CORE
Metadata, citation and similar papers at core.ac.uk
Provided by Syracuse University Research Facility and Collaborative Environment

Syracuse University
SURFACE

Dissertations - ALL
SURFACE

December 2019

Homology over a Complete Intersection Ring via the Generic Hypersurface

Eric Jeffrey Ottman
Syracuse University

Follow this and additional works at: https://surface.syr.edu/etd

Part of the Physical Sciences and Mathematics Commons

Recommended Citation

Ottman, Eric Jeffrey, "Homology over a Complete Intersection Ring via the Generic Hypersurface" (2019). *Dissertations - ALL*. 1131.
https://surface.syr.edu/etd/1131

This Dissertation is brought to you for free and open access by the SURFACE at SURFACE. It has been accepted for inclusion in Dissertations - ALL by an authorized administrator of SURFACE. For more information, please contact surface@syr.edu.

ABSTRACT

We study homological properties and constructions for modules over a complete intersection ring $Q/(f_1, \ldots, f_c)$ by way of the related generic hypersurface ring $Q[T_1, \ldots, T_c]/(f_1T_1 + \cdots + f_cT_c)$. The advantage of this approach is that over a hypersurface ring, free resolutions are eventually 2-periodic, given by matrix factorizations, and are thus relatively easy to understand. We approach this relationship in two ways. First, we give a correspondence between the two rings in the graded setting, where existing results are insufficient for preserving graded structures. As an application, we use this correspondence to move a functor appearing in a theorem of Orlov to the generic hypersurface setting. Second, we shift out of the graded setting to discuss the relationship between Tor groups over these rings, inspired by recent work of Bergh and Jorgensen, and building on cohomological results of Burke and Walker. This second part takes place in a scheme-theoretic context, so we develop some machinery that provides a sort of "global Tor" for complexes of sheaves that can be compared to the usual Tor for modules.

Homology over a Complete Intersection Ring via the Generic Hypersurface

by

Eric Ottman

B.S., University of Rochester, 2010

M.S., Syracuse University, 2018

Dissertation

Submitted in partial fulfillment of the requirements for the degree of

Doctor of Philosophy in Mathematics

Syracuse University

December 2019

Copyright © Eric Ottman 2019

All Rights Reserved

ACKNOWLEDGMENTS

First, I would like to thank Naomi Jochnowitz, without whose guidance and support I almost certainly would not have ended up on the path that led me here. I would also like to thank all of the professors at Syracuse University who, against all odds, collectively helped me grow into something vaguely resembling a decent mathematician. Most importantly, I want to thank my advisor, Claudia Miller, whose infectious and seemingly limitless kindness and enthusiasm for mathematics have been and continue to be a constant source of knowledge, inspiration, and joy.

Contents

Chapter 1

Introduction

The work presented in this thesis is centered around exploring relationships between complete intersection rings and their generic hypersurface rings. Let R be a complete intersection ring, i.e.,

$$R = \frac{Q}{(f_1, \ldots, f_c)},$$

where Q is a regular local ring and $f_1, \ldots, f_c$ a regular sequence on Q. The generic hypersurface ring of R is

$$\frac{S}{(W)} = \frac{Q[T_1, \ldots, T_c]}{(f_1 T_1 + \cdots , f_c T_c)},$$

with S and W defined in the obvious ways.

We aim to develop a framework for moving back and forth between properties of an R-module M and those of the associated $S/(W)$-module $M' := M[T_1, \ldots, T_c]$, building on recent work of Burke and Walker ([6] and [7]). These papers, extending a theorem of Orlov in [12], establish, in a more general setting, an equivalence of categories

$$\Psi : [MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)] \to \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$$

between the singularity category $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, which captures so-called stable homological behavior over the complete intersection R, and the matrix factorization category $[MF]$, which does the same for the generic hypersurface $S/(W)$. We will formally define these categories in Chapter 2. Burke and Walker also give an explicit description of the inverse of Ψ when R is a complete intersection.

The thesis is structured as follows: In Chapter 2 we recall most of the background definitions and results that are needed. After this, the thesis contains results in two different directions. The first direction involves a functor defined by Orlov (independent of the equivalence of categories described above), and modified slightly by Burke and Stevenson, typically referred to as the Orlov embedding, that can be used to understand the homological properties of graded modules when R is a graded ring; in Section 3.1, we give a result that facilitates explicit computations of the embedding (Proposition 3.1.1) and Section 3.2 consists of an example in which we use this proposition to compute the image of the Orlov embedding for a particular ring. In Chapter 4 we define a modified version of Ψ (Construction 4.3.1) which admits a compatible version of the Orlov embedding on matrix factorizations. This compatible version of the Orlov embedding is given as Definition 4.4.3. A more detailed summary of Chapter 4 can be found in Section 4.1. In Chapter 5, we give some additional scheme-theoretic background in Section 5.2 before proving a result allowing for Tor groups to be passed along Ψ and its inverse in the non-graded setting in Section 5.3. The main result is Theorem 5.3.1, which provides an isomorphism between notions of Tor. Section 5.4 consists of adaptions to Tor groups of results of Burke and Walker for Ext groups and uses these results to obtain Corollary 5.4.5, which relates vanishing of Tor groups over complete intersections to vanishing of Tor over the generic hypersurface. A more detailed summary of Chapter 5 is given in Section 5.1.

Chapter 2

Background

While all of the original results of this thesis are for commutative rings, some of the machinery used is valid even for noncommutative rings, so we will present it in that generality. As such, in Sections 2.1-2.4, unless otherwise specified we do not assume that rings are commutative.

2.1 Categorical constructions

Throughout the thesis we will work with various localizations of categories, so we first recall the construction of the localization of a category.

2.1.1 Definition. A class W of morphisms in a category $\mathcal{C}$ is called a *multiplicative system* if it satisfies the following properties:

(1) W contains all identity morphisms.

(2) W is closed under composition, i.e., if $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ are in W, then so is $A \xrightarrow{g \circ f} C$.

(3) Any diagram $A' \xleftarrow{f} A \xrightarrow{u} B$ in $\mathcal{C}$ with $f \in W$ can be extended to a commutative

diagram

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"] & B \arrow[d, "g"] \\ A' \arrow[r, "u'"] & B' \end{tikzcd}$$

in $\mathcal{C}$, with $g \in W$. Additionally, the statement obtained by reversing all arrows holds.

(4) For any two morphisms $u, v : A \to B$ in $\mathcal{C}$ and a fixed object $Z \in \mathcal{C}$, the existence of $f : Z \to A$ in W with $uf = vf$ implies that there exists $g : B \to C$ (for some $C \in \mathcal{C}$) in W with $gu = gv$. Also, given a fixed object $C \in \mathcal{C}$, existence of $g : B \to C$ in W with $gu = gv$ implies that there exists $f : Z \to A$ (for some $Z \in \mathcal{C}$) in W with $uf = vf$.

Given a category $\mathcal{C}$ and a multiplicative system W of morphisms in $\mathcal{C}$, one may construct the localization $\mathcal{C}[W^{-1}]$. The motivation behind the construction is similar to that of localizations of commutative rings; by localizing at W one makes all of the morphisms in W invertible (and thus makes the morphisms in W into isomorphisms).

2.1.2 Definition. The objects of $\mathcal{C}[W^{-1}]$ are the same as those of $\mathcal{C}$.

Given any two objects X and Y of $\mathcal{C}$, the morphisms from X to Y in $\mathcal{C}[W^{-1}]$ are defined to be equivalence classes of diagrams

$$X \xleftarrow{f} X' \to Y,$$

with X' any object of $\mathcal{C}$, f a morphism in W, and the unlabeled arrow any morphism in $\mathcal{C}$. Such diagrams are typically referred to as roofs.

Two roofs $X \xleftarrow{f} X' \xrightarrow{u} Y$ and $X \xleftarrow{g} X'' \xrightarrow{v} Y$ are equivalent if there exists a commutative diagram

$$\begin{tikzcd} & X' \arrow[dl, "f"'] \arrow[dr, "u"] & \\ X & X''' \arrow[l, "h"'] \arrow[r, "w"] \arrow[u] \arrow[d] & Y \\ & X'' \arrow[ul, "g"] \arrow[ur, "v"'] & \end{tikzcd}$$

in $\mathcal{C}$, with $h \in W$.

The composition of two roofs $X \xleftarrow{f} X' \xrightarrow{u} Y$ and $Y \xleftarrow{g} Y' \xrightarrow{v} Z$ is defined as follows: by property (3) of Definition 2.1.1 of a multiplicative system, there exists a commutative diagram

$$\begin{tikzcd}
 & & X'' \arrow[dl, "f'"'] \arrow[dr, "v'"] & & \\
 & X' \arrow[dl, "f"'] \arrow[dr, "u"] & & Y' \arrow[dl, "g"'] \arrow[dr, "v"] & \\
X & & Y & & Z
\end{tikzcd}$$

in $\mathcal{C}$, with $f' \in W$. The composition is defined to be the roof

$$X \xleftarrow{f \circ f'} X'' \xrightarrow{v \circ v'} Z.$$

A morphism $g : X \to Y$ in $\mathcal{C}$ may be represented in $\mathcal{C}[W^{-1}]$ by the roof

$$X \xleftarrow{\mathrm{id}} X \xrightarrow{g} Y,$$

and if $g \in W$ then its inverse in $\mathcal{C}[W^{-1}]$ is represented by the roof

$$Y \xleftarrow{g} X \xrightarrow{\mathrm{id}} X.$$

2.1.3 Definition. A *translation functor* (or *shift functor*) on a category $\mathcal{C}$ is an automorphism $T : \mathcal{C} \to \mathcal{C}$; for an object $X \in \mathcal{C}$, $T^n X$ is typically written as $X[n]$.

A *triangle* (X, Y, Z, u, v, w) is a collection of three objects $X, Y, Z \in \mathcal{C}$ and morphisms $X \xrightarrow{u} Y, Y \xrightarrow{v} Z$, and $Z \xrightarrow{w} X[1]$. Such a triangle is usually written as

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1].$$

A *triangulated category* is an additive category $\mathcal{C}$ equipped with a shift functor and a class of triangles, called *distinguished triangles*, satisfying the following properties:

(1) For any object X, the triangle

$$X \xrightarrow{\mathrm{id}} X \to 0 \to X[1]$$

is distinguished.

For any morphism $u : X \to Y$, there is an object Z, called a *mapping cone* of u, fitting into a distinguished triangle

$$X \xrightarrow{u} Y \to Z \to X[1].$$

Any triangle isomorphic to a distinguished triangle is distinguished, i.e., for any distinguished triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and isomorphisms $X \xrightarrow{f} X', Y \xrightarrow{g} Y'$, and $Z \xrightarrow{h} Z'$,

$$X' \xrightarrow{guf^{-1}} Y' \xrightarrow{hvg^{-1}} Z' \xrightarrow{f[1]wh^{-1}} X'[1]$$

is a distinguished triangle.

(2) If

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a distinguished triangle, then so are

$$Y \xrightarrow{v} Z \xrightarrow{w} X[1] \xrightarrow{-u[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{u} Y \xrightarrow{v} Z.$$

(3) Given two distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

and

$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} X'[1]$$

and morphisms $X \xrightarrow{f} X'$ and $Y \xrightarrow{g} Y'$ such that $gu = u'f$, there exists a map $h : Z \to Z'$ such that the following diagram commutes:

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, "h"] & X[1] \arrow[d, "f[1]"] \\
X' \arrow[r, "u'"] & Y' \arrow[r, "v'"] & Z' \arrow[r, "w'"] & X'[1].
\end{tikzcd}$$

(4) Given distinguished triangles

$$X \xrightarrow{u} Y \xrightarrow{j} Z' \xrightarrow{k} X[1],$$

$$Y \xrightarrow{v} Z \xrightarrow{l} X' \xrightarrow{i} Y[1],$$

$$X \xrightarrow{vu} Z \xrightarrow{m} Y' \xrightarrow{n} X[1],$$

there exists a distinguished triangle

$$Z' \xrightarrow{f} Y' \xrightarrow{g} X' \xrightarrow{h} Z'[1]$$

such that $l = gm$, $k = nf$, $h = j[1]i$, $ig = u[1]n$, and $fj = mv$.

If $\mathcal{C}$ is a triangulated category, one has the following important special case of localization.

2.1.4 Definition. Given a triangulated category $\mathcal{C}$ and a triangulated subcategory $\mathcal{D}$ of $\mathcal{C}$, the *Verdier quotient* (or *Verdier localization*) $\mathcal{C}/\mathcal{D}$ is defined to be the localization of $\mathcal{C}$ at the class of morphisms whose mapping cones are objects of $\mathcal{D}$.

The basic idea of the Verdier quotient is similar to that of a quotient ring; the following shows that every object of $\mathcal{D}$ is isomorphic to 0 in $\mathcal{C}/\mathcal{D}$, so in some sense the quotient operation kills $\mathcal{D}$. For an object $D \in \mathcal{D}$, the cones of the zero maps $0 \to D \to 0$ are also in $\mathcal{D}$ (this is an immediate consequence of Definition 2.1.3, (1)); therefore

$$D \leftarrow 0 \to 0$$

and

$$0 \leftarrow 0 \to D$$

represent inverse isomorphisms between D and 0 in $\mathcal{C}/\mathcal{D}$.

We now define two categories that will form the primary framework of the rest of the thesis: the (bounded) derived category and the (bounded) singularity category of a ring R.

2.1.5 Definition. Given a ring R, the *bounded derived category* of R, denoted $\mathrm{D^b}(R)$, is constructed as follows. First, one defines the bounded homotopy category $K^{\mathrm{b}}(R)$, whose objects are complexes of R-modules with bounded, finitely generated cohomology and whose morphisms are chain maps modulo homotopy (in other words, homotopic maps are identified). Then $\mathrm{D^b}(R)$ is defined to be the localization of $K^{\mathrm{b}}(R)$ at the class of quasi-isomorphisms.

It typically suffices to think of $\mathrm{D^b}(R)$ simply as a version of the category of chain complexes of R-modules in which quasi-isomorphic complexes have been made isomorphic; in particular, complexes are isomorphic to their resolutions in $\mathrm{D^b}(R)$.

Let X be a complex of R-modules with bounded and finitely generated cohomology, viewed as an object in $\mathrm{D^b}(R)$. The differential of X will always be denoted ∂_X. We denote by $X[i]$ the complex with $X[i]_n = X_{i+n}$ for all n and differential $\partial_{X[i]} = (-1)^i \partial_X$.

Given a map of complexes of R-modules $f : X \to Y$, recall that the *(mapping) cone* of f, denoted $\mathbf{C}(f)$, is the complex with $\mathbf{C}(f)_n = X_{n-1} \bigoplus Y_n$ for all n, and $\partial_{\mathbf{C}(f)}$ given by the matrix

$$\partial_{\mathbf{C}(f)} = \begin{bmatrix} -\partial_X & 0 \\ -f & \partial_Y \end{bmatrix},$$

i.e.,

$$\partial_{\mathbf{C}(f)}(x, y) = (-\partial_X(x), \partial_Y(y) - f(x)).$$

These usual shift and cone operations turn out to define a triangulated structure on $\mathrm{D}^{\mathrm{b}}(R)$, so we may make the following definition.

2.1.6 Definition. Let Perf R denote the full triangulated subcategory of $\mathrm{D}^{\mathrm{b}}(R)$ whose objects are perfect complexes (i.e., complexes that are quasi-isomorphic to bounded complexes of finitely generated projective modules). The *(bounded) singularity category* of R, denoted $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, is defined to be the Verdier quotient

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R) := \frac{\mathrm{D}^{\mathrm{b}}(R)}{\mathrm{Perf}\ R}.$$

The most important feature of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ is that its perfect complexes are isomorphic to 0. This is the motivation behind the terminology; the singularity category ignores objects of finite projective dimension (i.e., nonsingular behavior). In particular, every R-module is isomorphic in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ to all of its syzygies (with corresponding homological shifts).

2.2 Complete resolutions

For this section, when we refer to a ring as Gorenstein we mean that it has finite injective dimension as both a left module and a right module over itself.

Objects of the singularity category $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ are represented by complete resolutions, defined as follows:

2.2.1 Definition. An acyclic complex X of R-modules is called *totally acyclic* if the dual complex $\mathrm{Hom}_R(X, R)$ is also acyclic.

2.2.2 Definition. A *complete resolution* of a complex X of R-modules is a diagram

$$T \xrightarrow{\tau} P \xrightarrow{\pi} X,$$

where T is a totally acyclic complex of projective modules, $P \xrightarrow{\pi} X$ is a projective resolution, τ is a chain map, and $\tau_i : T_i \to P_i$ is an isomorphism for $i \gg 0$. We will often simply refer to T as a complete resolution of M.

In [4], Buchweitz gives a way of constructing a complete resolution of a given object of $\mathrm{D}^{\mathrm{b}}(R)$ when R is Gorenstein, which we reproduce here.

2.2.3 Construction (Buchweitz, [4], §5.6). Given an object X of $\mathrm{D}^{\mathrm{b}}(R)$, one may construct a complete resolution $\mathbf{CR}(X)$ of X as follows: Choose projective resolutions $\mathbf{P}(X) \xrightarrow{\simeq} X$ and $\mathbf{P}(\mathbf{P}(X)^*) \xrightarrow{\varphi_X} \mathbf{P}(X)^*$, where $(-)^*$ denotes $\mathrm{Hom}_R(-, R)$. Denote by $\mathbf{N}(X)$ the composition

$$\mathbf{N}(X) : \mathbf{P}(X) \xrightarrow{\cong} \mathbf{P}(X)^{**} \xrightarrow{\varphi_X^*} \mathbf{P}(\mathbf{P}(X)^*)^*,$$

where the first arrow is the canonical isomorphism. Then

$$\mathbf{CR}(X) := \mathbf{C}(\mathbf{N}(X))[1]$$

is a complete resolution of X.

Complete resolutions are unique up to homotopy equivalence.

For the remainder of this section, let R be commutative.

In the case when R is a hypersurface, i.e. $R = Q/(f)$, where Q is a regular local ring and f is a non-zero divisor, the structure of complete resolutions is fully understood; by [8] they are given by matrix factorizations.

2.2.4 Definition. If Q is any ring, a *matrix factorization* of an element $f \in Q$ is a diagram

$$F \xrightarrow{d_1} G \xrightarrow{d_0} F,$$

where F and G are finitely generated free Q-modules, and the compositions $d_0 \circ d_1$ and $d_1 \circ d_0$ are both multiplication by f. We will sometimes abbreviate this diagram as (d_1, d_0) when the modules F and G are clear from context.

We will denote by $MF(Q, f)$ the category whose objects are matrix factorizations of f over Q, and for which given two matrix factorizations $E = (F \xrightarrow{d_1} G \xrightarrow{d_0} F)$ and $E' = (F' \xrightarrow{d_1'} G' \xrightarrow{d_0'} F')$, the morphisms $E \to E'$ are pairs (u, v) of homomorphisms $u : F \to F'$ and $v : G \to G'$ making the following diagram commute:

$$\begin{tikzcd} F \arrow[r, "d_1"] \arrow[d, "u"] & G \arrow[r, "d_0"] \arrow[d, "v"] & F \arrow[d, "u"] \\ F' \arrow[r, "d_1'"] & G' \arrow[r, "d_0'"] & F'. \end{tikzcd}$$

It follows from results in [8] that if $R = Q/(f)$ is a hypersurface ring, a minimal complete resolution of any R-module M has the form

$$\mathbf{F}(d_1, d_0) : \ \cdots \to \overline{F} \xrightarrow{\overline{d_1}} \overline{G} \xrightarrow{\overline{d_0}} \overline{F} \xrightarrow{\overline{d_1}} \overline{G} \xrightarrow{\overline{d_0}} \overline{F} \to \cdots$$

for some matrix factorization (d_1, d_0) of f in Q, where $\overline{(-)}$ represents reduction modulo f.

