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Spatial sound intensity vectors in spherical harmonic domain

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Abstract: Sound intensity is a fundamental quantity describing acoustic wave fields and it contains both energy and directivity information. It is used in a variety of applications such as source localization, reproduction, and power measurement. Until now, intensity is defined at a point in space, however given sound propagates over space, knowing its spatial distribution could be more powerful. This paper formulates spatial sound intensity vectors in spherical harmonic domain such that the vectors contain energy and directivity information over continuous spatial regions. These representations are derived with finite sets of closed form coefficients enabling ease of implementation.

[PG]
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1. Introduction

Intensity is a fundamental quantity in acoustics that is defined as the power carried by sound waves per unit area in a direction perpendicular to that area. It indicates the rate of energy flow and also gives a measure of direction of energy flow. Therefore, sound intensity has both magnitude and direction components, and that is why it is also referred to as "intensity vector."

Sound intensity is useful for localization of sources, reproduction of sound fields, measurement of sound power, measurement of transmission loss, identification of transmission paths, etc.[1] A practical example for sound intensity based applications is seen in high speed trains, where noise sources are measured and identified to create a quiet environment for passengers.[2] The reason why sound intensity is useful for a range of applications as mentioned above is because it plays an important role in directional psychoacoustics. As shown by Gerzon, the human ability to localize is related to the ratio of the sound intensity vector gain to the total energy.[3] Using this relationship, Arteaga[4] and Scaini and Argeaga,[5] propose a novel method for sound reproduction using intensity matching.[4,5] In addition, intensity is an effective tool to estimate direction of arrival (DOA) because there is no need to compute a spatial cost function by directly computing the direction of energy flow.[6] To improve the accuracy of DOA estimation, an augmented intensity vector is proposed by exploiting higher order spherical harmonics.[7]

All the applications mentioned above are based on sound intensity at a single point or several points, but these applications can be extended and facilitated by using spatial sound intensity vectors. For example, sound intensity based reproduction can be realized over space instead of at a single point using spatial intensity matching so that the original sound can be reproduced over a large region for more listeners. Besides, spatial sound intensity vectors facilitate sound intensity measurement over space that is necessary in most of the aforementioned applications. Currently, the dominating and economical method of measuring sound intensity over space is based on the combination of two pressure microphones,[8,9] which requires performing a measurement in the volume of interest point by point. While this process can be done automatically by an industrial robot, the design and implementation of it with high accuracy is comparably time-consuming and costly. In such instances, spherical harmonics can

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play a role because they cover the entire space together with the radial functions. Therefore, if we can generate a spherical harmonic decomposition of the spatial sound intensity, the sound intensity at any point in space is readily available.

The main contribution of this paper is a theoretical derivation and proof of a spherical harmonic based representation for sound intensity over a continuous spatial region. The closed form expressions of spatial sound intensity are given and finite modes of spherical harmonics in each expression are indicated. We choose the spherical harmonic domain also because spherical harmonics are spatial basis functions that can be used to describe a variety of acoustics based functions[10,11] in the three-dimensional space.

2. Problem formulation

Consider a point $\boldsymbol{x} = (r, \theta, \phi)$ in a homogeneous medium in space. The sound pressure at $\boldsymbol{x}$ is $P(\boldsymbol{x}, k)$ and the particle velocity in spherical coordinates is $\boldsymbol{V}(\boldsymbol{x}, k) = [V_r(\boldsymbol{x}, k),\ V_\theta(\boldsymbol{x}, k),\ V_\phi(\boldsymbol{x}, k)]$, where k is the wavenumber. The intensity relationship for the steady state field is defined as[12]

$$\boldsymbol{I}(\boldsymbol{x}, k) = P^*(\boldsymbol{x}, k)\boldsymbol{V}(\boldsymbol{x}, k), \tag{1}$$

where $\boldsymbol{I}(\boldsymbol{x}, k) = [I_r(\boldsymbol{x}, k), I_\theta(\boldsymbol{x}, k), I_\phi(\boldsymbol{x}, k)]$ is the sound intensity vector in spherical coordinates and $*$ stands for complex conjugate. All of the above intensity components are defined on a sphere and therefore can be decomposed in terms of spherical harmonic functions[13]

$$I_D(\boldsymbol{x}, k) = \sum_{p=0}^{\infty}\sum_{q=-p}^{p} S_{pq}^{(D)}(k, r)\, Y_{pq}(\theta, \phi); \quad D = \{r, \theta, \phi\}, \tag{2}$$

where $S_{pq}^{(D)}(k, r)$ are intensity coefficients in the D direction and $Y_{pq}(\theta, \phi) = A_{pq}P_{pq}(\cos\theta)e^{iq\phi}$ is the spherical harmonic of order p and degree q with $A_{pq} = [(2p+1)(p-q)!/(4\pi(p+q)!)]^{1/2}$, where $P_{pq}(\cos\theta)$ are the associated Legendre functions.

