Distributed Orientation Localization of Multi-agent Systems in 3-dimensional Space with Direction-only Measurements

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***Abstract*— When a group of agents such as unmanned aerial vehicles are operating in 3-dimensional space, their coordinated action in pursuit of some group objective generally requires all agents to share a common coordinate frame or orientations of the coordinate axes of agents up to an unknown coordinate rotation common to all agents, which are simply referred to as having *common coordinate axis orientations*. Given coordinate axes that are initially unaligned, this paper considers the process of using direction measurements between agent pairs (obtained in their own coordinate frames) to achieve orientation localization, i.e. determination of common coordinate axis orientations, the calculations all being distributed. The process builds on the initial determination of relative orientations of agent pairs in a common coordinate basis. Distributed differential equations then allow determination of a common set of coordinate axis orientations, uniquely up to a common rotation transformation, which can itself be determined if and only if one or more agents have access to global coordinates.**

I. Introduction

Many formation control problems involving e.g. unmanned aerial vehicles require substantial coordination between the vehicles that is only possible when the agents have a common understanding of certain physical quantities [1]–[4]. For example, a very common requirement is for all agents to have the same view as to the directions of north/south, east/west, and up/down, i.e. the coordinate frames in which each agent views the world are required to be aligned, (or equivalently, the orientation of the agent's coordinate frame with respect to global coordinates needs to be determined). However, unless agents are equipped with e.g. a compass sensor, this may not be possible. Indeed, while most agents are likely to have sensors to allow inertial navigation, even if the frames are aligned at $t = 0$, the inevitable drift means that after some finite period, alignment of frames cannot be assumed.

Without the common orientation information, either global orientation estimation [2] or coordinate frame alignment [3], [4] is required to achieve the target formation. The former method provides a global convergence with more communications. There has been considerable past work dealing with this problem; such orientation estimation schemes have attracted interest recently, see e.g. [2], [5], [6]. The algorithms are attractive when they exhibit global convergence, as is typical, or is at least almost always the case. In [2], the authors propose an orientation estimator by defining a vector auxiliary variable for each agent. They further extend the orientation estimation method for arbitrary dimensional spaces, which still guarantees almost global convergence, by using relative orientation information [5]. When the relative orientation measurements are affected by noise, an efficient maximum likelihood estimator, which is locally optimal, is proposed in [6]. In such orientation estimation schemes, the orientations of the coordinate axes of all agents are estimated up to an unknown coordinate rotation common to all agents. The orientations of the coordinate axes of agents up to an unknown coordinate rotation common to all agents are simply called *common coordinate axis orientations*, and the process of determining the common coordinate axis orientations is referred to as *orientation localization*.

The unknown coordinate rotation just mentioned can be determined if there exists at least one agent that knows the true orientation of its coordinate axes. This allows the orientations of the coordinate axes of other agents to be precisely computed by utilizing kinematic relationships between the agents, a process which may be called *absolute orientation localization*.

For a two-dimensional (2-D) ambient space, orientation localization laws using angles of arrival between triplets of nodes are proposed in [8] and an orientation localization method utilizing orientation knowledge of a few nodes is presented in [7]. The authors in [9] further proposed a least-squared optimization problem to achieve orientation localization by exploiting kinematic relationships among the orientations of nodes. In 3-dimensional space (3-D), some necessary and sufficient conditions are provided for orientation localizability of triangular sensing networks in [9], without providing a distributed orientation localization law. Orientation localization schemes using relative orientations, which are measured by a vision-based technique, are investigated in [10], [11].

In 2-D, it is straightforward to see how two neighboring agents observing each other might determine a common view of their relative orientation, within an unknown constant rotation common to both, see e.g. [2], [3]. Each agent maintains a (possibly body-fixed) coordinate frame and measures the orientation of its neighbour agent (assuming direction sensing technology). In any common frame, the measured angles must differ by precisely π radians. Hence a rotation of the coordinate axes of one agent can be made to ensure that after rotation, the difference is overcome. For an n agent network, one has to put together in a distributed fashion a

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collection of such calculations.

How to do something like this in a 3-dimensional ambient space is less clear, and the presentation of an algorithm is one of the contributions of this paper. The key is to use not a single pair of inter-agent direction measurements, but rather direction measurements between three agents forming a triangle (a 'triangular sensing network'). The proposed method can then be extended to the n agent case for many graphs by using an analog of Henneberg extensions [13]. The computations involving a triangular sensing network are closely related to the work of [12], but in distinction from that work, we calculate the relative orientation by defining some basic rotations which align two associated coordinate frames, before extending the process to n agent systems in a manner allowing determination of consistent orientations for all agents simultaneously.

There are in fact three basic steps in our procedure;

(a) For a given pair of agents i and j observing the direction of each other, we compute two rotations whose composition aligns the x-axes of the two coordinate frames.

(b) Using direction measurements to and from a third agent k, a third rotation is determined which achieves the desired alignment of the other two coordinate axes and provides the rotation matrix or relative orientation matrix, call it $\mathbf{R}_{ij}$, which transforms the directions of the original coordinate axes of one agent into the directions of the original coordinate axes of the other.

(c) The set of such relative orientation matrices is then used in a differential equation to determine a consistent set of orientations defined by rotation matrices $\mathbf{R}_i$, the orientations being relative to some presumed global coordinate basis; the orientations are only known up to a common constant rotation, which will be determinable if at least any one agent has access to global coordinates.

Unsurprisingly, in order for the above steps to proceed satisfactorily, certain graphical conditions need to be satisfied. Figure 1 depicts, necessarily with oversimplification, much of the above procedure. The relative orientation computation referred to in the caption can only be determined when a third agent is involved (see Fig 2a). The third step is analogous to a method proposed in our previous work [5], which is extended here to a matrix form with matrix auxiliary variables.

This paper is organized as follows. Section II contains preliminaries and problem formulation. The procedure for calculating relative orientation using direction-only information in 3-D is presented in Section III. In Section IV, we propose two orientation localization schemes by using relative orientation information. Finally, concluding remarks are given in Section V.