2.2.5 Definition. The Hom groups in the singularity category are called *stable Ext groups* (denoted $\widehat{\mathrm{Ext}}$); more precisely, for any two objects X and Y of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$,

$$\widehat{\mathrm{Ext}}^q_R(X, Y) := \mathrm{Hom}_{\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)}(X, Y[q]).$$

Alternatively, by [7], Lemma B.6, if X admits a complete resolution one may define stable

Ext groups (and hence describe the Hom groups of the singularity category) using complete resolutions: if T is a complete resolution of X, one has

$$\widehat{\operatorname{Ext}}^q_R(X,Y) := H^q(\operatorname{Hom}_R(T,Y)).$$

In particular, one sees from this definition that for $q \gg 0$, conventional Ext and stable Ext coincide, since for $q \gg 0$, T coincides with a projective resolution of X. Similarly we may define stable Tor groups, but there is no tensor analog of the first categorical Hom definition given above; so we simply have, when X admits a complete resolution,

2.2.6 Definition. For objects X and Y of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, the *stable Tor* groups are

$$\widehat{\operatorname{Tor}}^R_q(X,Y) := H_q(T \otimes_R Y),$$

with T a complete resolution of X.

2.3 Graded rings and modules

Chapters 3 and 4 concern graded rings, so in this section we establish general conventions and notations for working in the graded setting.

Whenever we refer to R as a "graded ring" we mean that $R = \bigoplus_{i \geq 0} R_i$ is a positively graded (left and right) noetherian ring. For the entirety of this section R is assumed to be graded.

When R is not commutative, "R-module" will mean right R-module. The category of *finitely generated* graded R-modules and degree-zero homomorphisms will be denoted $\operatorname{gr} R$. For each $i \in \mathbb{Z}$, we denote by $\operatorname{gr}_{\geq i} R$ the full subcategory of $\operatorname{gr} R$ whose objects are graded R-modules M for which $M = M_{\geq i}$ (equivalently, for which $M_{<i} = 0$).

If $M = \bigoplus M_i$ is a graded R-module, we denote by $M(k)$ the graded R-module with $M(k)_i = M_{i+k}$ for all i. It is easy to prove ([5], Lemma 3.8, for example) that if P is any finitely generated graded projective R-module, then there exist integers $n, m_1, \ldots, m_n$ such that

$$P \cong \bigoplus_{k=1}^{n} P_k \otimes_{R_0} R(m_k), \tag{2.1}$$

where each P_k is a projective right R_0-module. In other words, every such module is a direct sum of projective modules that are each generated in a single graded degree. If one insists that all m_k are distinct (which is possible because direct sums of projectives are projective), then $n, m_1, \ldots, m_n$ are unique (up to reordering). Given such a graded projective module P, decomposed in this way, and any integer i, we define

$$P_{\prec i} = \bigoplus_{m_k > -i} P_k \otimes_{R_0} R(m_k);$$

in other words, $P_{\prec i}$ consists precisely of the summands generated in graded degree less than i. Similarly, $P_{\succcurlyeq i}$ is defined to be the module consisting of the summands generated in graded degree greater than or equal to i. If P is a free module, one may think of the operation $(-)_{\succcurlyeq i}$ as truncating away all R-summands generated in graded degree less than i.

Given a complex X of finitely generated graded projective R-modules, define $X_{\prec i}$ to be the complex with $(X_{\prec i})_n = (X_n)_{\prec i}$ and the obvious induced differential, and define $X_{\succcurlyeq i}$ similarly. Given a map $f : X \to Y$ of complexes of finitely generated graded projective R-modules, we will denote by $f_{\prec i}$ the composition $X \xrightarrow{f} Y \twoheadrightarrow Y_{\prec i}$ (where the second arrow is the projection), and we will denote by ${}_{\prec i}f$ the restricted map $X_{\prec i} \to Y$ (i.e., the composition $X_{\prec i} \hookrightarrow X \xrightarrow{f} Y$). We have analogous definitions for ${}_{\succ i}f$, $f_{\succ i}$, etc., and we can combine these notations; for instance, ${}_{\prec i}f_{\prec i}$ denotes the composition $X_{\prec i} \hookrightarrow X \xrightarrow{f} Y \twoheadrightarrow Y_{\prec i}$.

We will occasionally need to impose a bit of extra structure on our graded projective resolutions in this chapter and the next, as follows:

2.3.1 Definition. Given a complex X of graded R-modules, let $\mathbf{P}(X)$ denote a graded projective resolution of X, with the property that for any integer i, there exists a k_i such that $(\mathbf{P}(X)_k)_{\prec i} = 0$ for all $k \geq k_i$.

By [5], Lemma 3.10, if R_0 has finite global dimension then such a resolution always exists.

Informally, this condition requires that given any fixed graded degree i, the modules of $\mathbf{P}(X)$ must all be generated in degrees greater than i when one looks far enough to the left.

The definition of a matrix factorization (Definition 2.2.4) must be modified slightly in the graded setting, as follows:

2.3.2 Definition. If Q is any graded ring, a *(graded) matrix factorization* of a homogeneous element $f \in Q_n$ is a diagram

$$F \xrightarrow{d_1} G \xrightarrow{d_0} F(n),$$

where F and G are finitely generated graded free Q-modules, and the compositions $d_0 \circ d_1$ and $d_1(n) \circ d_0$ are both multiplication by f. We will sometimes abbreviate this diagram as (d_1, d_0) when the modules F and G are clear from context.

Graded matrix factorizations of f over Q form a category $MF_{\mathrm{gr}}(Q, f)$, defined analogously to the category $MF(Q, f)$ (see Definition 2.2.4); a morphism $E \to E'$ consists of degree zero homomorphisms $F \xrightarrow{u} F'$ and $G \xrightarrow{v} G'$ such that the obvious diagram commutes.

In the graded setting the complete resolution corresponding to a graded matrix factorization (d_1, d_0) has the following form:

$$\mathbf{F}(d_1, d_0) : \cdots \to \overline{F} \xrightarrow{\overline{d_1}} \overline{G} \xrightarrow{\overline{d_0}} \overline{F}(n) \xrightarrow{\overline{d_1}(n)} \overline{G}(n) \xrightarrow{\overline{d_0}(n)} \overline{F}(2n) \to \cdots ,$$

where $\overline{(-)}$ represents reduction modulo f.

2.3.3 Definition. Given any two complexes X, Y of graded R-modules, we set

$$\mathrm{Hom}^*_{\mathrm{gr}\ R}(X, Y) := \bigoplus_{n \in \mathbb{Z}} \mathrm{Hom}_{\mathrm{gr}\ R}(X, Y(n)).$$

In other words, $\mathrm{Hom}^*_{\mathrm{gr}\ R}(X,Y)$ consists of all homogeneous chain maps (which are just homogeneous homomorphisms if X and Y are modules) of all graded degrees, as opposed to only those of degree zero. If R is commutative, then $\mathrm{Hom}^*_{\mathrm{gr}\ R}(X,Y)$ is a graded R-module.

2.4 The Orlov embedding

The contents of this section (and of Section 3.1) are valid only for graded rings R for which R_0 has finite global dimension and with the following property:

2.4.1 Definition. A (not necessarily commutative) graded ring R is *Artin-Schelter Gorenstein* (or *AS-Gorenstein*) if it has finite graded injective dimension as both a left and a right module over itself, and if

$$\mathbf{R}\mathrm{Hom}^*_{\mathrm{gr}\ R}(R_0, R) \cong R_0[n](a)$$

for some integers n and a, in both $\mathrm{D^b}(\mathrm{gr}\ R)$ and $\mathrm{D^b}(\mathrm{gr}\ R^{\mathrm{op}})$.

The integer a is called the *a-invariant* of R.

The main result of Chapter 3 concerns the so-called Orlov embedding. Before explaining this terminology, we first recall its definition; the following formulation is due to Burke and Stevenson in [5].

2.4.2 Definition. For each $i \in \mathbb{Z}$, define the functor $\mathbf{b}_i : \mathrm{D^b_{sg}}(\mathrm{gr}\ R) \to \mathrm{D^b}(\mathrm{gr}_{\geq i}\ R)$ by

$$\mathbf{b}_i(X) = (\mathbf{P}((\mathbf{P}(X)_{\succcurlyeq i})^*)^*)_{\prec i}$$

for any object $X \in \mathrm{D^b_{sg}}(\mathrm{gr}\ R)$ (in other words, for any object of $\mathrm{D^b}(\mathrm{gr}\ R)$, viewed as an object of $\mathrm{D^b_{sg}}(\mathrm{gr}\ R)$). Here $\mathbf{P}(X)$ denotes a projective resolution as in Definition 2.3.1.

The next result concerns semiorthogonal decompositions, which are defined as follows:

2.4.3 Definition. A *semiorthogonal decomposition* of a triangulated category $\mathcal{T}$ is a pair of full triangulated subcategories $\mathcal{A}$ and $\mathcal{B}$ such that the inclusion functor $i : \mathcal{A} \to \mathcal{T}$ has a left adjoint and $\mathrm{Hom}_{\mathcal{T}}(X, A) = 0$ for all A in $\mathcal{A}$ if and only if X is in $\mathcal{B}$. Such a decomposition is written as

$$\mathcal{T} = (\mathcal{A}, \mathcal{B})$$

A sequence of full triangulated subcategories $(\mathcal{D}_1, \ldots, \mathcal{D}_n)$ of $\mathcal{T}$ is a *semiorthogonal decomposition* if for each $i = 1, \ldots, n-1$, there is a semiorthogonal decomposition

$$\mathcal{T} = (\langle \mathcal{D}_1, \ldots, \mathcal{D}_i \rangle, \langle \mathcal{D}_{i+1}, \ldots, \mathcal{D}_n \rangle),$$

where $\langle \mathcal{D}_1, \ldots, \mathcal{D}_i \rangle$ denotes the thick subcategory generated by $\mathcal{D}_1, \ldots, \mathcal{D}_i$ (a thick subcategory is one that is closed under taking direct summands).

These functors are significant for their role in the following famous result, originally due to Orlov in [13]. We give the version in [5], which is slightly more general than Orlov's original formulation. We explain the relevant terminology after giving the statement.

2.4.4 Theorem (Orlov, Burke-Stevenson; [5], 6.4). *Let $A = \bigoplus_{i \geq 0} A_i$ be a positively graded noetherian AS-Gorenstein ring with A_0 of finite global dimension, but not necesssarily commutative. We assume in addition that A satisfies condition χ. Let a be the a-invariant of A.*

(1) If $a > 0$, then for any $i \in \mathbb{Z}$ there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\ X) = \big(\mathcal{O}(-i-a+1), \ldots, \mathcal{O}(-i), \widetilde{\mathcal{B}}_i\big),$$

where $\mathcal{O}(j)$ is the image of $A(j)$ in $\mathrm{coh}\ X$ and $\mathcal{B}_i$ is the image of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A)$ under the fully faithful functor $\mathbf{b}_i : \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_{\geq i}\ A)$.

(2) If $a < 0$, then for any $i \in \mathbb{Z}$ there is a semiorthogonal decomposition

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A) = \big(pA_0(-i), \ldots, pA_0(-i+a+1), p\mathbf{R}\Gamma_{\geq i-a}\mathrm{D}^{\mathrm{b}}(\mathrm{coh}\ X)\big),$$

where $p : \mathrm{D}^{\mathrm{b}}(\mathrm{gr}_{\geq i}\ A) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A)$ is the canonical quotient.

(3) If $a = 0$, then for any $i \in \mathbb{Z}$ the functors $\widetilde{(-)}\mathbf{b}_i : \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\ X)$ and $p\mathbf{R}\Gamma_{\geq i} : \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\ X) \to \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ A)$ are inverse equivalences.

We now give definitions of the various categories and functors involved in the statement. Condition χ is a technical condition which we will not discuss in detail here; [5], Lemma 6.2 shows that it is satisfied by any flat Gorenstein algebra over a commutative ring. In Orlov's version of the result, coh X is the category of coherent sheaves on a particular projective scheme; in this context, one may instead define it to be the Verdier quotient category

$$\mathrm{coh}\ X := \frac{\mathrm{gr}\ A}{\mathrm{tors}\ A},$$

where tors A is the full subcategory of gr A consisting of what Burke and Stevenson refer to as torsion A-modules, i.e., A-modules which are annihilated by $A_{\geq n}$ for some $n \geq 1$. Since gr A consists of finitely generated graded A-modules, each object of gr A resides in $\mathrm{gr}_{\geq i}\ A$ for some $i \in \mathbb{Z}$. In light of this, each subcategory $\mathrm{gr}_{\geq i}\ A$ contains all of the objects of gr A up to shift. In particular, for each $i \in \mathbb{Z}$ one has an equivalence

$$\mathrm{coh}\ X \xrightarrow{\cong} \frac{\mathrm{gr}_{\geq i}\ A}{\mathrm{tors}_{\geq i}\ A}, \tag{2.2}$$

where $\mathrm{tors}_{\geq i}\ A := \mathrm{gr}_{\geq i}\ A \ \cap\ \mathrm{tors}\ A$. The functor $\Gamma_{\geq i}$ may then be defined as the right adjoint of the quotient functor $\mathrm{gr}_{\geq i} \to \mathrm{coh}\ X$ (as shown in (2.2); this right adjoint exists essentially by [5], Proposition 4.5). The functor $\widetilde{(-)}$ appearing in (3) is the functor $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ A) \to \mathrm{D}^{\mathrm{b}}(\mathrm{coh}\ X)$ induced by the localization functor $\mathrm{gr}\ A \to \mathrm{coh}\ X$.

2.5 The generic hypersurface and the Orlov correspondence

For the remainder of this chapter, all rings are commutative. Furthermore, Q will be a local (or graded) ring with maximal ideal (or homogeneous maximal ideal) $\mathfrak{m}$, $f_1, \ldots, f_c$ a (homogeneous) regular sequence on Q contained in $\mathfrak{m}^2$, and

$$R := \frac{Q}{(f_1, \ldots, f_c)}.$$

We set $S = Q[T_1, \ldots, T_c]$ and $W = f_1T_1 + \cdots + f_cT_c \in S$.

2.5.1 Definition. The *generic hypersurface ring* of R is the ring

$$\frac{S}{(W)} = \frac{Q[T_1, \ldots, T_c]}{(f_1T_1 + \cdots + f_cT_c)}.$$

2.5.2 Remark. The name "generic hypersurface" is motivated by the fact that substituting elements of Q for each of the variables $T_1, \ldots, T_c$ yields a quotient of Q by an element of the ideal $(f_1, \ldots, f_c)$. Such quotients can be thought of as hypersurfaces that lie between Q and R.

The primary connection between R and its generic hypersurface is given by work of Orlov (outside of the graded setting) in [12]. The specific details are not needed for the majority of the thesis, so we give only a brief overview here and provide more detail in Chapter 5. Orlov establishes an equivalence of categories

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{Proj}\ S/(W)) \longrightarrow \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R).$$

In [6], Burke and Walker extend this equivalence to a homotopy category of graded matrix factorizations of W over $\mathbb{P}_Q^{c-1}$, denoted $[MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)$:

$$\Psi : [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)] \xrightarrow{\cong} \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R).$$

The definition of the matrix factorization category $[MF(\mathbb{P}_Q^{c-1}), \mathcal{O}(1), W]$ is rather technical and is not needed in the statements or proofs of the results in this thesis; we give the formal definition, in full generality, in Section 2.7, but really one may simply think of it as the correct scheme-theoretic analog of the usual homotopy category of matrix factorizations of W over S.

2.6 The Eisenbud-Shamash construction

In this section, we remain in the setting of Section 2.5.

For a quotient $R = Q/(f_1, \ldots, f_c)$ with Q any local ring and $f_1, \ldots, f_c$ a regular sequence, Shamash, in [15], gives a construction of a free resolution of an R-module M from a free resolution $\mathbf{G}$ of M over Q, and a system of homotopies for $f_1, \ldots, f_c$ on $\mathbf{G}$, i.e., a collection of degree 1 endomorphisms $s_i : \mathbf{G} \to \mathbf{G}$ for $1 \leq i \leq c$ such that each s_i is a nulhomotopy for multiplication by f_i. This Shamash construction requires the additional property that $s_i^2 = 0$ for all i. In [8, §7], Eisenbud generalizes this construction by removing this extra condition using so-called higher homotopies, as defined below (Definition 2.6.1). We reproduce the latter construction here. Additionally, we will not insist that the complex $\mathbf{G}$ be a resolution of a module; we may apply the Eisenbud-Shamash construction to *any* bounded complex of finitely generated free Q-modules. This construction will be our main tool for studying the bounded derived category of a complete intersection ring.

Throughout the rest of this section, we use the following multi-indexing conventions. For each element $J = (a_1, \ldots, a_c) \in \mathbb{N}^c$, set $|J| = \sum_1^c a_i$. To simplify the statement of the following definition, we abbreviate the tuple $(0, \ldots, 1, \ldots, 0)$, with 1 in the ith position, as simply i, and the tuple $(0, \ldots, 0)$ as 0.

2.6.1 Definition. Let $\mathbf{G}$ be a complex of Q-modules. A *system of higher homotopies* for $f_1, \ldots, f_c$ on $\mathbf{G}$ is a family

$$\boldsymbol{\sigma} = \{\sigma^J | J \in \mathbb{N}^c\}$$

of endomorphisms of $\mathbf{G}$, where σ^J has degree $2|J| - 1$, satisfying the following conditions:

(1) $\sigma^0 = \partial_{\mathbf{G}}$

(2) $\sigma^0 \sigma^i + \sigma^i \sigma^0 = f_i 1_{\mathbf{G}}$

(3) $\displaystyle\sum_{J'+J''=J} \sigma^{J'} \sigma^{J''} = 0$ for all $J \in \mathbb{N}^c, |J| \geq 2$.

By [8], Theorem 7.1, if $\mathbf{G}$ is a Q-free resolution of some module M which is annihilated by $f_1, \ldots, f_c$, then such a system of higher homotopies on $\mathbf{G}$ always exists.

Let D be the graded $Q[T_1, \ldots, T_c]$-module $\mathrm{Hom}^*_{\mathrm{gr}\,Q}(Q[T_1, \ldots, T_c], Q)$, where $T_1, \ldots, T_c$ are variables of degree -2. If we denote by τ_i the dual of T_i in D, note that each τ_i has degree 2 and that the $Q[T_1, \ldots, T_c]$-module structure on D is given by

$$T_i(\tau_1^{m_1} \cdots \tau_c^{m_c}) = \begin{cases} \tau_1^{m_1} \cdots \tau_i^{m_i - 1} \cdots \tau_c^{m_c} & \text{if} \quad m_i \neq 0 \\ 0 & \text{if} \quad m_i = 0 \end{cases}. \tag{2.3}$$

Armed with this machinery and notation, we now describe the Eisenbud-Shamash construction. Let $\mathbf{G}$ be a bounded complex of finitely generated free Q-modules, equipped with a system of higher homotopies $\boldsymbol{\sigma}$ for $f_1, \ldots, f_c$. Consider the graded R-module

$$(\overline{\mathbf{G}} \otimes_R \overline{D})^\natural$$

where $\overline{(-)}$ is $- \otimes_Q R$ and $(\overline{\mathbf{G}} \otimes \overline{D})^\natural$ denotes the underlying graded module of $\overline{\mathbf{G}} \otimes \overline{D}$, ignoring the differential. There is an endomorphism $\partial : \overline{\mathbf{G}} \otimes \overline{D} \to \overline{\mathbf{G}} \otimes \overline{D}$ given by

$$\partial = \sum_{J \in \mathbb{N}^c} \sigma^J \otimes T^J, \tag{2.4}$$

where $T^J = T_1^{a_1} \cdots T_c^{a_c}$ for $J = (a_1, \dots, a_c)$. Note that ∂ is homogeneous of degree -1; for each $J \in \mathbb{N}^c$, σ^J has degree $2|J| - 1$ and T^J has degree $-2|J|$. Furthermore, one can show that $\partial^2 = 0$.

2.6.2 Definition. We denote by $\mathbf{G}\{\boldsymbol{\sigma}\}$ the complex with underlying modules given by the graded components of $(\overline{\mathbf{G}} \otimes \overline{D})^{\natural}$ and differential ∂. The complex $\mathbf{G}\{\boldsymbol{\sigma}\}$ is sometimes called the *Eisenbud-Shamash complex* of $(\mathbf{G}, \boldsymbol{\sigma})$.