Our objective is to derive complete sets of closed form intensity coefficients $S_{pq}^{(D)}(k, r)$ related to each r, θ, ϕ component of the sound intensity vector.

3. Particle velocities in the spherical harmonic domain

In spherical harmonic domain, the sound pressure at $\boldsymbol{x}$ in a source-free region is given by

$$P(\boldsymbol{x}, k) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr)\, Y_{nm}(\theta, \phi), \tag{3}$$

where $\alpha_{nm}(k)$ are pressure coefficients and $j_n(\cdot)$ is the nth order spherical Bessel of the first kind.

The particle velocity $V_D(\boldsymbol{x}, k)$ at $\boldsymbol{x}$, in the direction D, is related to the sound pressure by[14]

$$V_D(\boldsymbol{x}, k) = \frac{i}{k\rho_0 c}\frac{\partial P(\boldsymbol{x}, k)}{\partial D}, \tag{4}$$

where ρ_0 is the medium density and c is the speed of propagation. Therefore, from Eq. (3) the particle velocity in the r, θ, and ϕ directions can be derived as

$$V_r(\boldsymbol{x}, k) = \frac{i}{k\rho_0 c}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \alpha_{nm}(k) j_n'(kr)\, Y_{nm}(\theta, \phi), \tag{5a}$$

$$V_\theta(\boldsymbol{x}, k) = \frac{i}{k\rho_0 c}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} \alpha_{nm}(k) j_n(kr) A_{nm} P_{nm}'(\cos\theta) e^{im\phi}, \tag{5b}$$

$$V_\phi(\boldsymbol{x}, k) = \frac{i}{k\rho_0 c}\sum_{n=0}^{\infty}\sum_{m=-n}^{n} im\alpha_{nm}(k) j_n(kr)\, Y_{nm}(\theta, \phi), \tag{5c}$$

with

$$j_n'(kr) = \frac{nkj_{n-1}(kr) - (n+1)kj_{n+1}(kr)}{2n+1}, \tag{6}$$

$$P'_{nm}(\cos\theta) = \frac{(n-m+1)P_{(n+1)m}(\cos\theta) - (n+1)\cos\theta P_{nm}(\cos\theta)}{\sin\theta}; \quad \theta \neq \{0, \pi\}, \tag{7}$$

$$P'_{nm}(\cos 0) = P'_{nm}(1) = \begin{cases} -n(n+1)/2, & \text{if } m = 1, \\ 1/2, & \text{if } m = -1, \\ 0, & \text{otherwise,} \end{cases} \tag{8}$$

and $P'_{nm}(\cos\pi) = P'_{nm}(-1) = (-1)^n P'_{nm}(1)$ according to the property of associated Legendre function.

4. Derivation of the spatial sound intensity vector coefficients

Substituting Eqs. (5a), (5b), and (5c), separately, with Eq. (3) into Eq. (1), we can get

$$I_r(\boldsymbol{x}, k) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} R_{nmn'm'}(k,r)\, Y^*_{nm}(\theta,\phi)\, Y_{n'm'}(\theta,\phi), \tag{9a}$$

$$I_\theta(\boldsymbol{x}, k) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} A_{n'm'} T_{nmn'm'}(k,r)\, Y^*_{nm}(\theta,\phi) P'_{n'm'}(\cos\theta) e^{im'\phi}, \tag{9b}$$

$$I_\phi(\boldsymbol{x}, k) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} im' T_{nmn'm'}(k,r)\, Y^*_{nm}(\theta,\phi)\, Y_{n'm'}(\theta,\phi), \tag{9c}$$

where $R_{nmn'm'}(k,r) = i\alpha^*_{nm}(k)\alpha_{n'm'}(k) j_n(kr) j'_{n'}(kr)/(k\rho_0 c)$ and $T_{nmn'm'}(k,r) = i\alpha^*_{nm}(k) \times \alpha_{n'm'}(k) j_n(kr) j_{n'}(kr)/(k\rho_0 c)$.