II. Preliminaries and Problem Formulation

In this paper we use the following notations. Given two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$, their dot product is denoted by $\mathbf{x} \cdot \mathbf{y}$ and $(\mathbf{x}, \mathbf{y})$ denotes the plane defined by $\mathbf{x}$ and $\mathbf{y}$. The symbol Σ represents a global coordinate frame and the symbol ${}^k\Sigma$ with the superscript index k denotes the k-th local coordinate frame. Let $\mathbf{1}_n = [1, \ldots, 1]^T \in \mathbb{R}^n$ be the vector of all ones, and $\mathbf{I}_d$ denotes the $d \times d$ identity matrix. Let $\otimes$ be the Kronecker product. The set of rotation matrices in $\mathbb{R}^d$ is denoted by $SO(d) = \{\mathbf{Q} \in \mathbb{R}^{d\times d} \mid \mathbf{Q}\mathbf{Q}^T = \mathbf{I}_d, det(\mathbf{Q}) = 1\}$.

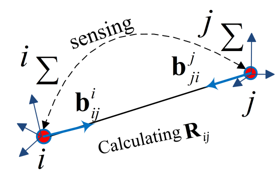


Fig. 1: The agents i and j respectively measure the directions $\mathbf{b}^i_{ij}$ and $\mathbf{b}^j_{ji}$ in local coordinate frames. Using these measurements, they would like to decide the relative orientation $\mathbf{R}_{ij}$. Then, with the calculated $\mathbf{R}_{ij}$, they would like to decide the orientations $\mathbf{R}_i$ and $\mathbf{R}_j$.

A. Directional vector and orientation of agent

Consider n single-integrator modeled agents in d-dimensional space

$$\dot{\mathbf{p}}^i_i = \mathbf{u}^i_i, \; i = 1, \ldots, n, \tag{1}$$

where $\mathbf{p}^i_i \in \mathbb{R}^d$ and $\mathbf{u}^i_i \in \mathbb{R}^d$ denote the position and control input of agent i, respectively, expressed in its body-fixed coordinate frame ${}^i\Sigma$. We define the unit directional vector pointing from agent i toward its neighbor j along the direction of $\mathbf{p}_{ij}$ ($\mathbf{p}_{ij} = \mathbf{p}_j - \mathbf{p}_i$) as

$$\mathbf{b}_{ij} \triangleq \frac{\mathbf{p}_j - \mathbf{p}_i}{\|\mathbf{p}_j - \mathbf{p}_i\|} = \frac{\mathbf{p}_{ij}}{\|\mathbf{p}_{ij}\|}.$$

The directional vector with the reverse direction is $\mathbf{b}_{ji} = -\mathbf{b}_{ij}$ which points from agent j toward i. The direction from agent i to j expressed in ${}^i\Sigma$ is denoted as $\mathbf{b}^i_{ij}$.

Orientation or attitude of agent i in $\mathbb{R}^d$ can be characterized by a square, orthogonal matrix $\mathbf{R}_i \in SO(d)$ whose column vectors represent the coordinates of the orthogonal bases of the i-th local coordinate frame expressed in the global coordinate frame. Thus, $\mathbf{R}_i$ can be understood as the rotation matrix which rotates the global coordinate system, i.e., Σ, to the local coordinate frame ${}^i\Sigma$.

B. Graph theory

An interaction graph characterizing an interaction topology of a multi-agent network is denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, where, $\mathcal{V} = \{1, \ldots, n\}$ denotes the vertex set and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ denotes the set of edges of $\mathcal{G}$. An edge is defined by the ordered pair $e_k = (i, j), k = 1, \ldots, m, m = |\mathcal{E}|$. The graph $\mathcal{G}$ is said to be undirected if $(i, j) \in \mathcal{E}$ implies $(j, i) \in \mathcal{E}$, i.e. if j is a neighbor of i, then i is also a neighbor of j. If the graph $\mathcal{G}$ is directed, $(i, j) \in \mathcal{E}$ does not necessarily imply $(j, i) \in \mathcal{E}$. The set of neighboring agents of i is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$. The Laplacian matrix $\mathbf{L} = [l_{ij}]$ associated with $\mathcal{G}$ is defined as $l_{ij} = -1$ for $(i, j) \in \mathcal{E}, \; i \neq j$, $l_{ii} = -\sum_{j\in\mathcal{N}_i} l_{ij}, \; \forall i = 1, \ldots, n$, and $l_{ij} = 0$ otherwise. The graph $\mathcal{G}$ contains a rooted-in spanning tree if there exists at least one node reachable by at least one directed path from every other nodes.

C. Gram-Schmidt orthonormalization procedure (GSOP)

For a set of d independent vectors $\mathcal{Z} = \{\mathbf{z}_1, \ldots, \mathbf{z}_d\}$in $\mathbb{R}^d$, the Gram-Schmidt orthonormal process (GSOP), which constructs d orthonormal column vectors of $\mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_d] \in \mathbb{R}^{d\times d}$ from $\mathcal{Z}$, is defined as follows

$$\begin{aligned}
&\mathbf{v}_1 := \mathbf{z}_1, && \mathbf{q}_1 := \mathbf{v}_1/\|\mathbf{v}_1\|,\\
&\mathbf{v}_2 := \mathbf{z}_2 - \langle \mathbf{z}_2, \mathbf{q}_1\rangle \mathbf{q}_1, && \mathbf{q}_2 := \mathbf{v}_2/\|\mathbf{v}_2\|,\\
&\quad\cdots && \quad\cdots\\
&\mathbf{v}_d := \mathbf{z}_d - \textstyle\sum_{k=1}^{d-1}\langle \mathbf{z}_d, \mathbf{q}_k\rangle \mathbf{q}_k, && \mathbf{q}_d := \alpha\mathbf{v}_d/\|\mathbf{v}_d\|,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the inner product, and the coefficient α is chosen such that $det(\mathbf{Q}) = +1$ as $\alpha := sign\left(det([\mathbf{q}_1, \ldots, \mathbf{q}_{d-1}, \mathbf{v}_d/\|\mathbf{v}_d\|])\right)$.

Remark 1: The orthonormality of column vectors of $\mathbf{Q}$ and $det(\mathbf{Q}) = +1$ imply that $\mathbf{Q} \in SO(d)$. In $\mathbb{R}^3$, $\mathbf{Q}$ contains coordinates of bases of a right-handed Cartesian coordinate frame.

