	0.22013	0.7581713
11	0.48224	0.52983037	36	0.20418	0.76994571
12	0.4586	0.6382574	37	0.20389	0.67408264
13	0.45837	0.80597	38	0.19535	0.41759683
14	0.432	0.2520265	39	0.1947	0.68898008
15	0.43067	0.37266718	40	0.18248	0.82117398
16	0.38623	0.79358615	41	0.17444	0.33890645
17	0.36792	0.19972853	42	0.16867	0.63497574
18	0.35492	0.32368825	43	0.16638	0.16224351
19	0.35374	0.94736709	44	0.15374	0.47778872
20	0.32799	0.50704873	45	0.14542	0.23702317
21	0.32589	0.7105828	46	0.1387	0.49406539
22	0.30411	0.92616787	47	0.12764	0.45956048
23	0.30352	0.64768258	48	0.12217	0.75210134
24	0.29988	0.51815068	49	0.11418	0.66488509
25	0.2905	0.47170285	50	0.08636	0.80257928

Table 3.1: Values of quality and appeal for the products in the independent case. Recall that the values of the appeal in the anti-correlated setting are given by $a_i = 1 - q_i$.

Position	Visibility	Position	Visibility
1	0.83	25	0.16583292
2	0.75	26	0.15370582
3	0.69	27	0.13640378
4	0.62	28	0.13084858
5	0.58	29	0.12666812
6	0.48	30	0.12429217
7	0.44	31	0.12362827
8	0.4	32	0.11847651
9	0.37	33	0.10675012
10	0.35	34	0.1001895
11	0.338	35	0.10377821
12	0.321	36	0.10192779
13	0.317	37	0.10484361
14	0.31063943	38	0.10609265
15	0.2750814	39	0.11420125
16	0.25493054	40	0.1260095
17	0.25148059	41	0.13163135
18	0.23254506	42	0.14843575
19	0.22517471	43	0.15040223
20	0.22429915	44	0.15529018
21	0.21502087	45	0.1699023,
22	0.19038769	46	0.17265442
23	0.18407585	47	0.17825863
24	0.18185429	48	0.18851792
25	0.17013229	50	0.22057129

Table 3.2: Values of the visibilities for each position $j \in \{1, \ldots, n\}$.

Chapter 4

Sublinear social signals over a model of consumer choice with position bias

This chapter is reproduced with minor changes from:

Maldonado, F.; Van Hentenryck, P.; Berbeglia, G.; and Berbeglia, F., 2018. Popularity signals in trial-offer markets with social influence and and position bias. *European Journal of Operational Research*, 266(2), pp. 775-793.

Motivation In this chapter we consider only static rankings (like the quality ranking) but we add complexity, including a family of social signal functions indexed by a parameter $r > 0, r \neq 1$ (known as the social signal or network effect parameter), where the functions are given by $f(x) = x^r$. The main objective is to study whether we can represent a market evolution to something different than a monopoly for a particular product, having on the other hand, a better distributed and fairer market. We prove that with probability 1, when $r < 1$, the vector of market shares converges to a distribution, where all the products have a strictly positive chance of being purchased. We analyse the properties of this equilibrium, such as stability and monotony in terms of the parameters. Many computational experiments are developed, to show the behaviour of the market equilibrium and its properties in terms of the parameter r.

4.1 Model

In the previous chapter we discussed in detail a model where the social signal is linear $f(x) = x$, under two different ranking policies, a static ranking Q-rank, and a dynamic ranking P-rank. In both cases the expected asymptotic market behaviour is to reach a monopoly for the highest quality product, having a predictable market. However a monopoly is not always a desirable outcome. This chapter explores the use of other social signals that may lead to a *fairer* distribution of the market, but remaining predictable.

The primary objective of this chapter is to understand what happens to the Trial-Offer market when the social signal is given by $f(x) = x^r$, and $r > 0$, $r \neq 1$.

The chapter contains both theoretical and simulation results and its contributions can be summarised as follows:

1. When $r < 1$ and a static ranking is used, *the market converges to a unique equilibrium, which we characterise analytically*. In the equilibrium, the market shares depend only on the product qualities q_i and no monopoly occurs. Moreover, a product of higher quality receives a larger market share than a product of lower quality, introducing a notion of fairness in the market and reducing the inequalities introduced by a linear social signal.

2. When $r > 1$ and a static ranking is used, the equilibria can be characterised similarly. However, contrary to the case $r < 1$, the equilibria that are not monopolies can be shown to be unstable under certain conditions. As a result, the market will typically go to a monopoly for some product: Which product wins the entire market share depends on the initial condition and the early dynamics.

3. Agent-based simulations show that the market converges quickly towards an equilibrium when using sublinear social signals ($0 < r < 1$) and the quality ranking. They also show that the quality ranking outperforms the popularity ranking in maximising the efficiency of the market. The popularity ranking is also shown to have some significant drawbacks in some settings.

These theoretical results indicate that, when the social influence signal is a sublinear function ($r < 1$) of the market share and a static ranking of the products (e.g., the quality ranking) is used, the market is entirely predictable, depends only on the product quality, and does not lead to a monopoly. This contrasts with the case of $r = 1$ where the market is entirely predictable but goes to a monopoly for the product of highest quality (assuming the quality ranking); and the case of $r > 1$ where the market becomes unpredictable (even with a static ranking). As a result, sublinear social signals provide a way to balance market efficiency and the inequalities introduced by social influence. In particular, with sublinear social signals and a static ranking, markets do not exhibit a Matthew effect where the winner takes all, and remain predictable.

The remaining of this chapter is organised as follows. Section 4.2 derives the equilibria for the market as a function of the social signal and also presents the convergence results. Section 4.3 reports the results from the agent-based simulation. Section 4.4 discusses some additional results on sublinear signals.

4.2 Equilibria of Trial-Offer Markets

In this section we will analyse how the market shares $\{\phi^t\}_{t>0}$ evolve over time for various functions f given a static ranking σ. We are particularly interested in studying the asymptotic behaviour of $\{\phi^t\}_{t>0}$ for the cases where $f(x) = x^r$ with $r > 0$.

For instance, when $r = 0.5$, the social signal displays the square root of the number of past purchases. For notational simplicity, we assume that the ranking is fixed and is the identity function $\sigma_i = i$ and omit it from the formulas. We also notice that if the qualities and visibilities also satisfy $q_1 \geq \ldots \geq q_n$ and $v_1 \geq \ldots \geq v_n$, we obtain the quality ranking proposed in Van Hentenryck et al. [2016], however these results hold for any static ranking.

We recall from Lemma 3.1 that for the setting presented in this chapter, when the social signal is given by $f(x) = x^r, r > 0, r \neq 1$, the probability that the next purchase is product i is given by

$$\pi_i(\phi) \; = \frac{v_{\sigma_i} q_i (\phi_i)^r}{\sum_{j=1}^n v_{\sigma_j} q_j (\phi_j)^r},$$

where, $\sigma : [n] \to [n]$ is an assignment of n products to n positions in the ranking, in the context of this chapter where we use a static ranking, the visibilities v_{σ_i} do not change over time.

This section characterises the equilibria and the asymptotic behaviour of the continuous dynamics

$$\frac{d\phi^t}{dt} = \pi(\phi^t) - \phi^t, \quad (\phi^t \in \Delta^n), \tag{4.1}$$

which is associated with the RMA

$$\phi^{k+1} = \phi^k + \gamma^{k+1}(F(\phi^k) + U^{k+1}), \tag{4.2}$$

where $\gamma^{k+1} = \frac{1}{D^k+1}$, $F(\phi) = \pi(\phi) - \phi$, and $U^{k+1} = e^k - \mathbb{E}[e^k | \mathcal{F}^k]$.

For simplicity, we remove the visibilities by stating $\hat{q}_j = v_{\sigma_j} q_j$. We are interested in the case where $f(x) = x^r$ with $r > 0, r \neq 1$ (since the case $r = 1$ has been settled in the previous chapter). Let Q be the set of positive market shares, this is, $Q = \{i \in [n] : \phi_i \neq 0\}$, clearly $Q \neq \emptyset$ since $\sum_{i=1}^n \phi_i = 1$.

Theorem 4.1. *Let $r > 0$, and $r \neq 1$. Any equilibria ϕ for Equation* (4.1) *has coordinates*

$$\phi_i = \frac{\hat{q}_i^{\frac{1}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}} \quad \textit{if } i \in Q,$$

and zero otherwise (i.e., if $i \in [n] \setminus Q$).

Proof. An equilibrium to (4.1) must satisfy $\pi_i(\phi) = \phi_i$, i.e.,

$$\frac{\hat{q}_i(\phi_i)^r}{\sum_{j=0}^n \hat{q}_j(\phi_j)^r} = \phi_i.$$

For $i \in Q$, we have

$$\hat{q}_i(\phi_i)^{r-1} = \sum_{j \in Q} \hat{q}_j(\phi_j)^r$$

and, for all $i, k \in Q$, we also have

$$\hat{q}_i(\phi_i)^{r-1} = \sum_{j \in Q} \hat{q}_j(\phi_j)^r = \hat{q}_k(\phi_k)^{r-1},$$

which is equivalent to

$$\hat{q}_i(\phi_i)^{r-1} = \hat{q}_k(\phi_k)^{r-1} \Leftrightarrow \phi_i = \left(\frac{\hat{q}_k}{\hat{q}_i}\right)^{\frac{1}{r-1}} \phi_k. \tag{4.3}$$

By summing for all $i \in Q$, we obtain

$$1 = \sum_{i \in Q} \phi_i = \frac{\phi_k}{\hat{q}_k^{1/(1-r)}} \sum_{i \in Q} \hat{q}_i^{1/(1-r)},$$

and hence

$$\phi_k = \frac{\hat{q}_k^{1/(1-r)}}{\sum_{i \in Q} \hat{q}_i^{1/(1-r)}}.$$

It remains to prove ϕ is indeed an equilibrium, i.e., $\pi(\phi) = \phi$. This is equivalent to prove that $\pi_i(\phi) = \phi_i$ for all $i \in \{1, \ldots, n\}$. The result is obvious if $i \in [n] \setminus Q$ ($\phi_i = 0 \Rightarrow \pi_i(\phi) = 0$). If $i \in Q$, then

$$\begin{aligned}
\pi_i(\phi) &= \frac{\hat{q}_i(\phi_i)^r}{\sum_{j \in Q} \hat{q}_j(\phi_j)^r} \\
&= \frac{\hat{q}_i(\hat{q}_i^{1/(1-r)})^r}{\sum_{j \in Q} \hat{q}_j(\hat{q}_j^{1/(1-r)})^r} * \frac{\left(\sum_{j \in Q} \hat{q}_j^{1/(1-r)}\right)^r}{\left(\sum_{j \in Q} \hat{q}_j^{1/(1-r)}\right)^r} \\
&= \frac{\hat{q}_i^{[1+r/(1-r)]}}{\sum_{j \in Q} \hat{q}_j^{[1+r/(1-r)]}} \\
&= \frac{\hat{q}_i^{1/(1-r)}}{\sum_{j \in Q} \hat{q}_j^{1/(1-r)}} = \phi_i.
\end{aligned}$$

□

Note that, when $|Q| = n$, the equilibrium lives in the interior of the simplex $int(\Delta^n)$ (all its coordinates are strictly positive). We use ϕ^* to denote this equilibrium (sometimes we call it the inner equilibrium). When $|Q| = 1$, then the equilibrium is one of the vertices of the simplex. Finally, the cases $1 < |Q| < n$ cover the other possible equilibria (for example $\phi = (3/4, 1/4, 0, \ldots, 0)$).

Observe also that the equilibrium $\phi^* \in int(\Delta^n)$ for the case $0 < r < 1$ has some very interesting properties: It is unique, which means that the market is completely predictable. Moreover, if $\hat{q}_i \geq \hat{q}_j$, then $\phi_i^* \geq \phi_j^*$, which endows the market with a basic notion of *fairness*. Finally, the market is not a monopoly: All the market shares are strictly positive for the equilibrium ϕ^*.

Our next result characterises the ICT of the continuous dynamics. We start with a useful lemma which indicates that sub-markets can also be modelled as RMAs.

Lemma 4.1. *Consider a Trial-Offer market defined by n items and the sub-market obtained by considering only $n-1$ items. Then this sub-market can also be modelled by an RMA.*

Proof. Let $\Phi^t = [\phi_1^t, \phi_2^t, \cdots, \phi_n^t]$ be the market share for the n-item Trial-Offer market at stage t. Consider a new process $\{\Psi^t\}_{t\geq 0}$ consisting of $n-1$ products only. We show that this process can also be modelled as a RMA. The key is to prove that the probability of purchasing product j in stage t follows Equation (3.4). Consider any item $i \in \{1, .., n\}$ such that $\phi_i^t \neq 1$. Without loss of generality, assume that $i = n$, define

$$\psi_i^t = \frac{\phi_i^t}{1-\phi_n^t}, \quad (i < n),$$

and consider the following events:

- $A = \{\text{product } n \text{ is not purchased at stage } t\}$
- $B = \{\text{product } j \neq n \text{ is purchased at stage } t\}$.

Since $B \subseteq A$, $Pr[B \cap A] = Pr[B] = \dfrac{\hat{q}_j(\phi_j^t)^r}{\sum_{i=1}^n \hat{q}_i(\phi_i^t)^r}$. On the other hand

$$Pr[A] = 1 - \frac{\hat{q}_n(\phi_n^t)^r}{\sum_{i=1}^n \hat{q}_i(\phi_i^t)^r} = \frac{\sum_{j=1}^{n-1} \hat{q}_j(\phi_j^t)^r}{\sum_{i=1}^n \hat{q}_i(\phi_i^t)^r},$$

and therefore

$$Pr[B|A] = \frac{Pr[B \cap A]}{Pr[A]} = \frac{\hat{q}_j(\phi_j^t)^r}{\sum_{i=1}^{n-1} \hat{q}_i(\phi_i^t)^r} \cdot \frac{(1-\phi_n^t)^r}{(1-\phi_n^t)^r} = \frac{\hat{q}_j(\psi_j^t)^r}{\sum_{i=1}^{n-1} \hat{q}_i(\psi_i^t)^r}.$$

Since $\psi_i^t \geq 0$ and $\sum_{i=1}^{n-1} \psi_i^t = \sum_{i=1}^{n-1} \frac{\phi_i^t}{1-\phi_n^t} = \frac{1}{1-\phi_n^t} \sum_{i=1}^{n-1} \phi_i^t = 1$, the ψ_i^t are well-defined market shares. Since the evolution of ψ^t depends on the probability $Pr[B|A]$, one can obtain a similar formula to (4.2). Indeed, observe that on the event A, we have that for every $i = 1, \ldots, n-1$, $\psi_i^{k+1} = \frac{(D^k+1)\psi_i^k}{D^k+1} - \frac{\psi_i^k}{D^k+1} + \frac{\hat{e}_i^k}{D^k+1}$, with $\mathbb{E}[\hat{e}^k|\mathcal{F}^k] = Pr[B|A]$. Hence, $\{\psi^t\}_{t\geq 0}$ can be modelled as an $n-1$ dimensional RMA. □

We are now in position to prove the main result of this chapter. The theorem considers the case where $\phi^0 \in int(\Delta^n)$, which is the case when the product appeals are strictly positive. It proves that, under this condition, the ICT set of Equation (4.1) consists of a single equilibrium ϕ^*.

Theorem 4.2. *Under the social signal $f(x) = x^r, 0 < r < 1$ with $\phi^0 \in int(\Delta^n)$, the RMA $\{\phi^t\}_{t>0}$ converges to ϕ^* almost surely.*

Proof. Analogously to what we did in the previous chapter, we will first study the long term behaviour of the trajectories of the continuous dynamic, proving that they converge to some equilibria (in this case a unique equilibrium that is asymptotically stable). Finally, using Theorem 2.6 we will conclude that the RMA (3.7) that describes the discrete dynamic converges almost surely to that equilibrium.

Indeed, the proof studies the asymptotic behaviour of the solutions of the following ODE:

$$\frac{d\phi^t}{dt} = \pi(\phi^t) - \phi^t. \tag{4.4}$$

Equation (4.4) is equivalent to

$$\frac{d\phi_i^t}{dt} = \frac{\hat{q}_i(\phi_i^t)^r}{\sum_j \hat{q}_j(\phi_j^t)^r} - \phi_i^t, \quad i \in \{1, \cdots, n\}; 0 < t < \infty.$$

Hence, we have the following equivalences:

$$\begin{aligned}
&\frac{\hat{q}_i(\phi_i^t)^r}{\sum_j \hat{q}_j(\phi_j^t)^r} = \frac{d\phi_i^t}{dt} + \phi_i^t, \\
&\frac{1}{\sum_j \hat{q}_j(\phi_j^t)^r} = \frac{1}{\hat{q}_i(\phi_i^t)^r}[\frac{d\phi_i^t}{dt} + \phi_i^t] \\
&\frac{1}{\hat{q}_i(\phi_i^t)^r}[\frac{d\phi_i^t}{dt} + \phi_i^t] = \frac{1}{\hat{q}_j(\phi_j^t)^r}[\frac{d\phi_j^t}{dt} + \phi_j^t] \quad \forall i, j \in \{1, \cdots, n\}, \\
&\hat{q}_i^{-1}(\phi_i^t)^{-r}[\frac{d\phi_i^t}{dt} + \phi_i^t] = \hat{q}_j^{-1}(\phi_j^t)^{-r}[\frac{d\phi_j^t}{dt} + \phi_j^t], \\
&\hat{q}_i^{-1}[(\phi_i^t)^{-r}\frac{d\phi_i^t}{dt} + (\phi_i^t)^{1-r}] = \hat{q}_j^{-1}[(\phi_j^t)^{-r}\frac{d\phi_j^t}{dt} + (\phi_j^t)^{1-r}], \\
&e^{(1-r)t}(1-r)\hat{q}_i^{-1}[(\phi_i^t)^{-r}\frac{d\phi_i^t}{dt} + (\phi_i^t)^{1-r}] = e^{(1-r)t}(1-r)\hat{q}_j^{-1}[(\phi_j^t)^{-r}\frac{d\phi_j^t}{dt} + (\phi_j^t)^{1-r}], \\
&\frac{d}{dt}\left[e^{(1-r)t}\hat{q}_i^{-1}(\phi_i^t)^{1-r}\right] = \frac{d}{dt}\left[e^{(1-r)t}\hat{q}_j^{-1}(\phi_j^t)^{1-r}\right],
\end{aligned}$$

where the sixth equivalence is obtained by multiplying both sides with $\mu(t) = (1-r)e^{(1-r)t}$. Notice also that as $\phi_i^0 > 0$, then for any finite time $t > 0$, $\phi_i^t > 0$. Taking the integral $\int_0^t dt$ of the last expression gives

$$e^{(1-r)t}\hat{q}_i^{-1}(\phi_i^t)^{1-r} - \hat{q}_i^{-1}(\phi_i^0)^{1-r} = e^{(1-r)t}\hat{q}_j^{-1}(\phi_j^t)^{1-r} - \hat{q}_j^{-1}(\phi_j^0)^{1-r} \tag{4.5}$$

and hence

$$\frac{(\phi_i^t)^{1-r}}{\hat{q}_i} - \frac{(\phi_j^t)^{1-r}}{\hat{q}_j} = e^{(r-1)t}\left[\frac{(\phi_i^0)^{1-r}}{\hat{q}_i} - \frac{(\phi_j^0)^{1-r}}{\hat{q}_j}\right]. \tag{4.6}$$