4.1 Intensity coefficients in the r and $\boldsymbol{\phi}$ *directions*

In this section, we derive the closed form expressions of intensity coefficients in the r and ϕ directions and introduce the following theorem.

Theorem 1. *The intensity coefficients in the r direction* $S^{(r)}_{pq}(k,r)$ *and* ϕ *direction* $S^{(\phi)}_{pq}(k,r)$ *can be expressed separately as*

$$S^{(r)}_{pq}(k,r) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} (-1)^{m+q} C_{nn'p} W_1 W_2 R_{nmn'm'}(k,r), \tag{10a}$$

$$S^{(\phi)}_{pq}(k,r) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} im'(-1)^{m+q} C_{nn'p} W_1 W_2 T_{nmn'm'}(k,r), \tag{10b}$$

where

$$C_{nn'p} = \sqrt{(2n+1)(2n'+1)(2p+1)/4\pi}, \quad W_1 = \begin{pmatrix} n & n' & p \\ 0 & 0 & 0 \end{pmatrix}$$

and

$$W_2 = \begin{pmatrix} n & n' & p \\ -m & m' & -q \end{pmatrix}$$

denoting Wigner 3-j symbols.

Proof. Multiplying both sides of Eq. (2) by $Y^*_{pq}(\theta,\phi)$ and integrating with respect to θ and ϕ,

$$S^{(D)}_{pq}(k,r) = \int_0^{2\pi}\int_0^{\pi} I_D(\boldsymbol{x},k)\, Y^*_{pq}(\theta,\phi) \sin\theta d\theta d\phi; \quad D = \{r, \theta, \phi\}. \tag{11}$$

In the r direction, substituting Eq. (9a) into Eq. (11),

$$S^{(r)}_{pq}(k,r) = \int_0^{2\pi}\int_0^{\pi} \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} R_{nmn'm'}(k,r)\, Y^*_{nm}(\theta,\phi)\, Y_{n'm'}(\theta,\phi)\, Y^*_{pq}(\theta,\phi) \sin\theta d\theta d\phi. \tag{12}$$

The integral of products of three spherical harmonics is given by[15]

$$\int_0^{2\pi}\int_0^{\pi} Y_{l_1 m_1}(\theta,\phi)\, Y_{l_2 m_2}(\theta,\phi)\, Y_{l_3 m_3}(\theta,\phi) \sin\theta d\theta d\phi = \sqrt{\frac{(2l_1+1)(2l_2+1)(2l_3+1)}{4\pi}} \times \begin{pmatrix} l_1 & l_2 & l_3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix}, \tag{13}$$

as well as $Y^*_{nm}(\theta,\phi) = (-1)^m Y_{n(-m)}(\theta,\phi)$. Substituting Eq. (13) into Eq. (12) completes the proof of Eq. (10a). Similarly, substituting Eq. (13), as well as Eq. (9c), into Eq. (11) completes the proof of Eq. (10b).

4.2 Intensity coefficients in the θ direction

The derivation of coefficients in the θ direction is more complicated than the other two due to the partial derivative of the associated Legendre function. With regard to intensity coefficients in the θ direction, we derive the following theorem.

Theorem 2. *The intensity coefficients in the θ direction $S^{(\theta)}_{pq}(k,r)$ can be expressed as*

$$S^{(\theta)}_{pq}(k,r) = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\sum_{n'=0}^{\infty}\sum_{m'=-n'}^{n'} A_{nm}A_{n'm'}A_{pq}\mathcal{P}_{nmn'm'pq}\mathcal{E}_{mm'q}T_{nmn'm'}(k,r), \tag{14}$$

where $\mathcal{E}_{mm'q} = 2\pi$ when $m'-m-q=0$, otherwise $\mathcal{E}_{mm'q}=0$, and $\mathcal{P}_{nmn'm'pq} = (n'-m'+1)\mathcal{G}_1 - (n'+1)\mathcal{G}_2$ with