D. Problem formulation

Let $\mathbf{R}_k \in SO(3)$ be the orientation of agent k, for all $k = 1, \ldots, n$. Let $\mathbf{R}_{ij}$ be the relative orientation of j-th local coordinate frame ${}^j\Sigma$ with respect to the i-th local coordinate frame ${}^i\Sigma$ which is calculated by

$$\mathbf{R}_{ij} = \mathbf{R}_i^{-1}\mathbf{R}_j = \mathbf{R}_i^T\mathbf{R}_j. \quad (2)$$

For any nonzero vector $\mathbf{x} \in \mathbb{R}^3$ we have the following relationship

$$\mathbf{x}^k = \mathbf{R}_k^T\mathbf{x},\ \forall k = 1, \ldots, n. \quad (3)$$

We first address the problem of calculating the relative orientation $\mathbf{R}_{ij}$ based on direction information.

Problem 1: For two neighboring agents i and j in $\mathbb{R}^3$ possessing common neighbors, compute the relative orientation, i.e., $\mathbf{R}_{ij}$, based on direction information $\mathbf{b}_{ij}^i$ and $\mathbf{b}_{ji}^j$, and for some $k \in \mathcal{N}_i \cap \mathcal{N}_j$, the directions $\mathbf{b}_{ik}^i$ and $\mathbf{b}_{jk}^j$.

It is noteworthy that in *Problem 1* the two agents i and j and a common neighboring agent, k, form a *triangular sensing network* (see Fig. 2a). In the following section, we will show that the relative orientation $\mathbf{R}_{ij}$ can be calculated from direction-only information of the *triangular sensing network* by deriving some basic rotations which align two associated coordinate frames.

For a general n-agent system, suppose that the interaction topology is given by a connected undirected graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Specifically, if $(i, j) \in \mathcal{E}$, agent i computes the relative orientation $\mathbf{R}_{ij}$ by using $\mathbf{b}_{ij}^i, \mathbf{b}_{ji}^j, \mathbf{b}_{ik}^i$, and $\mathbf{b}_{jk}^j$ of a triangular sensing network, which includes i, j, and the third agent k (see Fig. 2a). Note that agent j needs to send the information $\mathbf{b}_{ji}^j$ and $\mathbf{b}_{jk}^j$ to agent i. For each agent $i \in \mathcal{V} = \{1, \ldots, n\}$, the second task formalized below in *Problem* 2 is to determine its coordinate frame orientation, which is denoted as $\hat{\mathbf{R}}_i \in \mathrm{SO}(3)$, by using the relative orientations $\mathbf{R}_{ij}$, $j \in \mathcal{N}_i$, and auxiliary matrices, which will be defined in the following section, communicated from neighbor agents j of agent i. Note that since the graph $\mathcal{G}$ is connected and undirected, it contains at least one rooted-in spanning tree.

The second problem investigated in this work is now formally stated.

Problem 2: Consider a system of n agents whose interaction graph is undirected and connected. Using the relative orientation information (2), design an orientation localization law to compute quantities $\hat{\mathbf{R}}_i \to (\mathbf{Q}^\infty)^T\mathbf{R}_i$ where $\mathbf{Q}^\infty \in \mathrm{SO}(3)$ is an unknown rotation matrix common to all agents.

Remark 2: It is evident that the objective of the orientation localization law in the *Problem 2* is to estimate $\mathbf{R}_i$, up to a common coordinate rotation $(\mathbf{Q}^\infty)^T$, $\forall i = 1, \ldots, n$, an objective called *relative orientation localization* in this paper.

Finally, we study an absolute orientation localization problem based on relative orientation information and global orientation knowledge of one or more anchor nodes.

Problem 3: For a system of n agents whose interaction graph $\mathcal{G}$ is undirected and connected, design an absolute orientation localization law such that $\hat{\mathbf{R}}_i \to \mathbf{R}_i$ $(i = 1, \ldots, n)$, asymptotically, by using relative orientations (2) and global orientation knowledge of one or more anchor nodes.

III. Relative Orientation Calculation using Direction-only Measurements

In this section, we obtain inter-neighbor relative orientation from some kinematic and geometrical relationships, by which two neighboring agents' coordinate systems can be aligned. Consider a directional vector connecting two neighboring agents. Let $\mathbf{Q}_{\mathbf{b}_{ij}} \in \mathrm{SO}(3)$ be a rotation about the directional vector $\mathbf{b}_{ij}$ which preserves coordinates of the directional vector. Then, we have

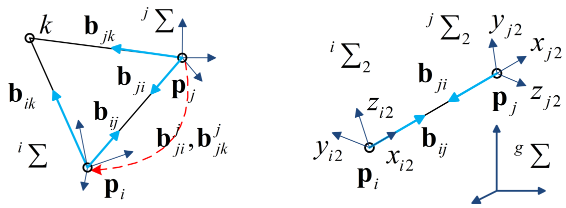
$$\mathbf{Q}_{\mathbf{b}_{ij}}\mathbf{b}_{ij} = \mathbf{b}_{ij},$$

from which it is evident that $\mathbf{b}_{ij}$ is an eigenvector of $\mathbf{Q}_{\mathbf{b}_{ij}}$ corresponding to the eigenvalue $\lambda = 1$. Consequently, the coordinate frame ${}^i\Sigma$ is determined up to the coordinate rotation $\mathbf{Q}_{\mathbf{b}_{ij}}$ with regard to the given direction $\mathbf{b}_{ij}$. Figure 2b graphically illustrates the ambiguity when aligning two inter-neighbor coordinate systems by using only a pair of directional vectors, $\mathbf{b}_{ij}$ and $\mathbf{b}_{ji}$. That is, even if the x-axes of the two neighboring coordinate systems are transformed to be aligned with $\mathbf{b}_{ij}$ and $-\mathbf{b}_{ji}$, respectively, the other axes are not (at least normally) aligned with each other. So, when the x-axes are aligned, the relative orientation of the resulted coordinate systems is determined by a rotation about the directional vector (see Fig. 2b). Consequently, to secure an alignment between two coordinate systems, we need to have one more rotation. The following subsections provide a procedure to align two coordinate systems by consecutive basic rotations.