Consider Equation (4.6):

- if, for some $i \neq j$, $\dfrac{(\phi_i^0)^{1-r}}{\hat{q}_i} = \dfrac{(\phi_j^0)^{1-r}}{\hat{q}_j}$, then $\dfrac{(\phi_i^t)^{1-r}}{\hat{q}_i} = \dfrac{(\phi_j^t)^{1-r}}{\hat{q}_j}$, for all t;

- if $\dfrac{(\phi_i^0)^{1-r}}{\hat{q}_i} \neq \dfrac{(\phi_j^0)^{1-r}}{\hat{q}_j}$, then the right-hand side of Equation (4.6) goes to zero as $t \to \infty$ (because $r < 1$) and hence the left-hand side of (4.6) also goes to zero:

$$\lim_{t\to\infty} \frac{(\phi_i^t)^{1-r}}{\hat{q}_i} - \frac{(\phi_j^t)^{1-r}}{\hat{q}_j} = 0. \tag{4.7}$$

We now prove by induction that the limits for the market shares exist. Consider first the case of 2 products. Since $\phi_2^t = (1 - \phi_1^t)$, the market is completely characterised by the value of ϕ_1^t and hence we can use an one-dimensional RMA and, by Theorem 1 in [Renlund, 2010], the RMA converges since $F(x) = \pi(x) - x$ is a continuous function and ϕ_1^t is bounded. Assume now that a RMA with $k-1$ products converges and consider a market with k products. By Lemma 4.1, given a k-dimensional RMA $\Phi^t = [\phi_1^t, \phi_2^t, \cdots, \phi_k^t]$, we can create a $k-1$ dimensional RMA $\{\Psi^t\}_{t\geq 0}$ given by $\psi_i^t = \frac{\phi_i^t}{1-\phi_k^t}$ $(i < k)$. By induction hypothesis, $\psi_i = \lim_{t\to\infty} \psi_i^t$ exists for all $i < k$ and therefore Equation (4.6) is equivalent to

$$\frac{(\phi_k^t)^{1-r}}{\hat{q}_k(1-\phi_k^t)^{1-r}} - \frac{(\psi_i^t)^{1-r}}{\hat{q}_i} = \frac{e^{(r-1)t}}{(1-\phi_k^t)^{1-r}} \left[\frac{(\phi_k^0)^{1-r}}{\hat{q}_k} - \frac{(\phi_i^0)^{1-r}}{\hat{q}_i} \right]. \tag{4.8}$$

Observe that, if $\lim_{t\to\infty} \phi_k^t = 1$, then $\lim_{t\to\infty} \phi_j^t = 0$ for all $j \neq k$, and the market shares converge to one of the possible equilibria (i.e., a monopoly of the product k). Otherwise, the right-hand side of (4.8) goes to 0 when $t \to \infty$ and $\lim_{t\to\infty} \dfrac{(\psi_i^t)^{1-r}}{\hat{q}_i}$ exists. Hence $\lim_{t\to\infty} \dfrac{(\phi_k^t)^{1-r}}{\hat{q}_k(1-\phi_k^t)^{1-r}}$ also exists.

Now denote by ϕ_j the limit of ϕ_j^t for all $j \in \{1, \cdots, n\}$. Using Equation (4.7), the following equation holds for all $i, j \in \{1, \cdots, n\}$:

$$\frac{\phi_i^{1-r}}{\hat{q}_i} = \frac{\phi_j^{1-r}}{\hat{q}_j}. \tag{4.9}$$

Observe that, if there exists $l \in \{1, \cdots, n\}$ such that $\phi_l = 0$, Equation (4.9) implies that $\phi_i = 0$ for all i which is impossible since they add up to 1. Hence the limit process has strictly positive components and Equation (4.9) is equivalent to

$$\phi_i = \frac{\phi_j}{\hat{q}_j^{1/(1-r)}} \hat{q}_i^{1/(1-r)} \tag{4.10}$$

which is the equation that defines ϕ^* in Theorem 4.1 (see Equation (4.3)). As a result, when $\phi^0 \in int(\Delta^n)$, the only ICT set for the ODE (4.4) is the equilibrium ϕ^* and, by

Theorem 2.6, the RMA given by Equation (4.2) converges almost surely to ϕ^*. □

Consider now the case $r > 1$ for which Theorem 4.1 still characterises the equilibria. In this case, the dynamic behaviour is completely different due to the strength of the social signal. It is however possible to prove that the ICT set of the RMA $\{\phi^t\}_{t>0}$ consists only of equilibria.

Theorem 4.3. *Consider the social signal $f(x) = x^r$ with $r > 1$. The RMA $\{\phi^t\}_{t\geq 0}$ converges almost surely to one of the equilibria $\phi \in Z_F := \{x \in \Delta^n : \pi(x) - x = 0\}$.*

Proof. Again, the procedure is same, however in this case, there are many possible equilibria (a finite set), therefore the conclusion will be that the RMA converges almost surely to one of them.

The analysis of the ODE is the same as in Theorem 4.2 since the only restriction in the proof is $r \neq 1$. However, the interpretation of Equation (4.6) changes when $r > 1$.

We define $H_{i,t_0} := \dfrac{(\phi_i^{t_0})^{1-r}}{\hat{q}_i}$ for all $1 \leq i \leq n$, $t_0 \geq 0$, and order the products in decreasing order of H_{i,t_0}. Let $h : \{1,..,n\} \to \{1,..,n\}$ be the permutation that defines this order and denotes by h^{-1} its inverse function, i.e., $h^{-1}(i) = j$ means that product j is in the i-th position in permutation h. We have that $H_{h^{-1}(1),t_0} \geq \cdots \geq H_{h^{-1}(n),t_0}$, which characterises the starting configuration for time t_0. Define the following sets:

- $Q_0(t_0) = \{i \in \{1,..,n-1\} : H_{h^{-1}(i),t_0} = H_{h^{-1}(i+1),t_0}\}$,
- $Q_1(t_0) = \{i \in \{1,..,n-1\} : H_{h^{-1}(i),t_0} > H_{h^{-1}(i+1),t_0}\}$,

and consider the following case analysis:

i) If $|Q_0| = n-1$, then $H_{h^{-1}(i),t_0} = H_{h^{-1}(i+1),t_0}$ for all $1 \leq i \leq n-1$. By Equation (4.6), $\dfrac{(\phi^t_{h^{-1}(i)})^{1-r}}{\hat{q}_{h^{-1}(i)}} = \dfrac{(\phi^t_{h^{-1}(i+1)})^{1-r}}{\hat{q}_{h^{-1}(i+1)}}$, for all $t > 0$ and for all $1 \leq i \leq n-1$, which leads again to the inner equilibrium ϕ^*.

ii) If $0 < |Q_0| < n-1$, select $i \notin Q_0$. Equation (4.6) implies that

$$\lim_{t\to\infty} \frac{(\phi^t_{h^{-1}(i)})^{1-r}}{\hat{q}_{h^{-1}(i)}} - \frac{(\phi^t_{h^{-1}(i+1)})^{1-r}}{\hat{q}_{h^{-1}(i+1)}} = \infty,$$

because $r > 1$ and hence $e^{(r-1)t} \to \infty$ when $t \to \infty$. It follows that $\lim_{t\to\infty} \phi^t_{h^{-1}(i)} = 0$ and the RMA necessarily converges to one of the equilibria that live in the boundary of the simplex, but they are not monopolies (see Theorem 4.1).

iii) If $|Q_0| = 0$ then $|Q_1| = n-1$, Using a similar reasoning as in case ii), it follows that $\lim_{t\to\infty} \phi^t_{h^{-1}(i)} = 0$ for all $1 \leq i \leq n-1$ and, since $\phi^t \in \Delta^n$ for all t, $\lim_{t\to\infty} \phi^t_{h^{-1}(n)} = 1$.

As a result, the only ICT for the differential equation (4.4) are equilibria and, by Theorem 2.4, the RMA $\{\phi^t\}_{t\geq t_0}$ converges almost surely to one of them, for any finite t_0 starting point. Clearly, any starting configuration can be attained (with positive probability) for some suitable finite $t_0 \geq 0$ steps. □

It is important to observe that, in the case $r > 1$, the initial conditions, i.e., the initial appeals and how the market evolves early on, affect the entire dynamics. This is in contrast with the case $r < 1$ for which the long-term behaviour only depends of the product qualities. This has fundamental consequences for the predictability and efficiency of the market. We will show now that, when $r > 1$, the inner equilibrium ϕ^* is always unstable. The result will follow as corollary of the following theorem.

Theorem 4.4. *Consider the equilibria given by*

$$\hat{\phi}_i = \frac{\hat{q}_i^{\frac{1}{1-r}}}{\sum_{j\in Q}\hat{q}_j^{\frac{1}{1-r}}} \quad \text{if } i \in Q \text{ and } \hat{\phi}_i = 0 \text{ if } i \in [n]\setminus Q$$

with $Q = \{i \in [n] : \phi_i \neq 0\}$. The trace of the Jacobian matrix, $tr(JF(\hat{\phi}))$, where $F(x) = \pi(x) - x$, is given by

$$tr(JF(\hat{\phi})) = 2r[|Q| - 1] - n.$$

Proof. Consider the trace of the Jacobian at $\hat{\phi}$, i.e.,

$$tr(JF(\hat{\phi})) = \sum_{i=1}^{n} \frac{\partial F_i(\hat{\phi})}{\partial \phi_i}.$$

Observe that, for $k \neq i$, $\frac{\partial \hat{\phi}_k}{\partial \hat{\phi}_i} = -1$, since $\sum_j \hat{\phi}_j = 1$ and thus $\hat{\phi}_k = 1 - \sum_{j\neq k}\hat{\phi}_j$. We have

$$\begin{aligned}
\frac{\partial F_i(\hat{\phi})}{\partial \phi_i} &= \frac{\partial}{\partial \phi_i}\left[\frac{\hat{q}_i f(\phi_i)}{\sum_k \hat{q}_k f(\phi_k)} - \phi_i\right](\hat{\phi}) \\
&= \frac{\hat{q}_i f'(\hat{\phi}_i)}{\sum_k \hat{q}_k f(\hat{\phi}_k)} - \underbrace{\frac{\hat{q}_i f(\hat{\phi}_i)}{\sum_k \hat{q}_k f(\hat{\phi}_k)}}_{\pi_i(\hat{\phi})} \frac{\hat{q}_i f'(\hat{\phi}_i) - \sum_{k\neq i}\hat{q}_k f'(\hat{\phi}_k)}{\sum_k \hat{q}_k f(\hat{\phi}_k)} - 1 \\
&= \frac{1}{\sum_k \hat{q}_k f(\hat{\phi}_k)}\left[\hat{q}_i f'(\hat{\phi}_i) + \hat{\phi}_i\left(-\hat{q}_i f'(\hat{\phi}_i) + \sum_{k\neq i}\hat{q}_k f'(\hat{\phi}_k)\right)\right] - 1 \\
&= \frac{1}{\sum_k \hat{q}_k f(\hat{\phi}_k)}\left[(1-\hat{\phi}_i)\hat{q}_i f'(\hat{\phi}_i) + \hat{\phi}_i\left(\sum_{k\neq i}\hat{q}_k f'(\hat{\phi}_k)\right)\right] - 1,
\end{aligned}$$

where we used that $\pi_i(\hat{\phi}) = \hat{\phi}_i$ (since $\hat{\phi}$ is an equilibrium) to move from the second to the third equality. Now, when $f(x) = x^r$ for $r > 1$, $f'(x) = rx^{r-1}$ and we have

$$\frac{\partial F_i(\hat{\phi})}{\partial \phi_i} = \frac{1}{\sum_k \hat{q}_k(\hat{\phi}_k)^r}\left[(1-\hat{\phi}_i)\hat{q}_i r(\hat{\phi}_i)^{r-1} + \hat{\phi}_i\left(\sum_{k\neq i}\hat{q}_k r(\hat{\phi}_k)^{r-1}\right)\right] - 1. \tag{4.11}$$

If $i \in [n] \setminus Q$, then $\hat{\phi}_i = 0$ and $\dfrac{\partial F_i(\hat{\phi})}{\partial \phi_i} = -1$. If $i \in Q$, it follows that

$$\frac{\partial F_i(\hat{\phi})}{\partial \phi_i} = \frac{r}{\sum_{k \in Q} \hat{q}_k \left(\frac{\hat{q}_k^{\frac{1}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}} \right)^r} \left[(1-\hat{\phi}_i)\hat{q}_i \left(\frac{\hat{q}_i^{\frac{1}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}} \right)^{r-1} + \hat{\phi}_i \left(\sum_{k \in Q \setminus \{i\}} \hat{q}_k \left(\frac{\hat{q}_k^{\frac{1}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}} \right)^{r-1} \right) \right] - 1$$

$$= \frac{r}{\sum_{k \in Q} \hat{q}_k \frac{\hat{q}_k^{\frac{r}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}}} \left[(1-\hat{\phi}_i)\hat{q}_i\hat{q}_i^{-1} + \hat{\phi}_i \left(\sum_{k \in Q \setminus \{i\}} \hat{q}_k \hat{q}_k^{-1} \right) \right] - 1.$$

Since $\sum_{k \in Q} \hat{q}_k \frac{\hat{q}_k^{\frac{r}{1-r}}}{\sum_{j \in Q} \hat{q}_j^{\frac{1}{1-r}}} = 1$, we have $\dfrac{\partial F_i(\hat{\phi})}{\partial \phi_i} = r\left[1 - \hat{\phi}_i + \hat{\phi}_i(|Q|-1)\right] - 1 = r[1 + (|Q|-2)\hat{\phi}_i] - 1$. As a result, the trace of the Jacobian at $\hat{\phi}$ is given by

$$\begin{aligned} tr(JF(\hat{\phi})) = \sum_{i=1}^{n} \frac{\partial F_i(\hat{\phi})}{\partial \phi_i} &= \sum_{i \in Q} (r[1 + (|Q|-2)\hat{\phi}_i] - 1) + \sum_{i \in N \setminus Q} (-1) \\ &= r[|Q| + (|Q|-2) \sum_{i \in Q} \hat{\phi}_i] - |Q| - (|N| - |Q|) \\ &= 2r[|Q| - 1] - n. \end{aligned}$$

□

Corollary 4.1. *Under a social signal $f(x) = x^r, r > 1$, the inner equilibrium ϕ^* is unstable.*

Proof. By Theorem (4.4), we have that $tr(JF(\phi^*)) = 2r[|Q|-1] - n$. Since ϕ^* has n non-zero market shares, it follows that $tr(JF(\phi^*)) = 2r[n-1] - n = (r-1)n + r(n-2) > 0$, since $r > 1$ and $n \geq 2$. As a result, by Theorem (2.1) there exists an eigenvalue $\lambda = \lambda(\hat{\phi})$ satisfying $Re(\lambda) > 0$. And using Theorem (2.2), ϕ^* is unstable. □

Remark 4.1. Theorem (4.4) can also be used to show that many other equilibria are unstable: They simply need to have enough non-zero market shares to satisfy $2r[|Q|-1] > n$. Moreover, the theorem can also be used to show that, for any equilibrium ϕ that is not a monopoly, there exists $r > 1$ that makes ϕ unstable. It suffices to choose $r > \frac{n}{2(|Q|-1)}$. For instance, for $n = 4$, all the equilibria but the monopolies are unstable as soon as $r > 2$.

4.3 Agent-Based Simulation Results

We now report results from an agent-based simulation to highlight and complement the theoretical analysis. The agent-based simulation uses the setting from [Abeliuk

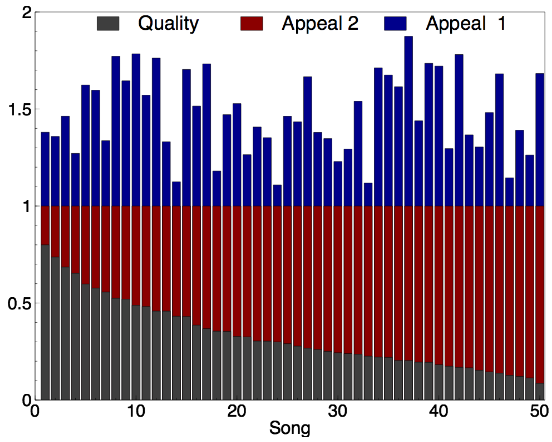

Figure 4.1: The quality q_i (grey) and appeal a_i (red and blue) of song i in the two settings. The settings only differ in the appeal of songs, and not in the quality of songs. In the first setting, the quality and the appeal for the songs were chosen independently according to a Gaussian distribution normalised to fit between 0 and 1. The second setting explores an extreme case where the appeal is anti-correlated with the quality used in setting 1. In this second setting, the appeal and quality of each song sum to 1.

et al., 2015], which used a dataset to emulate an environment similar to the MusicLab. The setting consists of 50 songs with the values of qualities and appeals specified in Appendix A [Dataset]. As mentioned in the introduction, the MusicLab is a trial-offer market where participants can try a song and then decide to download it. The generative model of the MusicLab [Krumme et al., 2012] uses the consumer choice preferences described in Section 3.2.

From this section and onwards, we assume that, it each period, a new customer arrives and may or may not buy a product based on the probability (quality) of the product tried. (Note that, in the earlier sections, each new period began when a product was purchased). The reason for this change is our interest in quantifying the expected number of purchases per period, and how it changes depending on different ranking policies. We use the expected number of purchases per period as way to measure the *market efficiency*. This view obviously does not change any result from the previous sections.

The Simulation Setting The agent-based simulation aims at emulating the MusicLab: Each simulation consists of L iterations (L simulated users) and, at each iteration $t : 0 < t < L$,

1. the simulator randomly selects a song i according to the probabilities $P_i(\sigma, \phi)$, where σ is the ranking proposed by the policy under evaluation and ϕ represents the market shares;

2. the simulator randomly determines, with probability q_i, whether selected song i is downloaded. In the case of a download, the simulator increases the num-

ber of downloads of song i, i.e., $d_i^{t+1} = d_i^t + 1$, changing the market shares. Otherwise, $d_i^{t+1} = d_i^t$.

Every $t > 0$ iterations, a new list σ may be recomputed if the ranking policy is dynamic (e.g., the popularity ranking). In this chapter, the simulation setting focuses mostly on two policies for ranking the songs:

- The *quality ranking* (Q-RANK) that assigns the songs in decreasing order of quality to the positions in decreasing order of visibility (i.e., the highest quality song is assigned to the position with the highest visibility and so on);
- The *popularity ranking* (D-RANK) that assigns the songs in decreasing order of popularity (i.e., d_i^t) to the positions in decreasing order of visibility (i.e., the most popular song is assigned to the position with the highest visibility and so on);

Note that the popularity ranking was used in the original MusicLab, while the quality ranking is a static policy: the ranking remains the same for the entire simulation. The simulation setting, which aims at being close to the MusicLab experiments, considers 50 songs and simulations with L=10^5 iterations unless stated otherwise. The songs are displayed in a single column. The analysis in [Krumme et al., 2012] indicated that participants are more likely to try songs higher in the list. More precisely, the visibility decreases with the list position, except for a slight increase at the bottom positions. The chapter also uses two settings for the quality and appeal of each song, which are depicted in Figure 4.1. In the first setting, the quality and the appeal were chosen independently according to a Gaussian distribution normalised to fit between 0 and 1. The second setting explores an extreme case where the appeal is anti-correlated with quality: The quality is the same as in the first setting but the appeal is chosen such that the sum of appeal and quality is 1.