$$\begin{aligned}\mathcal{G}_1 = {} & H(n,m)H(n'+1,m')H(p,q)\mathcal{G}\Bigg(\frac{m+m'+q+1}{2},\frac{4-\delta_{m+n}-\delta_{m'+n'+1}-\delta_{p+q}}{2};\\ & \frac{1+m-n-\delta_{m+n}}{2},\frac{m'-n'-\delta_{m'+n'+1}}{2},\frac{1+q-p-\delta_{p+q}}{2};\frac{2+m+n}{2}\\ & -\frac{\delta_{m+n}}{2},\frac{3+m'+n'-\delta_{m'+n'+1}}{2},\frac{2+p+q-\delta_{p+q}}{2};\ m+1,m'+1,q+1\Bigg),\end{aligned} \tag{15a}$$

$$\begin{aligned}\mathcal{G}_2 = {} & H(n,m)H(n',m')H(p,q)\mathcal{G}\Bigg(\frac{m+m'+q+1}{2},\frac{5-\delta_{m+n}-\delta_{m'+n'}-\delta_{p+q}}{2};\\ & \frac{1+m-n-\delta_{m+n}}{2},\frac{1+m'-n'-\delta_{m'+n'}}{2},\frac{1+q-p-\delta_{p+q}}{2};\frac{2+m+n}{2}\\ & -\frac{\delta_{m+n}}{2},\frac{2+m'+n'-\delta_{m'+n'}}{2},\frac{2+p+q-\delta_{p+q}}{2};\ m+1,m'+1,q+1\Bigg),\end{aligned} \tag{15b}$$

where $H(n,m) = (-1)^m(n+m)!/[2^m m!(n-m)!]$,

$$\delta_M = \begin{cases} 1, & \text{if } M \text{ is even} \\ 0, & \text{if } M \text{ is odd}\end{cases} \tag{16}$$

and

$$\begin{aligned}\mathcal{G}(\alpha,\beta;-n_1,-n_2,-n_3;a_1,a_2,a_3;c_1,c_2,c_3) = {} & \sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3}\frac{(-n_1)_{j_1}(a_1)_{j_1}}{(c_1)_{j_1}j_1!}\frac{(-n_2)_{j_2}(a_2)_{j_2}}{(c_2)_{j_2}j_2!}\frac{(-n_3)_{j_3}}{(c_3)_{j_3}}\\ & \times\frac{(a_3)_{j_3}}{j_3!}\frac{\left[(-1)^{2\beta+1}+1\right]B(j_1+j_2+j_3+\alpha,\beta)}{2},\end{aligned} \tag{17}$$

with

$$(a)_j = \begin{cases} 1, & \text{if } j=0 \\ a(a+1)\dots(a+j-1), & \text{if } j=1,2,\dots,\end{cases} \tag{18}$$

and $B(\cdot)$ is the beta function.

Proof. By substituting Eq. (9b) into Eq. (11), Eq. (14) can be achieved with

$$\mathcal{P}_{nmn'm'pq} = \int_{-1}^{1} P_{nm}(\cos\theta)P'_{n'm'}(\cos\theta)P_{pq}(\cos\theta)d\cos\theta, \tag{19a}$$

$$\mathcal{E}_{mm'q} = \int_0^{2\pi} e^{-im\phi}e^{im'\phi}e^{-iq\phi}d\phi = \begin{cases} 2\pi, & \text{if } m'-m-q=0 \\ 0, & \text{otherwise.}\end{cases} \tag{19b}$$

In order to calculate the integral of Eq. (19a), we begin with the Euler integral,

$$\int_0^1 (1-x)^{\alpha-1}x^{\beta-1}dx = B(\alpha,\beta) \quad (\mathrm{Re}\,\alpha>0, \mathrm{Re}\,\beta>0). \tag{20}$$

From Eq. (20), we can obtain

$$\int_{-1}^{1}(1-t^2)^{\alpha-1}t^{2\beta-1}dt = \frac{\left[(-1)^{2\beta+1}+1\right]B(\alpha,\beta)}{2}\quad (\mathrm{Re}\,\alpha>0, \mathrm{Re}\,\beta>0). \tag{21}$$