A. Alignment between x-axis and directional vector

To compute the two successive basic rotations [18] which align the x-axis of the coordinate frame ${}^i\Sigma$ with the direction $\mathbf{b}_{ij}$, we first decompose $\mathbf{b}_{ij}$ into $\mathbf{b}_{ij} = \mathbf{b}_{ij||} + \mathbf{b}_{ij\perp}$ such that $\mathbf{b}_{ij||}$ lies in (x_i, y_i) plane and $\mathbf{b}_{ij\perp}$ is perpendicular to the plane (x_i, y_i) (see Fig. 3a). Let $\left\{\mathbf{e}_{x_i}^i, \mathbf{e}_{y_i}^i, \mathbf{e}_{z_i}^i\right\}$ be the standard basis of ${}^i\Sigma$; then $\mathbf{b}_{ij||}^i$ and $\mathbf{b}_{ij\perp}^i$ can be defined as

$$\mathbf{b}_{ij\perp}^i = [(\mathbf{b}_{ij}^i)^T\mathbf{e}_{z_i}^i]\mathbf{e}_{z_i}^i,\ \text{and}\ \mathbf{b}_{ij||}^i = \mathbf{b}_{ij}^i - \mathbf{b}_{ij\perp}^i.$$



(a) Direction-based sensing and communication (b) Orientation alignment based on directional vectors

Fig. 2: Direction-based relative orientation measurement in 3-D.

The signed angle θ_i between the $\mathbf{e}_{x_i}$ and $\mathbf{b}_{ij||}$ is obtained as

$$\theta_i = atan2\big((\mathbf{e}^i_{x_i} \times \mathbf{b}^i_{ij||}) \cdot \mathbf{e}^i_{z_i}, \mathbf{e}^i_{x_i} \cdot \mathbf{b}^i_{ij||}\big). \tag{4}$$

If ${}^i\Sigma_1$ is obtained from ${}^i\Sigma$ by the rotation $\mathbf{Q}_z(\theta_i)$ with the rotation matrix $\mathbf{Q}_z(\theta_i)$ that is determined as in Fig. 3b, then the signed angle ψ_i between two vectors $\mathbf{b}_{ij||}$ and $\mathbf{b}_{ij}$ is given as

$$\psi_i = atan2\left((\mathbf{e}^{i1}_{x_{i1}} \times \mathbf{b}^{i1}_{ij}) \cdot \mathbf{e}^{i1}_{y_{i1}}, \mathbf{e}^{i1}_{x_{i1}} \cdot \mathbf{b}^{i1}_{ij}\right), \tag{5}$$

where $\mathbf{b}^{i1}_{ij} = \mathbf{Q}_z(\theta_i)^T \mathbf{b}^i_{ij}$. Note that Fig. 3 describes the two-steps procedure. First, rotate ${}^i\Sigma$ about z_i by the angle θ_i, followed by the rotation of ψ_i about the new y-axis. Consequently, the x-axis of the transformed coordinate system ${}^i\Sigma_2$ becomes aligned with $\mathbf{b}_{ij}$ and the combined rotation transformation is obtained as $\mathbf{Q}_z(\theta_i)\mathbf{Q}_y(\psi_i)$.

B. Full coordinate frame alignment using triangular sensing

Note that to align two coordinate frames in 3-D, generically three successive basic rotations are required [18]. Via the approach of the previous subsection, two consecutive basic rotations[1] are derived to make the x-axes of two coordinate frames become aligned with the direction connecting two neighboring agents. Thus, we need to have one more rotation. To add one more rotation, we will introduce a supplementary directional information from a triangular sensing graph.

Suppose that the two neighboring agents, i and j, sense directions to the third agent, k (see Fig. 2a). To define the third rotation that can be combined with the two aforementioned rotations, we first decompose $\mathbf{b}^{i2}_{ik}$, which is determined by

$$\mathbf{b}^{i2}_{ik} = [\mathbf{Q}_z(\theta_i)\mathbf{Q}_y(\psi_i)]^T \, \mathbf{b}^i_{ik} \tag{6}$$

into $\mathbf{b}^{i2}_{ik\perp} = [(\mathbf{b}^{i2}_{ik})^T \mathbf{e}^{i2}_{x_{i2}}]\mathbf{e}^{i2}_{x_{i2}}$ and $\mathbf{b}^{i2}_{ik||} = \mathbf{b}^{i2}_{ik} - \mathbf{b}^{i2}_{ik\perp}$, such that $\mathbf{b}_{ik\perp} \perp (y_{i2}, z_{i2})$ and $\mathbf{b}_{ik||} \in (y_{i2}, z_{i2})$ (see Fig. 4). Note that $\mathbf{b}_{ik||}$ is the projection of $\mathbf{b}_{ik}$ onto the plane perpendicular to the directional vector $\mathbf{b}_{ij}$, which is aligned with the x-axis in ${}^i\Sigma_2$. The signed angle φ_i between $\mathbf{e}_{z_{i2}}$ and $\mathbf{b}_{ik||}$ is computed similarly to (4) as

$$\varphi_i = atan2\big((\mathbf{e}^{i2}_{z_{i2}} \times \mathbf{b}^{i2}_{ik||}) \cdot \mathbf{e}^{i2}_{x_{i2}}, \mathbf{e}^{i2}_{z_{i2}} \cdot \mathbf{b}^{i2}_{ik||}\big), \tag{7}$$

which is also the angle between $\mathbf{e}_{z_{i2}}$ and the plane $(\mathbf{b}_{ik}, \mathbf{b}_{ij})$. Hence, the rotation about x_{i2} by the angle φ_i transforms ${}^i\Sigma_2$ into ${}^i\Sigma_3$ such that $z_{i3} \equiv \mathbf{b}_{ik||}$. Finally, the rotation which transforms ${}^i\Sigma$ into ${}^i\Sigma_3$ is the successive multiplication of $\mathbf{Q}_z(\theta_i)$, $\mathbf{Q}_y(\psi_i)$ and $\mathbf{Q}_x(\varphi_i)$, i.e.,

$$\mathbf{Q}_i = \mathbf{Q}_z(\theta_i)\mathbf{Q}_y(\psi_i)\mathbf{Q}_x(\varphi_i), \tag{8}$$

which has the same form as a ZYX-Euler angle transformation [18].

Lemma 1: Consider the triangular sensing topology in Fig. 2a. The coordinate system ${}^i\Sigma_3$ obtained by transforming ${}^i\Sigma$ by a rotation of $\mathbf{Q}_i$ defined in (8) is uniquely determined by $\mathbf{b}_{ij}$ and $\mathbf{b}_{ik}$.

