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# Blaschke Decompositions on Weighted Hardy Spaces and the Unwinding Series

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**Abstract**

Recently, several authors have considered a nonlinear analogue of Fourier series in signal analysis, referred to as either the unwinding series or adaptive Fourier decomposition. In these processes, a signal is represented as the real component of the boundary value of an analytic function $F : \partial\mathbb{D} \to \mathbb{C}$, and by performing an iterative method to obtain a sequence of Blaschke decompositions, the signal can be efficiently approximated using only a few terms. To better understand the convergence of these methods, the study of Blaschke decompositions on weighted Hardy spaces was initiated by Coifman and Steinerberger, under the assumption that the complex valued function $F$ has an analytic extension to $\mathbb{D}_{1+\epsilon}$ for some $\epsilon > 0$. This provided bounds on weighted Hardy norms involving a *single* zero, $\alpha \in \mathbb{D}$, of $F$ and its Blaschke decomposition. That work also noted that in many specific examples, the unwinding series of $F$ converges at an exponential rate to $F$, which when coupled with an efficient algorithm to compute a Blaschke decomposition, has led to the hope that this process will provide a new and efficient way to approximate signals.

In this work, we accomplish three things. Firstly, we continue the study of Blaschke decompositions on weighted Hardy Spaces for functions in the larger space $\mathcal{H}^2(\mathbb{D})$ under the assumption that the function has finitely many roots in $\mathbb{D}$. This is meaningful, since there are many functions that meet this criterion but do not extend analytically to $\mathbb{D}_{1+\epsilon}$ for any $\epsilon > 0$, for example $F(z) = \log(1-z)$. By studying the growth rate of the weights, we improve the bounds provided by Coifman and Steinerberger to obtain new estimates containing *all* roots of $F$ in $\mathbb{D}$. This provides us with new insights into Blaschke decompositions on classical function spaces including the Hardy-Sobolev spaces and weighted Bergman spaces, which

correspond to making specific choices for the aforementioned weights. Further, we state a sufficient condition on the weights for our improved bounds to hold for *any* function in the Hardy space, $\mathcal{H}^2(\mathbb{D})$, in particular functions with an infinite number of roots in $\mathbb{D}$. Second, we compare the Fourier series and the unwinding series: we show that there are many examples of functions whose unwinding series converges much faster than the Fourier series, but there are also functions for which the Fourier and unwinding series are term wise equal. From the latter, we show the existence of functions that have unwinding series that do not converge exponentially. Lastly, we discuss an efficient algorithm for computing Blaschke decompositions, and apply this algorithm to verify our theoretical results and to gain a better understanding of the underlying mechanics of the unwinding series.

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# Chapter 1

# Introduction

In many fields, the analytic Hardy Space $\mathcal{H}^p(\mathbb{D})$, where $1 \leq p \leq \infty$, has been studied due to its well behaved nature when compared to the larger Lebesgue space, $L^p(\partial\mathbb{D})$. One of the most fundamental results for the $\mathcal{H}^p(\mathbb{D})$ spaces is the Blaschke decomposition (factorization) theorem. Simply put, given a function $F \in \mathcal{H}^p(\mathbb{D})$, we can decompose

$$F = B \cdot G$$

where $|B(z)| \leq 1$ and $G(z) \neq 0$ for any $z \in \mathbb{D}$. In this factorization, the function $B$ is a Blaschke product with the same zeros as $F$ in $\mathbb{D}$, and $G$ is a function that is also in the space $\mathcal{H}^p(\mathbb{D})$. In the past few decades, this theorem was utilized to create an iterative method to express a $2\pi$ periodic real valued signal, $s$, as the real part of a summation involving Blaschke products [26]. These summations are commonly referred to as the "unwinding series" of $s$.

The idea is that given a signal $s : [0, 2\pi] \to \mathbb{R}$ that satisfies certain regularity conditions, we can use the Hilbert transform of $s$ to create a complex, analytic function $F \in \mathcal{H}^2(\mathbb{D})$ whose real part agrees with $s$ on $\partial\mathbb{D}$. That is,

$$F(e^{i\cdot}) = s(\cdot) + i\mathbf{H}(s(\cdot)),$$

where $\mathbf{H}(s)$ is the Hilbert transform of $s$. From there, the unwinding series of $F$ can be produced.