4.3.1 Convergence

We first illustrate the convergence of the market for various popularity signals ($r < 1$) using the quality ranking. In order to visualise the results, we focus on only 5 songs, where the qualities, appeals, and visibilities are given by

q = [0.80, 0.72, 0.68, 0.65, 0.60]
a = [0.38, 0.35, 0.46, 0.27, 0.62]
v = [0.80, 0.75, 0.69, 0.62, 0.58].

The simulation is run for 10^5 iterations for the social signals $f(x) = x^r (r \in \{0.1, 0.25, 0.5, 0.75\})$ and Figure 4.2 depicts the simulation results. Observe that the equilibrium ϕ^* (dashed lines) changes because it depends of the value of r. Interestingly, for social signals with $r \leq 0.5$, the convergence of the process seems to occur around 10^4 time steps (iterations) even when they start with a strong initial distortion due to the appeals of the songs. The simulations show clear differences in behaviour depending

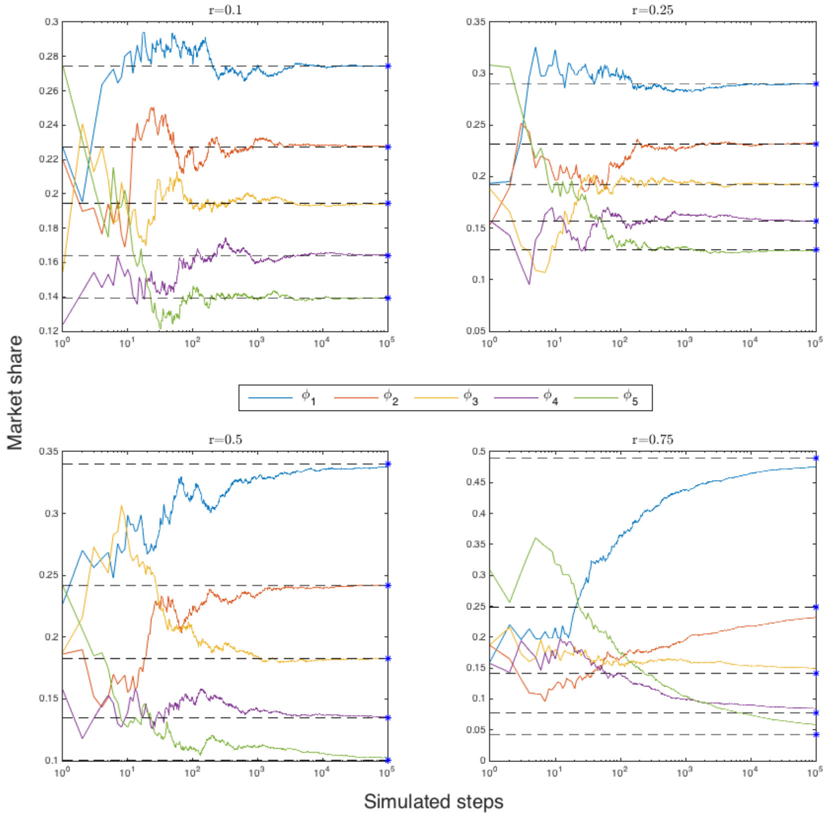

Figure 4.2: Evolution of market shares of 5 songs using a social signal $f(x) = x^r, \quad r \in \{0.1, 0.25, 0.5, 0.75\}$. Dashed lines are the values of the equilibrium for each song.

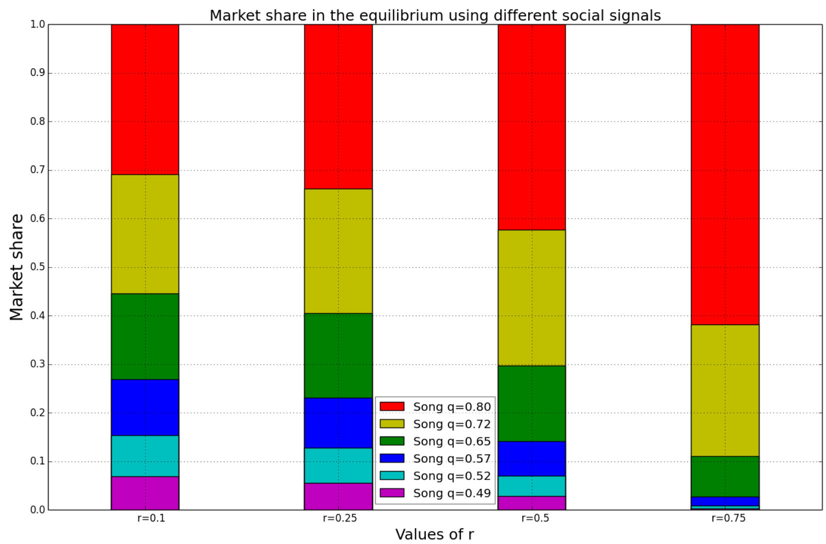

Figure 4.3: Market shares of 6 songs and their qualities, using a social signal $f(x) = x^r, \quad r \in \{0.1, 0.25, 0.5, 0.75\}$.

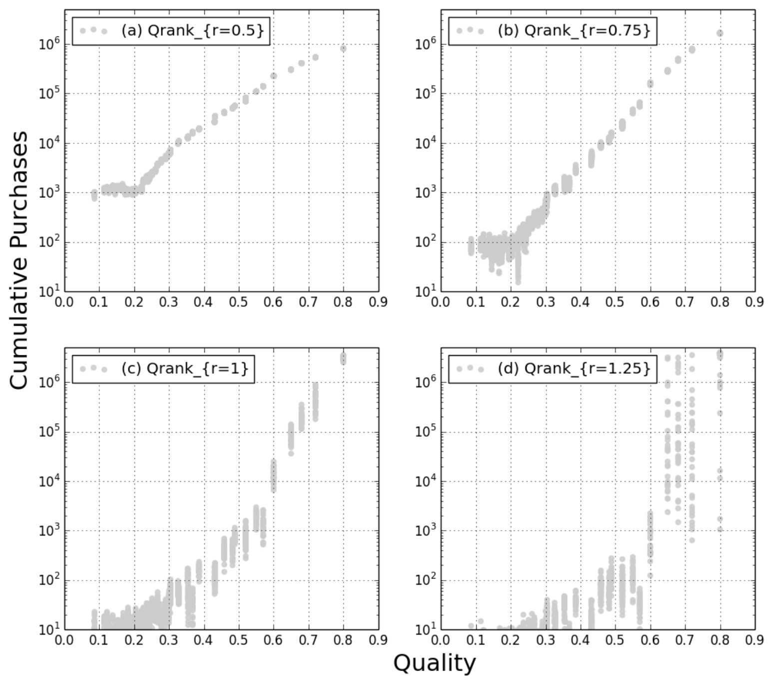

Figure 4.4: Distribution of downloads versus the qualities, using social signals $f(x) = x^r, \quad r \in \{0.5, 0.75, 1, 1.25\}$. The results are for the first setting where the quality and appeal of each song are chosen independently. The songs are ordered by increasing quality along the x-axis. The y-axis is the number of downloads.

on r and, when r moves closer to 1, the market tends to exhibit a monopolistic behaviour for the song with the best quality.

Figure 4.3 shows how the market is distributed in the equilibrium among 6 songs. The qualities, appeals, and visibilities are given by

q = [0.80, 0.72, 0.65, 0.57, 0.52, 0.49],
a = [0.38, 0.36, 0.27, 0.60, 0.77, 0.78],
v = [0.80, 0.75, 0.62, 0.48, 0.40, 0.35],

and the social signals are of the form $f(x) = x^r$ $(r \in \{0.1, 0.25, 0.5, 0.75\})$. Each stacked bar represents the proportion of the market for the 6 songs for a given social signal. Songs with better qualities (i.e., the top 2 songs represented in red and yellow respectively) have larger market shares and their market shares increase with r. In contrast, the market shares of the lower-quality songs (i.e., cyan and purple respectively) decrease when r increases. These results indicate that social influence has a beneficial effect on the market: it drives customers towards the better products, while not going to a monopoly as long as $r < 1$.

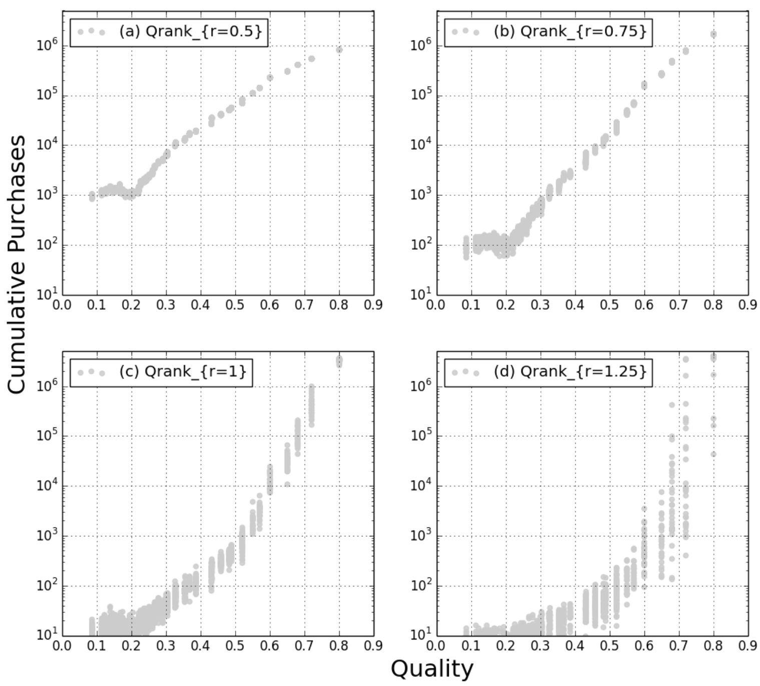

Figure 4.5: Distribution of downloads versus the qualities, using social signals $f(x) = x^r, \quad r \in \{0.5, 0.75, 1, 1.25\}$. The results are for the first setting where the quality and appeal of each song are anti-correlated. The songs are ordered by increasing quality along the x-axis. The y-axis is the number of downloads.

4.3.2 Market Predictability

This section depicts the predictability of the market for various values of r and the number of downloads per song as a function of its quality. Figures 4.4 and 4.5 depict the results for the two quality/appeal settings discussed previously. The figures display the results of 40 experiments for each setting with 1 million arrivals. Each experiment contributes 50 data points, i.e., the number of downloads for each song, and all the data points for the 40 experiments are displayed in the figures.

In the plots, the x-axis represents the song qualities and the y-axis the number of downloads. A dot at location (q, d) indicates that the song with quality q had d downloads in an experiment. Obviously, there can be several dots at the same location. For $r \in \{0.5, 0.75, 1\}$, the market is highly predictable and there is little variation in the song downloads. For $r = 1$, the market converges to a monopoly for the song of highest quality, confirming the results from [Abeliuk et al., 2015; Van Hentenryck et al., 2016]. Finally, for $r = 1.25$, the market exhibits significant unpredictability, as suggested by the theoretical results. In this case, the equilibria are monopolies for various songs but it is hard to predict which song will dominate the market.

Note also that the unpredictability of the market increases significantly for $r = 1.25$ when the appeal and quality of the songs are anti-correlated. This is not the case for $r \in \{0.5, 0.75\}$. To evaluate the statistical significance of these results, we measure the market unpredictability as suggested by Salganik et al. [2006]. The unpredictability un_i for product i is defined as the average difference in market share for that product over the 40 experiments:

$$un_i = \frac{1}{\binom{40}{2}} \sum_{w=1}^{40} \sum_{w^*=w+1}^{40} |\phi_{i,w} - \phi_{i,w^*}|,$$

where $\phi_{i,w}$ is the final market share of product i in experiment w. We then computed the overall unpredictability for each social signal $r \in \{0.5, 0.75, 1, 1.25\}$: $U = \frac{\sum_{j=1}^{n} un_j}{n}$.

Figure 4.6 shows the average unpredictability U and the standard deviation for the different social signals, using the same data as in Figures 4.4 and 4.5 (Figure 4.6 a and Figure 4.6 b respectively). We also performed Mann-Whitney U tests, comparing the values of U for pairs of social signals. In all cases, a social signal $r < 1$ is significantly more predictable than the signal $r = 1.25$ (p-value<0.05). Comparisons between $r = 0.5$ and $r = 0.75$ and $r = 0.75$ and $r = 1$ also show statistically significant differences in unpredictability. For instance, for the anti-correlated setting, the p-values for the various pairwise comparisons (*first column* is less unpredictable than *second column*) are given in Table 4.1

social signal	social signal	p-value
0.5	0.75	0.0029
0.5	1	8.73e-07
0.5	1.25	4.24e-10
0.75	1	0.0003
0.75	1.25	2.10e-07
1	1.25	0.0022

Table 4.1: p-values of the hypothesis: first column is less predictable than second column. Case a_i, q_i anti-correlated.

For the independent setting, Table 4.2 shows the pairwise comparisons that are also statistically significant in that case:

social signal	social signal	p-value
0.5	1	0.0414
0.5	1.25	0.0034
0.75	1.25	0.0266

Table 4.2: p-values of the hypothesis: first column is less predictable than second column. Case a_i, q_i independent.

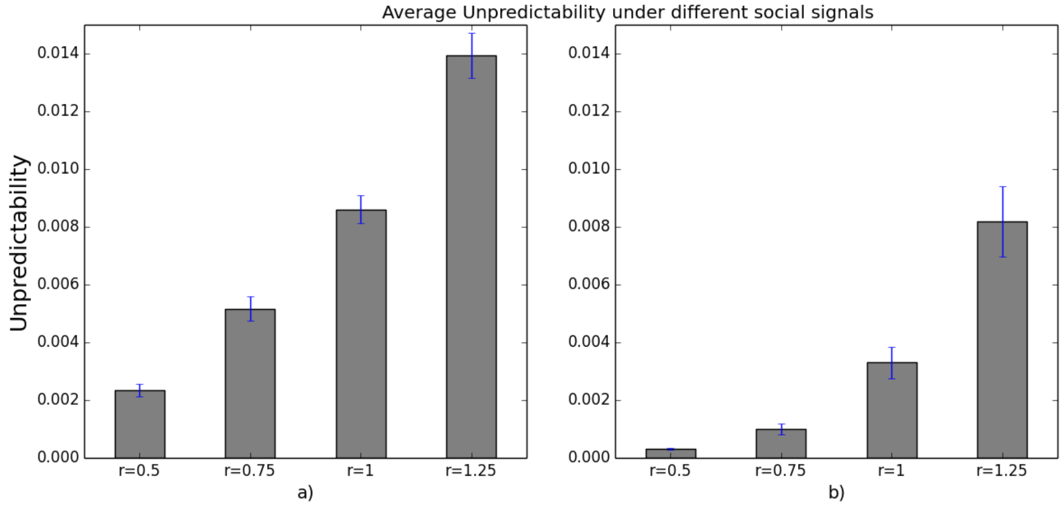

Figure 4.6: Average unpredictability (grey bars), using social signals $f(x) = x^r$, $r \in \{0.5, 0.75, 1, 25\}$. a) shows the results for the independent setting, and b) for the anti-correlated setting. Both cases consist of 40 experiments with 1 million iterations each. Blue lines represent the respective standard deviations.

Figure 4.7 compares the predictability of Q-rank and D-rank for the first setting of Quality/Appeal. For each ranking, two different social signals were used ($r = 0.5$ and $r = 1$) and the figure displays the result of 50 experiments, consisting in 1 million iterations. Two phenomena can be observed. First, sublinear signals seem to help the D-rank, making the outcome less chaotic (first column). Second, Q-rank clearly performs better than D-rank and exhibits much less unpredictability.

4.3.3 Performance of the Market

Figures 4.8 and 4.9 report results about the performance of the markets as a function of the social influence signals. The figures report the average number of downloads over time among 50 experiments, for the quality and popularity rankings as a function of the social signals. There are a few observations that deserve mention.

1. For the quality ranking, the expected number of downloads increases with the strength of the social signal as r approaches 1. The equilibrium when $r = 1$ is optimal asymptotically and assigns the entire market share to the song of highest quality. When $r = 2$, the situation is more complicated. The figure shows that the market efficiency can further improve if $r = 2$. However, when the simulation is run for more iterations (a result not shown in the figure), the market efficiency decreases slightly compared to $r = 1$, which is consistent with the theory since there is no guarantee that the monopoly for $r > 1$ is for the song of highest quality.

2. The popularity ranking is always dominated by the quality ranking and the benefits of the quality ranking increase as r approaches 1 from below.

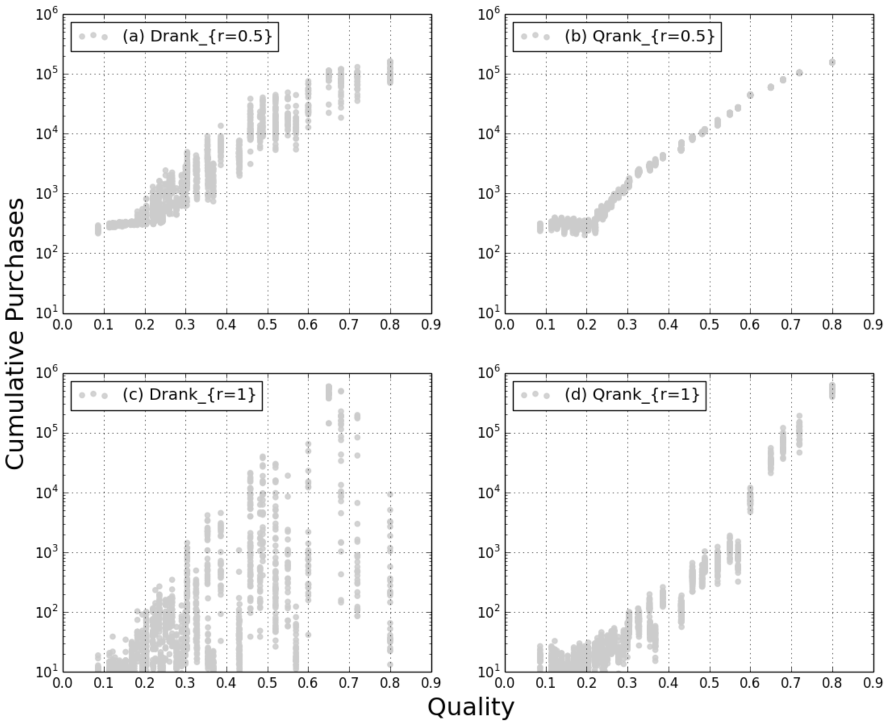

Figure 4.7: Distribution of purchases versus product qualities for 50 experiments with 1 million users. Figures (a) and (b) use a social signal $f(x) = x^{0.5}$, Figure (a) shows the results for the popularity ranking and Figure (b) for the quality ranking. Figures (c) and (d) use the social signal $f(x) = x$, Figure (c) shows the results for the popularity ranking and Figure (d) for the quality ranking.