The following is well known; in the case that $\mathbf{G}$ is a resolution of an R-module, it is due to Eisenbud ([8], Theorem 7.2). In the general case, Eisenbud's proof still carries through verbatim as long as one replaces all instances of M with the complex $H(\mathbf{G})$ (which is called $H(\mathbf{F})$ in his notation), with 0 differential.

2.6.3 Proposition. *For any* $(\mathbf{G}, \boldsymbol{\sigma})$ *as in the above construction,*

$$H(\mathbf{G}) \cong H(\mathbf{G}\{\boldsymbol{\sigma}\}).$$

2.6.4 (Eisenbud-Shamash in the graded case). If Q is a graded ring, $f_1, \dots, f_c$ homogeneous, and $\mathbf{G}$ a bounded complex of finitely generated graded free Q-modules (i.e., if $\partial_{\mathbf{G}}$ is homogeneous of graded degree zero), we may preserve this grading in the Eisenbud-Shamash complex by assigning the following grading to $\mathbf{G}\{\boldsymbol{\sigma}\}$: Assign to elements of R their pre-existing graded degree in R, and assign to each τ_i the graded degree deg f_i.

One can see that ∂ is homogeneous of graded degree zero as follows: Since $\sigma^0 = \partial_{\mathbf{G}}$ has graded degree zero, by condition (2) of Definition 2.6.1 each σ^i is a map of graded degree deg f_i. Since each τ_i has degree deg f_i, by (2.3) each T_i is a map of graded degree $-$deg f_i, so we see that $\sigma^i \otimes T_i$ is a map of graded degree 0. By induction using condition (3) of Definition 2.6.1, in fact all of the summands in (2.4) have graded degree 0, so ∂ indeed has graded degree zero.

2.7 The homotopy category of matrix factorizations

In this section we give the definition, first given in [14], of the homotopy category of matrix factorizations, following [7]. For the entirety of this section, let X be a Noetherian separated scheme, $\mathcal{L}$ a line bundle on X, and W a global section of $\mathcal{L}$. For any coherent sheaf $\mathcal{G}$ on X, we denote by $\mathcal{G}(n)$ the twist $\mathcal{G} \otimes_{\mathcal{O}_X} \mathcal{L}^{\otimes n}$, where $\mathcal{L}^{\otimes -n} := \mathcal{H}\mathrm{om}_{\mathcal{O}_X}(\mathcal{L}^{\otimes n}, \mathcal{O}_X)$ for all $n > 0$.

We first require a scheme-theoretic definition of matrix factorization; the following definition first appeared in [14].

2.7.1 Definition. A *matrix factorization* $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ of the triple $(X, \mathcal{L}, W)$ consists of a pair of locally free coherent sheaves $\mathcal{E}_1, \mathcal{E}_0$ on X and morphisms e_1, e_0 such that $e_0 \circ e_1 = e_1(1) \circ e_0 = W \cdot \mathrm{Id}$. Such matrix factorizations form a category, denoted $MF(X, \mathcal{L}, W)$, with morphisms defined in the same way as for matrix factorizations over a ring; in other words, given matrix factorizations $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ and $\mathbb{F} = (\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_0 \xrightarrow{f_0} \mathcal{F}_1(1))$ of $(X, \mathcal{L}, W)$, a morphism $\mathbb{E} \to \mathbb{F}$ in $MF(X, \mathcal{L}, W)$ consists of a pair of maps $\mathcal{E}_0 \to \mathcal{F}_0$ and $\mathcal{E}_1 \to \mathcal{F}_1$ such that the two obvious squares commute.

The morphisms in the homotopy category of matrix factorizations are defined using the following construction:

2.7.2 Definition. Let $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ and $\mathbb{F} = (\mathcal{F}_1 \xrightarrow{f_1} \mathcal{F}_0 \xrightarrow{f_0} \mathcal{F}_1(1))$ be objects of $MF(X, \mathcal{L}, W)$. The *mapping complex* of $\mathbb{E}$ and $\mathbb{F}$, denoted $\mathcal{H}\mathrm{om}_{\mathrm{MF}}(\mathbb{E}, \mathbb{F})$, is the following complex of locally free sheaves:

$$\cdots \xrightarrow{\partial^0(-1)} \begin{matrix} \mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{F}_1) \\ \oplus \\ \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{F}_0(-1)) \end{matrix} \xrightarrow{\partial^{-1}} \begin{matrix} \mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{F}_0) \\ \oplus \\ \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{F}_1) \end{matrix} \xrightarrow{\partial^0} \begin{pmatrix} \mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{F}_1) \\ \oplus \\ \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{F}_0(-1)) \end{pmatrix} (1) \xrightarrow{\partial^{-1}(1)} \cdots$$

The term $\mathcal{H}\mathrm{om}(\mathcal{E}_0, \mathcal{F}_0) \oplus \mathcal{H}\mathrm{om}(\mathcal{E}_1, \mathcal{F}_1)$ resides in degree 0, and the differentials are defined as

$$\partial^{-1} := \begin{bmatrix} (f_1)_* & -e_0^* \\ -e_1^* & (f_0)_* \end{bmatrix} \quad \text{and} \quad \partial^0 := \begin{bmatrix} (f_0)^* & e_0^* \\ e_1^* & (f_1)_* \end{bmatrix}.$$

One can easily see that given objects $\mathbb{E}$ and $\mathbb{F}$ of $MF(X, \mathcal{L}, W)$,

$$\mathrm{Hom}_{MF(X,\mathcal{L},W)}(\mathbb{E}, \mathbb{F}) = Z^0(\Gamma(X, \mathcal{H}\mathrm{om}_{\mathrm{MF}}(\mathbb{E}, \mathbb{F}))),$$

where $\Gamma(X, \mathcal{H}\mathrm{om}_{\mathrm{MF}}(\mathbb{E}, \mathbb{F}))$ denotes the complex obtained by applying the global sections functor degreewise to $\mathcal{H}\mathrm{om}_{\mathrm{MF}}(\mathbb{E}, \mathbb{F})$ and Z^0 denotes the cycles in degree zero.

One defines the *naive homotopy category* of matrix factorizations of $(X, \mathcal{L}, W)$ to have the same objects as $MF(X, \mathcal{L}, W)$, and, given any two objects $\mathbb{E}$ and $\mathbb{F}$, morphisms given by

$$\mathrm{Hom}_{[MF(X,\mathcal{L},W)]_{\mathrm{naive}}}(\mathbb{E}, \mathbb{F}) := H^0(\Gamma(X, \mathcal{H}\mathrm{om}_{\mathrm{MF}}(\mathbb{E}, \mathbb{F}))).$$

2.7.3 Definition. An object $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ of $[MF(X, \mathcal{L}, W)]_{\mathrm{naive}}$ is *locally contractible* if, for each $x \in X$, the matrix factorization (of W_x over $\mathcal{O}_{X,x}$)

$$\mathbb{E}_x = ((\mathcal{E}_1)_x \xrightarrow{(e_1)_x} (\mathcal{E}_0)_x \xrightarrow{(e_0)_x} (\mathcal{E}_1(1))_x)$$

is isomorphic to zero in $[MF(\mathrm{Spec}\ \mathcal{O}_{X,x}, \mathcal{L}_x, W_x)]_{\mathrm{naive}}$.

It is shown in [6] that $[MF(X, \mathcal{L}, W)]_{\mathrm{naive}}$ is a triangulated category and that the full subcategory consisting of locally contractible objects is thick, so one may define the homotopy category of matrix factorizations as follows:

2.7.4 Definition. The homotopy category of matrix factorizations of $(X, \mathcal{L}, W)$, denoted $[MF(X, \mathcal{L}, W)]$, is the Verdier quotient

$$[MF(X, \mathcal{L}, W)] := \frac{[MF(X, \mathcal{L}, W)]_{\mathrm{naive}}}{\text{locally contractible objects}}.$$

Chapter 3

Computing the Orlov Embedding

3.1 Computing the Orlov embedding

In this chapter, R is a commutative graded (AS-)Gorenstein ring.

The nature of the definition of the functors $\mathbf{b}_i$ is somewhat abstract at face value. However, it turns out that applying $\mathbf{b}_i$ to a complex of R-modules is the same as constructing a complete resolution of X and performing an appropriate graded truncation. This is made explicit by the following result, which we suspect is known to Burke and Stevenson, but to our knowledge is not acknowledged or proved explicitly in the literature.

3.1.1 Proposition. *For any complex X in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$, viewed as an object of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(\mathrm{gr}\ R)$, $\mathbf{b}_i(X) = \mathbf{CR}(X)_{\succcurlyeq i}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$.*

Proof. We proceed by fixing i and carefully working through Definition 2.4.2 of $\mathbf{b}_i(X)$. First note that $(\mathbf{P}(X)_{\succcurlyeq i})^* = (\mathbf{P}(X)^*)_{\preccurlyeq i}$.

Denote by Q the cone of the map

$$(\varphi_X)_{\succ i}[1] : (\mathbf{P}(\mathbf{P}(X)^*)[1] \to ((\mathbf{P}(X)^*)_{\succ i})[1].$$

(Recall that φ_X is a quasi-isomorphism $\mathbf{P}(\mathbf{P}(X)^*) \to \mathbf{P}(X)^*$.) More explicitly, we have

$$Q_n = (\mathbf{P}(\mathbf{P}(X)^*))_n \oplus ((\mathbf{P}(X)^*)_{n+1})_{\succ i},$$

$$\partial_Q = \begin{bmatrix} -\partial_{\mathbf{P}(\mathbf{P}(X)^*)[1]} & 0 \\ -(\varphi_X)_{\succ i} & {}_{\succ i}(\partial_{(\mathbf{P}(X)^*)[1]})_{\succ i} \end{bmatrix} = \begin{bmatrix} \partial_{\mathbf{P}(\mathbf{P}(X)^*)} & 0 \\ -(\varphi_X)_{\succ i} & -{}_{\succ i}(\partial_{\mathbf{P}(X)^*})_{\succ i} \end{bmatrix}.$$

Now we consider the map

$$\phi : Q \to (\mathbf{P}(X)^*)_{\preccurlyeq i},$$

defined by

$$\phi = (\varphi_X)_{\preccurlyeq i} \oplus -{}_{\succ i}(\partial_{\mathbf{P}(X)^*})_{\preccurlyeq i}.$$

The cone of ϕ is as follows:

$$\mathbf{C}(\phi)_n = (\mathbf{P}(\mathbf{P}(X)^*))_{n-1} \oplus ((\mathbf{P}(X)^*)_n)_{\succ i} \oplus ((\mathbf{P}(X)^*)_n)_{\preccurlyeq i},$$

$$\partial_{\mathbf{C}(\phi)} = \begin{bmatrix} -\partial_{\mathbf{P}(\mathbf{P}(X)^*)} & 0 & 0 \\ -(\varphi_X)_{\succ i} & {}_{\succ i}(\partial_{\mathbf{P}(X)^*})_{\succ i} & 0 \\ -(\varphi_X)_{\preccurlyeq i} & {}_{\succ i}(\partial_{\mathbf{P}(X)^*})_{\preccurlyeq i} & {}_{\preccurlyeq i}(\partial_{\mathbf{P}(X)^*})_{\preccurlyeq i} \end{bmatrix}.$$

Notice that all of the direct sum components of $\partial_{\mathbf{P}(X)^*}$ are accounted for in this matrix, except for those in the truncation ${}_{\preccurlyeq i}(\partial_{\mathbf{P}(X)^*})_{\succ i}$. However, note that since graded projective modules may be decomposed uniquely as in (2.1), we may view $\partial_{\mathbf{P}(X)^*}$ as a collection of homogeneous maps between the projective R_0-modules P_k. Since the differential must have graded degree zero, there can be no nonzero maps from a given summand to a summand generated in larger graded degree (because, for example, a homogeneous component of the boundary map of degree d must send a degree k generator to a degree k element, which then must reside in a summand generated in degree $k - d$). Therefore ${}_{\prec i}(\partial_{\mathbf{P}(X)})_{\succcurlyeq i} = 0$ because $\partial_{\mathbf{P}(X)}$ is represented by a sequence of matrices all of whose entries all have nonnegative graded degree. Dualizing, we have ${}_{\preccurlyeq i}(\partial_{\mathbf{P}(X)^*})_{\succ i} = 0$. So in fact, the entire boundary map of

$\partial_{\mathbf{P}(X)^*}$ is accounted for in $\partial_{\mathbf{C}(\phi)}$. Therefore, by combining the second and third summands of each $\mathbf{C}(\phi)_n$, we see that $\mathbf{C}(\phi)$ is precisely the cone of the quasi-isomorphism φ_X; hence ϕ is a quasi-isomorphism as well, so we may choose $\mathbf{P}((\mathbf{P}(X)_{\succcurlyeq i})^*)$ to be Q.

Next, since $Q_{-n} = (\mathbf{P}(\mathbf{P}(X)^*))_{-n} \oplus ((\mathbf{P}(X)^*)_{-n+1})_{\succ i}$, we observe that

$$(Q^*)_n = (\mathbf{P}(\mathbf{P}(X)^*)^*)_n \oplus ((\mathbf{P}(X)^{**})_{n-1})_{\prec i},$$
$$\partial_{Q^*} = \begin{bmatrix} \partial_{\mathbf{P}(\mathbf{P}(X)^*)^*} & -(\varphi_X^*)_{\prec i} \\ 0 & -_{\prec i}(\partial_{\mathbf{P}(X)^{**}})_{\prec i} \end{bmatrix}.$$

By definition, $\mathbf{b}_i(X) = (Q^*)_{\prec i}$; more precisely,

$$\mathbf{b}_i(X)_n = ((Q^*)_{\prec i})_n = ((\mathbf{P}(\mathbf{P}(X)^*)^*)_n)_{\prec i} \oplus ((\mathbf{P}(X)^{**})_{n-1})_{\prec i},$$
$$\partial_{\mathbf{b}_i(X)} = \partial_{(Q^*)_{\prec i}} = \begin{bmatrix} {}_{\prec i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\prec i} & -_{\prec i}(\varphi_X^*)_{\prec i} \\ 0 & -_{\prec i}(\partial_{\mathbf{P}(X)^{**}})_{\prec i} \end{bmatrix}.$$

Recall from Construction 2.2.3 that a complete resolution of X is given by the shifted cone of the map $\mathbf{N}(X) : \mathbf{P}(X) \to \mathbf{P}(\mathbf{P}(X)^*)^*$ defined as the composition of the canonical isomorphism $\mathbf{P}(X) \to \mathbf{P}(X)^{**}$ with φ_X^*. So to complete the proof, consider the truncated complete resolution $\mathbf{CR}(X)_{\succcurlyeq i} = \mathbf{C}(\mathbf{N}(X))_{\succcurlyeq i}[1] \cong \mathbf{C}(\varphi_X^*)_{\succcurlyeq i}[1]$. Explicitly,

$$(\mathbf{CR}(X)_{\succcurlyeq i})_n \cong ((\mathbf{P}(X)^{**})_n)_{\succcurlyeq i} \oplus ((\mathbf{P}(\mathbf{P}(X)^*)^*)_{n+1})_{\succcurlyeq i},$$
$$\partial_{\mathbf{CR}(X)_{\succcurlyeq i}} = -\begin{bmatrix} -_{\succcurlyeq i}(\partial_{\mathbf{P}(X)^{**}})_{\succcurlyeq i} & 0 \\ -_{\succcurlyeq i}(\varphi_X^*)_{\succcurlyeq i} & {}_{\succcurlyeq i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\succcurlyeq i} \end{bmatrix} = \begin{bmatrix} {}_{\succcurlyeq i}(\partial_{\mathbf{P}(X)^{**}})_{\succcurlyeq i} & 0 \\ {}_{\succcurlyeq i}(\varphi_X^*)_{\succcurlyeq i} & -_{\succcurlyeq i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\succcurlyeq i} \end{bmatrix}.$$

Now we define the map

$$\psi : \mathbf{CR}(X)_{\succcurlyeq i} \to \mathbf{b}_i(X),$$

given by

$$\psi = \begin{bmatrix} {}_{\succcurlyeq i}(\varphi_X^*)_{\prec i} & {}_{\succcurlyeq i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\prec i} \\ {}_{\succcurlyeq i}(\partial_{\mathbf{P}(X)^{**}})_{\prec i} & 0 \end{bmatrix}.$$

Then the mapping cone of ψ is given by

$$\mathbf{C}(\psi)_n \cong ((\mathbf{P}(X)^{**})_{n-1})_{\succcurlyeq i} \oplus ((\mathbf{P}(\mathbf{P}(X)^*)^*)_n)_{\succcurlyeq i} \oplus ((\mathbf{P}(\mathbf{P}(X)^*)^*)_n)_{\prec i} \oplus ((\mathbf{P}(X)^{**})_{n-1})_{\prec i},$$

$$\partial_{\mathbf{C}(\psi)} = \begin{bmatrix} -{}_{\succcurlyeq i}(\partial_{\mathbf{P}(X)^{**}})_{\succcurlyeq i} & 0 & 0 & 0 \\ -{}_{\succcurlyeq i}(\varphi_X^*)_{\succcurlyeq i} & {}_{\succcurlyeq i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\succcurlyeq i} & 0 & 0 \\ -{}_{\succcurlyeq i}(\varphi_X^*)_{\prec i} & -{}_{\succcurlyeq i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\prec i} & {}_{\prec i}(\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*})_{\prec i} & -{}_{\prec i}(\varphi_X^*)_{\prec i} \\ -{}_{\succcurlyeq i}(\partial_{\mathbf{P}(X)^{**}})_{\prec i} & 0 & 0 & -{}_{\prec i}(\partial_{\mathbf{P}(X)^{**}})_{\prec i} \end{bmatrix}.$$

The ${}_{\prec i}(-)_{\succcurlyeq i}$ truncations of $\partial_{\mathbf{P}(X)^{**}}$, $\partial_{\mathbf{P}(\mathbf{P}(X)^*)^*}$, and φ_X^* that are missing from this matrix are all zero, by precisely the same argument as before. So by interchanging the positions of the second and fourth summands, we now see that $\mathbf{C}(\psi) = \mathbf{C}(\varphi_X^*) \cong \mathbf{CR}(X)[-1]$. Since $\mathbf{CR}(X)$ is exact, this implies that ψ is a quasi-isomorphism, so we have $\mathbf{b}_i(X) = \mathbf{CR}(X)_{\succcurlyeq i}$ in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$. Since $(-)_{\succcurlyeq i}$ necessarily truncates out all of $(-)_{<i}$ by definition, in fact $\mathbf{CR}(X)$ is in the subcategory $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}_{\geq i}\ R)$, from which the result follows.

□

3.2 Hypersurface example

In this section we use the result of Proposition 3.1.1 to explicitly compute the images of the functors $\mathbf{b}_i$ for a particular graded hypersurface ring. In light of the role these functors play in describing the structure of the bounded derived category as in Theorem 2.4.4, the ability to compute them explicitly is potentially a useful tool. By Proposition 3.1.1, this computation amounts to simply performing appropriate graded truncations on complete resolutions, which in the hypersurface case correspond precisely to matrix factorizations.

We consider the hypersurface ring

$$R = k[x, y]/(x^3 + xy^3).$$

This ring is known as the E_7 hypersurface singularity, and belongs to an often-studied class of hypersurface rings (called ADE singularities) that are known to have finite representation type, i.e., they have, up to isomorphism, only finitely many indecomposable maximal Cohen-Macaulay (MCM) modules. Such modules can be realized as cokernels of matrix factorizations. In [16], Yoshino gives a list of all the minimal matrix factorizations of $x^3 + xy^3$ over $Q = k[x, y]$, and thus of all the indecomposable MCM R-modules.

We assign a grading to $Q = k[x, y]$ (which induces a grading on R) in which x has degree 3 and y has degree 2, in order to make $x^3 + xy^3$ into a homogeneous element. Since every complete resolution of an R-module M has the form $\mathbf{F}(d_1, d_0)$ as in the final paragraph of Section 2.3, using Yoshino's list of matrix factorizations, given below, we may essentially compute the full image of $\mathbf{b}_i$ in $\mathrm{D^b}(\mathrm{gr}_{\geq i}\ R)$.