Proof: The transformed coordinate frame ${}^i\Sigma_3$ ensures x_{i3} points along the direction of $\mathbf{b}_{ij}$, and z_{i2} points along $\mathbf{b}_{ik||}$ which is also uniquely determined. The y_{i2} axis is obtained from the right-hand rule for coordinate systems; thus, it is also determined uniquely. ■

By repeating the alignment procedure for the neighboring agent of i, i.e., agent j, using two directional vectors $-\mathbf{b}^j_{ji}$ and $\mathbf{b}^j_{jk}$, a rotation matrix is obtained similarly to (8) as

$$\mathbf{Q}_j = \mathbf{Q}_z(\theta_j)\mathbf{Q}_y(\psi_j)\mathbf{Q}_x(\varphi_j). \tag{9}$$

Since three agents define a plane in 3-D, the projections of $\mathbf{b}_{ik}$ and $\mathbf{b}_{jk}$ onto any planes perpendicular to $\mathbf{b}_{ij}$ (or $\mathbf{b}_{ji}$) are parallel, i.e., $\mathbf{b}_{ik||} || \mathbf{b}_{jk||}$. We now are ready to state the main result of this section.

Theorem 1: Consider the triangular sensing topology in Fig. 2a, and the alignment transformations (8) and (9). Then, the relative orientation between two agents i and j is obtained as

$$\mathbf{R}_i^T \mathbf{R}_j = \mathbf{Q}_i \mathbf{Q}_j^T \tag{10}$$

Proof: Under the rotation (9), the coordinate frame ${}^j\Sigma$ is transformed into ${}^j\Sigma_3$, which has x_{j3} and y_{j3} aligned with $-\mathbf{b}_{ji} = \mathbf{b}_{ij}$ and $\mathbf{b}_{jk||}$, respectively. Moreover, since $\mathbf{b}_{ik||}$ and $\mathbf{b}_{jk||}$ are parallel and their directions are the same, two coordinate frames ${}^i\Sigma_3$ and ${}^j\Sigma_3$ are identical. For a nonzero vector $\mathbf{x} \in \mathbb{R}^3$, we have the following relationships

$$\mathbf{x}^i = \mathbf{Q}_i \mathbf{x}^{i3}, \;\; \mathbf{x}^j = \mathbf{Q}_j \mathbf{x}^{j3}.$$

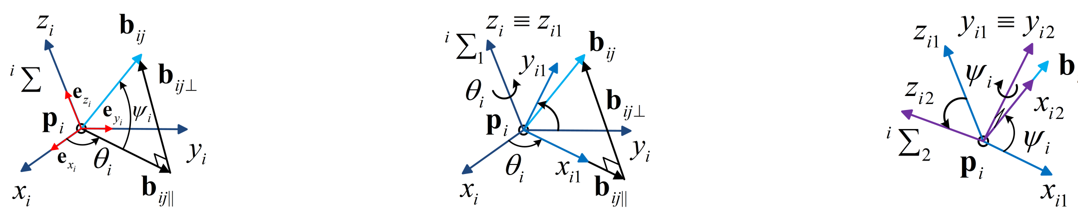
Since $\mathbf{x}^{i3} = \mathbf{x}^{j3}$, we obtain $\mathbf{Q}_i^T \mathbf{x}^i = \mathbf{Q}_j^T \mathbf{x}^j$. Using the relationships in (3) we have

$$\mathbf{x}^i = \mathbf{R}_i^T \mathbf{x}, \;\; \mathbf{x}^j = \mathbf{R}_j^T \mathbf{x}.$$

Thus $\mathbf{Q}_i^T \mathbf{R}_i^T = \mathbf{Q}_j^T \mathbf{R}_j^T \iff \mathbf{R}_i^T \mathbf{R}_j = \mathbf{Q}_i \mathbf{Q}_j^T$, which completes the proof. ■

Remark 3: The proposed method of calculating relative orientation of two neighboring agents which have a common neighboring agent, uses only local direction information. Thus, it is fully distributed. For the general case with n agents, the method can be extended by way of similar approaches as Henneberg extensions given suitable restrictions on the graph. In particular, consider a graph grown from an initial triangle with relative orientations obtainable as described above, in the following manner, which is akin to a sequence of Henneberg vertex estensions of a particular type. We can add a pair of two neighboring agents that have a common neighbor agent in the initial triangle (see Fig.

[1]The basic rotation is a rotation about one of the coordinate axes.



(a) Directional vector in 3-D (b) Rotation of $\mathbf{Q}_{z_i}(\theta_i)$ (c) Rotation of $\mathbf{Q}_{y_{i1}}(\psi_i)$

Fig. 3: Alignment of x-axis and directional vector by two consecutive basic rotations.

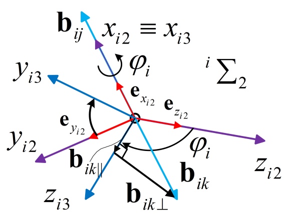
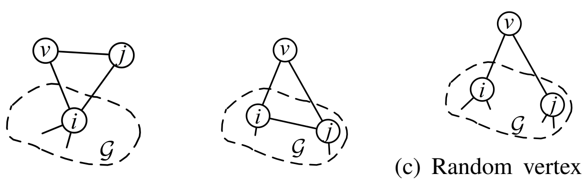


Fig. 4: The third rotation $\mathbf{Q}_{x_{i2}}(\varphi_i)$ that transforms ${}^i\Sigma_2$ into ${}^i\Sigma_3$ such that $z_{i3} \in (\mathbf{g}, \mathbf{b}_{ij})$ and $x_{i3} \equiv \mathbf{b}_{ij}$.



(a) Vertex extension. (b) Edge extension. (c) Random vertex addition.

Fig. 5: Extension procedure of the relative orientation measurement graph.

5a). Then, the newly added pair of agents and the agent in the initial triangle can define another triangular graph. This extension can be called a *triangular vertex extension*. On the other hand, we can add one single agent that has two edge connections to two neighboring agents (see Fig. 5b). Then, the newly added agent and two neigboring agents define another triangle. So, we can find relative orientations between agents in the new triangle. This extension can be called a *triangular edge extension*.

Remark 4: If the orientations of all nodes in the graph $\mathcal{G}$ are localized with regard to a common reference coordinate frame, i.e., ${}^c\Sigma$, (which is studied in treating *Problem 2*), we can add a new agent (let say v) to $\mathcal{G}$ as follows. Connect v to an arbitrary pair of nodes i and j as illustrated in Fig. 5c. Now, suppose i and j respectively send the direction information $\mathbf{b}^c_{iv} = \hat{\mathbf{R}}_i \mathbf{b}^i_{iv}$ and $\mathbf{b}^c_{jv} = \hat{\mathbf{R}}_j \mathbf{b}^j_{jv}$ to v. Then, v can compute its orientation, $\hat{\mathbf{R}}_v$, with regard to ${}^c\Sigma$ by applying a similar method to that of the above alignment process by using $\{\mathbf{b}^c_{iv}, \mathbf{b}^c_{jv}, \mathbf{b}^v_{vi}, \mathbf{b}^v_{vj}\}$. The inter-agent relative orientation follows directly by applying (2). An application of this *vertex addition process* in network orientation localization and formation control will be presented in our future work.