3. The popularity ranking in the second setting when $r = 2$, obtains nearly a third of the expected downloads than the quality ranking.

4.4 Additional Observations on Sublinear Social Signals

The Benefits of Social Influence A linear social signal has been shown to be beneficial to the market efficiency, maximising the expected number of downloads. This result was proved by Abeliuk et al. [2015] for the performance ranking and by Van Hentenryck et al. [2016] for any static ranking such as the quality ranking. Unfortunately, sublinear social signals are not always beneficial to the market in that sense, as one can see in Example 4.1. Consider, once again, the quality ranking and assume that $q_1 \geq \ldots \geq q_n$. When there is no social signal, by following the Multinomial Logit model of Krumme et al. [2012] who define two sets of probabilities, $\pi^{SI}_{i,t}$ and π^{I}_{i}, that capture the probability of trying product i with and without social influence.

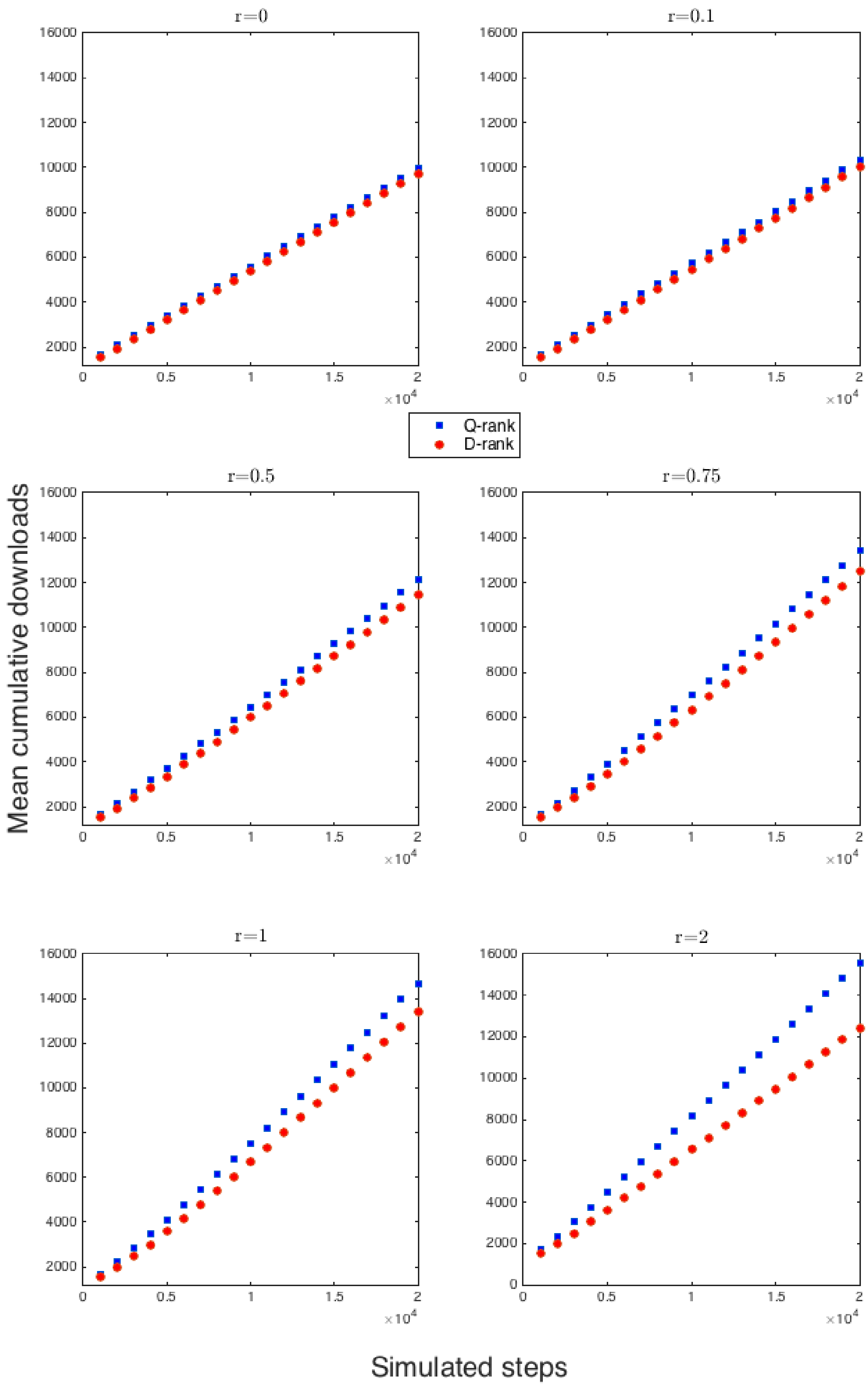

Figure 4.8: The Average Number of Downloads over Time for the Quality and Popularity Rankings for Various Social Signals in the First Setting for Song Appeal and Quality.

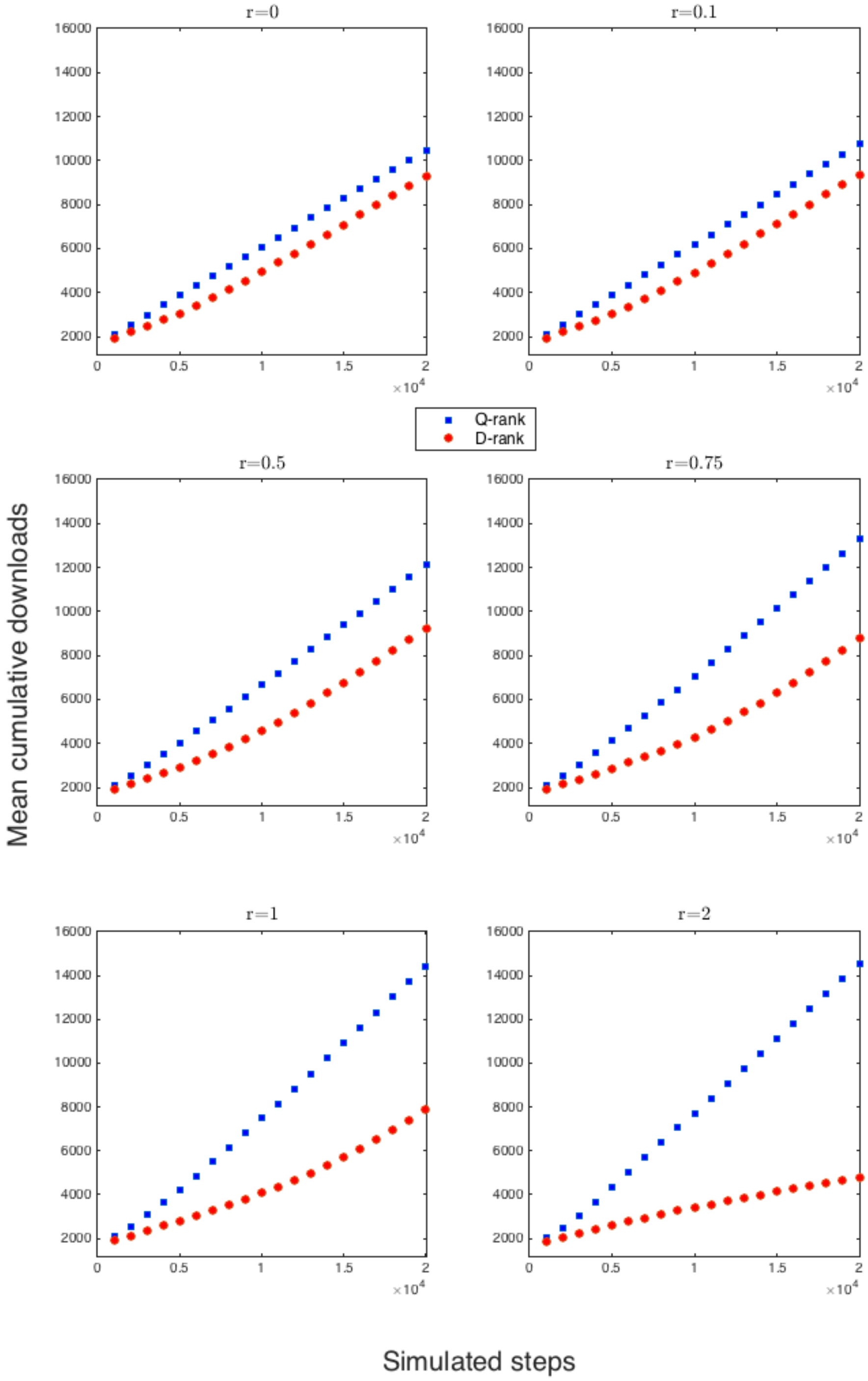

Figure 4.9: The Average Number of Downloads over Time for the Quality and Popularity Rankings for Various Social Signals in the Second Setting for Song Appeal and Quality.

These probabilities are defined as:

$$\pi_{i,t}^{SI} = \frac{v_{\sigma(i)}(\alpha a_i + d_i^t)}{\sum_{j=1}^{n} v_{\sigma(j)}(\alpha a_j + d_j^t)}, \qquad \pi_i^I = \frac{v_{\sigma(i)} a_i}{\sum_{j=1}^{n} v_{\sigma(j)} a_j}, \tag{4.12}$$

where α is a parameter to calibrate the strength of the social signal (e.g., $\alpha = 200$ for the MusicLab experiments). Equation (3.1) allows us to recover the formulae (4.12) via some linear transformation of the identity function: $f(\phi_i) = \beta\phi_i + \alpha a_i$, with $\beta = \sum_j d_j$ or $\beta = 0$ for each case:

$$\pi_i^I = \frac{v_i a_i}{\sum_{j=1}^{n} v_j a_j}.$$

In this case, the expected number of purchases per period is

$$\sum_{i=1}^{n} \pi_i^I q_i.$$

On the other hand, with a social signal, the probability of trying product i at time t is

$$\mathcal{P}_i(\phi^t) = \frac{v_i f(\phi_i^t)}{\sum_{j=1}^{n} v_j f(\phi_j^t)}$$

and the expected number of purchases per period at the equilibrium is given by

$$\sum_{i=1}^{n} \mathcal{P}_i(\phi^*) q_i = \sum_{i=1}^{n} \frac{v_i q_i f(\phi_i^*)}{\sum_{j=1}^{n} v_j f(\phi_j^*)}.$$

The following example shows that, under a sublinear social signal, the expected number of purchases (per period) at equilibrium, i.e., $\sum_{i=1}^{n} \mathcal{P}_i(\phi^*) q_i$, can be lower than the expected number of purchases when no social signal is used, i.e., $\sum_{i=1}^{n} \pi_i^I q_i$.

Example 4.1. *Consider a 2-dimensional Trial-Offer market with social signal $f(x) = x^{0.5}$, where the qualities, visibilities, and appeals are given by*

- $q_1 = 1$, $q_2 = 0.4$,
- $v_1 = 1$, $v_2 = 1$,
- $a_1 = 1$, $a_2 = 0.3$.

The expected number of purchases at equilibrium for the case with social signal is given by

$$\frac{v_1 q_1 (\phi_1^*)^r + v_2 q_2 (\phi_2^*)^r}{v_1 (\phi_1^*)^r + v_2 (\phi_2^*)^r} = \frac{v_1 q_1 (v_1 q_1)^{r/(1-r)} + v_2 q_2 (v_2 q_2)^{r/(1-r)}}{v_1 (v_1 q_1)^{r/(1-r)} + v_2 (v_2 q_2)^{r/(1-r)}} = \frac{1 + (0.4)^2}{1 + 0.4} \sim 0.83,$$

while, for the case without social signal, it is given by

$$\frac{v_1 q_1 a_1 + v_2 q_2 a_2}{v_1 a_1 + v_2 a_2} = \frac{1 + 0.3(0.4)}{1 + 0.3} \sim 0.86.$$

This simple example, in which the qualities and appeals are positively correlated, shows that if customers follow a sublinear social influence signal ($r = 0.5$), the market efficiency gets reduced by around 3 percent (with respect to not showing them the social signal). In contrast, when $r = 1$, social influence drives the market towards a monopoly, which leads to an asymptotically optimal market that assigns the entire market share to the highest quality product (which may be undesirable in practice). Note that, once the qualities and appeals have been recovered (using, say, Bernoulli sampling as suggested in [Abeliuk et al., 2015]), one could potentially decide whether to use the social influence (in case it is sublinear $r < 1$): simply compare the expected number of purchases in both settings, using the equilibrium for the social influence case and the formula for the case with no social signal.

Another measures could be also implemented whenever is detected that social influence can harm the firm's profit. For example in [Hu et al., 2015] the authors propose among their solutions, an **incluencer recruitment**, which consist of offering the product to the *correct people* so they can drive the initial trends.

Optimality of the Quality Ranking When $r = 1$, it has been shown that the quality ranking is optimal asymptotically: it maximises the expected number of purchases Van Hentenryck et al. [2016]. If another static ordering is used, the market will converge to the product that has the highest quality when scaled by its visibility. However, when $0 < r < 1$, the quality ranking is no longer guaranteed to be optimal asymptotically.

Example 4.2. *Consider a 3-dimensional Trial-Offer market with a social signal $f(x) = x^r, r = 0.3$, and the following values for qualities and visibilities:*

- $q_1 = 1$, $q_2 = 0.261$, $q_3 = 0.002$,
- $v_1 = 1$, $v_2 = 0.720$, $v_3 = 0.229$,

then, using quality ranking we would end up with an expected number of purchases at equilibrium, given by

$$\begin{aligned}\sum_{i=1}^{n} \mathcal{P}_i(\phi^*)q_i &= \frac{v_1q_1(v_1q_1)^{r/(1-r)} + v_2q_2(v_2q_2)^{r/(1-r)} + v_3q_3(v_3q_3)^{r/(1-r)}}{v_1(v_iq_i)^{r/(1-r)} + v_2(v_2q_2)^{r/(1-r)} + v_3(v_3q_3)^{r/(1-r)}} \\ &= \frac{1 + (0.720 * 0.261)^{10/7} + (0.229 * 0.002)^{10/7}}{1 + 0.720(0.720 * 0.261)^{3/7} + 0.229(0.229 * 0.002)^{3/7}} \sim 0.8026,\end{aligned}$$

on the other hand, if we decide to place the third product (quality $q_3 = 0.002$) in the second position, and the second product (quality $q_2 = 0.261$) in the third position of the ranking, we get

$$\begin{aligned}\sum_{i=1}^{n} \mathcal{P}_{\sigma_i}(\phi^*)q_i &= \frac{v_1q_1(v_1q_1)^{r/(1-r)} + v_2q_3(v_2q_3)^{r/(1-r)} + v_3q_2(v_3q_2)^{r/(1-r)}}{v_1(v_iq_i)^{r/(1-r)} + v_2(v_2q_3)^{r/(1-r)} + v_3(v_3q_2)^{r/(1-r)}} \\ &= \frac{1 + (0.720 * 0.002)^{10/7} + (0.229 * 0.261)^{10/7}}{1 + 0.720(0.720 * 0.002)^{3/7} + 0.229(0.229 * 0.261)^{3/7}} \sim 0.9154.\end{aligned}$$

The intuition behind the previous example is that if there exists a product which is much better than the rest, the best decision is to exhibit it in the first position and place, in the second position, the lowest quality product to make the first product even more appealing. It is an open problem to determine whether there is a polynomial-time algorithm to find an optimal ranking.

Chapter 5

Pricing strategies under a multinomial logit model with network effects

This chapter is reproduced from a paper in preparation.

Maldonado, F.; Berbeglia, G.; Van Hentenryck, P., 2019. Compete or collaborate: pricing strategies under a multinomial logit model with network effects.

5.1 Introduction

In this chapter we aim to study seller's pricing strategies based on a model of consumer choice, where the purchasing decisions are affected by past consumption. In general terms we assume that the willingness to purchase is influenced by the (known) intrinsic utility of the products, their prices, and network effects as a function of consumption history. The consumers can purchase the product $i \in \{1, \dots, n\}$ that maximises their expected utility or they can choose to leave the market without making any purchase, what is called choosing the *no purchase option*.

We also aim to represent how ineffective transactions affect the purchasing decisions, motivated for markets like eBay, where after each transaction the users can give an evaluation to the seller in the categories of positive, negative and neutral. New consumers can observe how many transactions a seller has made and a reputation score that penalises the negative feedback, providing extra information about transactions where the consumers were not satisfied (e.g, Cabral and Hortacsu [2010]). In order to capture those ineffective transactions with our model, we consider that the no purchase option also presents network effects. In this way we are able to keep track of consumers that do not buy anything in the market, or equivalently they buy similar products somewhere else.

The main contributions of this chapter can be summarised in the following way:

- **Non-linear network effects in a consumer choice model:** We propose a variation of the Multinomial Logit Model for consumer choice where we incorporate non-linear network effects, representing in this way, market interactions where consumers only see a score function of the past consumption. Since the probability of choosing the available products (or the no purchase option) dynamically changes over time due to the network effects, we apply stochastic approximation techniques to prove that such probability converges almost surely to an asymptotic stationary distribution, that represents the market share of each product in the long run.

- **Monopolistic and competitive pricing are analysed:** For a market with n sellers, we model their expected revenues based on the asymptotic market share distribution and the displayed prices. First, we study the case where sellers act collaboratively, adopting a monopolistic pricing strategy to maximise the overall expected revenue. We show that the market share of the no purchase option is decreasing in terms of the network parameter $r : 0 < r < 1$, and that the overall expected revenue is increasing in that parameter as long as r is large enough. We then study the case where sellers compete, inducing a price competition game that has a unique pure Nash Equilibrium (we also provide an algorithm to compute it). We finally compare experimentally and analytically both cases, incorporating the consumers' perspective into the analysis.

The rest of the chapter is structured as follows: Section 5.2 details the proposed consumer choice model. Section 5.3 focuses on the sellers acting collaboratively, in opposition to Section 5.4 where they compete. Finally, Section 5.5 compares the strategies defined in the previous two sections, showing how they affect/benefit the consumers.

Across the whole chapter we include some numerical examples based on synthetic data, complementing the theoretical results and providing some extra insights. We also include an Appendix where more experiments are shown.

5.2 Model

We consider a market with n sellers, $n \geq 2$, where each seller $i \in \{1, ..., n\}$ owns one indivisible product with infinite supply (e.g., digital goods like e-books). For notational convenience we also call i to the product of seller i.

Once the sellers have fixed the prices for their products, sequential consumers arrive and decide to buy one of the n products or not to buy anything. We define a discrete time $k \geq 1$ as the arrival of consumer k to the market. Notice that this is different to the previous chapters, where the discrete times were defined by new purchases. We assume that consumers' decisions are affected by the intrinsic utility of the products, the prices and some network effects related to the popularity of the products.