The homology computations are easily done by hand for matrices of reasonable small rank; with Macaulay2, one may quickly compute homologies for larger matrices.

Note that given a matrix factorization (d_1, d_0), the pair (d_0, d_1) is also a matrix factorization, and $\mathbf{F}(d_0, d_1)$ is simply a graded and homological shift of $\mathbf{F}(d_1, d_0)$; thus for the purposes of this example the order of the maps does not matter. Furthermore, to obtain all minimal *graded* matrix factorizations of $x^3 + xy^3$, one must consider all graded shifts of the minimal matrix factorizations listed here. Our computation of $\mathbf{b}_i$ for all $i \in \mathbb{Z}$ but for a fixed grading on each matrix factorization is the same (up to graded shift) as computing $\mathbf{b}_i$ for a *fixed* $i \in \mathbb{Z}$, but for all possible graded shifts, since $(\mathbf{b}_i(E))(n) = \mathbf{b}_{i-n}(E(n))$ for any matrix factorization E and any $i, n \in \mathbb{Z}$.

We now list and examine each of the seven minimal matrix factorizations of $x^3 + xy^3$, which we will denote by $\{E^i\}_{i=1}^7$. For each matrix factorization, we give one possible graded complete resolution it produces (all others are simply graded shifts of this chosen resolution) and investigate its graded truncations in detail. To simplify and shorten the exposition, all results given are understood to be valid only up to some possible graded and/or homological shift.

For brevity, we now assign notation to all of the matrices that make up the matrix factorizations $E^1, \dots, E^7$:

$$\alpha = [x], \qquad \beta = [x^2 + y^2],$$

$$\gamma = x \begin{bmatrix} x & y \\ y^2 & -x \end{bmatrix}, \qquad \delta = \begin{bmatrix} x & y \\ y^2 & -x \end{bmatrix},$$

$$\varphi_1 = \begin{bmatrix} x & y \\ xy^2 & -x^2 \end{bmatrix}, \qquad \psi_1 = \begin{bmatrix} x^2 & y \\ xy^2 & -x \end{bmatrix},$$

$$\varphi_2 = \begin{bmatrix} x & y^2 \\ xy & -x^2 \end{bmatrix}, \qquad \psi_2 = \begin{bmatrix} x^2 & y^2 \\ xy & -x \end{bmatrix},$$

$$\xi_1 = \begin{bmatrix} xy^2 & -x^2 & -x^2y \\ xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{bmatrix}, \qquad \eta_1 = \begin{bmatrix} y & 0 & x \\ -x & xy & 0 \\ 0 & -x & y \end{bmatrix},$$

$$\xi_2 = \begin{bmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \\ xy^2 & xy & x^2 \end{bmatrix}, \qquad \eta_2 = \begin{bmatrix} x & 0 & y \\ -xy & x^2 & 0 \\ 0 & -xy & x \end{bmatrix},$$

$$\xi_3 = \begin{bmatrix} \gamma & \epsilon \\ 0 & \delta \end{bmatrix}, \qquad \text{and } \eta_3 = \begin{bmatrix} \delta & -\epsilon \\ 0 & \gamma \end{bmatrix},$$

where

$$\epsilon = \begin{bmatrix} y & 0 \\ 0 & y \end{bmatrix}.$$

We now compute the image of each $\mathbf{b}_i$, one matrix factorization at a time.

(1) $E^1 = (\alpha, \beta)$

A corresponding graded complete resolution $\mathbf{F}(\alpha, \beta)$ is

$$\cdots \to R(-9) \xrightarrow{\alpha} R(-6) \xrightarrow{\beta} R \xrightarrow{\alpha} R(3) \xrightarrow{\beta} R(9) \to \cdots$$

The key observation in directly computing $\mathbf{b}_i$ is that for any $i, n \in \mathbb{Z}$, since $R(n)$ is generated in degree $-n$ we have

$$R(n)_{\succcurlyeq i} = \begin{cases} R(n) & \text{if } i \leq -n \\ 0 & \text{otherwise} \end{cases}$$

So for $i = 0$, we obtain the truncation

$$\mathbf{F}(\alpha, \beta)_{\succcurlyeq 0} : \cdots \to R(-9) \xrightarrow{\alpha} R(-6) \xrightarrow{\beta} R \to 0.$$

It is clear that this complex is exact everywhere except the rightmost position because $\mathbf{F}(\alpha, \beta)$ is exact, so it is isomorphic in the derived category to $\operatorname{coker} \beta \cong R/(x^2 + y^3)$. The result is identical for $i = -1$ and $i = -2$, and for $i = 9$, for example, we get the same result, but with a graded shift (by 9, which is the degree of $x^3 + xy^3$) and a homological shift (by 2).

Up to shifts there is only one other possible outcome, which occurs, for example, when $i = 6$. In this case we see that up to shifts $\mathbf{F}(\alpha, \beta)_{\succcurlyeq 6}$ is a free resolution of $\operatorname{coker} \alpha = R/(x)$.

(2) $E^2 = (\gamma, \delta)$

A corresponding graded complete resolution $\mathbf{F}(\gamma,\delta)$ is

$$\cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\gamma} \begin{matrix} R(-4) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\delta} \begin{matrix} R(-1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\gamma} \begin{matrix} R(5) \\ \bigoplus \\ R(6) \end{matrix} \to \cdots$$

Up to shifts, there are four possible results of taking $\mathbf{F}(\gamma,\delta)_{\succcurlyeq i}$; two of the them (for example, $i=3$ and $i=0$, respectively) result in taking brutal truncations of the complex $\mathbf{F}(\gamma,\delta)$ in a particular degree and so clearly yield free resolutions of $\operatorname{coker}\gamma$ and $\operatorname{coker}\delta$, respectively.

When $i=1$ we have

$$\mathbf{F}(\gamma,\delta)_{\succcurlyeq 1} : \cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy \\ xy^2 & -x^2 \end{bmatrix}} \begin{matrix} R(-4) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R(-1) \to 0$$

The cokernel in the rightmost position is clearly nonzero, so we examine the homology in the next position. It is easily verified that the kernel of $\begin{bmatrix} x & y \end{bmatrix}$ is generated by $(y,-x)$ and (x^2, xy^2), but $(y,-x)$ is clearly not in the image of γ, so there is nonzero homology in the $R(-4) \bigoplus R(-3)$ position (which is in fact isomorphic to $(R/(x))\,(-6)$). Therefore, $\mathbf{F}(\gamma,\delta)_{\succcurlyeq 1}$ represents an object in the image of $\mathbf{b}_1$ which cannot be represented by a single module.

When $i=-5$ we have

$$\mathbf{F}(\gamma,\delta)_{\succcurlyeq -5} : \cdots \to \begin{matrix} R(-4) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x & y \\ y^2 & -x \end{bmatrix}} \begin{matrix} R(-1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & xy \end{bmatrix}} R(5) \to 0$$

Again the rightmost cokernel is clearly nonzero. The kernel of $\begin{bmatrix} x^2 & xy \end{bmatrix}$ is generated by $(y, -x)$ and (x, y^2), so this complex is in fact exact in the $R(-1) \bigoplus R$ position. Thus the complex is equal in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$ to the cokernel of the last map, (which is isomorphic to $(R/(x^2, xy))(5)$). This is a module whose minimal complete resolution is given by $\mathbf{F}(\gamma, \delta)$.

(3) $E^3 = (\varphi_1, \psi_1)$

From here on, the computations are very similar to those already described, so we provide less detail.

A corresponding graded complete resolution $\mathbf{F}(\varphi_1, \psi_1)$ is

$$\cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(-7) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\psi_1} \begin{matrix} R(-1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\varphi_1} \begin{matrix} R(2) \\ \bigoplus \\ R(6) \end{matrix} \to \cdots$$

Up to shifts, there are again four possible results of taking $\mathbf{F}(\varphi_1, \psi_1)_{\succcurlyeq i}$; two of them (for example, $i = 3$ and $i = 0$, respectively) yield free resolutions of $\operatorname{coker} \varphi_1$ and $\operatorname{coker} \psi_1$, respectively.

When $i = 1$ we have

$$\mathbf{F}(\varphi_1, \psi_1)_{\succcurlyeq 1} : \cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\begin{bmatrix} x & y \\ xy^2 & -x^2 \end{bmatrix}} \begin{matrix} R(-7) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & y \end{bmatrix}} R(-1) \to 0$$

It is easily verified that the kernel of $\begin{bmatrix} x^2 & y \end{bmatrix}$ is generated by $(y, -x^2)$ and (x, xy^2), so this complex is exact in the $R(-7) \bigoplus R(-3)$ position, so in fact is equal in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$ to $(R/(x^2, y))(-1)$.

When $i=-2$ we have

$$\mathbf{F}(\varphi_1,\psi_1)_{\succcurlyeq -2}: \cdots \to \begin{matrix} R(-7) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & y \\ xy^2 & -x \end{bmatrix}} \begin{matrix} R(-1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R(2) \to 0$$

As already established, the kernel of $\begin{bmatrix} x & y \end{bmatrix}$ is generated by $(y,-x)$ and (x,xy^3), so this complex is exact in the $R(-1)\bigoplus R$ position, so is equal in $\mathrm{D^b}(\mathrm{gr}\ R)$ to $(R/(x,y))(2)$ in $\mathrm{D^b}(\mathrm{gr}\ R)$.

From E^3 we have obtained, up to shifts, just four different modules.

(4) $E^4=(\varphi_2,\psi_2)$

A corresponding graded complete resolution $\mathbf{F}(\varphi_2,\psi_2)$ is

$$\cdots \to \begin{matrix} R(-8) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(-5) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\psi_2} \begin{matrix} R(1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\varphi_2} \begin{matrix} R(4) \\ \bigoplus \\ R(6) \end{matrix} \to \cdots$$

Up to shifts, there are four possible results of taking $\mathbf{F}(\varphi_2,\psi_2)_{\succcurlyeq i}$; two of them (for example, $i=3$ and $i=-1$, respectively) yield free resolutions of $\operatorname{coker}\varphi_2$ and $\operatorname{coker}\psi_2$, respectively. When $i=0$ we have

$$\mathbf{F}(\varphi_2,\psi_2)_{\succcurlyeq 0}: \cdots \to \begin{matrix} R(-8) \\ \bigoplus \\ R(-9) \end{matrix} \xrightarrow{\begin{bmatrix} x & y^2 \\ xy & -x^2 \end{bmatrix}} \begin{matrix} R(-5) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} xy & -x \end{bmatrix}} R \to 0$$

It is easy to see that the kernel of $\begin{bmatrix} xy & -x \end{bmatrix}$ is generated by $(1,y)$ and $(y^2,-x^2)$, so the homology in the $R(-5) \bigoplus R(-3)$ position is nonzero (and in fact is isomorphic to $(R/(x))\,(-5))$.

When $i = -4$ we have

$$\mathbf{F}(\varphi_2, \psi_2)_{\succcurlyeq -4} \cdots \to \begin{matrix} R(-5) \\ \bigoplus \\ R(-3) \end{matrix} \xrightarrow{\begin{bmatrix} x^2 & y^2 \\ xy & -x \end{bmatrix}} \begin{matrix} R(1) \\ \bigoplus \\ R \end{matrix} \xrightarrow{\begin{bmatrix} x & y^2 \end{bmatrix}} R(4) \to 0$$

The kernel of $\begin{bmatrix} x & y^2 \end{bmatrix}$ is generated by $(y^2, -x)$ and (x^2, xy), so this complex is exact in the $R(1) \bigoplus R$ position, so in fact is equal in $\mathrm{D^b}(\mathrm{gr}\ R)$ to $(R/(x, y^2))(4)$.

From E^4, as from E^2, we have obtained, up to shifts, three modules and one complex that cannot be represented in $\mathrm{D^b}(\mathrm{gr}\ R)$ by a module.

(5) $E^5 = (\xi_1, \eta_1)$

For the matrix factorizations of rank 3 and rank 4, we simply summarize the results, rather than give explicit representatives of homology classes.

A corresponding graded complete resolution $\mathbf{F}(\xi_1, \eta_1)$ is

$$\cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \\ \bigoplus \\ R(-11) \end{matrix} \xrightarrow{\xi_1} \begin{matrix} R(-3) \\ \bigoplus \\ R(-5) \\ \bigoplus \\ R(-4) \end{matrix} \xrightarrow{\eta_1} \begin{matrix} R(-1) \\ \bigoplus \\ R \\ \bigoplus \\ R(-2) \end{matrix} \xrightarrow{\xi_1} \begin{matrix} R(6) \\ \bigoplus \\ R(4) \\ \bigoplus \\ R(5) \end{matrix} \to \cdots$$

Up to shifts, there are now six possible results of taking $\mathbf{F}(\xi_1, \eta_1)_{\succcurlyeq i}$. As usual, two of them (for example, $i = 0$ and $i = 3$, respectively) yield free resolutions of $\operatorname{coker} \eta_1$ and $\operatorname{coker} \xi_1$, respectively.

Three of the remaining four options turn out to yield cokernels of the appropriate maps. Up to shifts, we have:

$$\mathbf{F}(\xi_1,\eta_1)_{\geqslant 1} \cong \operatorname{coker}\begin{bmatrix} y & 0 & x \\ 0 & -x & y \end{bmatrix}$$
$$\mathbf{F}(\xi_1,\eta_1)_{\geqslant 4} \cong \operatorname{coker}\begin{bmatrix} xy & y^2 & -x^2 \\ x^2 & xy & xy^2 \end{bmatrix}$$
$$\mathbf{F}(\xi_1,\eta_1)_{\geqslant 5} \cong \operatorname{coker}\begin{bmatrix} xy & y^2 & -x^2 \end{bmatrix}.$$

The final option, for example $i = 2$, results in a complex with nonzero homology in both of the positions affected by the corresponding truncation.

(6) $E^6 = (\xi_2, \eta_2)$

A corresponding graded complete resolution $\mathbf{F}(\xi_2,\eta_2)$ is

$$\begin{array}{ccccccccc} & R(-11) & & R(-5) & & R(-2) & & R(4) & \\ & \bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & \\ \cdots \to & R(-9) & \xrightarrow{\xi_2} & R(-6) & \xrightarrow{\eta_2} & R & \xrightarrow{\xi_2} & R(3) & \to \cdots \\ & \bigoplus & & \bigoplus & & \bigoplus & & \bigoplus & \\ & R(-10) & & R(-4) & & R(-1) & & R(5) & \end{array}$$

Up to shifts, there are again six possible results of taking $\mathbf{F}(\xi_2,\eta_2)_{\geqslant i}$. As usual, two of them (for example, $i = 0$ and $i = 4$, respectively) yield free resolutions of $\operatorname{coker}\eta_2$ and $\operatorname{coker}\xi_2$, respectively.

Two of the remaining four options turn out to yield cokernels of the appropriate maps.

Up to shifts, we have:

$$\mathbf{F}(\xi_2,\eta_2)_{\geqslant 1} \cong \operatorname{coker}\begin{bmatrix} x & 0 & y \\ 0 & -xy & x \end{bmatrix}$$
$$\mathbf{F}(\xi_2,\eta_2)_{\geqslant 5} \cong \operatorname{coker}\begin{bmatrix} x^2 & -y^2 & -xy \\ xy & x & -y^2 \end{bmatrix}.$$

The other two options, for example $i = 2$ and $i = 6$, both result in complexes with nonzero homology in both of the positions affected by the corresponding truncations.

(7) $E^7 = (\xi_3, \eta_3)$

A corresponding graded complete resolution $\mathbf{F}(\xi_3, \eta_3)$ is

$$\cdots \to \begin{matrix} R(-10) \\ \bigoplus \\ R(-9) \\ \bigoplus \\ R(-6) \\ \bigoplus \\ R(-5) \end{matrix} \xrightarrow{\xi_3} \begin{matrix} R(-4) \\ \bigoplus \\ R(-3) \\ \bigoplus \\ R(-3) \\ \bigoplus \\ R(-2) \end{matrix} \xrightarrow{\eta_3} \begin{matrix} R(-1) \\ \bigoplus \\ R \\ \bigoplus \\ R(3) \\ \bigoplus \\ R(4) \end{matrix} \xrightarrow{\xi_3} \begin{matrix} R(5) \\ \bigoplus \\ R(6) \\ \bigoplus \\ R(6) \\ \bigoplus \\ R(7) \end{matrix} \to \cdots$$

Up to shifts, there are now seven possible results of taking $\mathbf{F}(\xi_3, \eta_3)_{\geqslant i}$. As usual, two of them (for example, $i = -4$ and $i = 2$, respectively) yield free resolutions of $\operatorname{coker} \eta_3$ and $\operatorname{coker} \xi_3$, respectively.

Three of the remaining five options turn out to yield cokernels of the appropriate maps.

Up to shifts, we have:

$$\mathbf{F}(\xi_3,\eta_3)_{\succcurlyeq -3} \cong \operatorname{coker}\begin{bmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \\ 0 & 0 & x^2 & xy \end{bmatrix}$$
$$\mathbf{F}(\xi_3,\eta_3)_{\succcurlyeq 0} \cong \operatorname{coker}\begin{bmatrix} x & y & -y & 0 \\ y^2 & -x & 0 & -y \end{bmatrix}$$
$$\mathbf{F}(\xi_3,\eta_3)_{\succcurlyeq 3} \cong \operatorname{coker}\begin{bmatrix} x^2 & xy & y & 0 \\ xy^2 & -x^2 & 0 & y \\ 0 & 0 & x & y \end{bmatrix}.$$

The remaining two options (for example, $i = 1$ and $i = 4$) both yield complexes with nonzero homology in both positions affected by the corresponding truncations.

Chapter 4

The Orlov Correspondence in the Graded Case

4.1 Motivation and summary of results

The work presented in this chapter was motivated by a desire to find a way to pass the Orlov embeddings $\mathbf{b}_i$ back and forth across an equivalence of categories originally defined by Orlov in [12] and extended by Burke and Walker in [6] to an equivalence

$$\Psi : [MF(\mathbb{P}_Q^{c-1}, W)] \to \mathrm{D}_{\mathrm{sg}}^{\mathrm{b}}(R),$$

where $\mathrm{D}_{\mathrm{sg}}^{\mathrm{b}}(R)$ is the singularity category of R and $[MF(\mathbb{P}_Q^{c-1}, W)]$ is the homotopy category of matrix factorizations of W over Proj $S = \mathbb{P}_Q^{c-1}$, for an appropriate notion of homotopy. We will not give the definition of this homotopy category here; for the purposes of this chapter one can simply think of it as the homotopy category of matrix factorizations of W over S. Specifically, the goal was to define a version of $\mathbf{b}_i$ on matrix factorizations of W that is compatible with Ψ and its inverse. However, this turns out to require some adjustments, since Ψ does not preserve graded structures in any way. Thus the majority of this chapter is dedicated to reformulating the functor Ψ to work in the graded setting.

The first problem is that for the purposes of this chapter, this geometric formulation is insufficient. We aim to compute with various graded truncations of graded modules, and it is well known that over $\mathbb{P}_Q^{c-1}$, one has an equality of coherent sheaves $\widetilde{M} \cong \widetilde{M_{\geq i}}$ (where $\widetilde{N}$ is the sheaf associated to a graded S-module N as in [10], II.5) for all graded S-modules M and all $i \in \mathbb{Z}$. In other words, categories of coherent sheaves do not distinguish between a module and its graded truncations. For this reason we give a purely algebraic correspondence in Construction 4.3.1. This necessitates a new definition of Ψ on the level of modules, rather than coherent sheaves (and which takes graded structures into account). The second problem is that objects over the generic hypersurface must be *bigraded*, i.e. graded both via the given grading on R and the standard grading on the generic hypersurface (where elements of Q have degree 0 and each T_i has degree 1). Finally, the category $MF_{\mathrm{bg}}(S, W)$ of (bigraded) matrix factorizations still turns out to be insufficient to encode the necessary grading information - instead we define a version of Ψ on a certain subcategory $\mathcal{D}$ of the larger category $LF_{\mathrm{bg}}(S, W)$, the category of (bigraded) *linear factorizations* of W over S (a linear factorization is essentially a matrix factorization whose modules are not necessarily free; see Definition 4.2.4).