Into Eq. (21) we add the product of three hypergeometric polynomials defined by

$$F(-n,a;c;x) = \sum_{j=0}^{n}\frac{(-n)_j(a)_j}{(c)_j j!}x^j, \tag{22}$$

to obtain

$$\int_{-1}^{1}(1-t^2)^{\alpha-1}t^{2\beta-1}F(-n_1,a_1;c_1;1-t^2)F(-n_2,a_2;c_2;1-t^2)F(-n_3,a_3;c_3;1-t^2)dt$$
$$=\sum_{j_1=0}^{n_1}\sum_{j_2=0}^{n_2}\sum_{j_3=0}^{n_3}\frac{(-n_1)_{j_1}(a_1)_{j_1}}{(c_1)_{j_1}j_1!}\frac{(-n_2)_{j_2}(a_2)_{j_2}}{(c_2)_{j_2}j_2!}\frac{(-n_3)_{j_3}(a_3)_{j_3}}{(c_3)_{j_3}j_3!}\frac{\left[(-1)^{2\beta+1}+1\right]B(j_1+j_2+j_3+\alpha,\beta)}{2}, \tag{23}$$

and then the relation between the associated Legendre function and the hypergeometric function can be given as[16]

$$P_{nm}(t) = H(n,m)\left[\delta_{m+n}(1-t^2)^{m/2}F\left(\frac{m-n}{2},\frac{1+m+n}{2};m+1;1-t^2\right)\right.$$
$$\left.+\delta_{m+n+1}(1-t^2)^{m/2}tF\left(\frac{1+m-n}{2},\frac{2+m+n}{2};m+1;1-t^2\right)\right]. \tag{24}$$

Substituting Eq. (24) into Eq. (19a) together with Eq. (23) completes the proof.

Although the proofs of intensity coefficients in spherical domain are self-sufficient, we have verified the theory by simulations, but not present due to lack of available space.

5. Truncation theorem

The pressure representation, Eq. (3), has an infinite number of modes, however, we can truncate this series expansion to a finite number within the region of interest due to the properties of the spherical Bessel function. Therefore, Eq. (3) can be truncated to $N=\lceil keR/2\rceil$ (Refs. 17 and 18) terms as

$$P_N(\boldsymbol{x},k) = \sum_{n=0}^{N}\sum_{m=-n}^{n}\alpha_{nm}(k)j_n(kr)Y_{nm}(\theta,\phi), \tag{25}$$

where k is the wave number and R is the radius of the region of interest.

Likewise, it would be beneficial to truncate the intensity expression, Eq. (2). To describe the spatial sound intensity vectors in the region of interest with radius R, the order of sound intensity expressions in the r and ϕ directions is $P=2N$ because of

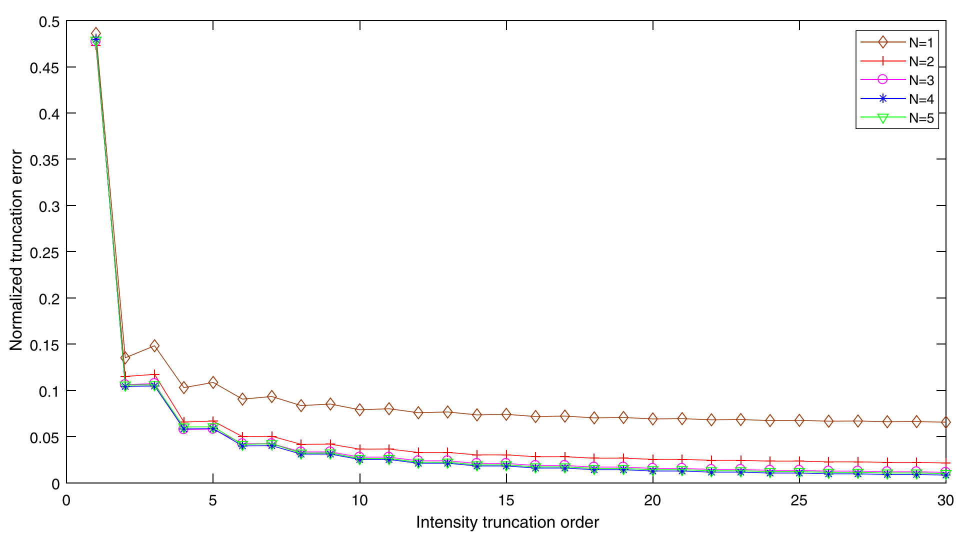

Fig. 1. (Color online) The relationship between normalized truncation error and intensity truncation order P_θ for various pressure truncation orders N.