C. Coordinate frame alignment using a common direction information instead of triangular sensing

If two neighboring agents share a common sense of a direction, then the third rotation can be derived using this common vector. For example, if the agents know the direction information of the earth's magnetic field, which is measured in each agent's local coordinate frame, it can be utilized for determining the third rotation. The idea of using the earth's magnetic field direction is motivated by the empirical observation that birds are able to sense the magnetic field to navigate in a homing direction [14], [15]. Additionally, the earth's magnetic field direction information can be measured by using low-cost magnetometers. By using the supplementary direction information, the complexity of relative sensing can be reduced significantly, compared with the triangulation sensing (sub)networks. However, the earth's magnetic field is a global information and it is vulnerable to magnetic interferences in real applications.

IV. Orientation Localization based on relative orientation information

This section presents a relative orientation localization scheme and an absolute orientation localization law using relative orientation and additional information in the second case. As part of the scheme, we will define a nonsingular matrix auxiliary variable for each agent and its localized orientation will be derived from the auxiliary variable by the Gram-Schmidt orthonormalization procedure (GSOP). We establish an almost globally exponentially convergence of the localized orientations to the real orientations up to a common orientation. Under the absolute orientation localization law, the computed orientations of all agents globally exponentially converge to the true orientations.

A. Relative Orientation Localization

For each agent i, we introduce a nonsingular matrix auxiliary variable $\mathbf{P}_i \in \mathbb{R}^{3\times 3}$. Given $\mathbf{P}_i$, let the orientation of agent i be computed as $\hat{\mathbf{R}}_i^T$ via the Gram-Schmidt orthonormalization procedure (GSOP), which is denoted as $\hat{\mathbf{R}}_i^T = \text{GSOP}(\mathbf{P}_i)$. Then a distributed orientation estimator for each agent is proposed as

$$\dot{\mathbf{P}}_i(t) = \textstyle\sum_{j\in\mathcal{N}_i} (\mathbf{R}_{ij}\mathbf{P}_j(t) - \mathbf{P}_i(t)), \tag{11}$$

where $\mathbf{R}_{ij} = \mathbf{R}_i^T \mathbf{R}_j$ is obtained by (10) and the initial value $\mathbf{P}_i(0)$ is assigned to be nonsingular; but can otherwise be assigned randomly. Using the stacked matrix of matrix auxiliary variables $\mathbf{P} = [\mathbf{P}_1^T, \ldots, \mathbf{P}_n^T]^T \in \mathbb{R}^{3n\times 3}$, (11) can be further written as

$$\dot{\mathbf{P}}(t) = -\mathbf{M}\mathbf{P}(t), \tag{12}$$

where $\mathbf{M} = \mathbf{D}(\mathbf{L}\otimes \mathbf{I}_3)\mathbf{D}^T$, $\mathbf{L}$ is Laplacian matrix of $\mathcal{G}$, and $\mathbf{D} = diag(\mathbf{R}_1^T, \ldots, \mathbf{R}_n^T)$. To study the convergence of $\mathbf{P}$, it is useful to introduce a coordinate transformation and new variables $\mathbf{S}_i$ related to $\mathbf{P}_i$. In fact, we set $\mathbf{P}_i = \mathbf{R}_i^T\mathbf{S}_i$ and define a stacked matrix of transformed variable $\mathbf{S} = [\mathbf{S}_1^T, \ldots, \mathbf{S}_n^T]^T \in \mathbb{R}^{3n\times 3}$. Then (12) can be written as

$$\dot{\mathbf{S}}(t) = -(\mathbf{L}\otimes \mathbf{I}_3)\mathbf{S}(t). \tag{13}$$

Let $[\mathbf{S}]_k \in \mathbb{R}^{3n}$ be the k^{th} column vector of $\mathbf{S}$, for $k \in \{1,2,3\}$. Since the graph $\mathcal{G}$ is connected, it can be shown that $\mathbf{S}(t)$ converges to the equilibrium set $\mathcal{E}_{\mathbf{S}} = \{\mathbf{S} = [\mathbf{S}_1^T, \ldots, \mathbf{S}_n^T]^T \in \mathbb{R}^{3n\times 3} : \mathbf{S}_1 = \mathbf{S}_2 \ldots = \mathbf{S}_n\}$. Furthermore, it is noticed that $\mathbf{P}_i$ and $\mathbf{S}_i$ are related by a coordinate rotation, for the determination of orientation, we have to avoid the convergence of each column vector of $\mathbf{S}_i$ to the zero vector and the linear dependence of the steady-state column vectors of $\mathbf{S}_i$. The convergence property of (13) is provided in the following theorem.

Theorem 2: Suppose that the graph $\mathcal{G}$ is connected. Then, $\mathbf{S}(t)$ in the dynamics (13) globally exponentially converges to $(\mathbf{1}_n \otimes \mathbf{I}_3)\mathbf{S}^\infty \in \mathcal{E}_{\mathbf{S}}$, where $\mathbf{S}^\infty \triangleq Ave(\mathbf{S}_i(0)) \in \mathbb{R}^{3\times 3}$.

Proof: The proof is straightforward from the facts that $\text{rank}(\mathbf{L}\otimes\mathbf{I}_3) = 3n-3$, $\text{null}(\mathbf{L}\otimes\mathbf{I}_3) = \text{Range}(\mathbf{1}_n\otimes\mathbf{I}_3)$, and $(\mathbf{1}_n\otimes\mathbf{I}_3)^T\mathbf{S}(t)$ is invariant under (13). ■

The *Theorem 2* implies that $\lim_{t\to\infty}\mathbf{S}_i(t) = \lim_{t\to\infty}\mathbf{R}_i\mathbf{P}_i(t) = \mathbf{S}^\infty$, or $\lim_{t\to\infty}\mathbf{P}_i(t) = \mathbf{R}_i^T\mathbf{S}^\infty,\ \forall i \in \{1,\ldots,n\}$. Since $\hat{\mathbf{R}}_i^T$ is derived from $\mathbf{P}_i$ by the GSOP, we can make the following lemma.