After making these adjustments, we obtain the following, which combines all the main results of this chapter:

Theorem. *There exist functors $\Phi : \mathcal{D} \to \mathrm{D^b}(\mathrm{gr}\ R)$ and $\mathbf{b}_i^g : \mathcal{D} \to \mathcal{D}$ such that the square*

$$\begin{tikzcd} MF_{\mathrm{bg}}(S,W) \arrow[r, "\overline{\Phi}"] \arrow[d, "\mathbf{b}_i^g"] & \mathrm{D^b_{sg}}(\mathrm{gr}\ R) \arrow[d, "\mathbf{b}_i"] \\ \mathcal{D} \arrow[r, "\Phi"] & \mathrm{D^b}(\mathrm{gr}_{\geq i}\ R) \end{tikzcd}$$

commutes, where $\overline{\Phi}$ is the restriction of Φ to the full subcategory $MF_{\mathrm{bg}}(S, W)$ of $\mathcal{D}$ followed by the localization functor $\mathrm{D^b}(\mathrm{gr}\ R) \to \mathrm{D^b_{sg}}(\mathrm{gr}\ R)$.

The functors $\mathbf{b}_i^g$ are simple graded truncations (see Definition 4.4.3), so they potentially allow one to understand the less concretely defined $\mathbf{b}_i$ functors more easily by instead working in the matrix factorization category.

4.2 Construction of matrix factorizations

Recall that we denote by Hom^* the graded module consisting of homomorphisms of all graded degrees:

$$\operatorname{Hom}^*_{\mathrm{gr}\ R}(X,Y) := \bigoplus_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathrm{gr}\ R}(X, Y(n)).$$

We now discuss the process of building a matrix factorization of W over S whose sheafification over $\mathbb{P}^{c-1}_Q$ corresponds via the Burke-Walker equivalence Ψ to a given module or complex over R. In [7], §6, Burke and Walker give a way to construct a graded matrix factorization $E = E(M, \mathbf{G}, \boldsymbol{\sigma})$ of $W = f_1T_1 + \ldots + f_cT_c$ over $S = Q[T_1, \ldots, T_c]$ from an R-module M, using the data of a finite free resolution $\mathbf{G}$ of M over Q and a system of higher homotopies $\boldsymbol{\sigma} = \{\sigma^J | J \in \mathbb{N}^c\}$ for $f_1, \ldots, f_c$. In fact they construct a matrix factorization of W over S and then sheafify to obtain a sequence of coherent sheaves over the scheme $\mathbb{P}^{c-1}_Q$, but as discussed in the previous section, this approach does not allow for the graded truncations that we will need to perform, so we omit the sheafification step. Furthermore, with virtually no modification their construction can be applied to the more general situation where we take as input only an arbitrary bounded complex of finitely generated free Q-modules $\mathbf{G}$ and a system of higher homotopies $\boldsymbol{\sigma}$, so we instead present it in that setting.

For simplicity, here we describe the construction using only the standard grading on S, in which elements of Q have degree 0 and each T_i has degree 1; the adjustments necessary to preserve a nontrivial grading on Q are detailed immediately after the basic construction.

Given the data $(\mathbf{G}, \boldsymbol{\sigma})$, we construct the matrix factorization $E(\mathbf{G}, \boldsymbol{\sigma}) = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(1))$ as follows: The S-modules E_1 and E_0 are

$$E_1 := \bigoplus_{j\in\mathbb{Z}} \mathbf{G}_{2j+1} \otimes_Q S(j) \text{ and } E_0 := \bigoplus_{j\in\mathbb{Z}} \mathbf{G}_{2j} \otimes_Q S(j).$$

These direct sums are finite because $\mathbf{G}$ is bounded. For each $J \in \mathbb{N}^c$, there are maps

$$\mathbf{G}_{2j+1} \otimes S(j) \xrightarrow{\sigma^J \otimes T^J} \mathbf{G}_{2j+2|J|} \otimes S(j+|J|) \text{ and}$$
$$\mathbf{G}_{2j} \otimes S(j) \xrightarrow{\sigma^J \otimes T^J} \mathbf{G}_{2j+2|J|-1} \otimes S(j+|J|),$$

which are used to define the maps $e_1 : E_1 \to E_0$ and $e_0 : E_0 \to E_1(1)$, component-wise, as follows:

$$(e_1)_j = \sum_{J \in \mathbb{N}^c} \sigma^J \otimes T^J : \mathbf{G}_{2j+1} \otimes_Q S(j) \to \bigoplus_i \mathbf{G}_{2i} \otimes_Q S(i) = E_0,$$
$$(e_0)_j = \sum_{J \in \mathbb{N}^c} \sigma^J \otimes T^J : \mathbf{G}_{2j} \otimes_Q S(j) \to \bigoplus_i \mathbf{G}_{2i-1} \otimes_Q S(i) = E_1(1). \tag{4.1}$$

If $\mathbf{G}$ is known to be a Q-resolution of an R-module M, we will write $E(M, \mathbf{G}, \boldsymbol{\sigma})$ in place of $E(\mathbf{G}, \boldsymbol{\sigma})$ for emphasis. Note that if $\mathbf{G}$ is a resolution of a Q-module N, then the presence of a system of higher homotopies implies that N is annihilated by $(f_1, \dots, f_c)$ and is hence an R-module.

4.2.1(Nontrivial gradings on Q). We may modify the construction of $E(\mathbf{G}, \boldsymbol{\sigma})$ to preserve a nontrivial grading on Q. To do so, we first introduce a second grading on $S = Q[T_1, \dots, T_c]$:

4.2.2 Definition. We define the following bigraded ring structure on S: we assign to an element of $Q \subset S$ the bidegree $(0, q)$, where q is the element's graded degree as an element of the graded ring Q, and we assign to each variable T_i the bidegree $(1, -\deg\, f_i + \sum_{j=1}^{c} \deg\, f_j)$. For clarity and brevity, we will henceforth refer to the first-component grading as G1 and the second-component grading as G2.

This choice of grading for the variables T_i is motivated by the fact that $W = f_1 T_1 + \cdots + f_c T_c$ must be a homogeneous element of S in order for the notion of graded matrix factorization to make sense. Note that the G2 degree of T_i is the sum of the degrees of

$f_1, \ldots, f_c$, but with f_i omitted. This degree choice ensures that W is homogeneous of bidegree $(1, \sum_{j=1}^{c} \deg\, f_j)$. If Q has only the trivial grading, then grading G2 is also trivial.

If $\mathbf{G}$ is a bounded complex of graded free Q-modules (with graded maps), we get a bigraded matrix factorization $E(\mathbf{G}, \boldsymbol{\sigma})$ as follows: To each copy of $S(0)$ that appears in the above construction of $E(\mathbf{G}, \boldsymbol{\sigma})$, assign the second, or G2, shift to be the one on the corresponding copy of Q in $\mathbf{G}$. Then the bidegrees of the components of the maps e_1 and e_0 force a unique second grading on all the copies of $S(i)$ for $i \neq 0$.

4.2.3 For the remainder of this chapter, $R = Q/(f_1, \ldots, f_c)$ is a *graded* complete intersection ring. The goal now is to define two functors: one analogous to $\mathbf{b}_i$ on matrix factorizations of $W = f_1T_1 + \cdots + f_cT_c$ over $S = Q[T_1, \ldots, T_c]$ and one that associates to each matrix factorization an object in $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$, in such a way that the obvious square commutes.

However, the notion of matrix factorization turns out to be too restrictive for this purpose.

4.2.4 Definition. Given a graded ring A and a homogeneous element $X \in A_d$, a *(graded) linear factorization* of X over A is a sequence $E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d)$, where E_0, E_1 are finitely generated graded A-modules and the compositions $e_1(d) \circ e_0$ and $e_0 \circ e_1$ are both equal to multiplication by X. We will typically write a linear factorization $E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(d)$ simply as (e_1, e_0), when the modules E_0 and E_1 are clear from context. We define a category $LF(A, X)$ (or $LF_{\mathrm{gr}}(A, X)$) whose objects are (graded) linear factorizations of X over A, with morphisms defined analogously to those in $MF(A, X)$ (or $MF_{\mathrm{gr}}(A, X)$).

Note that this definition is precisely that of a graded matrix factorization, except that E_0 and E_1 are not required to be free modules. Consequently, the category $MF(A, X)$ (respectively $MF_{\mathrm{gr}}(A, X)$) is a full subcategory of $LF(A, X)$ (respectively $LF_{\mathrm{gr}}(A, X)$).

In order to work in the graded setting, we must require that all linear factorizations are *bigraded* with respect to the bigrading given in Definition 4.2.2, i.e., are graded linear factorizations in each degree independently. We will denote the category of such bigraded linear factorizations of W over S by $LF_{\mathrm{bg}}(S,W)$. Similarly, we denote by $MF_{\mathrm{bg}}(S,W)$ the full subcategory whose objects are matrix factorizations.

For reasons that will later become evident, it suffices to restrict our attention to a particular type of linear factorization, as follows.

4.2.5 Definition. We denote by $\mathcal{D}$ the full subcategory of $LF_{\mathrm{bg}}(S,W)$ whose objects are those linear factorizations (e_1, e_0) for which:

- E_0 and E_1 are isomorphic to direct sums of bigraded shifts of monomial ideals of S (i.e., ideals of S generated by monomials in $T_1, \ldots, T_c$), each of which contains the corresponding bigraded shift of $S_{\geq n}$ (where the subscript refers to the G1-grading) for some $n \in \mathbb{Z}$. In other words, E_0 and E_1 are isomorphic to direct sums of bigraded shifts of $(T_1, \ldots, T_c)$-primary monomial ideals.

- In general, the maps e_1, e_0 in such a linear factorization may be viewed as sums of homogeneous maps between direct summands (each of which is a monomial ideal); we require that for $e_0, e_1)$ to be in $\mathcal{D}$, each of these maps must be multiplication by a homogeneous element of S (with respect to both gradings).

The reason for this definition is that the objects of $\mathcal{D}$ are precisely those linear factorizations that arise when one kills a finite collection of initial monomials in each of E_1 and E_0, as this will be precisely the operation we wish to perform on matrix factorizations to obtain our analog of $\mathbf{b}_i$; see Definition 4.4.3. In particular, we note that $MF_{\mathrm{bg}}(S,W)$ is a full subcategory of $\mathcal{D}$.

4.3 Main results

We now define a functor

$$\Phi : \mathcal{D} \longrightarrow \mathrm{D^b}(\mathrm{gr}\ R)$$

with the property that $\Phi(E(M,\mathbf{G},\boldsymbol{\sigma}))\cong M$, where $E(M,\mathbf{G},\boldsymbol{\sigma})$ is the matrix factorization associated to any finite Q-free resolution of an R-module M via the construction given in Section 4.2. We prove, among other things, that this property holds (Proposition 4.3.3) and that Φ is indeed a well-defined functor $\mathcal{D}\to \mathrm{D^b}(\mathrm{gr}\ R)$ (Theorem 4.3.4) after giving the construction.

4.3.1 Construction. Until the last step of the construction, we refer only to the G1 grading on S, in which elements of Q have degree 0 and each T_i has degree 1 (in particular, in this grading the element W is homogeneous of degree 1).

Given a linear factorization $E=(e_1,e_0)$ in $\mathcal{D}$, we first construct the sequence

$$E^\infty : \cdots \xrightarrow{e_0(-1)} E_1(-1) \xrightarrow{e_1(-1)} E_0(-1) \xrightarrow{e_0} E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0(1)} E_1(1) \xrightarrow{e_1(1)} \cdots, \tag{4.2}$$

with E_0 in homological degree 0.

Since E_0 and E_1 are direct sums of graded shifts of monomial ideals in S, for each $i\in\mathbb{Z}$ the S-module E_i^∞ can be expressed as a direct sum

$$E_i^\infty = \bigoplus_{j\in\mathbb{Z},\lambda\in\Lambda} Q^{r_{i,j,\lambda}}\otimes_Q I_\lambda(j), \tag{4.3}$$

where Λ is some index set, each I_λ is a $(T_1,\dots,T_c)$-primary monomial ideal in S, and each $r_{i,j,\lambda}$ is a nonnegative integer. This expression is unique so long as we insist that all I_λ be distinct. In light of this, each of the maps e_0,e_1 (and thus each of the maps in the sequence E^∞) can be represented as a matrix where each entry has the form

$$\sum_{l\in\mathbb{N}^c} g_l\otimes T^l, \tag{4.4}$$

where each g_l is a map of free Q-modules and T^l is the monomial $T_1^{l_1}\cdots T_c^{l_c}$.

We next we define graded S-modules $\{\mathbf{E}_i\}_{i\in\mathbb{Z}}$ as follows:

$$\mathbf{E}_i := \bigoplus_{\lambda\in\Lambda} Q^{r_{i,0,\lambda}} \otimes_Q I_\lambda. \tag{4.5}$$

In other words, $\mathbf{E}_i$ consists of precisely the direct summands of E_i^∞ for which $j = 0$ in (4.3).

Now from $\{\mathbf{E}_i\}$ we define new graded S-modules $\{\mathbf{E}_i^\Gamma\}_{i\in\mathbb{Z}}$ by dualizing with respect to Q. For each $i \in \mathbb{Z}$ let

$$\mathbf{E}_i^\Gamma := \bigoplus_{\lambda\in\Lambda} Q^{r_{i,0,\lambda}} \otimes_Q D_\lambda, \tag{4.6}$$

where $D_\lambda = \operatorname{Hom}^*_{\text{gr }Q}(I_\lambda, Q)$ and all $r_{i,0,\lambda}$ and I_λ are as in (4.5). We denote by τ_i the dual in $D = \operatorname{Hom}^*_{\text{gr }Q}(S, Q)$ of T_i, for $1 \le i \le c$, so that τ^J is the dual in each D_λ of T^J for each monomial $T^J \in I_\lambda$. Observe that each τ_i has graded degree -1 in D (so that each $\tau^J \in I_\lambda$ has degree $-j$), and that each D_λ is in fact simply the quotient of the graded S-module $\operatorname{Hom}^*_{\text{gr }Q}(S, Q)$ by a submodule generated by monomials in the dual variables τ_i.

Finally, we define $\Phi(E)$ to be the complex $\mathbf{F}$ defined as follows. For each $i \in \mathbb{Z}$, let $\mathbf{F}_i$ be the graded R-module

$$\mathbf{F}_i = \bigoplus_{l\le 0} \left(\mathbf{E}^\Gamma_{i+2l}\right)_l \otimes_Q R, \tag{4.7}$$

where $\left(\mathbf{E}^\Gamma_{i+2l}\right)_l$ denotes the graded degree l component of the module $\mathbf{E}^\Gamma_{i+2l}$. The maps of $\mathbf{F}$ are given simply by deleting any necessary rows and/or columns of the matrices representing those of E^∞. Note that the entries of these matrices still have the form given in 4.4, but now each T_i acts on the dual $S^* = Q[\tau_1, \ldots, \tau_c]$, so the action is the one described in Section 2.6. It is perhaps not immediately clear that $\mathbf{F}$ is necessarily a complex, but the next three results (Propositions 4.3.2 and 4.3.3 and Theorem 4.3.4) will show that it is.

The graded R-module structure of $\Phi(E)$ is defined via the following graded Q-module structure on the modules $\{\mathbf{E}_i\}$ (which supercedes the existing G1- and G2-gradings): by construction each summand of each of the modules E_1, E_0 of the linear factorization E appears exactly once in exactly one $\mathbf{E}_i$, and no other summands appear; in each of these summands we assign to elements of Q their original G2 grading from E, and we set $|T_i| = -\deg f_i$ for $1 \leq i \leq c$ (and consequently $|\tau_i| = \deg f_i$). That the maps of $\mathbf{F}$ are graded follows from the fact that E^∞ is G2-graded.

Contained in the following proposition is the assertion that Φ is, in fact, analogous to Ψ on matrix factorizations associated to graded R-modules, as in Section 4.2. Proposition 4.3.3 will show that this result is more general than it may appear; every bigraded matrix factorization of W over S is of the form $E(\mathbf{G}, \boldsymbol{\sigma})$.

4.3.2 Proposition. *If $\mathbf{G}$ is a bounded complex of finitely generated free Q-modules with a system of higher homotopies $\boldsymbol{\sigma}$ for $f_1, \ldots, f_c$, we have*

$$\Phi(E(\mathbf{G}, \boldsymbol{\sigma})) \cong \mathbf{G}\{\boldsymbol{\sigma}\},$$

where $\mathbf{G}\{\boldsymbol{\sigma}\}$ is the Eisenbud-Shamash complex of $(\mathbf{G}, \boldsymbol{\sigma})$.

In particular, given an R-module M, viewed as an object of $\mathrm{D^b}(\mathrm{gr}\ R)$, along with a free resolution $\mathbf{G}$ of M over Q and a system of higher homotopies $\boldsymbol{\sigma}$, we have

$$\Phi(E(M, \mathbf{G}, \boldsymbol{\sigma})) \cong M.$$

Proof. We proceed by working through Construction 4.3.1 in detail; note that as in the construction, we use only grading G1 of $E(\mathbf{G}, \boldsymbol{\sigma})$, wherein elements of Q have degree 0 and each T_i has degree 1.

We begin by explicitly describing the graded Q-module structure, in each homological degree, of $E(\mathbf{G}, \boldsymbol{\sigma})$. First, it is clear that as a graded Q-module, the graded degree n part of S has the form

$$S_n \cong \bigoplus_{\substack{J \in \mathbb{N}^c, \\ |J| = n}} QT^J,$$

where QT^J is the Q-direct summand of S generated by the monomial $T^J = T_1^{a_1} \cdots T_c^{a_c}$ for

$J = (a_1, \dots, a_c)$. Recall from Section 4.2 that the graded S-modules of $E(\mathbf{G}, \boldsymbol{\sigma})$ are defined to be

$$E_1 := \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j+1} \otimes_Q S(j) \qquad \text{and} \qquad E_0 := \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j} \otimes_Q S(j).$$

Note that in this situation, the only monomial ideal I_λ that actually appears in Construction 4.3.1, (4.3) is all of S.

From this we see that the graded degree n parts of the corresponding graded Q-modules are given by

$$(E_1)_n \cong \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\substack{J \in \mathbb{N}^c, \\ |J| = n+j}} \mathbf{G}_{2j+1} \otimes_Q QT^J,$$

$$(E_0)_n \cong \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\substack{J \in \mathbb{N}^c, \\ |J| = n+j}} \mathbf{G}_{2j} \otimes_Q QT^J.$$

Now we can explicitly describe all of the modules of the sequence E^∞; we have

$$E^\infty_{2k+1} = E_1(-k) = \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j+1} \otimes_Q S(j-k),$$

$$E^\infty_{2k} = E_0(-k) = \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j} \otimes_Q S(j-k).$$

As graded Q-modules, we then have the following expressions for the graded degree n part of each E^∞_i:

$$(E^\infty_{2k+1})_n \cong \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\substack{J \in \mathbb{N}^c, \\ |J| = n+j-k}} \mathbf{G}_{2j+1} \otimes_Q QT^J,$$

$$(E^\infty_{2k})_n \cong \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\substack{J \in \mathbb{N}^c, \\ |J| = n+j-k}} \mathbf{G}_{2j} \otimes_Q QT^J.$$

Next, to obtain the modules $\{\mathbf{E}_i\}$ we take only the $j = k$ summand (i.e., the zero-shift summand) in each degree:

$$\mathbf{E}_{2k+1} = \mathbf{G}_{2k+1} \otimes_Q S \qquad \text{and} \qquad \mathbf{E}_{2k} = \mathbf{G}_{2k} \otimes_Q S.$$

As graded Q-modules, we have

$$(\mathbf{E}_{2k+1})_n \cong \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=n}} \mathbf{G}_{2k+1} \otimes_Q QT^J,$$

$$(\mathbf{E}_{2k})_n \cong \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=n}} \mathbf{G}_{2k} \otimes_Q QT^J.$$

Now we replace S by $S^* := \operatorname{Hom}^*_{\mathrm{gr}\ Q}(S, Q) \cong Q[\tau_1, \ldots, \tau_c]$ (called D_λ in Construction 4.3.1, (4.6)) to obtain the modules $\{\mathbf{E}_i^\Gamma\}$:

$$\mathbf{E}^\Gamma_{2k+1} = \mathbf{G}_{2k+1} \otimes_Q S^* \qquad \text{and} \qquad \mathbf{E}^\Gamma_{2k} = \mathbf{G}_{2k} \otimes_Q S^*.$$

As graded Q-modules, we have

$$\left(\mathbf{E}^\Gamma_{2k+1}\right)_n \cong \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=-n}} \mathbf{G}_{2k+1} \otimes_Q Q\tau^J,$$

$$\left(\mathbf{E}^\Gamma_{2k}\right)_n \cong \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=-n}} \mathbf{G}_{2k} \otimes_Q Q\tau^J.$$

Finally, the modules of $\mathbf{F}$ are as follows:

$$\mathbf{F}_{2k+1} = \bigoplus_{l\le 0} \left(\mathbf{E}^\Gamma_{2k+1+2l}\right)_l \otimes_Q R \cong \bigoplus_{l\le 0} \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=-l}} \mathbf{G}_{2(k+l)+1} \otimes_Q Q\tau^J \otimes_Q R,$$

$$\mathbf{F}_{2k} = \bigoplus_{l\le 0} \left(\mathbf{E}^\Gamma_{2k+2l}\right)_l \otimes_Q R \cong \bigoplus_{l\le 0} \bigoplus_{\substack{J\in\mathbb{N}^c,\\ |J|=-l}} \mathbf{G}_{2(k+l)} \otimes_Q Q\tau^J \otimes_Q R.$$

But one easily sees that these modules are precisely those of the Eisenbud-Shamash complex $\mathbf{G}\{\boldsymbol{\sigma}\}$. The grading on $\mathbf{G}$ is preserved through both the construction of E and the application of Φ, and Φ assigns graded degree deg f_i to each τ_i, as does our graded Eisenbud-Shamash complex. Finally, it is clear that the maps of $\Phi(E)$ are identical to those of $\mathbf{G}\{\boldsymbol{\sigma}\}$, so in fact we have $\Phi(E) = \mathbf{G}\{\boldsymbol{\sigma}\}$.