We model this as a variation of a standard MNL model with network effects (see for example Du et al. [2016, 2018]), where we incorporate a non-linear network effect, reflecting a score function of the past purchases. This is done mainly for two reasons: first, we intend to use some results from previous chapters, having some assurances over the asymptotic behaviour of the market, and second, we aim to avoid multiplicity of price equilibria, a phenomenon that can be observed for example in Du et al. [2016]. Formally our model is defined as follows: the k-th consumer's utility obtained from purchasing product i is given by

$$u_i^k := u_i(r, g_i, d_i^k, p_i) = g_i + r\ln(d_i^k) - \beta_i p_i + \xi_i, \tag{5.1}$$

where g_i represents the intrinsic utility of product i (a measure of its quality); r is a constant that represents the strength of the network effect on the consumers ($0 < r < 1$); p_i is the price of the product and β_i is a price sensitivity parameter; d_i^k is its cumulative amount of purchases up to time k, that for notational convenience we initialised as $d_i^0 = 1$ for all $i \in \{1, \dots, n\}$ (this is equivalent to consider $\ln(d_i^k + 1)$, with $d_i^0 = 0$). Finally ξ_i is a random variable representing consumer specific idiosyncrasies.

We also consider a *dummy* product, $n+1$, representing the no purchase option, which we characterise with the parameters $g_{n+1} = 0$, $p_{n+1} = 0$, and $d_{n+1}^0 = 1$ which is increased by 1 every time a new consumer does not buy anything, keeping track of the ineffective exchanges between sellers and consumers. The utility for the no purchase option is then $u_{n+1}^k = r\ln(d_{n+1}^k) + \xi_{n+1}$. In Dhar [1997] the author shows several empirical studies where consumers decide for a no choice option, even when the available products have a good intrinsic utility. With our model we try to capture that type of phenomenon.

We denote $[n+1] := \{1, ..., n\} \cup \{n+1\}$, the set of products extended by the no purchase option. Let ϕ^k be the vector of market share at time k, this is $\phi_i^k = \frac{d_i^k}{\sum_{j\in[n+1]} d_j^k}$, for all $i \in [n+1]$. Under the assumption that $\{\xi_i\}_{i=1}^{n+1}$ are i.i.d random variables following a Gumbel distribution, and according to standard results for the Multinomial Logit model (see McFadden et al. [1973] for details), the probability that the $(k+1)$-th consumer purchases product i is given by

$$\pi_i^k = \pi_i(\phi^k, p, q, \beta) = \frac{(d_i^k)^r e^{g_i - \beta_i p_i}}{\sum_{j\in[n+1]} (d_j^k)^r e^{g_j - \beta_j p_j}} = \frac{(\phi_i^k)^r e^{g_i - \beta_i p_i}}{\sum_{j\in[n+1]} (\phi_j^k)^r e^{g_j - \beta_j p_j}}.$$

Where $\pi_{n+1}^k + \sum_{i=1}^n \pi_i^k = 1$ for all $k \geq 0$. We put $\pi^k = (\pi_1^k, \dots, \pi_n^k, \pi_{n+1}^k)$.

If we compare this probability distribution with the one from Chapter 4, we can establish some relations between the parameters that define each model in the following way. Let $\varrho_i = e^{g_i - \Gamma}$ and $v_i = e^{\Gamma - \beta_i p_i}$, where $\Gamma \geq 1$ is some fixed constant (for example $\Gamma = \max_{i\in\{1,...,n\}} g_i + 1$). We notice that $\varrho_i : 0 < \varrho_i < 1$ could represent the parameter of quality in the model from Chapter 4, while v_i could represent the visibility of position i in that model. We have in this way the relation of **higher price** $\Leftrightarrow$ **lower visibility**. Furthermore, the expected market shares in both models will depend on the values of $\varrho_i v_i = e^{g_i - \Gamma + \Gamma - \beta_i p_i} = e^{g_i - \beta_i p_i}$ (see Lemma 5.2).

However there is a big difference between the models. While in the model from Chapter 4, the visibilities are an intrinsic characteristic of the market (independent of the products), here, the prices are chosen by the sellers. In that sense, the pricing policy that will be developed in Section 5.4 can also be interpreted as an auction for positions in the market model from Chapter 4. Meanwhile Section 5.3 could be seen as a market maker choosing an optimal ranking.

In order to use results from previous chapters, the following Lemma establishes an important property that the market share ϕ^k satisfies:

Lemma 5.1. *The market share ϕ^k, satisfies the following recurrence:*

$$\phi^{k+1} = \phi^k + \gamma^{k+1}\left[\pi^k - \phi^k + U^{k+1}\right], \tag{5.2}$$

with $\gamma^{k+1} = \mathcal{O}(k^{-1})$ and U^{k+1} a martingale difference noise term (i.e., $\mathbb{E}[U^{k+1}|\phi^t, t \leq k] = 0$).

Proof. Following the idea of Chapter 4, consider that in each time step k (arrival of k-th consumer) either a product $i \in \{1, ..., n\}$ is purchased, or no product is purchased ($i = n+1$), then, defining $D^k := \sum_{j\in[n+1]} d_j^k = \sum_{j\in[n+1]} \sum_{t=1}^k d_j^t = k$, we have that $\phi^k = D^k \dfrac{\phi^k}{D^k} \Rightarrow \phi^{k+1} = \frac{D^k\phi^k + e^{k+1}}{D^{k+1}}$, with e^{k+1} a random ($n+1$- dimensional) variable with coordinates $(e^{k+1})_i = 1$ if product $i \in \{1, ..., n\}$ has been purchased at time $k+1$; $(e^{k+1})_{j\neq i} = 0$, and $(e^{k+1})_{n+1} = 1$ if no product is purchased by the consumer $k+1$.

Hence, clearly $\mathbb{E}[e^{k+1}|\phi^t, t \leq k] = \pi^k$, and considering $\gamma^{k+1} := \dfrac{1}{D^{k+1}} = \dfrac{1}{k+1}$, and $U^{k+1} := e^{k+1} - \mathbb{E}[e^{k+1}|\phi^t, t \leq k]$, we get the desired recurrence

$$\phi^{k+1} = \phi^k + \gamma^{k+1}\left[\pi^k - \phi^k + U^{k+1}\right].$$

□

As it was explained in the previous chapters, the asymptotic behaviour of the discrete dynamic (5.2) is related to the asymptotic behaviour of the continuous dynamic:

$$\dot{\phi}^t = \pi(\phi^t) - \phi^t, \quad \phi^t \in \Delta^{n+1}. \tag{5.3}$$

In Chapter 4, Theorem 4.1 shows that a dynamic like Equation (5.3) has only one equilibrium with all its coordinates strictly positive. Furthermore, Theorem 4.2 shows that under some conditions over the parameters, the Robbins-Monro algorithm (5.2) converges almost surely to that equilibrium. Using that result we can establish the following Lemma.

Lemma 5.2. *For any fixed price $\mathbf{p} = (p_1, ..., p_n, p_{n+1}) \in \mathbb{R}_+^n \times \{0\}$, fixed parameters β_i, g_i, and a network effect parameter $r : 0 < r < 1$, the market share ϕ^k converges almost surely to the unique equilibrium $\phi^* = (\phi_i^*)_{i\in[n+1]}$ given by*

$$\phi_i^* = \frac{\left(e^{g_i - \beta_i p_i}\right)^{1/(1-r)}}{\sum_{j\in[n+1]}\left(e^{g_j - \beta_j p_j}\right)^{1/(1-r)}}. \tag{5.4}$$

Furthermore, for every $i \in [n+1]$, $\pi_i(\phi^*) = \phi_i^*$(fixed point for the probability function π).

It is important to notice that according to Corollary 4.1 in Chapter 4, when $r > 1$, ϕ^* is an unstable equilibrium, hence the market converges to other equilibria (for example, monopolies for some product) with probability 1. Establishing pricing policies in those cases using the equilibrium ϕ^* as a decision variable does not make sense, therefore we will focus only on the cases where $0 < r < 1$. In some related research (e.g., Cui and Zhu [2016] and Wang and Wang [2016]) the authors do not consider upper bounds on the network parameters (but the market size is fixed). However, as Du et al. [2016] points out, higher values of those parameters can lead to suboptimal results (due to multiplicity of equilibria). In our case we compensate this limiting behaviour (bounding the network parameter to $(0, 1)$), allowing the purchases to grow freely (unbounded market size).

In the case that $0 < r < 1$ we notice that the term $\tau_i := g_i - \beta_i p_i$ affects directly the expected market share for each product, in particular the product with the highest value of τ_i gets the largest market share. In this way, if a high intrinsic utility product is too expensive, then the chances of being purchased decrease, or equivalently lower intrinsic utility products could increase their expected sales after a reduction on its price. Keeping this into consideration, we define the expected revenue for each seller in terms of the expected market share and the chosen prices.

Definition 5.1. *The expected revenue for seller i is given by*

$$w_i = w_i(r, q, p_i, p_{-i}) = p_i \phi_i^* = p_i \frac{\left(e^{g_i - \beta_i p_i}\right)^{1/(1-r)}}{\sum_{j \in [n+1]} \left(e^{g_j - \beta_j p_j}\right)^{1/(1-r)}}.$$

For notational convenience we assume without loss of generality that the intrinsic utilities are non-decreasingly ordered, this is, $g_1 \geq g_2 \geq \cdots \geq g_n > g_{n+1} = 0$, meaning that seller 1 has the highest intrinsic utility product. In the following two sections we will analyse two types of strategic decisions, that the sellers can follow based on their expected revenues w_i. In Section 5.3 we analyse the case of a coalition between the sellers where they adopt a monopolistic pricing strategy to maximise the overall expected revenue. Whereas in Section 5.4 we study the case where sellers compete on their prices to maximise their own expected revenues. We will pay special attention to the behaviour of the price and revenue in terms of the network parameter r, and when that's relevant we will make explicit the dependence (e.g., $p_i = p_i(r)$).

5.3 Monopolistic pricing

We consider in this section a setting where the sellers decide to act collaboratively. In this context the sellers choose their prices such that they maximise the overall expected revenue defined by

$$R(p) = \sum_{i=1}^{n} w_i = \sum_{i=1}^{n} p_i \phi_i^*.$$

Thus, we are interested in finding a price vector $\mathbf{p}^M := (p_1^M, \ldots, p_n^M)$, that we call *monopolistic price*, that satisfies

$$\mathbf{p}^M \in \underset{p \in \mathbb{R}_+^n}{\text{arg-max}}\, R(p).$$

In Theorem 5.1 we will deduce the conditions that $\mathbf{p}^M$ must satisfy to maximise $R(p)$, and in the special case of having the same price sensitivities for all the products, $\beta_i = \beta, \forall i \in \{1, \ldots, n\}$, we will provide a closed expression for this price using the Lambert W function (see Corless et al. [1996]), where in particular, for any non-negative x, $W(x)$ is defined as the solution of the equation

$$We^W = x.$$

If $x > 0$, then $W(x)$ is a positive continuous differentiable function, strictly increasing and concave. The use of the Lambert W function spans a wide range of applications, and particularly it has been used in Economics for pricing on discrete choice models (e.g., Li and Huh [2011] and Cui and Zhu [2016]).

Theorem 5.1. *The monopolistic price, $\mathbf{p}^M = (p_1^M, p_2^M, \ldots, p_n^M)$ that maximises $R(p)$ must satisfy that*

$$\frac{p_i^M}{1-r} - \frac{1}{\beta_i} = \frac{p_k^M}{1-r} - \frac{1}{\beta_k}, \text{ for every pair } i, k \in \{1, \ldots, n\}. \tag{5.5}$$

Furthermore, if the products have the same price sensitivity $\beta_i = \beta, \forall i \in \{1, \ldots, n\}$ then all the products have the same price $p_i^M = p^M$ given by

$$p^M = \frac{1-r}{\beta}\left[W\left(\frac{\sum_{i=1}^n e^{g_i/(1-r)}}{e}\right) + 1\right], \tag{5.6}$$

with $W()$ is the Lambert W function.

Before presenting the proof, it is important to notice how $\mathbf{p}^M$ compares against the monopolistic price obtained with other models.

Since the probability functions define how the models behave, we proceed to characterise the two models we will be comparing against (under our notation), using their probabilities.

Definition 5.2. *The probability π_i^C of choosing product $i \in \{1, \ldots, n\}$ for the classic MNL model (without network effects) is given by*

$$\pi_i^C = \frac{exp(g_i - \beta_i p_i)}{1 + \sum_{j=1}^n exp(g_j - \beta_j p_j)},$$

where g_i,β_i and p_i are defined as before. We put $\pi^C = (\pi_1^C, \ldots, \pi_{n+1}^C)$.

Definition 5.3. *For the MNL model with network effects defined in Du et al. [2016], the* *probability π_i^D of choosing product $i \in \{1, \ldots, n\}$ is given by*

$$\pi_i^D = \frac{exp(g_i - \beta_i p_i + \alpha_i \phi_i)}{1 + \sum_{j=1}^n exp(g_j - \beta_j p_j + \alpha_j \phi_j)},$$

where α_i is the network sensitivity of product i, and ϕ_i its market share. We put $\pi^D = (\pi_1^D, \ldots, \pi_{n+1}^D)$.

The following table summarises some of the comparisons we obtain when we consider the different probability models. The first column of the table contains the settings where we will be making the comparisons, the second column contains the conclusions given by our probability distribution $\pi = (\pi_1, \ldots, \pi_{n+1})$, where $\pi_i = \frac{\phi_i^r exp(g_i - \beta_i p_i)}{\sum_{j \in [n+1]} \phi_j^r exp(g_j - \beta_j p_j)}$. The third and fourth columns contain the results when π^C, π^D are used, respectively.

Setting	π	π^C	π^D
$\beta_i = \beta$ for all $i \in [n]$	Unique optimal price is to assign the same price to every product (uniform price).	Uniform price.	No explicit form for the optimal price.
	Market shares are increasing on the intrinsic utility of the products.	Market shares are increasing on the intrinsic utility of the products.	Since, in their model, the following expression must be constant $2\alpha\phi_i - \log(\phi_i) + g_i$, then if for some i, $\phi_i > \frac{1}{2\alpha}$, an increment on its intrinsic utility, would lead to a decrement of its market share.
$\beta_i = \beta$, $g_i = g$ for all $i \in [n]$	Uniform price	Uniform price	Uniform price if $\alpha < \hat{\alpha}$, for some $\hat{\alpha}$. Otherwise, uniform price for $n-1$ products, and one product with a lower price
	Uniform market share	Uniform market share	Uniform market share if $\alpha < \hat{\alpha}$, otherwise, the cheapest product has a larger market share.

Finally, it is worth mentioning that in our model, even if the price sensitivities are different, according to Equation (5.5), if $r \to 0$, then for all $i, k \in \{1, ..., n\}$, $p_i^M = p_k^M$.

However, in that case the highest intrinsic utility product gets a market share close to 1, while the rest of the products have a negligible share (a monopoly for the highest intrinsic utility product).

The proof for Theorem 5.1 is as follows.

Proof. To find the prices that optimise $R(p)$ we compute the gradient of R, $\nabla R(p)$, with coordinates $\frac{\partial R(p)}{\partial p_k}$ given by

$$\begin{aligned}
\frac{\partial R(p)}{\partial p_k} &= \sum_{i=1, i\neq k}^{n} \frac{\partial (p_i \phi_i^*)}{\partial p_k} + \frac{\partial (p_k \phi_k^*)}{\partial p_k} \\
&= \sum_{i=1, i\neq k}^{n} \frac{\beta_k}{1-r} p_i \phi_i^* \phi_k^* + \phi_k^* - \frac{\beta_k}{1-r} \phi_k^* p_k (1-\phi_k^*) \\
&= \phi_k^* \left[\frac{\beta_k}{1-r} \left(\sum_{i=1, i\neq k}^{n} p_i \phi_i^* + p_k \phi_k^* \right) + 1 - \frac{\beta_k}{1-r} p_k \right] \\
&= \phi_k^* \left[\frac{\beta_k}{1-r} R(p) + 1 - \frac{\beta_k}{1-r} p_k \right].
\end{aligned}$$

Imposing the first order conditions over $R(p)$ gives us

$$\frac{\partial R(p)}{\partial p_k} = 0 \Leftrightarrow \phi_k^* = 0 \vee \frac{R(p)}{1-r} = \frac{p_k}{1-r} - \frac{1}{\beta_k}, \qquad 1 \leq k \leq n,$$

However $\phi_k^* = 0 \Leftrightarrow p_k = \infty$, we conclude that for all pairs i, k, $1 \leq i, k \leq n$, the following equality must hold

$$\frac{p_k}{1-r} - \frac{1}{\beta_k} = \frac{p_i}{1-r} - \frac{1}{\beta_i},$$

which is the desired condition (5.5). Now, defining $z_k = \frac{\beta_k p_k}{1-r}$ (that we will call the normalised price for product k), Equation (5.5) is equivalent to

$$\frac{z_k - 1}{\beta_k} = \frac{z_i - 1}{\beta_i}, \quad \forall i, k \in [n]. \tag{5.7}$$

Equation (5.7) defines a pairwise relation. On the other hand the prices must also satisfy

$$\frac{R(p)}{1-r} = \frac{z_k - 1}{\beta_k}. \tag{5.8}$$

Now in the special case when $\beta_i = \beta$ for all the products, Equation (5.7) implies that all the prices are the same, $p_i = p$ for all $i \in \{1, \ldots, n\}$. Replacing this condition into

Equation (5.8) produces the following equivalences

$$\begin{aligned}
\frac{p}{1-r}\sum_{i=1}^{n}\phi_i^* &= \frac{z-1}{\beta} \\
z\phi_{n+1}^* &= 1 \qquad (5.9)\\
\frac{z}{1+e^{-z}\sum_{i=1}^{n} e^{g_i/(1-r)}} &= 1 \\
z-1 &= e^{-z}\sum_{i=1}^{n} e^{g_i/(1-r)} \qquad (5.10)\\
(z-1)e^{z-1} &= e^{-1}\sum_{i=1}^{n} e^{g_i/(1-r)} \\
\Rightarrow z^M &= W(e^{-1}\sum_{i=1}^{n} e^{g_i/(1-r)})+1. \qquad (5.11)
\end{aligned}$$

Finally replacing $p^M = \frac{(1-r)z^M}{\beta}$ we have our conclusion. □

We can easily notice that each coordinate of $\mathbf{p}^M := (p^M, \ldots, p^M)$ is increasing on each value of g_i for all $i \in \{1, \ldots, n\}$, this is, higher the intrinsic utility, higher the price. Theorem 5.2 summarises other properties related to the monopolistic price, and the monotonic behaviour of the revenue in terms of the network effect parameter r.

Theorem 5.2. *Let all the products have the same price sensitivity $\beta_i = \beta$, and consider a network effect parameter r, $0 < r < 1$, then the following statements hold true:*

1. *The market share of the no purchase option, $\phi_{n+1}^*(\mathbf{p}^M(r))$, is strictly decreasing in r.*

2. *The market share of the highest intrinsic utility product, $\phi_1^*(\mathbf{p}^M(r))$, is strictly increasing in r.*

3. *There exists $r^*, 0 < r^* < 1$ such that, the overall expected revenue $R(\mathbf{p}^M(r)) = \sum_{i=1}^{n} p_i^M(r)\phi_i^*(\mathbf{p}^M(r))$ is strictly increasing in r for all $r : r^* \leq r < 1$.*

Proof. Let $p^M(r)$ be the monopolistic price for each product, given by Theorem 5.1, and consider the normalised price $z^M(r) = \frac{\beta p^M(r)}{1-r}$.