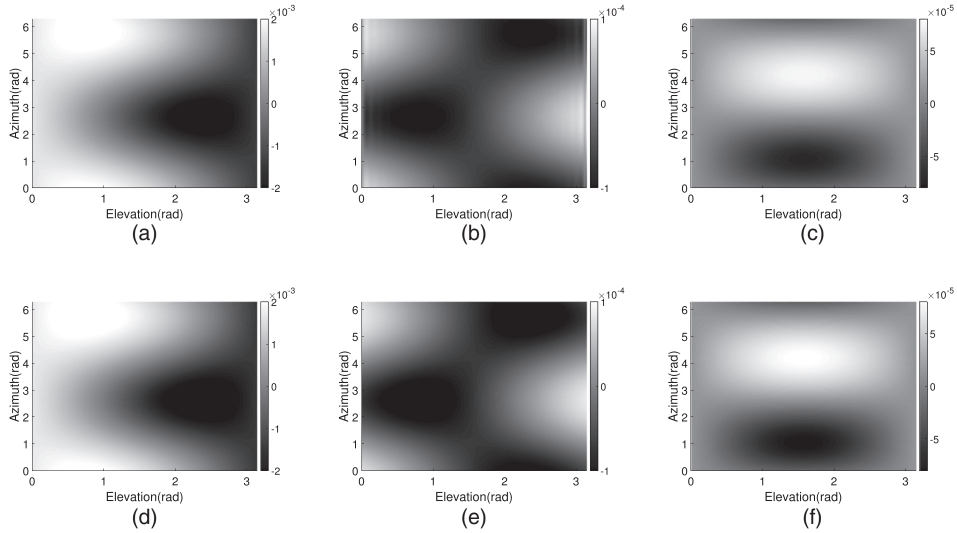

Fig. 2. Sound intensity on a sphere with radius of 0.05 m, generated by a plane wave from $(3\pi/4, 5\pi/6)$, with frequency 600 Hz. (a)–(c) Sound intensity in the r, θ, and ϕ directions, separately, calculated using the proposed theory, (d)–(f) sound intensity in the r, θ, and ϕ directions, separately, obtained from point by point measurement.

the selection rule of Wigner 3–j symbols that W_1 and W_2 equal zero when $p > n + n'$. The expressions in the r and ϕ directions can be rewritten as

$$I_{D'}(\boldsymbol{x}, k) = \sum_{p=0}^{P} \sum_{q=-p}^{p} S_{pq}^{(D')}(k, r) Y_{pq}(\theta, \phi); \quad D' = \{r, \phi\}. \tag{26}$$

However, it is quite different and hard to identify the active modes directly for sound intensity expression in the θ direction. In order to show the active modes, intuitively we calculate the normalized truncation error in terms of intensity truncation order P_θ for five different pressure truncation orders N determined by different radii of the region of interest, which is shown in Fig. 1. Note that the normalized truncation error is measured by

$$\epsilon(k) = \frac{\sum_{\forall \boldsymbol{x}} |I(\boldsymbol{x}, k) - \hat{I}(\boldsymbol{x}, k)|^2}{\sum_{\forall \boldsymbol{x}} |I(\boldsymbol{x}, k)|^2}, \tag{27}$$

where $\hat{I}(\boldsymbol{x}, k)$ is reconstructed sound intensity at point $\boldsymbol{x}$ using Eq. (2) with given truncation orders. Observe that the intensity truncation error becomes less and falls to an acceptable value as intensity truncation order increases no matter what N is. Also, as N grows, it has less influence on intensity truncation error.

6. Simulations

As mentioned in Sec. 1, the theory of spatial sound intensity vectors in a spherical harmonic domain can be largely useful in many applications including intensity measurement over continuous spatial regions, sound field reproduction, DOA estimation, etc. In this section we provide simulation results for the application of spatial intensity measurement, using pressure coefficients of a sound field (spherical harmonic domain). Figure 2 shows sound intensity on a sphere obtained from the proposed theory (note that the spherical pressure coefficients here were obtained using a spherical microphone array) against point by point measurement [note that point by point intensity was simulated using the theoretical expression for pressure due to a plane wave and the relationship in Eq. (1)]. We observe that reconstructed sound intensity vectors by using the proposed theory are similar to actual point by point measurements. Note that the performance of sound intensity in the θ direction is slightly worse than the other two, which is caused by the truncation error discussed in Sec. 5.

7. Conclusion

In this paper, we have defined and formulated the theory of spatial sound intensity vectors in spherical harmonic domain that is applicable to a variety of acoustic scenarios. The complete sets of closed form intensity coefficients are derived, and finite modes of spherical harmonics are suggested for practical implementations.

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