The second statement follows immediately from Proposition 2.6.3; when $\mathbf{G}$ is a resolution of M over Q, $\mathbf{G}\{\boldsymbol{\sigma}\}$ is a resolution of M over R.

□

It is not immediately clear that $\Phi(L)$ has bounded cohomology for an arbitrary linear factorization $L \in \mathcal{D}$; it follows from the next proposition that it does if L is a matrix factorization. In other words, the result implies that the restriction of Φ to $MF_{\mathrm{bg}}(S, W)$ maps to $\mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$. The case of general objects of $\mathcal{D}$ is established in Theorem 4.3.4.

4.3.3 Proposition. *Every object E in $MF_{\mathrm{bg}}(S, W)$ is of the form $E(\mathbf{G}, \boldsymbol{\sigma})$ for some bounded complex $\mathbf{G}$ of finitely generated free Q-modules and some system of higher homotopies $\boldsymbol{\sigma}$ for $f_1, \ldots, f_c$ on $\mathbf{G}$.*

Proof. In this proof we will again consider only the G1 grading on S for simplicity. Let $E = (E_1 \xrightarrow{e_1} E_0 \xrightarrow{e_0} E_1(1))$ be an object of $MF_{\mathrm{bg}}(S, W)$.

It is well known (first established in [8]) that the complex $E^\infty \otimes_S S/(W)$, with E^∞ as defined at the start of Construction 4.3.1, is a complete resolution of coker e_1 over $S/(W)$.

We first identify the complex $\mathbf{G}$. Let $\mathbf{G}_i$ be the degree-zero component of the module $\mathbf{E}_i$ defined in Construction 4.3.1, (4.5), i.e.,

$$\mathbf{G}_i := (\mathbf{E}_i)_0.$$

Then each $\mathbf{G}_i$ is a direct Q-summand of E_i^∞, so we may define the differential of $\mathbf{G}$ to be the one induced from the "differential" of E^∞ (which does not square to zero). As we have already observed in Construction 4.3.1, $\{\mathbf{E}_i\}$ collectively contains each S-summand of E_1 and E_0 precisely once; since E_1 and E_0 are finitely generated S-modules, $\mathbf{E}_i \neq 0$ for only finitely many $i \in \mathbb{Z}$, which implies that $\mathbf{G}$ is bounded. Since collectively the modules of $\mathbf{G}$ consist of finitely many Q-summands from each of finitely many copies of S, we further see that each $\mathbf{G}_i$ is a finitely generated Q-module. Finally, because the variables $T_1, \dots, T_c$ are not present in degree zero, we see that $\mathbf{G}$ is in fact a complex, since its differential is inherited from the "differential" on E^∞, whose square is multiplication by the homogeneous element $W \in S_1$.

Finally, we observe that the maps of E are necessarily formed from a system of higher homotopies on this choice of $\mathbf{G}$. Recall that the maps of $E(\mathbf{G}, \boldsymbol{\sigma})$ are precisely sums of maps of the form $\sigma^J \otimes T^J$; in light of this, we must define $\boldsymbol{\sigma} = \{\sigma^J\}_{J \in \mathbb{N}^c}$ as follows: the chain map $\sigma^J : \mathbf{G} \to \mathbf{G}[2|J| - 1]$ is the map obtained by replacing the monomial T^J by 1 and replacing all other monomials in the T_i by 0, in both e_1 and e_0; decomposing e_1 and e_0 into sums as in Section 4.2, (4.1) yields a map from each $\mathbf{G}_n$ to $\mathbf{G}_{n+2|J|-1}$. It remains to show that this choice of $\boldsymbol{\sigma}$ is indeed a system of higher homotopies on $\mathbf{G}$; in other words, to show that $\sigma^0 = \partial_{\mathbf{G}}$, that $\sigma^0\sigma^i + \sigma^i\sigma^0 = f_i 1_{\mathbf{G}}$ for $i = 1, \dots, c$, and that $\sum_{J'+J''=J} \sigma^{J'}\sigma^{J''} = 0$ for each J with $|J| \geq 2$. But these are all clear from the construction of $\boldsymbol{\sigma}$ and the fact that $e_0 \circ e_1$ and $e_1(1) \circ e_0$ are both multiplication by W; isolating the appropriate variables in W yields precisely these defining equations. □

4.3.4 Theorem. *For any object E of $\mathcal{D}$, $\Phi(E)$ has bounded cohomology, i.e., Φ is a well-defined functor $\mathcal{D} \longrightarrow \mathrm{D}^{\mathrm{b}}(\mathrm{gr}\ R)$.*

Proof. If E is a matrix factorization, i.e., $E \in MF_{\mathrm{bg}}(S, W)$, then the result follows immediately from Propositions 2.6.3 and 4.3.3.

For the general case, we note that any object E of $\mathcal{D}$ can be enlarged to a matrix factorization $\widehat{E} \in MF_{\mathrm{bg}}(S, W)$ by including more monomials; explicitly, $\widehat{E}$ may be constructed by replacing all of the I_λ (from Construction 4.3.1, (4.3)) in E with copies of S. Since by

Definition 4.2.5 the maps of E are already given by matrices with entries in S, one may use the same matrices to extend the maps from the monomial ideals I_λ to all of S. Then, as already established, $\Phi(\widehat{E})$ is an object of $\mathrm{D^b}(\mathrm{gr}\ R)$, i.e., it has bounded cohomology.

One may now recover E from $\widehat{E}$ by iteratively truncating away initial monomials from copies of S (each such truncation is the same as killing a copy of Q when one views S as a free Q-module), and this process has finitely many steps. By the definition of Φ, each of these truncations corresponds to the truncation of a single R-summand of $\Phi(\widehat{E})$, in a single homological degree. Therefore, $\Phi(E)$ differs from $\Phi(\widehat{E})$ in only finitely many degrees, so its cohomology must necessarily also be bounded. $\square$

4.4 Application: Orlov embeddings

Now that we are equipped with the functor $\Phi : \mathcal{D} \to \mathrm{D^b}(\mathrm{gr}\ R)$, we may realize our motivating goal by defining a functor $\mathbf{b}_i^g : \mathcal{D} \to \mathcal{D}$ whose restriction to the full subcategory $MF_{\mathrm{bg}}(S, W)$ (which, by a mild abuse of notation, we will also denote $\mathbf{b}_i^g$) makes the diagram

$$\begin{tikzcd} MF_{\mathrm{bg}}(S,W) \arrow[r, "\overline{\Phi}"] \arrow[d, "\mathbf{b}_i^g"] & \mathrm{D^b_{sg}}(\mathrm{gr}\ R) \arrow[d, "\mathbf{b}_i"] \\ \mathcal{D} \arrow[r, "\Phi"] & \mathrm{D^b}(\mathrm{gr}\ R) \end{tikzcd} \tag{4.8}$$

commute for $i \gg 0$, where $\overline{\Phi}$ is the composition of $\Phi : MF_{\mathrm{bg}}(S, W) \to \mathrm{D^b}(\mathrm{gr}\ R)$ with the localization functor $\mathrm{D^b}(\mathrm{gr}\ R) \to \mathrm{D^b_{sg}}(\mathrm{gr}\ R)$. Since $\mathbf{b}_i$ actually lands in the subcategory $\mathrm{D^b}(\mathrm{gr}_{\geq i}\ R)$, this is equivalent to commutativity of the diagram

$$\begin{tikzcd} MF_{\mathrm{bg}}(S,W) \arrow[r, "\overline{\Phi}"] \arrow[dr, "\Phi\mathbf{b}_i^g"'] & \mathrm{D^b_{sg}}(\mathrm{gr}\ R) \arrow[d, "\mathbf{b}_i"] \\ & \mathrm{D^b}(\mathrm{gr}_{\geq i}\ R). \end{tikzcd} \tag{4.9}$$

To facilitate the statement of the definition of $\mathbf{b}_i^g$, we introduce a third grading on S, which will be referred to as grading G3.

4.4.1 Definition. The G3-degree of each element of $Q \subset S$ is simply its G2-degree (i.e., its degree from the original grading on Q), and the G3-degree of each T_i is deg f_i.

4.4.2 Remark. The element W is not necessarily homogeneous in grading G3, so this grading cannot be applied to a linear factorization as a whole; in other words, when we refer to the G3-grading of a copy of S appearing in a linear factorization, it is understood to be independent of G1- or G2-graded shift.

4.4.3 Definition. The functor $\mathbf{b}_i^g : \mathcal{D} \to \mathcal{D}$ is defined as follows. For a linear factorization E of W over S, we first view E instead as a sequence of maps of graded Q-modules. Note that by the definition of the category $\mathcal{D}$, each E_j is in fact a graded free Q-module (of infinite rank). So we may define $(\mathbf{b}_i^g(E))_j$ to be the graded S-module $(E_j)_{\succcurlyeq i}$ (where E_j is truncated with respect to grading G3 as a graded free Q-module). The maps of $\mathbf{b}_i^g(E)$ are induced from those of E (the structure of the maps as described by Construction 4.3.1, (4.4) is unchanged by the truncations, so one may use precisely the same matrices to describe the new maps). Note that each of the finitely many copies of Q that are truncated away corresponds to a single initial monomial in some I_λ, so indeed $\mathbf{b}_i^g(E) \in \mathcal{D}$.

4.4.4 Proposition. *The diagram (4.8) commutes for all $i \gg 0$.*

Proof. By Proposition 3.1.1, we have that $\mathbf{b}_i(-) = \mathbf{CR}(-)_{\succcurlyeq i}$, so we may replace the right-hand vertical arrow by $\mathbf{CR}(-)_{\succcurlyeq i}$.

Fix an object $E \in MF_{\text{bg}}(S, W)$. By Propositions 4.3.2 and 4.3.3, $\Phi(E) = \mathbf{G}\{\boldsymbol{\sigma}\}$ for some $(\mathbf{G}, \boldsymbol{\sigma})$, so is a bounded-below complex of finitely generated free R-modules. Therefore it agrees with some complete resolution of itself in large homological degrees (one may simply take $\mathbf{P}(\Phi(E))$ to be $\Phi(E)$ in Construction 2.2.3. More formally, there is a choice of $\mathbf{CR}(\Phi(E))$ for which $\mathbf{CR}(\Phi(E))_j = \Phi(E)_j$ for $j \gg 0$.

The grading on $\mathbf{G}\{\boldsymbol{\sigma}\}$ is induced from the G3-grading on the copies of S appearing in $E(\mathbf{G}, \boldsymbol{\sigma})$, so with respect to this grading, the operation $\mathbf{b}_i^g(-) = (-)_{\succcurlyeq i}$ on $E(\mathbf{G}, \boldsymbol{\sigma})$ corresponds to $(-)_{\succcurlyeq i}$ on $\Phi(E(\mathbf{G}, \boldsymbol{\sigma}))$.

Since $\mathbf{G}$ is a bounded complex of finitely generated modules, it is clear by the construction of $\mathbf{G}\{\boldsymbol{\sigma}\}$ that for each n, there exists an $m_0 > n$ such that for all $m \geq m_0$, the minimum generator G3-degree of $(\mathbf{G}\{\boldsymbol{\sigma}\})_m$ is strictly greater than the maximum generator G3-degree of $(\mathbf{G}\{\boldsymbol{\sigma}\})_n$. Thus for i sufficiently large, $(-)_{\succcurlyeq i}$ truncates away all of the modules in homological degrees where $\Phi(E)$ differs from our choice of $\mathbf{CR}(\Phi(E))$. This completes the proof. $\square$

Chapter 5

Stable Homology in the Non-Graded Case

5.1 Summary of results

The aim of this chapter is to compare (non-graded) homology over a complete intersection to homology over its generic hypersurface. In [7], §2 and §3, Burke and Walker investigate how their equivalence of categories $\Psi : [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)] \to \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ described in Section 2.5 affects Ext groups. We obtain results on how it affects Tor groups, using the same basic approach.

We set

$$Y := \mathrm{Proj}\ (S/(W)).$$

Then one has the following commutative diagram of schemes; all of the maps are the ones induced on schemes by the obvious ring maps.

$$\begin{tikzcd} \mathbb{P}_R^{c-1} \arrow[r, "\beta"] \arrow[d, "\pi"] & Y \arrow[r, "\gamma"] & \mathbb{P}_Q^{c-1} \arrow[d] \\ \mathrm{Spec}\ R \arrow[rr] & & \mathrm{Spec}\ Q \end{tikzcd}$$

The first step in establishing a connection between Tor groups that translates across Ψ is to define an appropriate notion of Tor for complexes of coherent sheaves; the usual Tor sheaf is insufficient because the sheaf Tor functor does not commute with taking global sections, so it does not adequately produce global information. To solve this, we use the notion of hypercohomology, denoted by $\mathbb{H}$. Hypercohomology is defined via a bicomplex that incorporates both conventional homology of a complex and sheaf cohomology. This construction allows one to compute hypercohomology via two obvious spectral sequences, both of which involve often well-understood sheaf cohomology computations. The nature of this construction also makes it a useful tool for producing global versions of sheaf data, as the construction synthesizes information from an affine open cover, rather than using global sections.

Burke and Walker define a notion of stable Ext sheaf over Proj Y, denoted $\widehat{\mathcal{E}\mathrm{xt}}$, that is intended to be analogous to the definition of stable Ext for modules given in Section 2.2. We define a stable Tor sheaf $\widehat{\mathcal{T}\mathrm{or}}$ in an obvious way, analogously to the definition of $\widehat{\mathcal{E}\mathrm{xt}}$.

By applying the equivalence Ψ to Hom groups and using all of this technical machinery, Burke and Walker are able to give a short proof the following:

Proposition (Burke-Walker, [7], 3.8). *Let M and N be objects of $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, and set $\mathcal{M}$ (resp. $\mathcal{N}$) to be the image of $\beta_*\pi^*M$ (resp. $\beta_*\pi^*N$) in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(Y)$. For all $q \in \mathbb{Z}$ there are maps, natural in both arguments:*

$$\widehat{\mathrm{Ext}}^q_R(M,N) \cong \mathbb{H}^q(Y, \mathcal{H}\mathrm{om}_{\mathcal{O}_Y}(\gamma^*\mathbb{E}^\infty_M, \mathcal{N})) \to \Gamma(Y, \widehat{\mathcal{E}\mathrm{xt}}^q_{\mathcal{O}_Y}(\mathcal{M},\mathcal{N})).$$

The second map is an isomorphism for all $q \gg 0$.

In this statement $\mathbb{E}_M$ is a matrix factorization such that $\Psi(\mathbb{E}_M) = M$, and $\mathbb{E}^\infty_M$ represents the 2-periodic complex described by $\mathbb{E}_M$ (see Definition 5.2.4).

Proving analogous results for Tor poses extra difficulties, due to the fact that tensor products are not typically preserved by equivalences of categories. We use a different approach to directly compute $\mathbb{H}$ and to prove

Theorem. *In the setting of Proposition 5.1, we have maps*

$$\widehat{\operatorname{Tor}}_q^R(M,N) \cong \mathbb{H}^q(Y, \gamma^*\mathbb{E}_M^\infty \otimes_{O_Y} \mathcal{N})[c-1] \to \Gamma(Y, \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M},\mathcal{N}))[c-1].$$

The second map is an isomorphism for all $q \ll 0$.

This formulation is an immediate corollary of Theorem 5.3.1 and Proposition 5.4.3.

In [1] it is shown that in the present setting, vanishing of $\widehat{\operatorname{Tor}}_q^R(M,N)$ for $q \ll 0$ is equivalent to vanishing for all $q \in \mathbb{Z}$; in light of this, we obtain as a corollary a partial generic hypersurface version of a recent theorem of Bergh and Jorgensen relating Tor groups over R to those over hypersurfaces $Q/(f)$ with $f \in (f_1, \dots, f_c)$; that is, hypersurfaces that lie between R and Q ([3], Theorem 3.3); it appears as Corollary 5.4.5.

Corollary. *Let M and N be finitely generated R-modules, and M' and N' be the $S/(W)$-modules $M[T_1, \dots, T_c]$ and $N[T_1, \dots, T_c]$, respectively. If $\operatorname{Tor}_q^{S/(W)}(M', N') = 0$ for $q \gg 0$, then $\operatorname{Tor}_q^R(M,N) = 0$ for $q \gg 0$.*

5.2 Background

We now shift to the original scheme-theoretic setting of [12] and [7]; as such, we no longer work under that assumption that Q is graded.

For the entirety of this chapter, we work under the following standing assumptions (following [7]): All rings are commutative and noetherian. Q is any local ring (not necessarily regular) and

$$R = \frac{Q}{(f_1, \dots, f_c)},$$

with $f_1, \dots, f_c$ a regular sequence on Q contained in the square of the maximal ideal of Q. As before, set $S = Q[T_1, \dots, T_c]$ (with the standard grading), and $W = f_1T_1 + \cdots + f_cT_c \in S$.

Set

$$Y := \text{Proj } (S/(W)).$$

One has the following commutative diagram of ring maps; the horizontal arrows are the quotient maps and the vertical arrows are the canonical inclusions.

$$\begin{tikzcd}
R[T_1, \ldots, T_c] & S/(W) \arrow[l] & S = Q[T_1, \ldots, T_c] \arrow[l] \\
R \arrow[u] & & Q \arrow[ll] \arrow[u]
\end{tikzcd}$$

This diagram induces the following diagram of schemes:

$$\begin{tikzcd}
\mathbb{P}_R^{c-1} \arrow[r, "\beta"] \arrow[rr, bend left, "\delta"] \arrow[d, "\pi"] & Y \arrow[r, "\gamma"] & \mathbb{P}_Q^{c-1} \arrow[d] \\
\text{Spec } R \arrow[rr] & & \text{Spec } Q
\end{tikzcd} \tag{5.1}$$

Much of the content of this chapter is valid for more general schemes than those shown in this diagram, but we are always understood to be in this particular setting unless more general hypotheses are explicitly stated.