We begin with the Blaschke decomposition, which we rewrite as

$$F(z) = B_0(z) \cdot G_0(z).$$

Since $G_0 \in \mathcal{H}^2$, by adding and subtracting the term $G_0(0)$, we can introduce a root at the origin for the function $G_0(z) - G_0(0)$. This implies that we can further decompose

$$G_0(z) - G_0(0) = B_1(z) \cdot G_1(z).$$

Similarly, for any $n \geq 0$, we can iteratively obtain

$$G_n(z) - G_n(0) = B_{n+1}(z) \cdot G_{n+1}(z).$$

With all of this, through adding and subtracting the terms $G_j(0)$, for $0 \leq j \leq n$, we can expand $F$ into its *partial unwinding series*

$$
\begin{aligned}
F(z) &= B_0(z)G_0(z) && (1.1)\\
F(z) &= G_0(0)B_0(z) + B_0(z)(G_0(z) - G_0(0)) && (1.2)\\
F(z) &= G_0(0)B_0(z) + B_0(z)(B_1(z)G_1(z))) && (1.3)\\
F(z) &= G_0(0)B_0(z) + G_1(0)B_0(z)B_1(z) + B_0(z)B_1(z)(G_1(z) - G_1(0)) && (1.4)\\
&\vdots \\
F(z) &= G_0(0)B_0 + \cdots + G_n(0)\prod_{j=1}^{n} B_j(z) + \prod_{j=1}^{n} B_j(z)(G_n(z) - G_n(0)). && (1.5)
\end{aligned}
$$

By considering the real component of this series, we obtain the partial unwinding series for $s$. This formal iteration was first introduced by Nahon in his 2000 PhD thesis [26] where he also performed numerical tests on a few, specific examples of signals that potentially indicated that `(a.)` the formal unwinding series will indeed converge for any $F \in \mathcal{H}^2$, and that `(b.)` convergence of the unwinding series might always occur at an *exponential* rate.

In 2010 Qian [29] answered question `(a.)` in the affirmative: he proved convergence of the unwinding series for any $F \in \mathcal{H}^2$, but left question `(b.)` unanswered: the proof of convergence did not provide any insight into the rate of convergence of the unwinding series, other than that it is always at least as fast as the convergence rate of the Fourier series.

In 2017, Coifman and Steinerberger in [5] revisited those earlier results and `(a.)` obtained a new proof of Qian's result (convergence of the unwinding series), but also argued `(b.)` that the *rate* of convergence of the unwinding series may be better understood when the data $F$ is chosen in certain subspaces of $\mathcal{H}^2$ obtained by introducing suitable weights (*weighted Hardy spaces*). In particular, they proved new estimates for the weighted-$\mathcal{H}^2$ norm of the outer factor $G$ in the Blaschke decomposition of $F$ that takes into account *one* (any) root of $F$ in the unit disc, under the further assumption that $F$ is analytic in a disc with radius $1+\epsilon$.

In this dissertation, we relinquish the restrictive and unnatural (from the point of view of complex function theory) assumption that $F$ be analytic in the disc of radius $1+\epsilon$ and prove the following main results:

(i.) We obtain new estimates for the weighted-$\mathcal{H}^2$ norm of the outer factor $G$ that take into account *all* the roots of $F$ in the unit disc if $F$ in the weighted Hardy space has finitely many zeroes; in the case when $F$ has infinitely many zeroes, which is possible under our less restrictive assumptions, the issue of convergence of an infinite series arises, which we are able to resolve for a restricted family of weights that nonetheless give rise to spaces that are norm-equivalent to $\mathcal{H}^2$. These and related results are stated and proved in Chapter 3, which provides a more detailed exposition of the content of the manuscript [10].

(ii.) We observe that under Coifman-Steinerberger's more restrictive assumption that $F$ is analytic in a disc of radius $1+\epsilon$, convergence of Fourier series of non-polynomial $F \in \mathcal{H}^2$ (and thus of the unwinding series of $F$) will *always occur at an exponential rate*:

$$\lim_{n\to\infty} \|R_n\|_{H^2} \leq (1+c\epsilon)^{-n}, \qquad \text{where } 0<c<1$$

(here $R_n$ is the Fourier series remainder at step $n$) but

$$(1+c\epsilon) > 1 \iff \epsilon > 0.$$

Note that all the earlier numerical tests of Nahon [26] that gave exponential rate of convergence were performed on polynomials and *entire* functions (for which $\epsilon = +\infty$).

These and related results are proved in Chapter 4.

(iii.) By dropping the restrictive assumption that $F$ be analytic in a disc of radius $1+\epsilon$, we identify a family of functions in $\mathcal{H}^2(\mathbb{D})$ that do not extend analytically to a disc of radius $1+\epsilon$ for any $\epsilon > 0$ and have an unwinding series that converges non-exponentially. This provides a definitive answer to question (b.) above in the negative. This work is also carried out in Chapter 4.

Furthermore, in Chapter 5 we discuss algorithms to produce Blaschke decompositions and the partial unwinding series. We perform numerical experiments to test the sharpness of the bounds obtained in Chapter 3 and to verify the results of Chapter 4. We end this chapter with an experiment demonstrating the potential of the unwinding series in approximating signals. At the end of this dissertation, we offer concluding remarks and state the goals of future research.