1. We know that according to Equation (5.9), the market share for the no purchase option, $\phi_{n+1}^*(\mathbf{p}^M(r))$, must satisfy $z^M(r)\phi_{n+1}^*(\mathbf{p}^M(r)) = 1$. Clearly since $W()$ is an increasing function, Equation (5.11) implies that $z^M(r)$ is strictly increasing in terms of r, hence $\phi_{n+1}^*(\mathbf{p}^M)(r)$ must be strictly decreasing as a function of r.

2. We first compute the derivative of $z^M(r)$ with respect to r, indeed we use Equa-

tion (5.10) to obtain the $\frac{\partial z^M(r)}{\partial r}$ as follows

$$\frac{\partial z^M(r)}{\partial r} = -\frac{\partial z^M(r)}{\partial r} e^{-z^M(r)} \sum_{i=1}^{n} e^{g_i/(1-r)} + e^{-z^M(r)} \sum_{i=1}^{n} \frac{g_i e^{g_i/(1-r)}}{(1-r)^2}$$

$$\Rightarrow \frac{\partial z^M(r)}{\partial r} = \frac{1}{(1-r)^2} \frac{\sum_{i=1}^{n} e^{-z^M(r)} e^{g_i/(1-r)} g_i}{1+e^{-z^M(r)} \sum_{i=1}^{n} e^{g_i/(1-r)}} = \frac{1}{(1-r)^2} \sum_{i=1}^{n} \phi_i^*(\mathbf{p}^M(r)) g_i.$$

Now we consider the market share for the highest intrinsic utility product, $\phi_1^*(\mathbf{p}^M(r))$, and we take its first derivative with respect to r:

$$\begin{aligned}
\frac{\partial \phi_1^*(\mathbf{p}^M(r))}{\partial r} &= \frac{\frac{\partial e^{g_1/(1-r)}}{\partial r} e^{-z^M(r)} + \frac{\partial z^M(r)}{\partial r} e^{-z^M(r)} e^{g_1/(1-r)}}{1+e^{-z^M(r)} \sum_{i=1}^{n} e^{g_i/(1-r)}} - \\
&\quad \frac{e^{-z^M(r)} e^{g_1/(1-r)}}{(1+e^{-z^M(r)} \sum_{i=1}^{n} e^{g_i/(1-r)})^2} \left(e^{-z^M(r)} \sum_{i=1}^{n} \frac{\partial e^{g_i/(1-r)}}{\partial r} - \frac{\partial z^M(r)}{\partial r} e^{-z^M(r)} \sum_{i=1}^{n} e^{g_i/(1-r)} \right) \\
&= \frac{\phi_1^*(p^M(r))}{(1-r)^2} \left[g_1 - \sum_{j=1}^{n} \phi_j^*(\mathbf{p}^M(r)) g_j + [1 - \sum_{j=1}^{n} \phi_j^*(\mathbf{p}^M(r))] \sum_{j=1}^{n} \phi_j^*(\mathbf{p}^M(r)) g_j \right],
\end{aligned}$$

the only term that can be negative in the last equality is $g_1 - \sum_{j=1}^{n} \phi_j^*(\mathbf{p}^M) g_j$, but as $\sum_{j\in[n+1]} \phi_j^* = 1$, then $g_1 = \sum_{j\in[n+1]} \phi_j^* g_1 = \sum_{j=1}^{n} \phi_j^* g_1 + \phi_{n+1}^* g_1$, and since $g_1 \geq g_j$, for all $1 \leq j \leq n$ we have

$$g_1 - \sum_{j=1}^{n} \phi_j^*(\mathbf{p}^M) g_j = \sum_{j=1}^{n} (g_1 - g_j) \phi_j^*(\mathbf{p}^M) + \phi_{n+1}^*(\mathbf{p}^M) g_1 > 0.$$

In conclusion $\phi_1^*(p^M)$, the market share for the highest intrinsic utility product is strictly increasing in terms of r.

3. We first notice that if $p^M(r)$ is increasing in some interval $[r^*, r^{**})$, then the conclusion is direct, indeed, since $R(\mathbf{p}^M(r)) = p^M(r)(1 - \phi_{n+1}^*(\mathbf{p}^M(r)))$, taking the partial derivatives with respect to r gives an expression that it is always positive for $r : r^* < r < 1$. We assume then that $p^M(r)$ is decreasing for all $0 < r < 1$, in particular, we have that if for some $r_1 : 0 < r_1 < 1$, $g_i - \beta p^M(r_1) > 0$, then for all $r_2 : r_1 < r_2 < 1$, $g_i - \beta p^M(r_2) > 0$.

   Now, we know that the monopolistic price $p^M(r)$ is characterised by Equation (5.8) as follows:
   $$R(\mathbf{p}^M(r)) = p_i^M(r) - \frac{1-r}{\beta_i}.$$
   and in the special case where all β_i are the same, we have
   $$R(\mathbf{p}^M(r)) = \frac{1-r}{\beta} \left(\frac{\beta p^M(r)}{1-r} - 1 \right) = \frac{1-r}{\beta} (z^M(r) - 1).$$

   Hence, taking the derivative of $R(\mathbf{p}^M(r))$ with respect to r, is the same as computing the following

$$\frac{\partial R(\mathbf{p}^M(r))}{\partial r} = \frac{\partial \frac{1-r}{\beta}(z^M(r)-1)}{\partial r}$$
$$= \frac{1}{\beta}[(1-r)\frac{\partial z^M(r)}{\partial r} - (z^M(r)-1)].$$

But from previous computations we know that $\frac{\partial z^M(r)}{\partial r} = \dfrac{1}{(1-r)^2}\sum_{i=1}^n \phi_i^*(p^M(r))g_i$.

Therefore, using again that $\frac{1-r}{\beta}(z^M(r)-1) = R(\mathbf{p}^M(r)) = \sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r))p^M(r)$, we have

$$\frac{\partial R(\mathbf{p}^M(r))}{\partial r} = \frac{1}{\beta(1-r)}\left[\sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r))g_i - \sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r))\beta p^M(r)\right]$$
$$= \frac{1}{\beta}\left[\sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r))\left(\frac{g_i - \beta p^M(r)}{1-r}\right)\right],$$

where $\phi_i^*(\mathbf{p}^M(r)) = \dfrac{e^{\frac{g_i-\beta p^M(r)}{1-r}}}{1+\sum_{j=1}^n e^{\frac{g_j-\beta p^M(r)}{1-r}}}$.

For a fixed r we define the following sets:

$$N^-(r) = \{i \in \{1,...,n\} : g_i - \beta p^M(r) \le 0\},$$
$$N^+(r) = \{i \in \{1,...,n\} : g_i - \beta p^M(r) > 0\}.$$

Then we find the following equality

$$\sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r))\left[\frac{g_i - \beta p^M(r)}{1-r}\right] = \sum_{i\in N^-(r)} \frac{e^{\frac{g_i-\beta p^M(r)}{1-r}}\frac{g_i-\beta p^M(r)}{1-r}}{1+\sum_{j=1}^n e^{\frac{g_j-\beta p^M(r)}{1-r}}} + \sum_{i\in N^+(r)} \frac{e^{\frac{g_i-\beta p^M(r)}{1-r}}\frac{g_i-\beta p^M(r)}{1-r}}{1+\sum_{j=1}^n e^{\frac{g_j-\beta p^M(r)}{1-r}}}.$$

We notice that if r is close enough to 1, then for all $i \in N^-(r)$, $e^{\frac{g_i-\beta p}{1-r}}\frac{g_i-\beta p^M(r)}{1-r}$ is a small negative number, on the other hand for $i \in N^+(r)$, $e^{\frac{g_i-\beta p^M(r)}{1-r}}\frac{g_i-\beta p^M(r)}{1-r}$ is positive and can be arbitrarily large when $r \sim 1$. Necessarily there must exists r^* such that

$$\frac{1}{\beta}\left[\sum_{i=1}^n \phi_i^*(\mathbf{p}^M(r^*))\left(\frac{g_i - \beta p^M(r^*)}{1-r^*}\right)\right] > 0,$$

and as $p^M(r)$ is assumed to be decreasing, we can ensure that there will not be another change of monotony. In conclusion, $R(\mathbf{p}^M(r))$ is a strictly increasing function when $r^* < r < 1$.

□

The following example shows a small instance where we can see how the prices, market share and revenue are affected under different values of r.

Example 5.1. *Consider network parameters $r \in (0, 1)$, a price sensitivity $\beta_i = \beta = 0.1$ and intrinsic utilities given by $(g_1, g_2, g_3, g_4, g_5) = (0.9874, 0.6454, 0.4053, 0.2891, 0.03353)$. Figure 5.1 depicts the values of monopolistic price (p^M), the market share of the no purchase option ($\phi^*_{n+1}(\mathbf{p}^M)$) and the highest intrinsic utility product ($\phi^*_1(\mathbf{p}^M)$), and finally the overall revenue ($R(\mathbf{p}^M)$) as a function of r.*
Figure 5.2 shows the different expected revenues (the area of the rectangles) for each value of r, the total demand is defined as the sum of the expected market shares (not including the no purchase option), and the optimal prices are obtained using Theorem 5.1. As it can be observed, for lower values of r the prices are higher but the total demands are lower, the opposite effect is observed when r is close to 1.

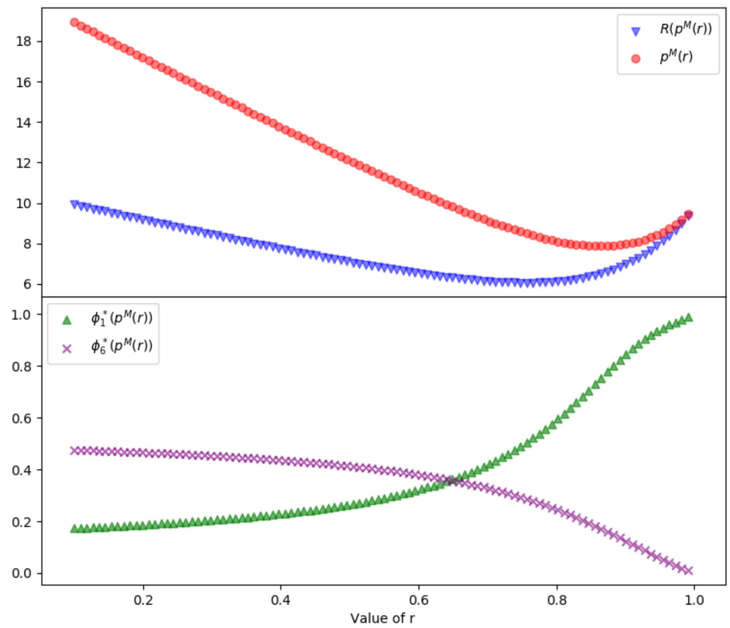

Figure 5.1: In the top figure, $R(\mathbf{p}^M(r))$ and $p^M(r)$ (blue and red respectively) are displayed for different values of the parameter $r : 0 < r < 1$. In the bottom figure, the market shares of the highest intrinsic utility product and the no purchase option are displayed (green and purple respectively).

In Section 5.4 we will study the case where the strategic sellers decide to compete to maximise their individual revenues, inducing a *price competition game*.

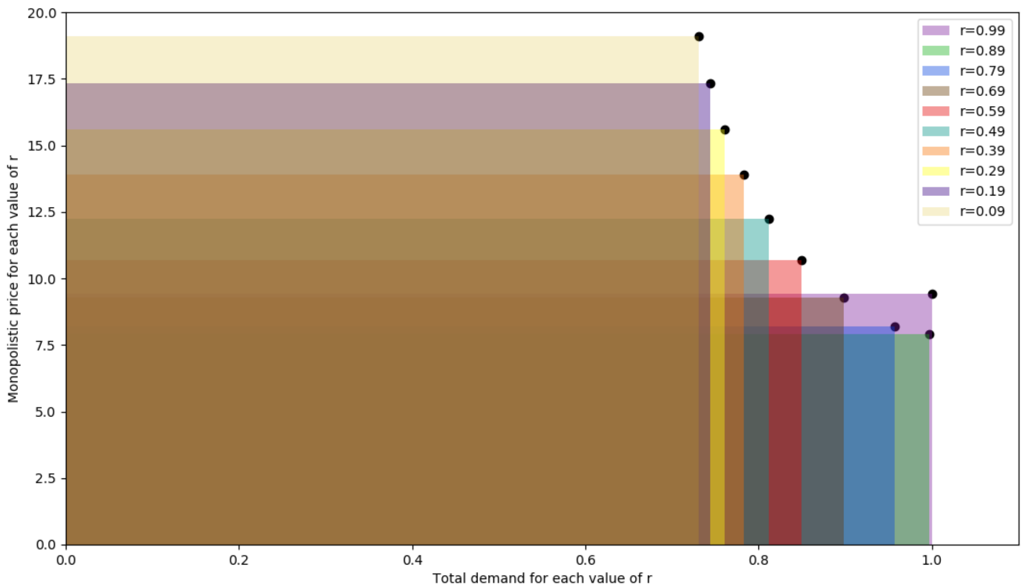

Figure 5.2: In the figure, the X axis represents the total demand (scaled up to 1) for the available products, while Y axis contains the prices. The area of each rectangle corresponds to the expected revenue for each value of r.

5.4 Price competition

We consider a complete information price competition game $\mathcal{G} = (\{1, \ldots, n\}, \mathbf{w}, S)$, where each player (seller) $i \in \{1, \ldots, n\}$ chooses as a strategy a price p_i for his product, in a common strategy space $S_i = [0, \infty)$. Let $S := \prod_{i=1}^{n} S_i = [0, \infty)^n$, and each element $\mathbf{p} \in S$ will be called a strategy profile.

The payoff received by player i after the strategy profile $\mathbf{p} = (p_i, p_{-i}) \in S$ is played, is given by $w_i(p) = p_i \phi_i^*(p)$, where $p_{-i} = (p_1, \ldots, p_{i-1}, p_{i+1}, \ldots, p_n)$. We define the joint payoff as $\mathbf{w} = (w_1, \ldots, w_n)$. Each player chooses the best response to the other sellers' strategies to maximise their payoff, hence our objective is to find a maximiser for $\mathbf{w}$. We consider the important notion of Nash Equilibrium in the following definition.

Definition 5.4. *A strategy profile $\mathbf{p}^* = (p_i^*, \ldots, p_n^*) \in S$ is a pure Nash Equilibrium (NE) for the game $\mathcal{G}$ if for each player i*

$$w_i(p_i^*, p_{-i}^*) \geq w_i(p_i, p_{-i}^*), \forall p_i \in S_i.$$

The following Theorem shows that there exists a unique pure NE for the game $\mathcal{G}$, given in an implicit form using the Lambert W function.

Theorem 5.3. *The price competition game $\mathcal{G}$ has a unique (pure) Nash Equilibrium, $\mathbf{p}^C = (p_1^C, \ldots, p_n^C) \in [0,\infty)^n$, with*

$$p_i^C = \frac{1-r}{\beta_i}\left[W\left(\frac{e^{g_i/(1-r)}}{e + \sum_{j=1, j\neq i}^{n} e^{\frac{g_j - \beta_j p_j^C}{1-r}+1}}\right) + 1\right], \quad \forall i : 1 \leq i \leq n. \tag{5.12}$$

We call $\mathbf{p}^C$, the competitive price.

Proof. We will proceed as follows: first we will show the conditions that the strategy profiles must satisfy in order to be critical points for the vector field $\mathbf{w} = (w_1, \ldots, w_n)$; second we will prove that these conditions are also sufficient, meaning that they describe the best response for each player; third we will show that the system of equations that define the best responses has a unique solution; and finally we will conclude.

Indeed, let us consider a vector $\mathbf{p} \in (0,\infty)^n$ and take the first order derivative of $w_i = p_i \phi_i^*(\mathbf{p})$ with respect to p_i for all $i \in [n+1]$, $i \neq n+1$ (where we are assuming a fixed intrinsic utility vector $\mathbf{g}$ and parameters β_i, r), this is,

$$\begin{aligned}
\frac{\partial w_i}{\partial p_i} &= \phi_i^* + p_i \frac{\partial \phi_i^*}{\partial p_i} \\
&= \phi_i^* + \frac{\beta_i p_i}{1-r}\left[(\phi_i^*)^2 - \phi_i^*\right] \\
&= \phi_i^* \left[1 - \frac{\beta_i p_i}{1-r}(1-\phi_i^*)\right]
\end{aligned}$$

Then $\frac{\partial w_i}{\partial p_i} = 0 \Leftrightarrow p_i = \frac{1-r}{\beta_i(1-\phi_i^*)} \vee \phi_i^* = 0$. Notice that $\phi_i^* = 0 \Leftrightarrow p_i = \infty$. The system of equations that define the possible equilibria are given by the conditions

$$\beta_i p_i = \frac{1-r}{1-\phi_i^*} \quad \text{for all } 1 \leq i \leq n. \tag{5.13}$$

Calling $z_i := \frac{\beta_i p_i}{1-r}$, the normalised price, $c_i := e^{g_i/(1-r)}$ and $M(z) := \sum_{j\in[n+1]} c_j e^{-z_j}$ (with $z_{n+1} = 0, c_{n+1} = 1$), we notice that $M(z)$ has the same value for all sellers $i \in [n+1]$, so in this context can be treated as a constant (for every set of values of prices, $M(z)$ has a fixed value). Equation (5.13) can be rewritten as follows:

$$\begin{aligned}
z_i &= \frac{M(z)}{\sum_{j=0, j\neq i}^{n} c_j e^{-z_j}}, \quad \text{for all } i \in [n] \qquad (5.14) \\
\Leftrightarrow (z_i - 1)e^{z_i} &= \frac{c_i}{\sum_{j=0, j\neq i}^{n} c_j e^{-z_j}} \\
\Leftrightarrow (z_i - 1)e^{z_i - 1} &= \frac{c_i}{\sum_{j=0, j\neq i}^{n} c_j e^{-z_j+1}} \\
\Rightarrow z_i - 1 &= W\left(\frac{c_i}{\sum_{j=0, j\neq i}^{n} c_j e^{-z_j+1}}\right), \quad \text{for all } i \in [n]. \qquad (5.15)
\end{aligned}$$

$W(\cdot)$ here is the Lambert W function. We have obtained a set of conditions that the critical points of $\mathbf{w}$ must satisfy. Moreover, z_i satisfying Equation (5.13) maximises the profit $w_i(z)$, indeed, we consider the second order condition for each function w_i

$$\frac{\partial^2 w_i(z)}{\partial p_i^2} = \frac{\partial}{\partial p_i}(\phi_i^* \left[1 - \frac{\beta_i p_i}{1-r}(1-\phi_i^*)\right]) = \phi_i^*(\phi_i^* - 1)\left[z_i - z_i^2 + 2z_i^2\phi_i^* + \frac{\beta_i}{1-r}\right] \tag{5.16}$$

and if z_i satisfies Equation (5.13), then $\phi_i^* = \frac{z_i - 1}{z_i}$, and replacing this into (5.16) we have

$$\frac{\partial^2 w_i(z)}{\partial p_i^2} = -\frac{z_i - 1}{z_i^2}\left[z_i^2 - z_i + \frac{\beta_i}{1-r}\right] < 0, \quad \forall i \in \{1, \dots, n\}.$$

Then if $\mathbf{p}^* = (p_1^*, \dots, p_n^*)$ is given by Equation (5.13) necessarily, $w(\mathbf{p}^*) \geq w(p_i, p_{-i}^*)$ for all $p_i \in S_i$ for all $i \in N$, this is, p^* is a pure NE.
Now, we claim that there is only one solution to the system of equations (5.14) (for each set of parameters $\mathbf{g}, \boldsymbol{\beta}, r$), defining a unique Nash Equilibrium for the price competition game $\mathcal{G}$.
Clearly the left hand side of (5.14) is increasing in z_i, and the right hand side of (5.14), $y_i(z) := \frac{M(z)}{\sum_{j=0, j\neq i}^{n} c_j e^{-z_j}} \in [1, \infty)$ is decreasing for every z_i, $0 < i \leq n$. Indeed, the denominator of $y_i(z)$ is constant in terms of z_i, and the numerator $M(z) = \sum_{j\in[n+1]} c_j e^{-z_j}$ is decreasing in z_i hence there exists a unique intersection of both curves, defining a vector solution $\mathbf{z}^* = (z_1^*, ..., z_n^*, z_{n+1}^*) = (z_1^*, ..., z_n^*, 0) \in (1, \infty)^n \times \{0\}$. Finally using that $z_i^* = \frac{\beta_i p_i^*}{1-r}$ into Equation (5.15), we find that the unique NE, $\mathbf{p}^C = (p_1^C, \dots, p_n^C)$ is given by

$$p_i^C = \frac{1-r}{\beta_i}\left[W\left(\frac{e^{g_i/(1-r)}}{e + \sum_{j=1, j\neq i}^{n} e^{\frac{g_i - \beta_j p_j^C}{1-r} + 1}}\right) + 1\right], \quad \forall i : 1 \leq i \leq n.$$

□

Remark 5.1. p_i^C is clearly increasing in terms of its associated intrinsic utility g_i (since $W()$ is increasing), and decreasing in terms of the others products' intrinsic utilities $g_j, j \neq i$. Also for all i, $p_i^C > \frac{1-r}{\beta_i}$.