We first recall Definition 2.7.1 of a matrix factorization over a scheme: if X is a Noetherian separated scheme, $\mathcal{L}$ a line bundle on X, and W a global section of $\mathcal{L}$, a *matrix factorization* $\mathbb{E} = (\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1))$ of the triple $(X, \mathcal{L}, W)$ consists of a pair of locally free coherent sheaves $\mathcal{E}_1, \mathcal{E}_0$ on X and morphisms e_1, e_0 such that $e_0 \circ e_1 = e_1(1) \circ e_0 = W \cdot \text{Id}$.

For the majority of this chapter, we are specifically interested in matrix factorizations of $(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)$, where $\mathcal{O}(1) = \mathcal{O}_{\mathbb{P}_Q^{c-1}}(1)$ is the usual Serre twisting sheaf and $W = f_1T_1 + \cdots + f_cT_c$.

We next recall the Orlov and Burke-Walker equivalences of categories, described in Section 2.5. The Orlov equivalence is valid in more generality than we have addressed thus far, so we first require the following definition.

5.2.1 Definition ([7], 2.6). Let $i : Z \hookrightarrow X$ be a closed immersion of finite flat dimension. An object $\mathcal{F}$ in $\mathrm{D^b}(Z)$ is *relatively perfect* on Z if $i_*\mathcal{F}$ is perfect on X. We write $\mathrm{RPerf}(Z \hookrightarrow X)$ for the full subcategory of $\mathrm{D^b}(Z)$ whose objects are relatively perfect on X.

The *relative singularity category of* i is defined to be the Verdier quotient

$$\mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Z \hookrightarrow X) := \frac{\mathrm{RPerf}(Z \hookrightarrow X)}{\mathrm{Perf}(Z)}.$$

Of special interest is the situation that $X = \mathrm{Spec}\ \mathrm{Q}$ and $Z = \mathrm{Spec}\ R$ with $R = Q/I$ (of finite projective dimension over Q); in this case we write $\mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Q \twoheadrightarrow R)$ in place of $\mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(\mathrm{Spec}\ R \hookrightarrow \mathrm{Spec}\ Q)$.

Note that a complex X of R-modules (with bounded cohomology) is in $\mathrm{RPerf}(Q \twoheadrightarrow R)$ if and only if it has a finite projective resolution over Q. It follows that if Q is regular, then $\mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Q \twoheadrightarrow R)$ is precisely $\mathrm{D}_{\mathrm{sg}}^{\mathrm{b}}(R)$.

5.2.2 (The Orlov and Burke-Walker equivalences). By [7], Corollary 2.11, for Q any local ring and $W = f_1T_1 + \cdots + f_cT_c$, there is an equivalence of categories Ψ which may be thought of as the following composition:

$$\Psi : [MF(\mathbb{P}_Q^{c-1}, \mathcal{O}(1), W)] \xrightarrow{\text{coker}} \mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Y \subset \mathbb{P}_Q^{c-1}) \xrightarrow{\cong} \mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Q \twoheadrightarrow R), \tag{5.2}$$

where the source category is defined as Definition 2.7.4, coker is the functor sending a matrix factorization $\mathbb{E} = (e_1, e_0)$ to $\operatorname{coker} e_1$, and the second arrow is the Orlov equivalence, which we will not describe in detail. The inverse of the Orlov equivalence is induced by $\beta_*\pi^* : \mathrm{RPerf}(Q \twoheadrightarrow R) \to \mathrm{RPerf}(Y \subset \mathbb{P}_Q^{c-1})$, with β and π as defined in diagram (5.1).

In fact, Burke and Walker show that coker is an equivalence in the more general setting of Definition 2.7.1:

$$[MF(X, \mathcal{L}, W)] \xrightarrow{\text{coker}} \mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Z \hookrightarrow X) \tag{5.3}$$

is an equivalence of categories. The precise definition of the source category $[MF]$ is given in Section 2.7, but is not needed for the remainder of this chapter; it suffices to know that we may associate matrix factorizations to objects of $\mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$ via an inverse of coker.

In the context of the Orlov equivalence (5.2) we will denote by $\mathbb{E}_M$ any fixed choice of $\Psi^{-1}(M)$ for $M \in \mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Q \twoheadrightarrow R)$, and in the more general context of (5.3) we will denote by $\mathbb{E}_{\mathcal{M}}$ any fixed choice of $\mathrm{coker}^{-1}(\mathcal{M})$ for $\mathcal{M} \in \mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$.

Note that one way to construct such matrix factorizations is by simply sheafifying an object of $MF_{\mathrm{gr}}(S, W)$, as in the next definition.

To accommodate the shift from free S-modules to locally free coherent sheaves on Y, we now use Burke and Walker's original construction of a matrix factorization of W over $\mathbb{P}^{c-1}_Q$ corresponding to a bounded complex $\mathbf{G}$ of finitely generated free Q-modules and a system of higher homotopies $\boldsymbol{\sigma}$, which they obtain by sheafifying $E(\mathbf{G}, \boldsymbol{\sigma})$ as defined in Section 4.2. More explicitly, given $(\mathbf{G}, \boldsymbol{\sigma})$, one constructs $\mathbb{E}(\mathbf{G}, \boldsymbol{\sigma}) \in MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$ as follows:

5.2.3 Definition (Burke-Walker, [7], 6.5). The matrix factorization $\mathbb{E}(\mathbf{G}, \boldsymbol{\sigma})$ is $\mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1)$, where

$$\mathcal{E}_1 := \widetilde{E_1} = \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j+1} \otimes_Q \mathcal{O}_{\mathbb{P}^{c-1}_Q}(j) \text{ and } \mathcal{E}_0 := \widetilde{E_0} = \bigoplus_{j \in \mathbb{Z}} \mathbf{G}_{2j} \otimes_Q \mathcal{O}_{\mathbb{P}^{c-1}_Q}(j),$$

and $e_1 = \widetilde{f_1}, e_0 = \widetilde{f_0}$, with f_1, f_0 the maps of $E(\mathbf{G}, \boldsymbol{\sigma})$ (renamed here to avoid abuse of notation):

$$(f_1)_j = \sum_{J \in \mathbb{N}^c} \sigma^J \otimes T^J : \mathbf{G}_{2j+1} \otimes_Q S(j) \to \bigoplus_i \mathbf{G}_{2i} \otimes_Q S(i) = E_0,$$

$$(f_0)_j = \sum_{J \in \mathbb{N}^c} \sigma^J \otimes T^J : \mathbf{G}_{2j} \otimes_Q S(j) \to \bigoplus_i \mathbf{G}_{2i-1} \otimes_Q S(i) = E_1(1). \tag{5.4}$$

As in Section 4.2, when $\mathbf{G}$ is known to be a resolution of some R-module M, we will write $\mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})$ in place of $\mathbb{E}(\mathbf{G}, \boldsymbol{\sigma})$ for emphasis. Burke and Walker prove that $\Psi(\mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})) \cong$

M in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$; we quote this result in full in the next section, as Proposition 5.3.2. Thus one may take $\mathbb{E}_M = \mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})$ as inverse of the cokernel functor in this setting.

5.2.4 Definition. As in the graded module case, given an object $\mathbb{E}$ of $MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$, we will denote by $\mathbb{E}^\infty$ the sequence

$$\cdots \longrightarrow \mathcal{E}_0(-1) \xrightarrow{e_0(-1)} \mathcal{E}_1 \xrightarrow{e_1} \mathcal{E}_0 \xrightarrow{e_0} \mathcal{E}_1(1) \xrightarrow{e_1(1)} \mathcal{E}_0(1) \longrightarrow \cdots$$

One problem with attempting to compare Ext (or Tor) groups in the present scheme-theoretic context is that on the $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ side of the Burke-Walker equivalence $[MF] \xrightarrow{\Psi} \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, objects are built out of modules, while on the $[MF]$ side they are built out of coherent sheaves, and it is not immediately clear how to correctly define Ext and Tor groups on both sides which are suitably compatible with Ψ. To address this problem for Ext, Burke and Walker, in both [6] and [7], use the notion of hypercohomology, denoted by $\mathbb{H}$; in [6] they prove that $\mathbb{H}^0$ can be used to describe the Hom-sets of their matrix factorization category (as an alternative to the description given by the definition of Verdier quotient). Hypercohomology is defined via a bicomplex whose total homology incorporates both conventional homology of a complex and Čech cohomology:

5.2.5 Definition. Given a complex $\mathbb{X}_\bullet$ of coherent sheaves on a projective scheme X, fix a finite affine open cover $\mathcal{U} = \{U_1, \ldots, U_n\}$ of X. One may construct a bicomplex by applying the Čech construction on $\mathcal{U}$ to each $\mathbb{X}_i$, as follows:

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_i \Gamma(U_i, \mathbb{X}_1) & \longrightarrow & \bigoplus_{i<j} \Gamma(U_{ij}, \mathbb{X}_1) & \longrightarrow & \bigoplus_{i<j<k} \Gamma(U_{ijk}, \mathbb{X}_1) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_i \Gamma(U_i, \mathbb{X}_0) & \longrightarrow & \bigoplus_{i<j} \Gamma(U_{ij}, \mathbb{X}_0) & \longrightarrow & \bigoplus_{i<j<k} \Gamma(U_{ijk}, \mathbb{X}_0) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \bigoplus_i \Gamma(U_i, \mathbb{X}_{-1}) & \longrightarrow & \bigoplus_{i<j} \Gamma(U_{ij}, \mathbb{X}_{-1}) & \longrightarrow & \bigoplus_{i<j<k} \Gamma(U_{ijk}, \mathbb{X}_{-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array} \tag{5.5}$$

Here $U_{ij} := U_i \cap U_j$ and $U_{ijk} := U_i \cap U_j \cap U_k$ and we define indices so that $\bigoplus_i \Gamma(U_i, \mathbb{X}_0)$ resides in degree 0 of the totalization. The horizontal maps are the usual Čech maps and the vertical maps are the direct sums of the maps on sections induced from the differential of $\mathbb{C}_\bullet$. Also note that since $\mathcal{U}$ is finite, each row is actually a bounded complex.

The qth *hypercohomology*, $\mathbb{H}^q(X, \mathbb{C}_\bullet)$, of the complex $\mathbb{C}_\bullet$ is defined to be the $-q$th homology of the totalization of this bicomplex. It is known (and easily verified) that hypercohomology is independent of the choice of affine open cover $\mathcal{U}$.

5.2.6 Remark. One could also write $\mathbb{H}_{-q}(X, \mathbb{C}_\bullet)$ for $\mathbb{H}^q(X, \mathbb{C}_\bullet)$, but we will index cohomologically, as this seems to make more sense in light of the terminology.

Hypercohomology may be computed via two obvious spectral sequences, both of which involve often well-understood sheaf cohomology computations and play a central role in the proofs of the results in this chapter. The nature of this construction also makes it a useful tool for producing global versions of sheaf data.

5.3 Main result

The next theorem is the main result of this section; it is analogous to, but less general than, a result of Burke and Walker for stable Ext groups ([7], 2.16); however, the proof is entirely different. Hom groups are an intrinsic part of a category, and as such are preserved by equivalences of categories (particularly by the Burke-Walker equivalence Ψ) by definition, which allows for a very brief proof. Tensor products do not share this nice behavior, so we instead resort to direct computations, aided by the use of some technical machinery developed in later sections of [7]. This theorem will then allow us to imitate several proofs for Ext groups in [7] to obtain analogous results for Tor groups, in Section 5.4.

For clarity, we first establish notational conventions for the notions of (co)homology that appear in this section: $\mathbb{H}$ denotes hypercohomology, H denotes sheaf cohomology, and $\mathcal{H}$ denotes the usual (co)homology of a complex. For easy reference, we also reproduce diagram (5.1) here.

$$\begin{tikzcd}
\mathbb{P}_R^{c-1} \arrow[r, "\beta"] \arrow[d, "\pi"] \arrow[rr, bend left, "\delta"] & Y \arrow[r, "\gamma"] & \mathbb{P}_Q^{c-1} \arrow[d] \\
\text{Spec } R \arrow[rr] & & \text{Spec } Q
\end{tikzcd}$$

5.3.1 Theorem. *Let $R = Q/(f_1, \ldots, f_c)$, where Q is any local ring and $f_1, \ldots, f_c$ is a regular sequence on Q. Let $S = Q[T_1, \ldots, T_c]$ and W the element $f_1T_1 + \cdots + f_cT_c$ of Q, and let $Y = \text{Proj } S/(W)$.*

For any R-modules M and N such that M has finite projective dimension over Q and for all $q \in \mathbb{Z}$, there is an isomorphism

$$\widehat{\text{Tor}}_q^R(M, N) \cong \mathbb{H}^{-q}(Y, \gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N})[c-1],$$

where $\mathcal{N} = \beta_*\pi^*N$ and $\mathbb{E}_M = \mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})$ for any choice of finite free resolution $\mathbf{G}$ of M over Q and system of higher homotopies $\boldsymbol{\sigma}$ on $\mathbf{G}$.

Proof. We begin by setting some notation. Set $\mathcal{U} = \{U_i\}_{i=1}^c$ to be the usual affine open cover of $\mathbb{P}_R^{c-1}$, i.e., $U_i = D_+(T_i) := \{\mathfrak{p} \in \mathbb{P}_R^{c-1} \mid T_i \notin \mathfrak{p}\}$. Then we have $\Gamma(U_i, \mathcal{O}_{\mathbb{P}_R^{c-1}}) = R[T_1, \dots, T_c]_{(T_i)}$, the graded degree 0 component of the localization $R[T_1, \dots, T_c]_{T_i}$ (see [10], II.2.5). By abuse of notation, we will also use $\mathcal{U}$ for the open cover $\{D_+(T_i)\}_{i=1}^c$ of Y. Furthermore, for a sequence $\mathbb{X}_\bullet$ of coherent sheaves we denote by $C^\bullet(\mathcal{U}, \mathbb{X}_\bullet)$ the degreewise Čech construction that produces diagram (5.5), and following [7], we define $\Gamma(\mathcal{U}, \mathbb{X}_\bullet) := \mathrm{Tot}(C^\bullet(\mathcal{U}, \mathbb{X}_\bullet))$ (so that hypercohomology of $\mathbb{X}_\bullet$ is precisely the homology of the complex $\Gamma(\mathcal{U}, \mathbb{X}_\bullet)$).

We first show that for any choice of $(\mathbf{G}, \boldsymbol{\sigma})$, $\Gamma(\mathcal{U}, \delta^*\mathbb{E}_M^\infty)$ is a complete resolution of $M[-c+1]$ over R. In other words, we show that the complex $\Gamma(\mathcal{U}, \delta^*\mathbb{E}_M^\infty)[c-1]$ is exact and that it agrees with some resolution of M in large degrees.

Since by definition $\mathbb{H}^q(\mathbb{P}_R^{c-1}, \delta^*\mathbb{E}_M^\infty) = \mathcal{H}^q(\Gamma(\mathcal{U}, \mathbb{E}_M^\infty))$, to show exactness we claim that the spectral sequence

$$E_2^{p,q} = \mathcal{H}^p(C^\bullet(\mathcal{U}, \mathcal{H}_q(\delta^*\mathbb{E}_M^\infty))) \Longrightarrow \mathbb{H}^{p-q}(\mathbb{P}_R^{c-1}, \delta^*\mathbb{E}_M^\infty) \tag{5.6}$$

which arises from first taking vertical homology of diagram (5.5) collapses to 0. First, note that a very slight alteration of [7], Remark 4.2, implies that $\delta^*\mathbb{E}_M^\infty \cong \beta^*\gamma^*\mathbb{E}_M^\infty$ is exact; for completeness we reproduce this argument here. It is known (essentially by [8], Proposition 5.1) that $\gamma^*\mathbb{E}_M^\infty$ is exact. Fix an integer n and consider the truncation $(\gamma^*\mathbb{E}_M^\infty)_{\geq n}$. The map β is locally complete intersection (because $f_1, \dots, \hat{f_i}, \dots, f_c$ is a regular sequence on $(S/(W))_{(T_i)}$ and f_i is a linear combination of $f_1, \dots, \hat{f_i}, \dots, f_c$ in $(S/(W))_{(T_i)}$), so β has finite flat dimension. Thus $\beta^*(\gamma^*\mathbb{E}_M^\infty)_{\geq n}$ remains exact in all degrees larger than n plus the flat dimension of β. Since this is true for all $n \in \mathbb{Z}$, we see that $\beta^*\gamma^*\mathbb{E}_M^\infty$ is exact. Thus the first page of the spectral sequence (5.6) is 0, so $E_\infty^{p,q} = 0$ and $\mathcal{H}(\Gamma(\mathcal{U}, \mathbb{E}_M^\infty)) = 0$ as desired.

Next we show that $\Gamma(\mathcal{U}, \delta^*\mathbb{E}_M^\infty)[c-1]$ agrees with a resolution of M in large degrees. We first give a brief summary of the technical machinery involved. Proposition 4.5 of [7] establishes an isomorphism of functors

$$\Psi(-) \xrightarrow{\cong} \Gamma(\mathcal{U}, \delta^\sharp(-)),$$

where $\Psi : [MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)] \to \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ is, as in Section 5.2, the composition of the equivalences

$$[MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)] \xrightarrow{\text{coker}} \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(Y) \xrightarrow{\text{Orlov}} \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R),$$

and where

$$\delta^\sharp(\mathbb{E}) := \delta^*(\mathbb{E}^\infty)_{\geq 0}[-c+1]$$

for any $\mathbb{E} \in MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$. The authors then define a functor $F : \mathcal{C}^{\mathrm{b}}(\mathbb{P}^{c-1}_R) \to \mathcal{C}(R)$ such that, by Proposition 4.13 of [7],

$$\Gamma(\mathcal{U}, \delta^\sharp(\mathbb{E})) \cong F((\delta^\sharp(\mathbb{E})_{\geq m})$$

in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ for any $\mathbb{E} \in MF(\mathbb{P}^{c-1}_Q, \mathcal{O}(1), W)$ and m sufficiently large. In light of the fact that $X \cong X_{\geq m}$ in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$ for any $X \in \mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$, for m sufficiently large we have established that

$$\Gamma(\mathcal{U}, \delta^*\mathbb{E}^\infty_M)[c-1]_{\geq m} \cong F((\delta^\sharp(\mathbb{E}_M))_{\geq m}).$$

Now we fix a minimal Q-free resolution $\mathbf{G}$ of M and a system of higher homotopies $\boldsymbol{\sigma}$ on $\mathbf{G}$, and set $\mathbb{E}_M = \mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})$. Using Proposition 5.3.2, quoted from [7] below, we get that $\Gamma(\mathcal{U}, \delta^*\mathbb{E}^\infty_M)$ agrees with the Eisenbud-Shamash complex $\mathbf{G}\{\boldsymbol{\sigma}\}$ in large degrees.

This establishes that $\Gamma(\mathcal{U}, \delta^*\mathbb{E}^\infty_M)[c-1]$ is a complete R-resolution of M, so we may use it to compute stable Tor over R:

$$\widehat{\mathrm{Tor}}^R_q(M, N) \cong \mathcal{H}_q(\Gamma(\mathcal{U}, \delta^*\mathbb{E}^\infty_M)[c-1] \otimes_R N).$$

The theorem now follows from the isomorphisms

$$
\begin{aligned}
\mathcal{H}_q(\Gamma(\mathcal{U}, \delta^*\mathbb{E}_M^\infty)[c-1] \otimes_R N)) &\cong \mathcal{H}_q(\Gamma(\mathcal{U}, \delta^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_{\mathbb{P}_R^{c-1}}} \pi^*N)[c-1]) \\
&\cong \mathcal{H}_q(\Gamma(\mathcal{U}, \gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N})[c-1] \\
&\cong \mathbb{H}^{-q}(Y, \gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N})[c-1],
\end{aligned}
$$

which arise as follows:

- For the first isomorphism, we note that (by [11], Proposition 5.1.12(b), for example) tensor products commute with taking affine sections; formally, for each affine open set $U_i \in \mathcal{U}$, we have $\Gamma(U_i, \delta^*\mathbb{E}_M^\infty \otimes_{\mathbb{P}_R^{c-1}} \pi^*N) \cong \Gamma(U_i, \delta^*\mathbb{E}_M^\infty) \otimes_R \Gamma(U_i, \pi^*N)$. The stated isomorphism is then due to the fact that the coherent sheaves that make up the complex $\delta^*\mathbb{E}_M^\infty$ are locally free on $\mathbb{P}_R^{c-1}$.
- For the second isomorphism, we first note that since $\delta^*\mathbb{E}_M^\infty \cong \beta^*\gamma^*\mathbb{E}_M^\infty$ is a complex of locally free coherent sheaves, the Projection Formula gives $\beta_*(\beta^*\gamma^*\mathbb{E}_M^\infty \otimes_{\mathbb{P}_R^{c-1}} \pi^*N) \cong \gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N}$. The isomorphism on homology follows from the fact that the sections of $\gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N}$ are clearly annihilated by $(f_1, \ldots, f_c)$.
- The final isomorphism is precisely the definition of hypercohomology.