# Chapter 2

# Background and Definitions

In this chapter, we begin by providing a list of important definitions with some examples. We also state several well known results with references to where the proofs of those results can be found. From there, we give a brief a review of existing literature on the topics of the unwinding series and Adaptive Fourier Decomposition. This final section is intended to help the reader understand the progress that has been made (and is currently being made) in the field.

## 2.1 Definitions, Classical Results, and Examples

Throughout this dissertation, we will be working with several classical function spaces. The assumption made in this section is that the reader has a basic understanding of complex analysis in one variable as well as a basic understanding of measure theory. The definitions of complex derivatives and analyticity on domains, uniform convergence, removable singularities, sets of measure 0 (including the notion of almost everywhere), singular measures, power series representations, $L^p$ spaces, Banach spaces, and Hilbert spaces will not be discussed in this section. The reader is directed to the books [11], [40], and [32] to brush up on these concepts, if needed, before reading further.

Many of the results we discuss in this section and many other interesting results we cannot include for the sake of time can also be found in [41], [13] and [9].

The following will be used for basic notation:

- We will denote complex discs $\mathbb{D}_r = \{z : |z| < r\}$. The unit disc ($r = 1$) will be denoted $\mathbb{D}$.
- The closure of a set $S$ will be denoted $cl(S)$.
- The boundary of a set $S$ will be denoted $\partial S$.
- The conjugate of a complex quantity $z$ will be denoted $\overline{z}$.
- The real and imaginary parts of a complex number will be denoted $\mathcal{R}e(z)$ and $Im(z)$ respectively.
- The space of analytic functions on a domain $\Omega$ will be denoted $\mathcal{O}(\Omega)$.

### 2.1.1 Blaschke Products

In the study of complex analysis, the topic of Blaschke products is usually discussed when introducing self maps of the unit disc, $B : \mathbb{D} \to \mathbb{D}$. Formally, we can define a Blaschke products in the following way.

**Definition 2.1.1.** *Let $a_n \in \mathbb{D} \setminus \{0\}$ be a collection of $N$ points, where $0 \leq N \leq \infty$, that satisfies*

$$\sum_{n=1}^{N} (1 - |a_n|) < \infty, \tag{2.1}$$

*and let $m \geq 0$ be a finite integer. A function $B : \mathbb{D} \to \mathbb{D}$ is a Blaschke Product if it is of the form*

$$B(z) = z^m \prod_{n=1}^{N} \frac{|a_n|}{a_n} \frac{a_n - z}{1 - \overline{a_n} z}. \tag{2.2}$$

**Remark 2.1.1.** *The Blaschke product $B(z) \equiv 1$ is often referred to as the trivial Blaschke product. We extend this convention to the polynomials $z^m$, where $m = 0, 1, \ldots$.*

**Remark 2.1.2.** *Equation* (2.1)*, called the Blaschke condition, is trivial if $N < \infty$. When $N = \infty$, this condition is meaningful and ensures that the Blaschke product is convergent for $z \in \mathbb{D}$.*

In Definition 2.1.1, the zeros of a Blaschke product (ignoring multiplicity) are at the points $a_n$, and $a_0 := 0$ if $m > 0$. In the case when $N$ is finite, we can see that the Blaschke product will be an $(N + m)$-1 covering of $\mathbb{D}$. Further, for every $|z| = 1$, we have $|B(z)| = 1$. In other words, points on the boundary of the unit disc are sent to points on the boundary of the unit disc. When $N = \infty$, we can similarly view the Blaschke product as an $\infty$-1 covering of $\mathbb{D}$. However, since the zero set, $\{a\}_n$, converges to $\partial\mathbb{D}$, we should not expect to have the condition:

$$\forall z \in \partial\mathbb{D}, \quad |B(z)| = 1.$$

In fact, when the set $\{z \in \partial\mathbb{D} : z = \frac{a_n}{|a_n|}\}$ is dense in $\partial\mathbb{D}$, for every $\theta \in [0, 2\pi)$, we can find a sequence $\{a_n^\theta\} \subset \mathbb{D}$ such that

$$a_n^\theta \to e^{i\theta}, \qquad \text{and} \qquad \lim_{n\to\infty} B(a_n^\theta) = 0.$$

To handle this, we need the notion of *nontangential limits*. This tool allows us to discuss the convergence of $B(z)$ when $|z| \to 1^-$ by restricting the angle in which we can approach points on the unit circle. By using nontangential limits, we can identify a meaningful extension of $B$ to the boundary of the unit disc, so that for almost every (with respect to the arc length measure) $|z| = 1, |B(z)| = 1$. This is proven rigorously in both [41] and [31].

There is rich theory on Blaschke products and their applications [14, 17, 20, 3, 27] and much of it comes from the study of Hardy spaces on the unit disc. This being the case, we now provide an overview of Hardy spaces.