Generally the competitive price for product i, p_i^C, depends on the coordinates of the other prices, thus there is no closed expression for each case. To overcome this issue, we propose the greedy Algorithm 1, to compute the value of the $\mathbf{p}^C$ for any set of parameters $\mathbf{g}, \boldsymbol{\beta}$, and r. We first consider the following definition: given a vector $x = (\phi_1, ..., \phi_n, \phi_{n+1})$, we consider the transformation $\Phi : \mathbb{R}^{n+1} \times \mathbb{R} \times \{1, ..., n\} \to \mathbb{R}^{n+1}$ that changes the i-th coordinate of x by a given real value a, this is, $\Phi_{i,x}(a) := \Phi(x, a, i) = (\phi_1, ..., \phi_{i-1}, a, \phi_{i+1}, ..., \phi_n, \phi_{n+1})$

In Lemma 5.3 we will show that Algorithm 1 always terminates. We will prove it, by exploiting the fixed point structure on how the normalised prices are defined.

Algorithm 1 Find equilibrium z^C given by Equation (5.15).

Require: Parameters: $r, c_i = e^{g_i/(1-r)}, 1 \leq i \leq n$, $\epsilon > 0$; Initial starting point: $z^0 \in \mathbb{R}^n_+ \times \{0\}$;
Ensure: A normalised equilibrium price $z \in \mathbb{R}^n_+ \times \{0\}$.
$z \leftarrow z^0$
repeat
  for $1 \leq i \leq n$ **do**
    $z \leftarrow \Phi_{i,z}(W(\frac{c_i}{\sum_{j=0,j\neq i}^n c_j e^{-z_j+1}}) + 1)$
  end for
until $\sqrt{\sum_{i=1}^n |z_i - (W(\frac{c_i}{\sum_{j=0,j\neq i}^n c_j e^{-z_j+1}}) + 1)|^2} < \epsilon$

Lemma 5.3. *Algorithm 1 is guaranteed to terminate, and its output is the normalised equilibrium price z^C.*

Proof. We consider the sequence $(z^k)_{k\in\mathbb{N}} \in \mathbb{R}^{n+1}$ created by each time the algorithm reaches the step 4, its coordinates are defined by the recurrence:

$$z_i^{k+1} = W(\frac{c_i}{\sum_{j=0,j\neq i}^n c_j e^{-z_j^k+1}}) + 1 \text{ for all } 1 \leq i \leq n, \text{ and } k \in \mathbb{N}. \tag{5.17}$$

$(z^k)_{k\in\mathbb{N}}$ is clearly bounded, hence the Bolzano-Weierstrass Theorem (Theorem 2.3) implies that z^k has a convergent subsequence z^{k_l} with $l \in \mathbb{N}$. Let z^C be the limit of z^{k_l}. As $W(\cdot)$ is a continuous function, we apply the limit when $l \to \infty$ in both sides of Equation (5.17). Necessarily z^C must satisfy that for any $i : 1 \leq i \leq n$, $z_i^C = W(\frac{c_i}{\sum_{j=0,j\neq i}^n c_j e^{-z_j^C+1}}) + 1$, hence Algorithm 1 terminates when it finds the Equilibrium z^C. □

The following Theorem shows the monotonic behaviour of the competitive price $\mathbf{p}^C$ in terms of the network effect parameter $r : 0 < r < 1$.

Theorem 5.4. *The competitive price $\mathbf{p}^C(r) = (p_1^C(r), \ldots, p_n^C(r)) \in [0,\infty)^n$ given by Equation* (5.12) *is decreasing as a function of the network effect parameter $r : 0 < r < 1$.*

Proof. Imposing the first order conditions over each expected revenue function $w_i(\mathbf{p^C(r)})$, gives us Equation (5.13), which is defined in the following way:

$$\beta_i p_i^C(r) = \frac{1-r}{1-\phi_i^*(\mathbf{p^C(r)})} \quad \text{for all } 1 \leq i \leq n,$$

or equivalently:

$$\frac{1-r}{\beta_i p_i^C(r)} = 1 - \phi_i^*(\mathbf{p^C(r)}) \quad \text{for all } 1 \le i \le n$$
$$\Rightarrow (1-r)\sum_{i=1}^{n} \frac{1}{\beta_i p_i^C(r)} = n - 1 + \phi_{n+1}^*(\mathbf{p^C(r)}). \tag{5.18}$$

Notice that as $0 < \phi_{n+1}^*(\mathbf{p^C(r)}) < 1$, then Equation (5.18) implies that

$$\frac{n-1}{1-r} < \sum_{i=1}^{n} \frac{1}{\beta_i p_i^C(r)} < \frac{n}{1-r}.$$

Clearly $\frac{n-1}{1-r}$ and $\frac{n}{1-r}$ are increasing in terms of r (and independent of the intrinsic utility parameters), necessarily $\sum_{i=1}^{n} \frac{1}{\beta_i p_i}$ is increasing, which implies that there exists a product $k \in \{1, \ldots, n\}$ such that $p_k^C(r)$ is decreasing, but by definition of the competitive price, we have

$$p_k^C(r) = \frac{1-r}{\beta_i}\left[W\left(\frac{e^{g_k/(1-r)}}{e + \sum_{j=1, j\neq k}^{n} e^{\frac{g_j - \beta_j p_j^C(r)}{1-r}+1}}\right) + 1\right], \quad \forall k : 1 \le k \le n.$$

Hence if any $p_k^C(r)$ decreases, in order to preserve the equilibrium, all the other coordinates must decrease as well, which proves the result. □

The following example shows the competitive prices for the case of 3 products with fixed intrinsic utilities, a fixed value of price sensitivities and 4 different values of network parameters, r.

Example 5.2. *Consider a set of network parameters given by $r \in \{0.2, 0.4, 0.6, 0.8\}$, and intrinsic utilities given by $(g_1, g_2, g_3) = (0.993, 0.480, 0.159)$, the competitive price equilibria $p^C = (p_1^C, p_2^C, p_3^C)$ are given in the following table.*

r	p_1^C	p_2^C	p_3^C
0.2	9.298	9.912	11.461
0.4	6.900	7.509	9.269
0.6	4.498	5.082	7.243
0.8	2.121	2.581	5.612

5.4.1 Homogeneous case

In this section we study a simplification of the general case where every product presents the same intrinsic utility. This case will allow us to study, from a theoretical point of view, the behaviour of the prices as a function of the network parameter r. We assume in this section that the values $g_i = g$ for all $i : 1 \le i \le n$, and we define for notational convenience $\hat{c} = e^{g/(1-r)}$. The following corollary is a direct consequence of Theorem 5.3 for the case where all products have the same intrinsic utility.

Corollary 5.1. *If all the products have the same intrinsic utility, $g_i = g$ for all $i \in N$, then the competitive price for the homogeneous case, $p^{CH} = (p_1^{CH}, \dots, p_n^{CH})$ is the unique pure NE for the game $\mathcal{G}$, and its coordinates are given by*

$$p_i^{CH} = \frac{1-r}{\beta_i}\left[W\Big(\frac{\hat{c}}{e + \hat{c}(n-1)e^{1-\frac{\beta_i p_i^{CH}}{1-r}}}\Big) + 1\right], \quad \forall 1 \le i \le n. \tag{5.19}$$

Proof. Thanks to Theorem 5.3, we know that the coordinates of the unique NE for the price competition are given by Equation (5.12). Now in the particular case where all the products have the same intrinsic utility, Equation (5.15) gets reduced to

$$z_i - 1 = W\Big(\frac{\hat{c}}{e + \hat{c}\sum_{j=1, j\neq i} e^{-z_j+1}}\Big), \quad \forall i : 1 \le i \le n,$$

which is completely symmetric for each z_i, therefore for all $1 \le i \le n$, it must hold $z_i = z$ for some $z > 1$. Consequently the previous Equation is equivalent to

$$\begin{aligned} z - 1 &= W\Big(\frac{\hat{c}}{e + \hat{c}(n-1)e^{-z+1}}\Big) \\ \Rightarrow p_i &= \frac{1-r}{\beta_i}\left[W\Big(\frac{\hat{c}}{e + \hat{c}(n-1)e^{1-\frac{\beta_i p_i}{1-r}}}\Big) + 1\right]. \end{aligned} \tag{5.20}$$

□

Remark 5.2. Even when the solution for z_i is given by a fixed value $z_i = z$ for all $1 \le i \le n$, the prices p_i can be different, due to the sensitivity parameter β_i. This phenomenon has also been studied in Ezra et al. [2017] where the authors analyse the problem of pricing identical items, that eventually leads to different prices depending on consumption patterns.

The following Theorem states similar properties to Theorem 5.2 but now for the case of the competitive price. We are able to prove some monotonic behaviour of the normalised price, the products' market share, and the market share of the no purchase option when the competitive homogeneous price is used. However, similar properties seem to hold also for the general case (see Example 5.3).

Theorem 5.5. *Under the assumption of homogeneity in the intrinsic utilities (i.e. $g_i = g$ for all $i \in \{1, ..., n\}$) , if we consider a network effect parameter r, $0 < r < 1$, then the following statements hold true:*

1. *The normalised competition price z^{CH} is increasing in terms of r.*
2. *Every product has the same market share $\phi_i^*(p^{CH}) = \phi^*(p^{CH})$ which is increasing in r.*
3. *The market share for the no purchase option, $\phi_{n+1}^*(p^{CH})$, is decreasing as a function of r.*

Proof. 1. We prove first that our normalised price z is decreasing in terms of r. Indeed, we notice that by definition of Lambert W function, Equation (5.20) is equivalent to

$$(z^{CH}-1)e^{z^{CH}}+z^{CH}\hat{c}(n-1)=n\hat{c}.$$

Taking the derivative with respect to r in both sides of the Equation, we find the following:

$$\begin{aligned} z^{CH}\frac{\partial z^{CH}}{\partial r}e^{z^{CH}}+\frac{\partial z^{CH}}{\partial r}\hat{c}(n-1)+z^{CH}(n-1)\frac{\partial \hat{c}}{\partial r}&=n\frac{\partial \hat{c}}{\partial r}\\ \frac{\partial z^{CH}}{\partial r}[z^{CH}e^{z^{CH}}+(n-1)\hat{c}]&=\frac{\hat{c}g[n-(n-1)z^{CH}]}{(1-r)^2}. \end{aligned} \tag{5.21}$$

On the other hand, according to Equation (5.20) we see that

$$z-1=\frac{\hat{c}}{e^{z}+\hat{c}(n-1)}<\frac{\hat{c}}{\hat{c}(n-1)}=\frac{1}{n-1},$$

and then $z<\dfrac{n}{n-1}$. Using this into Equation (5.21) we obtain that $\dfrac{\partial z}{\partial r}>0$, where z is a increasing function of r.

2. We notice that each market share in the equilibrium is given by $\phi_i^*(p^{CH})=\dfrac{\hat{c}e^{-z_i^{CH}}}{1+\hat{c}ne^{-z_i^{CH}}}=\dfrac{\hat{c}}{e^{z^{CH}}+\hat{c}n}:=\phi^*(p^{CH})$ which is independent of i, since all the normalised prices z_i are the same. Taking the derivative of x with respect to r give us the following equalities:

$$\begin{aligned} \frac{\partial \phi^*(p^{CH})}{\partial r}&=\frac{\partial}{\partial r}\left[\frac{\hat{c}}{e^{z^{CH}}+n\hat{c}}\right]\\ &=\frac{\hat{c}\phi^*(p^{CH})}{(1-r)^2}\left[\frac{z^{CH}e^{2z^{CH}}+(z^{CH}(n-1)-1)\hat{c}e^{z^{CH}}}{(z^{CH}e^{z^{CH}}+(n-1)\hat{c})(e^{z^{CH}}+n\hat{c})}\right]>0. \end{aligned}$$

Hence all the market share in the equilibrium are increasing in terms of the network parameter r.

3. Since $\phi_{n+1}^*(p^{CH})=1-\sum_{i=1}^{n}\phi_i^*(p^{CH})=1-n\phi^*(p^{CH})$, and $\phi^*(\cdot)$ is increasing in r, necessarily $\phi_{n+1}^*(p^{CH})$ must be decreasing.

□

Remark 5.3. Numerical simulations have shown us that similar conclusions from Theorem 5.5 in the points 1. and 3. (normalised price increasing and market share for no purchase option decreasing) seem to hold for the general competition case. However, we only have been able to observe it empirically (see for example Figures 5.3 to 5.6 in the Appendix).

The following example shows that Theorem 5.5 part 2. does not necessarily hold when the intrinsic utilities are different, where there are some products whose market shares decrease in terms of r. We also can observe that the market share for the highest intrinsic utility products seems to be increasing.

Example 5.3. *Consider a set of network parameters given byr $\in \{0.2, 0.4, 0.6, 0.8\}$, intrinsic utilities given by $(g_1, g_2, g_3) = (0.993, 0.480, 0.159)$, and a price sensitivity $\beta_i = \beta = 0.1$, the market share for each product and their respective expected revenue for each r are given in the following table.*

r	$\phi_1(p)$	$\phi_2(p)$	$\phi_3(p)$	$w_1(p)$	$w_2(p)$	$w_3(p)$
0.2	0.302	0.193	0.134	3.461	1.912	1.298
0.4	0.352	0.201	0.130	3.269	1.509	0.900
0.6	0.448	0.213	0.111	3.243	1.082	0.498
0.8	0.644	0.225	0.057	3.613	0.581	0.121

As we can observe from Examples 5.2 and 5.3, the highest intrinsic utility product $i = 1$ has in general a decreasing price, and increasing market share, which eventually leads to have a higher revenue when $r = 0.8$. In the following section we will compare the two different pricing strategies, including also the consumer's perspective.

5.5 Monopolistic vs Competitive

In this section we will compare the different pricing schemes where network effects are present, in absolute terms (which prices are higher) and in relative terms from the consumer's perspective. We assume from now on, that the products have the same price sensitivities $\beta_i = \beta$ for all $1 \leq i \leq n$. The following theorem recovers the intuitive result that the monopolistic price is higher than the competitive one.

Theorem 5.6. *For any set of parameters $g_i, i \in \{1, \ldots, n\}$, $0 \leq r < 1$ and $\beta > 0$, the monopolistic price p^M is higher than the competitive price, p_i^C for all $i \in \{1, \ldots, n\}$.*

Proof. We know that according to Equations 5.6, and 5.12, p^M and p_i^C are given respectively by

$$p^M = \frac{1-r}{\beta}\left[W\left(\frac{\sum_{i=1}^n e^{g_i/(1-r)}}{e}\right) + 1\right],$$

$$p_i^C = \frac{1-r}{\beta}\left[W\left(\frac{e^{g_i/(1-r)}}{e + \sum_{j\neq i}^n e^{1+\frac{g_j - \beta p_j^C}{1-r}}}\right) + 1\right].$$

Their respective vector forms are given by: $\mathbf{p}^M = (p^M, \ldots, p^M)$ and $\mathbf{p}^C = (p_1^C, \ldots, p_n^C)$. Comparing both expressions we have that for any set of parameters $g_i, i \in \{1, \ldots, n\}$,

$0 \leq r < 1$ and $\beta > 0$ and for any product $i \in \{1, \ldots, n\}$

$$p^M \geq p_i^C \Leftrightarrow \frac{1}{e}\left[\sum_{i=1}^{n} e^{g_i/(1-r)} - \frac{e^{g_i/(1-r)}}{1 + \sum_{j\neq i}^{n} e^{\frac{g_j - \beta p_j^C}{1-r}}}\right] \geq 0,$$

But,

$$\left[\sum_{i=1}^{n} e^{g_i/(1-r)} - \frac{e^{g_i/(1-r)}}{1 + \sum_{j\neq i}^{n} e^{\frac{g_j - \beta p_j^C}{1-r}}}\right] = \underbrace{e^{g_i/(1-r)}\left(1 - \frac{1}{1 + \sum_{j\neq i}^{n} e^{\frac{g_j - \beta p_j^C}{1-r}}}\right)}_{:=A} + \underbrace{\sum_{j\neq i}^{n} e^{g_j/(1-r)}}_{:=B}.$$

Clearly $B > 0$ and since $e^x > 0$ for any value of x, then for all $i \in \{1, \ldots, n\}$, $A > 0$. Consequently $p^M \geq p_i^C$ as desired. □

The following theorem shows that for any product, the consumer's expected utility obtained from purchasing it, is higher when the competitive price is used instead of the monopolistic price. This result is trivial when there is no network effects ($r = 0$) since the utility is a decreasing function of the price, however if we include the non-linear effect of past purchases the result is not necessarily obvious.