Lemma 2: Consider the rotation transformation $\mathbf{P}_i = \mathbf{R}_i^T\mathbf{S}_i$. If $\hat{\mathbf{R}}_i^T$ and $\mathbf{Q}_i$ are derived from $\mathbf{P}_i$ and $\mathbf{S}_i$ by the GSOP, respectively, then there holds

$$\hat{\mathbf{R}}_i^T = \mathbf{R}_i^T\mathbf{Q}_i. \tag{14}$$

Proof: Let $\mathbf{p}_k,\ \mathbf{s}_k,\ \hat{\mathbf{r}}_k$, and $\mathbf{q}_k \in \mathbb{R}^3$ be the k^{th} column vector of $\mathbf{P}_i$, $\mathbf{S}_i$, $\hat{\mathbf{R}}_i^T$ and $\mathbf{Q}_i$, respectively. In order to prove the equality (14) it is sufficient to show that $\hat{\mathbf{r}}_k = \mathbf{R}_i^T\mathbf{q}_k,\ \forall k = 1,2,3$. We show this by following the GSOP as follows:

$$\begin{aligned}
\hat{\mathbf{r}}_1 &= \mathbf{p}_1/\|\mathbf{p}_1\| = \mathbf{R}_i^T\mathbf{s}_1/\|\mathbf{R}_i^T\mathbf{s}_1\| = \mathbf{R}_i^T\mathbf{s}_1/\|\mathbf{s}_1\| = \mathbf{R}_i^T\mathbf{q}_1\\
\hat{\mathbf{r}}_2 &= normalize\{\mathbf{p}_2 - \langle\mathbf{p}_2, \mathbf{R}_i^T\mathbf{q}_1\rangle\mathbf{R}_i^T\mathbf{q}_1\}\\
&= normalize\{\mathbf{R}_i^T\mathbf{s}_2 - \langle\mathbf{R}_i^T\mathbf{s}_2, \mathbf{R}_i^T\mathbf{q}_1\rangle\mathbf{R}_i^T\mathbf{q}_1\}\\
&= \mathbf{R}_i^T normalize\{\mathbf{s}_2 - \langle\mathbf{s}_2, \mathbf{q}_1\rangle\mathbf{q}_1\} = \mathbf{R}_i^T\mathbf{q}_2\\
\hat{\mathbf{r}}_3 &= normalize\{\mathbf{p}_3 - \sum\nolimits_{k=1}^2 \langle\mathbf{p}_3, \mathbf{R}_i^T\mathbf{q}_k\rangle\mathbf{R}_i^T\mathbf{q}_k\}\\
&= normalize\{\mathbf{R}_i^T\mathbf{s}_3 - \textstyle\sum_{k=1}^2 \langle\mathbf{R}_i^T\mathbf{s}_3, \mathbf{R}_i^T\mathbf{q}_k\rangle\mathbf{R}_i^T\mathbf{q}_k\}\\
&= \mathbf{R}_i^T normalize\{\mathbf{s}_3 - \textstyle\sum_{k=1}^2 \langle\mathbf{s}_3, \mathbf{q}_k\rangle\mathbf{q}_k\} = \mathbf{R}_i^T\mathbf{q}_3'.\\
\hat{\mathbf{r}}_3 &= sign\{(det\left([\mathbf{R}_i^T\mathbf{q}_1, \mathbf{R}_i^T\mathbf{q}_2, \mathbf{R}_i^T\mathbf{q}_3']\right)\}\mathbf{R}_i^T\mathbf{q}_3'\\
&= \mathbf{R}_i^T sign\{det(\mathbf{R}_i^T)det\left([\mathbf{q}_1, \mathbf{q}_2, \mathbf{q}_3']\right)\}\mathbf{q}_3' = \mathbf{R}_i^T\mathbf{q}_3,
\end{aligned}$$

■

Algorithm 1: Orientation localization algorithm

Input: $\{\mathbf{R}_{ij}\}_{j\in\mathcal{N}_i}$, $\{\mathbf{P}_j(t)\}_{j\in\mathcal{N}_i}$
Output: $\hat{\mathbf{R}}_i$

1. Initialize $t = 0,\ \mathbf{P}_i(0) \in \mathbb{R}^{3\times 3}$
2. Compute $\mathbf{P}_i(t)$ by integrating (11)
3. $t = t + \Delta t$
4. Repeat *Steps 2 to 3*, until $||\mathbf{P}_i(t) - \mathbf{P}_i(t-\Delta t)|| < \epsilon$
5. Compute $\mathbf{P}_i^\infty$ from $\mathbf{P}_i$ by applying the GSOP
6. $\hat{\mathbf{R}}_i = (\mathbf{P}_i^\infty)^T = (\mathbf{Q}^\infty)^T\mathbf{R}_i$

The above lemma means that the GSOP is invariant under rotation.i.e., $\hat{\mathbf{R}}_i^T = \text{GSOP}(\mathbf{P}_i) = \text{GSOP}(\mathbf{R}_i^T\mathbf{S}_i) = \mathbf{R}_i^T\text{GSOP}(\mathbf{S}_i) = \mathbf{R}_i^T\mathbf{Q}_i$.

Theorem 3: Consider the computed orientation matrix $\hat{\mathbf{R}}_i^T \in \text{SO}(3)$ derived from $\mathbf{P}_i$ by the GSOP. Then there exists a certain rotation matrix $\mathbf{Q}^\infty \in \text{SO}(3)$ such that $\hat{\mathbf{R}}_i$ exponentially converges to $(\mathbf{Q}^\infty)^T\mathbf{R}_i,\ \forall i = 1,\ldots,n$, for almost all initial values $\mathbf{P}(0)$.

Proof: Let $\mathbf{Q}^\infty$ be derived from $\mathbf{S}^\infty = Ave(\mathbf{S}_i(0))$ by the GSOP. Then, it follows from *Theorem 2* and *Lemma 2* that $\hat{\mathbf{R}}_i$ exponentially converges to $(\mathbf{Q}^\infty)^T\mathbf{R}_i,\ \forall i = 1,\ldots,n$. The set of column vectors of initial matrix auxiliary variables leading to non-existence of solution, i.e., $\det(\mathbf{S}^\infty) = 0$, is a set of Lesbegue measure zero in $\mathbf{R}^{3n}$ [5]. This completes the proof. ■

Remark 5: From *Theorem 3*, the localization law (11) *almost globally asymptotically* solves the Problem 2.