□

Finally, we reproduce Proposition 6.6 of [7], which is used in the proof above.

5.3.2 Proposition (Burke-Walker). *Let M be a finitely generated R-module that has finite projective dimension over Q. Let $\mathbf{G}$ be a finite projective Q-resolution of M, $\boldsymbol{\sigma}$ a system of higher homotopies on $\mathbf{G}$, and $\mathbb{E} = \mathbb{E}(M, \mathbf{G}, \boldsymbol{\sigma})$. Then the complex $F(\delta^\sharp\mathbb{E})$ is exactly the standard resolution $\mathbf{G}\{\boldsymbol{\sigma}\}$. In particular, there is an isomorphism in $\mathrm{D}^{\mathrm{b}}_{\mathrm{sg}}(R)$,*

$$\Psi(\mathbb{E}) \cong M.$$

5.4 The stable Tor sheaf

We now work in the general setting described by Definition 5.2.1, as the definitions and results are no more difficult to state in that context. Unless otherwise stated, in this section X is any projective scheme over a Noetherian ring A of finite Krull dimension, $\mathcal{L} = \mathcal{O}(1)$, and W is any regular global section of $\mathcal{L}$. Finally, $\gamma : Z \hookrightarrow X$ is the embedding of the zero subscheme of W.

In [7], Burke and Walker define a notion of stable Ext sheaf analogous to that of the stable Ext groups:

5.4.1 Definition (Burke-Walker). For $\mathcal{M}$ in $\mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$, $\mathcal{N}$ a bounded complex of coherent sheaves on Z, and any integer $q \in \mathbb{Z}$, define

$$\widehat{\mathcal{E}\mathrm{xt}}^q_{\mathcal{O}_Z}(\mathcal{M}, \mathcal{N}) = \mathcal{H}^q \mathcal{H}\mathrm{om}_{\mathcal{O}_Z}(\gamma^* \mathbb{E}^\infty_{\mathcal{M}}, \mathcal{N}).$$

(Recall that given an object $\mathcal{M}$ of $\mathrm{RPerf}(Z \hookrightarrow X)$, $\mathbb{E}_{\mathcal{M}}$ denotes a choice of object of $MF(X, \mathcal{L}, W)$ whose cokernel is isomorphic in $\mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$ to $\mathcal{M}$.)

We modify this definition in a natural way to obtain an analogous notion of stable Tor sheaf:

5.4.2 Definition. In the setting of Definition 5.4.1,

$$\widehat{\mathcal{T}\mathrm{or}}^{\mathcal{O}_Z}_q(\mathcal{M}, \mathcal{N}) = \mathcal{H}_q(\gamma^* \mathbb{E}^\infty_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N}).$$

The following result is a stable Tor analog of a stable Ext result of Burke and Walker ([7], Proposition 3.8), and its proof is nearly identical.

5.4.3 Proposition. *Let $X, \mathcal{L}, W$, and Z be as above (so that in particular W is an arbitrary global section of $\mathcal{L}$). Let $\mathcal{M}$ be an object of $\mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$ and $\mathcal{N}$ be an object of $\mathrm{D}^{\mathrm{b}}(Z)$.*

For all $y \in Z$ and $q \in \mathbb{Z}$ there are isomorphisms

$$\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}) \cong \widehat{\mathcal{T}\mathrm{or}}_{q+2}^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})(1),$$
$$\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})_y \cong \widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Z,y}}(\mathcal{M}_y,\mathcal{N}_y),$$

that are natural in $\mathcal{M}$ and $\mathcal{N}$. For all q, there is a map

$$\mathbb{H}^{-q}(Z, \gamma^*\mathbb{E}_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N}) \longrightarrow \Gamma(Z, \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}))$$

that is natural in $\mathcal{M}$ and $\mathcal{N}$. This map is an isomorphism for $q \ll 0$.

Proof. The first isomorphism is an obvious consequence of the definition of $\widehat{\mathcal{T}\mathrm{or}}$ and the fact that $(\gamma^*\mathbb{E}_{\mathcal{M}})_{k+2} = (\gamma^*\mathbb{E}_{\mathcal{M}}(-1))_k$ for all $k \in \mathbb{Z}$.

For the second isomorphism, we may apply [7], Example B.5 (with $B = \mathcal{O}_{Z,y}$, $A = \mathcal{O}_{X,y}$, $\mathbb{E} = (\mathbb{E}_{\mathcal{M}})_y$, $M = \mathcal{M}_y$, and $T = \gamma^*(\mathbb{E}_{\mathcal{M}})_y)$ to see that $\gamma^*(\mathbb{E}_{\mathcal{M}})_y$ is a complete resolution of $\mathcal{M}_y$. Using this complete resolution to compute $\widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Z,y}}(\mathcal{M}_y,\mathcal{N}_y)$, we get

$$\begin{aligned}\widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Z,y}}(\mathcal{M}_y,\mathcal{N}_y) &\cong \mathcal{H}_q(\gamma^*(\mathbb{E}_{\mathcal{M}})_y \otimes_{\mathcal{O}_{Z,y}} \mathcal{N}_y)\\ &\cong \mathcal{H}_q(\gamma^*\mathbb{E}_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N})_y\\ &\cong \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})_y.\end{aligned}$$

The last map arises from the spectral sequence

$$E_2^{p,q} = H^p(Z, \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})) \Longrightarrow \mathbb{H}^{p-q}(Z, \gamma^*\mathbb{E}_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N}),$$

as $\Gamma(Z, \widehat{\mathcal{T}\mathrm{or}}_i^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})) \cong H^0(Z, \widehat{\mathcal{T}\mathrm{or}}_i^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}))$ appears as the $p = 0, q = i$ summand of $\mathbb{H}^{-i}(Z, \gamma^*\mathbb{E}_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N}) \cong \bigoplus_{p-q=-i} \mathbb{H}^{p-q}(Z, \gamma^*\mathbb{E}_{\mathcal{M}} \otimes_{\mathcal{O}_Z} \mathcal{N})$; the map in question is the projection onto this summand.

It remains to show that the map is an isomorphism for $q \ll 0$. Since $\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}) \cong$

$\widehat{\mathcal{T}\mathrm{or}}_{q+2}^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})(1)$, as q decreases, the spectral sequence involves computing cohomologies of increasing twists of the same two coherent sheaves. Since $\mathcal{L} = \mathcal{O}(1)$ is very ample, for all $q \ll 0$ Serre vanishing thus gives $H^i(\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N})) = 0$ for all $i > 0$. The result follows from the fact that the rows of the hypercohomology diagram (5.5) have finite length. $\square$

Applying this result to the setting of Theorem 5.3.1 (specifically $X = \mathbb{P}_Q^{c-1}, W = f_1T_1 + \cdots + f_cT_c, Z = Y$), we now have the following isomorphisms for $q \ll 0$ (with the first isomorphism valid for all $q \in \mathbb{Z}$):

$$\widehat{\mathrm{Tor}}_q^R(M,N) \cong \mathbb{H}^{-q}(Y, \gamma^*\mathbb{E}_M^\infty \otimes_{\mathcal{O}_Y} \mathcal{N})[c-1] \cong \Gamma(Y, \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M},\mathcal{N}))[c-1] \tag{5.7}$$

(where $\mathcal{M} = \beta_*\pi^*M$). Thus we have established a relationship between the stable Tor in $\mathrm{D}_{\mathrm{sg}}^{\mathrm{b}}(R)$ and a notion of stable Tor over Y.

The next proposition shows that stable Tor sheaves agree with the usual Tor sheaves in high degree. The statement is analogous to part of [7], Proposition 3.11. The proof is identical to theirs with Exts replaced by Tors, but we provide it here for completeness.

5.4.4 Proposition. *In the setting of Proposition 5.4.3, given any object $\mathcal{M}$ of* $\mathrm{RPerf}(Z \hookrightarrow X)$ *and any $\mathcal{N} \in \mathrm{D}^{\mathrm{b}}(Z)$, for each $q \gg 0$ there is an isomorphism*

$$\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}) \cong \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Z}(\mathcal{M}',\mathcal{N}),$$

where $\mathcal{M}'$ is the image of $\mathcal{M}$ in the localization $\mathrm{RPerf}(Z \hookrightarrow X)/\mathrm{Perf}(Z) = \mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Z \hookrightarrow X)$.

Proof. By [6], Lemma 5.2(1), $(\gamma^*\mathbb{E}_{\mathcal{M}}^\infty)_{\geq 0}$ is a locally free resolution of coker $\mathbb{E}_{\mathcal{M}}$ (which is isomorphic to $\mathcal{M}$ in $\mathrm{D}_{\mathrm{sg}}^{\mathrm{rel}}(Z \hookrightarrow X)$). By the definition of the stable Tor sheaf, it follows that

$$\widehat{\mathcal{T}\mathrm{or}}_n^{\mathcal{O}_Z}(\mathcal{M},\mathcal{N}) \cong \mathcal{H}_n(\gamma^*\mathbb{E}_{\mathcal{M}}^\infty \otimes_{\mathcal{O}_Z} \mathcal{N}) \cong \mathcal{T}\mathrm{or}_n^{\mathcal{O}_Z}(\mathrm{coker}\ \mathbb{E}_{\mathcal{M}},\mathcal{N})$$

for all $n \geq 1$.

By the definition of the Verdier quotient (see Definition 2.1.4), the fact that $\mathcal{M} \cong \operatorname{coker} \mathbb{E}_{\mathcal{M}}$ in $\mathrm{D}^{\mathrm{rel}}_{\mathrm{sg}}(Z \hookrightarrow X)$ is the same as the existence of a diagram

$$\mathcal{M} \leftarrow \mathcal{G} \rightarrow \operatorname{coker} \mathbb{E}_{\mathcal{M}} \tag{5.8}$$

with $\mathcal{G} \in \operatorname{RPerf}(Z \hookrightarrow X)$ and both arrows having cones that are objects of $\operatorname{Perf}(Z)$. Note that $\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(\mathcal{P}, \mathcal{N}) = 0$ for $q \gg 0$ when $\mathcal{P} \in \operatorname{Perf}(Z)$ since $\mathcal{P}$ has a bounded resolution by definition. Therefore, since $\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(-, \mathcal{N})$ is a triangulated functor (and hence commutes with taking cones), for $q \gg 0$ the cones of the arrows in the diagram

$$\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(\mathcal{M}, \mathcal{N}) \leftarrow \mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(\mathcal{G}, \mathcal{N}) \rightarrow \mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(\operatorname{coker} \mathbb{E}_{\mathcal{M}}, \mathcal{N})$$

obtained by applying $\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Z}(-, \mathcal{N})$ to (5.8) are 0, so both arrows are isomorphisms in $\operatorname{RPerf}(Z \hookrightarrow X)$, which gives the desired result. □

One application of the results of this chapter is the following partial generic hypersurface version of a recent theorem of Bergh and Jorgensen ([3], 3.3) that relates Tor groups over R to those over hypersurfaces that lie between R and Q, that is, rings $Q/(f)$ with f an arbitrary element of the ideal $(f_1, \ldots, f_c)$ that defines R. As explained in Section 2.5, in a sense the generic hypersurface stands in for all of these intermediate hypersurfaces; each intermediate hypersurface $Q/(a_1 f_1 + \cdots + a_c f_c)$ for $a_i \in Q$ is the result of substituting a_i for T_i in $S/(W)$.

5.4.5 Corollary. *Let R be a complete intersection, M and N be finitely generated R-modules, and M' and N' be the $S/(W)$-modules $M[T_1, \ldots, T_c]$ and $N[T_1, \ldots, T_c]$ respectively. If $\operatorname{Tor}_q^{S/(W)}(M', N') = 0$ for $q \gg 0$, then $\operatorname{Tor}_q^R(M, N) = 0$ for $q \gg 0$.*

Proof. Set $\mathcal{M} = \beta_* \pi^* M$ and $\mathcal{N} = \beta_* \pi^* N$. If $\operatorname{Tor}_q^{S/(W)}(M', N') = 0$ for all $q \gg 0$, then clearly $\mathcal{T}\mathrm{or}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = 0$ for all $q \gg 0$. Then Proposition 5.4.4 gives that $\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = 0$ for all $q \gg 0$, and in particular $\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})_y = 0$ for all $y \in Y$ and $q \gg 0$. Then by the second isomorphism in Proposition 5.4.3, we have $\widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Y,y}}(\mathcal{M}_y, \mathcal{N}_y) = 0$ for all $y \in Y$ and $q \gg 0$.

By [2], Proposition 1.6, localizations of modules of finite complete intersection dimension again have finite complete intersection dimension. Therefore $\mathcal{M}_y$ has finite CI-dimension for any $y \in Y$ and so by [1], Theorem 4.9, $\widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Y,y}}(\mathcal{M}_y, \mathcal{N}_y) = 0$ for $q \gg 0$ implies that in fact $\widehat{\mathrm{Tor}}_q^{\mathcal{O}_{Y,y}}(\mathcal{M}_y, \mathcal{N}_y) = 0$ for all $q \in \mathbb{Z}$, and in particular for all $q \ll 0$.

Now again by the second isomorphism in Proposition 5.4.3 we have $\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})_y = 0$ for all $y \in Y$ and $q \ll 0$, which clearly implies that $\widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N}) = 0$ for $q \ll 0$, and particularly that $\Gamma(Y, \widehat{\mathcal{T}\mathrm{or}}_q^{\mathcal{O}_Y}(\mathcal{M}, \mathcal{N})) = 0$ for $q \ll 0$. Next we may use the combination of Theorem 5.3.1 and Proposition 5.4.3 given by equation (5.7) above to get $\widehat{\mathrm{Tor}}_q^R(M, N) = 0$ for $q \ll 0$. One more application of [1], Theorem 4.9 gives that $\widehat{\mathrm{Tor}}_q^R(M, N) = 0$ for all q, and in particular for $q \gg 0$. Since stable Tor and ordinary Tor are isomorphic for $q \gg 0$, the result follows. $\square$

Bibliography

[1] L. L. Avramov and R.-O. Buchweitz, *Support varieties and cohomology over complete intersections*, Inventiones Mathematicae, 142 (2000), pp. 285–318.

[2] L. L. Avramov, V. N. Gasharov, and I. V. Peeva, *Complete intersection dimension*, Publications Mathématiques de l'IHÉS, 86 (1997), pp. 67–114.

[3] P. A. Bergh and D. A. Jorgensen, *A generalized Dade's Lemma for local rings*, Algebras and Representation Theory, (2017).

[4] R.-O. Buchweitz, *Maximal Cohen–Macaulay modules and Tate cohomology over Gorenstein rings*, Unpublished manuscript, (1987).

[5] J. Burke and G. Stevenson, *The derived category of a graded Gorenstein ring*, in Commutative algebra and noncommutative algebraic geometry. Vol. II, vol. 68 of Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, New York, 2015, pp. 93–123.

[6] J. Burke and M. E. Walker, *Matrix factorizations over projective schemes*, Homology Homotopy Appl., 14 (2012), pp. 37–61.

[7] ——, *Matrix factorizations in higher codimension*, Trans. Amer. Math. Soc., 367 (2015), pp. 3323–3370.

[8] D. Eisenbud, *Homological algebra on a complete intersection, with an application to group representations*, Trans. Amer. Math. Soc., 260 (1980), pp. 35–64.

[9] D. R. Grayson and M. E. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[10] R. Hartshorne, *Algebraic geometry*, Graduate Texts in Mathematics, Springer-Verlag, New York-Heidelberg, 1977.

[11] Q. Liu, *Algebraic Geometry and Arithmetic Curves*, Oxford graduate texts in mathematics, Oxford University Press, 2002.

[12] D. Orlov, *Triangulated categories of singularities, and equivalences between Landau-Ginzburg models*, Sbornik: Mathematics, 197 (2006), p. 1827.

[13] ——, *Derived categories of coherent sheaves and triangulated categories of singularities*, in Algebra, arithmetic, and geometry: in honor of Yu. I. Manin. Vol. II, vol. 270 of Progr. Math., Birkhäuser Boston, Inc., Boston, MA, 2009, pp. 503–531.

[14] A. Polishchuk and A. Vaintrob, *Matrix factorizations and singularity categories for stacks*, Ann. Inst. Fourier (Grenoble), 61 (2011), pp. 2609–2642.

[15] J. Shamash, *The Poincaré series of a local ring*, J. Algebra, 12 (1969), pp. 453–470.

[16] Y. Yoshino, *Cohen-Macaulay Modules over Cohen-Macaulay Rings*, London Mathematical Society Lecture Note Series, Cambridge University Press, 1990.

Eric Ottman

Curriculum Vitae

215 Carnegie Library
Syracuse, NY 13244
☏ *(585) 455-8273*
✉ *ejottman@syr.edu*

Education

2019 (expected) **Ph.D. Mathematics**, *Syracuse University*, Syracuse, NY.
Advisor: Claudia Miller
Title: Homology over a Complete Intersection Ring via the Generic Hypersurface

2018 **M.S. Mathematics**, *Syracuse University*, Syracuse, NY.

2010 **B.S. Mathematics**, *University of Rochester*, Rochester, NY.

Employment

2019-2020 **Visiting Lecturer**, *Rochester Institute of Technology*.

2010-2019 **Teaching Assistant**, *Syracuse University*.

Papers

Homology over a complete intersection ring via the generic hypersurface, *preprint in preparation*.

Selected Lectures

Future **Homology over a complete intersection via the generic hypersurface**, *AMS Southeastern Sectional Meeting, Special Session on Homological Methods in Algebra*, University of Florida, November 2019.

Homology over a complete intersection via the generic hypersurface, *AMS Eastern Sectional Meeting, Special Session on Commutative Algebra*, Binghamton University, October 2019.

2018 **Homology over a complete intersection via the generic hypersurface**, *Route 81 Conference on Commutative Algebra and Algebraic Geometry*, Syracuse University.

2017 **Axiomatic Number Theory and Gödel's incompleteness theorems**, *Mathematics Graduate Organization Colloquium*, Syracuse University.
Intended for a general graduate audience.

Axiomatic Number Theory and Gödel's incompleteness theorems, *New York Regional Graduate Mathematics Conference*, Syracuse University.
A shorter version of the same talk.

Teaching

Rochester Institute of Technology, as main instructor.

Calculus I (F19)

Linear Algebra (F19)

Syracuse University, as main instructor.
Responsible for all aspects of the class except the final exam.

Calculus for the Life Sciences I (F11, Summer '12, Summer '13)

Calculus for the Life Sciences II (S12)

Calculus I (F12, S13, F14, F16, F18)

Calculus II (F13, F15, F17)

Academic Excellence Workshops, Syracuse University, 2013-2019.
Responsible for writing problem sets and training student workshop leaders for the following courses:

Precalculus.

Calculus I, II, III.

Linear Algebra.

Differential Equations and Matrix Algebra for Engineers.

Service

2013-2019 **Academic Excellence Workshops Course Developer**, *Syracuse University*.
I was solely responsible for creating problem sets for weekly two-hour AEW problem sessions associated with the math classes taken by freshman and sophomore engineering students. I also held weekly meetings to train the undergraduate students who led the workshops. During my tenure as course developer, the program was substantially expanded in scope and size due to its popularity among students. The problems I wrote for the program have also been adapted by the mathematics department for use as review materials for calculus courses.

2016 **International Conference on Representations of Algebras**, *Syracuse University*.
Assisted with various organizational tasks throughout the workshop and conference.

Professional Memberships

- American Mathematical Society
- Mathematical Association of America
- Association for Women in Mathematics