Theorem 5.7. *For any product $i \in \{1, \ldots, n\}$, in the long run, the expected utility perceived by a customer after purchasing product i when the competitive price is used, is higher than the case when the monopolistic price is used.*

Proof. We want to prove that asymptotically $u_i^k(\mathbf{p}^C) - u_i^k(\mathbf{p}^M)$ is strictly positive, with $u_i^k(\mathbf{p})$ given by Equation (5.1). We know that by to Lemma 5.2, $\frac{d_i^k(\mathbf{p})}{k} \xrightarrow[a.s.]{} \phi_i^*(\mathbf{p})$, then

$$\begin{aligned} u_i^k(\mathbf{p}^C) - u_i^k(\mathbf{p}^M) &\xrightarrow[a.s.]{} r[\log(\phi_i^*(\mathbf{p}^C)) - \log(\phi_i^*(\mathbf{p}^M))] - \beta(p_i^C - p^M) \\ &= r\log\left[\frac{\phi_i^*(\mathbf{p}^C)}{\phi_i^*(\mathbf{p}^M)}\right] + \beta(p^M - p_i^C). \end{aligned}$$

According to Theorem 5.6, we know that $\beta(p^M - p_i^C) > 0$, on the other hand $\phi_i^*(\mathbf{p}) = \frac{e^{g_i/(1-r)}}{e^{\frac{\beta p_i}{1-r}} + \sum_j e^{g_j/(1-r)} e^{\frac{\beta(p_i - p_j)}{1-r}}}$, which is clearly decreasing in terms of p_i, then as $p^M > p_i^C$ for all $i \in \{1, .., n\}$, necessarily $r\log\left[\frac{\phi_i^*(\mathbf{p}^C)}{\phi_i^*(\mathbf{p}^M)}\right] > 0$ for all i, meaning that $r\log\left[\frac{\phi_i^*(\mathbf{p}^C)}{\phi_i^*(\mathbf{p}^M)}\right] + \beta(p^M - p_i^C) > 0$ as desired. □

The structure of the market share in the equilibrium (Equation(5.4)) implies that the highest market share would be assigned to the product with highest value of $\frac{g_i - \beta_i p_i}{1-r}$ which at least for the competitive price, $\mathbf{p}^C$ is a increasing function of r, meaning that in general, the consumer's utility associated to the product with highest intrinsic utility ($i = 1$), increases as r approaches to 1. The following example depicts this effect.

Example 5.4. *Consider a large enough amount of customers such that for any product $i \in \{1, \ldots, n\}$, $\frac{d_i^k}{k+1} = \phi_i^*(\mathbf{p})$, where $\phi_i^*(\mathbf{p})$ is the market share in the equilibrium (see Equation*(5.4)*) under a price p (competitive Nash Equilibrium and/or monopolistic price). Customer $k+1$ will then choose strategically a product $j = j(q, r, \beta, p, \xi)$ that maximises his expected utility of purchasing any product (or he will choose the no purchase option), this is, using formula* (5.1)*, we have*

$$j \in \underset{0 \leq i \leq n}{\text{arg-max}}\, \mathbb{E}[g_i + r \ln(d_i^k) - \beta p_i + \xi_i].$$

Where ξ_i, $i \in \{1, \ldots, n, n+1\}$ were chosen to be i.i.d random variables following a Gumbel distribution, in particular we have that $\mathbb{E}[\xi_i - \xi_j] = 0$ for all pairs $i, j \in \{1, \ldots, n, n+1\}$. Let $v_j(\mathbf{p}) := u_j(\mathbf{p}) - \xi_j$ and consider the following parameters: $\beta_i = \beta = 0.1$, $k = 10M$, and intrinsic utilities given by $(g_1, g_2, g_3) = (0.993, 0.480, 0.159)$ The following table summarises how the expected utilities, $\mathbb{E}[v_j]$, behave under different values of r. The second and third columns show which product, j is the one that maximises the expected utility, under the competitive and monopolistic pricing (j^C and j^M respectively). The fourth and fifth column show the respective competitive and monopolistic prices for those products. Finally, the last two columns show the expected values of v_{j^C} and v_{j^M} respectively.

r	j^C	j^M	$p^C_{j^C}$	p^M	$\mathbb{E}[v_{j^C}(\mathbf{p}^C)]$	$\mathbb{E}[v_{j^M}(\mathbf{p}^M)]$
0.2	1	1	11.461	15.498	2.831	2.395
0.4	1	1	9.269	12.523	6.097	5.721
0.6	1	1	7.243	9.798	9.458	9.167
0.8	1	1	5.613	7.934	12.974	12.791

Appendix

We present here some extra experimental results depicting the different behaviour of both pricing schemes (competitive price against monopolistic price). We use the following parameters: $g = (g_1, ..., g_5) = (0.850, 0.733, 0.416, 0.256, 0.139)$, $\beta_i = \beta = 0.1$.

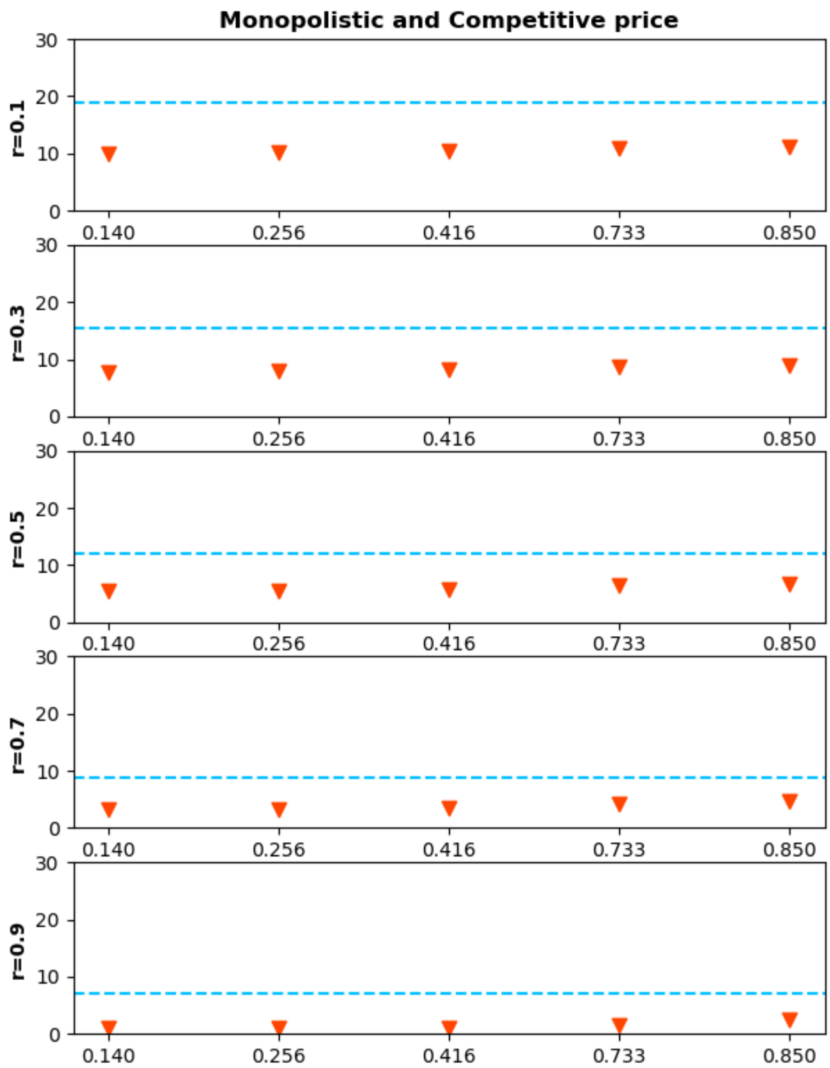

Figure 5.3: Comparison of prices, competition versus monopoly respect to the products' intrinsic utility (X axis). The red triangles are the competitive prices (NE) for each product, and the blue dotted line is the monopolistic price

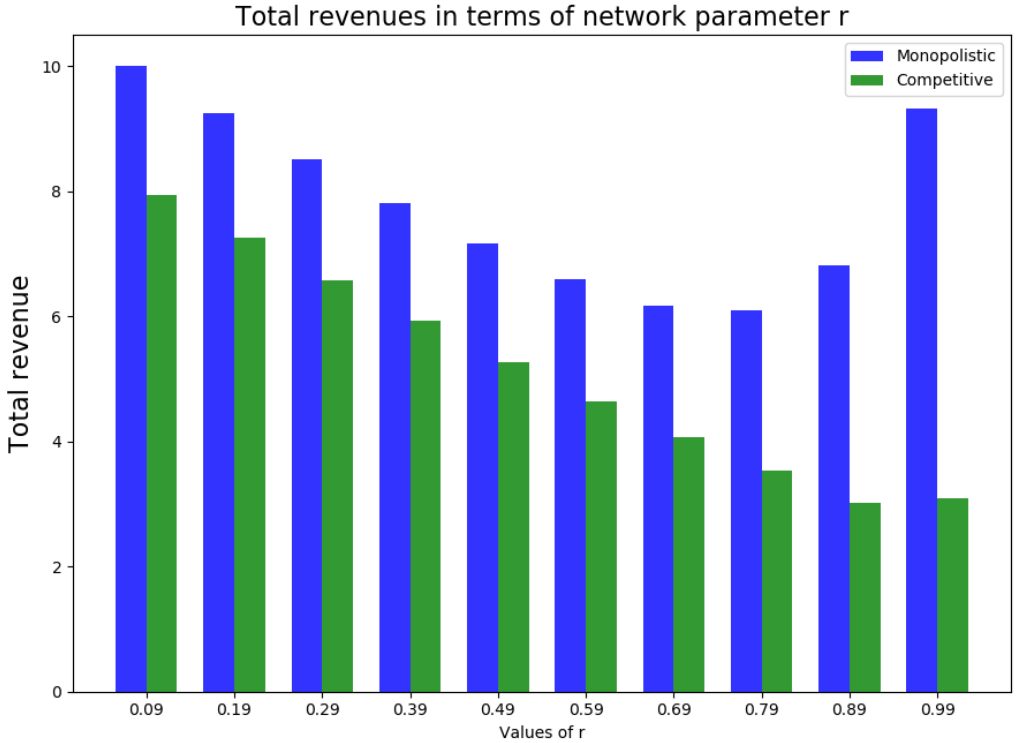

Figure 5.4: Comparison of total revenue perceived by the sellers: competition versus monopoly

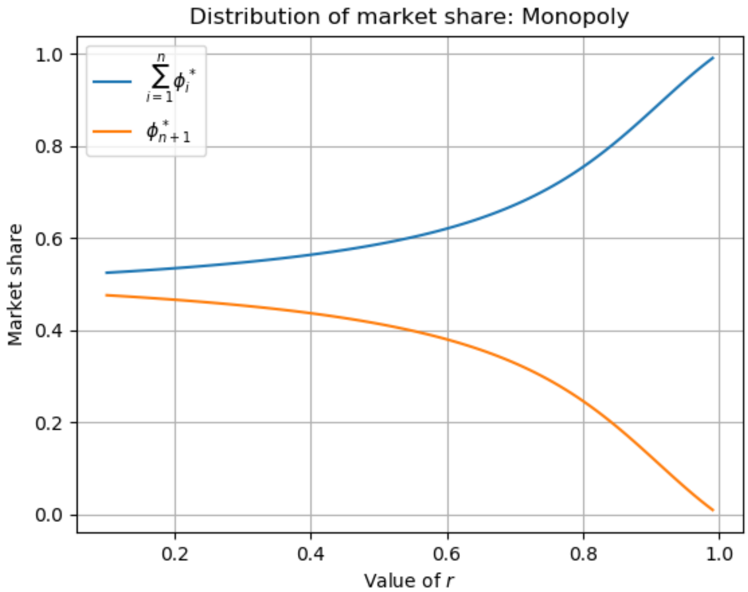

Figure 5.5: Comparison of total market shares assigned in the equilibrium for different values of r, and the respective market share for the no purchase option, when the monopolistic price is used.

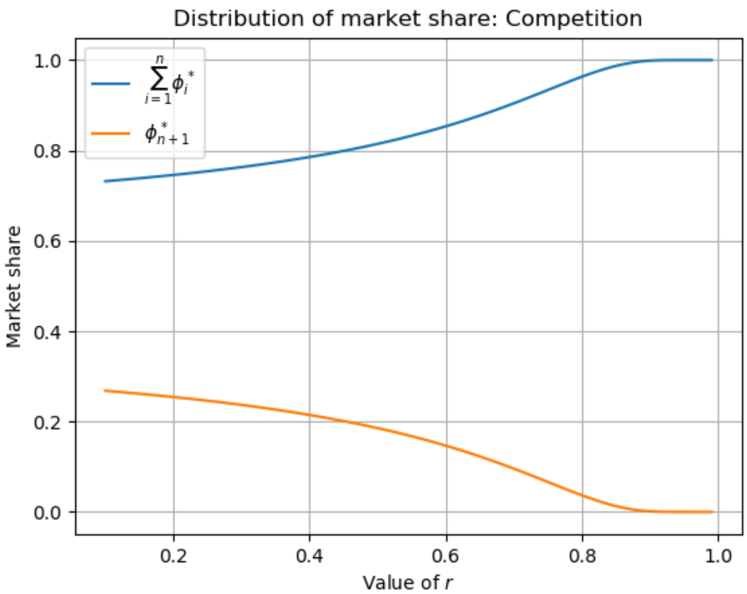

Figure 5.6: Comparison of total market shares assigned in the equilibrium for different values of r, and the respective market share for the no purchase option, when the competitive price is used.

Chapter 6

Conclusion

In this thesis we studied models for consumer choice based on a version of the Multinomial Logit model, where we incorporate effects of social influence and position biases. We also studied pricing strategies under these models.

In Chapters 3 and 4 we studied Trial-Offer cultural markets, which are ubiquitous in our societies and involve products such as books, songs, videos, clothes, and even newspaper articles. In these markets, participants are presented with products in a certain ranking. They can then try the products before deciding whether to purchase them or not. Social influence signals are widely used in such settings and help promote popular products to maximise market efficiency. However, it has been argued that social influence makes these markets unpredictable Salganik et al. [2006]. As a result, social influence is often presented in a negative light.

In this thesis, we have reconsidered this conventional wisdom. We have shown that, when products are presented to participants in a way that reflects their true quality, the market is both efficient and predictable. In particular, in Chapter 3 both, quality and performance rankings make the market to converge to a monopoly for the highest quality product, making the market both optimal and predictable asymptotically.

With the objective of remaining predictable, but with a better distributed market (different than a monopoly for a particular product) Chapter 4 studied how choice behaviour is affected under a family of social signals, when the products are presented using a static ranking (such as the quality ranking). The main result of that Chapter is to show that trial-offer markets, when the ranking of the products is fixed, converge to a unique equilibrium for sublinear social signals of the form $\phi_i^r, 0 < r < 1$, where ϕ_i represents the cumulative market share of product i. Of particular interest is the fact that the equilibrium does not depend on the initial conditions, e.g., the product appeals, but only depends on the product qualities. Moreover, when the products are ranked by quality, i.e., the best products are assigned the highest visibilities, the equilibrium is such that the better products receive the largest market shares, which increase as r increases for the best products (as long as $r < 1$). The equilibrium for a sublinear social signal contrasts with the case with $r = 1$, where the market goes to a monopoly for the highest quality product. In the sublinear case ($0 < r < 1$), the market shares reflect product quality but no product becomes a monopoly. The chapter also shows that, when $r > 1$, the market becomes more unpredictable. In particular, the *inner equilibrium*, which assigns a strictly positive market share to all products, is unstable and the market is likely to converge to a monopoly for some product. However, which product becomes the monopoly depends on the initial conditions, and early market interactions.

Simulation results on a setting close to the original MusicLab complemented the theoretical results. They show that the market converges quickly to the equilibrium for a sublinear social signal and that the convergence speed depends on the social signal strength. The simulation results also illustrate how the market shares of the highest (resp. lowest) quality products increase (resp. decrease) with r. As expected, when $r \leq 1$, the market is shown to be highly predictable, while it exhibits a lot of randomness when $r > 1$. The simulation results also show the benefits of social influence for market efficiency, and demonstrate that the quality ranking once again outperforms the popularity ranking.

Overall, these results shed a new light on the role of social influence in trial-offer markets and provide a comprehensive overview of the choices and trade-offs available to firms interested in optimising their markets with social influence. In particular, they show that social influence does not necessarily make markets unpredictable and is typically beneficial when the social signal is not too strong. Moreover, ranking the products by quality appears to be a much more effective policy than ranking products by popularity which may induce unpredictability and market inefficiency. The results also show that sublinear social signals give decision makers the ability to trade market efficiency for more balanced market shares.

Perhaps, the main contribution of this two chapters is to show that markets under social influence are very sensitive to various design choices. The findings in Salganik et al. [2006] used the popularity ranking, which significantly affected their conclusions about market unpredictability and efficiency. The theoretical and simulation results exposed here, show that the market is highly predictable when using any static ranking and $r \leq 1$. Moreover, the quality ranking is optimal asymptotically when $r = 1$ and dominates the popularity ranking in all our simulations which were modelled after the MusicLab. This does not diminish the value of the results by Salganik et al. [Salganik et al., 2006] who isolated potential pathologies linked to social influence. But this chapter shows that these pathologies are not inherent to the market but are a consequence of specific design choices in the experiment: the strength of the social signal and the ranking policy. Interestingly, it is only for a linear social signal that social influence can be shown to be always beneficial in expectation. Fortunately, for sublinear social signals, we can determine a priori if social influence is beneficial, given the analytic form of the equilibrium.

A key quantity in the model presented here is the value of r, which drives different dynamics and market behaviour. The original approach of this thesis was that the value of r could be decided by the market maker, displaying for example the square root of the downloads as a social signal. However, an equivalent approach would be to study for a particular market (with social signals), how the purchases evolve over time and find the best r that could represent this evolution. We could proceed as follows: set up a time frame (e.g., a month), for each new arrival collect the following data

[time, ranking, cumulative-download, product-clicked, purchased? yes/no,], And based on a number of observations we could estimate the best curve that fits the evolution of the purchases (linear, sublinear growth,...). Using for this purpose tools such as (max) log-likelihood.

Finally in Chapter 5 we have designed a model for consumer choice, based on a MNL model with non-linear network effects. We studied a multi-seller pricing problem where sellers can collaborate or compete, finding in each case a unique equilibrium price (monopolistic price and Nash Equilibrium respectively). We also studied the monotonic behaviour of the market shares, prices and revenues in terms of the network parameter r (both theoretically and numerically). We finally compared both pricing strategies from the consumer's perspective, recovering for our model some well known results from the traditional MNL, such as that the monopolistic price is higher than the competitive one, and that the utility perceived by the consumers is higher when the competitive price is used. We also analysed numerically how increasing the network parameter r generates higher utilities for the consumer.

Some interesting questions remain open, for example, the revenue for the highest intrinsic utility product, w_1, in the competitive case seems to increase with the value of r, as long as r is large enough, however we still do not have a formal proof or characterisation of this phenomenon. It would be also interesting to study the optimal pricing decisions when $r \geq 1$. Answering those questions would help to find the best value of r such that both consumers and sellers are benefited from network effects.

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