The orientation localization procedure for each agent is given in *Algorithm 1*.

Remark 6: The orientation localization scheme outlined in the *Algorithm 1* is conducted to compute the orientation of agents simultaneously up to an unknown common rotation. For any graph constructed by the *triangular vertex extension* or *triangular edge extension*, or combination of these extensions, the orientation can be determined simultaneously. Thus, it is important to note that the extensions defined in the *Remark 3* characterize a class of graphs that can be solved for the orientation localization. For such graphs, orientations of the coordinate frames of agents (apart from a common unknown additional transformation) can be simultaneously determined by (11).

B. Orientation localization with one or more anchor nodes

We assume that at least one of the root nodes (say agent 1) knows its orientation $\mathbf{R}_1$. Consider the augmented graph $\bar{\mathcal{G}} = (\mathcal{V}, \bar{\mathcal{E}})$ obtained from $\mathcal{G}$ by removing all the out-going edges from node 1.

Lemma 3: The augmented graph $\bar{\mathcal{G}}$ obtained from $\mathcal{G}$ by removing all the out-going edges from node 1 contains a rooted-in spanning tree.

Proof: The result is straightforward since $\mathcal{G}$ is connected and the removing of the out-going edges of node 1 preserves all in-coming paths from the other nodes to node 1. Thus, there exists a directed path from any other node to node 1, that is, $\bar{\mathcal{G}}$ has a rooted-in spanning tree. ■

The orientation localization scheme (11) now becomes

$$\dot{\hat{\mathbf{R}}}_i^T(t) = \sum_{j\in\mathcal{N}_i} (\mathbf{R}_{ij}\hat{\mathbf{R}}_j^T(t) - \hat{\mathbf{R}}_i^T(t)), \quad (15)$$

where $\mathcal{N}_i$ is defined in $\bar{\mathcal{G}}$.

Theorem 4: If the graph $\mathcal{G}$ is connected, with one root node knowing its orientation, the localized orientations $\hat{\mathbf{R}}_i^T(t)$ globally asymptotically converge to the true orientations $\mathbf{R}_i^T$, as $t \to \infty$, under the update law (15), for all $i = 1, \ldots, n$.

Proof: By using the rotation transformation $\hat{\mathbf{R}}_i^T = \mathbf{R}_i^T\mathbf{S}_i$ we rewrite (15) in the same form of (13) as

$$\dot{\mathbf{S}}(t) = -(\bar{\mathbf{L}} \otimes \mathbf{I}_3)\mathbf{S}(t), \quad (16)$$

where $\bar{\mathbf{L}}$ is the Laplacian of $\bar{\mathcal{G}}$. From the Theorem 1 in [16] and *Lemma 3*, the dynamics (16) are globally exponentially stable. From the Lemma 1 in [17], we have the steady-state solution $\lim_{t\to\infty} \mathbf{S}_i(t) = \sum_{k=1}^n \bar{\upsilon}_k \mathbf{I}_3\mathbf{S}_k(0)$, where $\bar{\Upsilon}_l = [\bar{\upsilon}_1, \ldots, \bar{\upsilon}_n] \in \mathbb{R}^{1\times n}$, $\sum_{k=1}^n \bar{\upsilon}_k = 1, \bar{\upsilon}_k \geq 0, \forall k = 1, \ldots, n$, is the left eigenvector of $\bar{\mathbf{L}}$ associated with the zero eigenvalue, for all $i = 1, \ldots, n$. Since the node 1 has no neighbors, it does not update its orientation, i.e., $\dot{\mathbf{S}}_1 = 0$. Thus, $\lim_{t\to\infty} \mathbf{S}_1(t) = \mathbf{S}_1(0) = \mathbf{R}_1\hat{\mathbf{R}}_1^T = \mathbf{I}_3$. It follows that $\mathbf{S}_i$ converges to $\mathbf{I}_3$; that is, $\hat{\mathbf{R}}_i$ globally asymptotically converges to $\mathbf{R}_i, \ \forall i \in \{2, \ldots, n\}$. ■

Remark 7: The underlying graph topologies in the relative orientation calculation of Section III and in the orientation localization of Section IV are simply assumed undirected, but only connected. Actually, we can separate the graph topologies for the relative orientation calculation and for the orientation localization. In the process of computing the relative orientation calculation $\mathbf{R}_i^T\mathbf{R}_j$, the neighboring agents i and j need to compute $\mathbf{Q}_i$ and $\mathbf{Q}_j$ respectively. In this process, they need to sense each other and need to sense the third common node k. Thus, in terms of sensing, they need to measure relative states a little closely. But, after computing $\mathbf{Q}_i$ and $\mathbf{Q}_j$, the graph topology becomes simpler. In fact, in the computation of $\mathbf{P}_i$ in (11), since we can achieve a consensus in (11) based on properties of a directed graph topology (i.e., strongly connected, or directed rooted tree), agent i may not need to send its information $\mathbf{P}_i$ to the incoming neighboring agents, although it needs to send $\mathbf{P}_i$ to outgoing neighboring agents. Thus, the topology for computing $\mathbf{R}_i^T\mathbf{R}_j$ and the topology for updating (11) could be different. We will address this issue in a more systematic way in our future work.

V. Conclusion

In this paper, we have presented a simple procedure for determining the relative orientation between the coordinate frames of two neighboring agents in 3-D by using direction-only information of a triangulation sensing network which includes a common neighboring agent that can be observed by the two agents. A *triangular extension procedure* was then described for relative orientation computation of some more general multi-agent networks. In addition, we proposed two distributed orientation localization laws by defining a matrix auxiliary variable for each agent. The orientations of the coordinate frames of the agents are derived from the auxiliary variables by applying the Gram-Schmidt orthonormalization procedure. Under the proposed orientation localization laws, orientations are almost globally asymptotically determined up to a common unknown coordinate rotation. If an agent knows the global orientation of its coordinate frame, this common matrix can be determined.

Acknowledgments

This work was supported, in part, by the National Research Foundation (NRF) of Korea under the grants NRF-2016M1B3A1A01937575 and NRF-2017R1A2B3007034, and in part, by the Australian Research Council under grants DP-130103610 and DP-160104500, and Data61-CSIRO.

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