### 2.1.2 Hardy Spaces

We now recall the definition of analytic Hardy spaces in [9].

**Definition 2.1.2.** *The Hardy space of analytic functions on the complex unit disc, denoted $\mathcal{H}^p(\mathbb{D})$ for $0 < p < \infty$, is the collection of functions, $F$, analytic on $\mathbb{D}$ that satisfy*

$$\|F\|_{\mathcal{H}^p(\mathbb{D})} := \sup_{0<r<1} \left( \frac{1}{2\pi} \int_0^{2\pi} |F(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty. \tag{2.3}$$

*The Hardy space $\mathcal{H}^\infty(\mathbb{D})$ is the collection of functions, $F$, analytic on the unit disc that satisfy*

$$\|F\|_{\mathcal{H}^\infty(\mathbb{D})} := \sup_{|z|<1} |F(z)| < \infty. \tag{2.4}$$

Throughout this work, the notation $\mathcal{H}^p$ will be used instead of $\mathcal{H}^p(\mathbb{D})$, as the domain will be assumed to be the unit disc. There are a multitude of results regarding Hardy spaces, and we will list a few well known results that stem from the fact that the boundary values of $\mathcal{H}^p$ functions are in the space $L^p(\partial\mathbb{D}, d\theta)$ when $p \geq 1$.

- If $1 \leq p \leq q \leq \infty$, then $\mathcal{H}^q \subset \mathcal{H}^p$.

- For any $1 \leq p \leq \infty, \mathcal{H}^p$ is a Banach space.

- $\mathcal{H}^2$ is a Hilbert space, with inner product

$$\langle f, g\rangle = \lim_{r\to 1}\int_0^{2\pi} f(re^{i\theta})\overline{g(re^{i\theta})}d\theta. \tag{2.5}$$

- The set of functions $\{F : F \in \mathcal{O}(\mathbb{D}_{1+\epsilon}), \text{where } \epsilon > 0\}$ is a proper subset of $\mathcal{H}^p$, for every $p \geq 1$.

- Functions in $L^p(\partial\mathbb{D}, d\theta)$ can have positively and negatively indexed Fourier coefficients. That is, we can express $F \in L^p$ as

$$F(z) = \sum_{n=-\infty}^{\infty} a_n e^{in\theta}.$$

  The Hardy space $\mathcal{H}^p$ adds the more restrictive condition that $F$ has no negatively indexed Fourier coefficients.

With these facts, we have more information about Blaschke products. As was mentioned earlier, a Blaschke product maps $\mathbb{D}$ to $\mathbb{D}$. Therefore, any Blaschke product, $B$, satisfies

$$B \in \mathcal{H}^\infty.$$

By the first bulleted item, this tells us that Blaschke products are in the space $\mathcal{H}^p$ for any $1 \leq p \leq \infty$. Further, for any $1 \leq p \leq \infty$,

$$\|B\|_{\mathcal{H}^p} = 1.$$

There are several other important results regarding Hardy Spaces which we will now discuss in more detail. The first provides a useful method for calculating the $\mathcal{H}^2$ norm of functions.

**Theorem 2.1.1.** *Let $F \in \mathcal{H}^2$ have Fourier series*

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

*Then,*

$$\|F\|_{\mathcal{H}^2}^2 = \sum_{n=0}^{\infty} |a_n|^2. \tag{2.6}$$

The proof of this result can be found in Chapter 7 of [41]. Throughout much of this work, we will use this identity to rewrite the $\mathcal{H}^2$ norms of functions using only the Fourier series coefficients.

The next result, referred to as the *Decomposition Theorem* or the *Factorization Theorem*, is one of the most essential results for this dissertation. Its original proof can be attributed to F. Riesz, and is found in many locations, including Chapter 7 of [41].

**Theorem 2.1.2.** *Suppose $F \in \mathcal{H}^p(\mathbb{D})$, for some $0 < p \leq \infty$. Then if $F$ is not identically 0, it can be factored*

$$F = B \cdot G, \tag{2.7}$$

*where $B$ is a Blaschke product, $G \in \mathcal{H}^p$ has no roots in $\mathbb{D}$, and $\|G\|_{\mathcal{H}^p} = \|F\|_{\mathcal{H}^p}$.*

This theorem tells us that every meaningful function in the Hardy space $\mathcal{H}^p$ can be decomposed into the product of a Blaschke product and a function that contains no roots in $\mathbb{D}$. To illustrate the decomposition theorem, we show how a degree 3 polynomial can be factored in this way.

**Example 2.1.1.** *Decompose the function*

$$F(z) = z(z-2)\left(\frac{i}{2} + z\right)$$

*using Theorem 2.1.2.*

***Solution:*** *$F(z)$ has 2 zeros in $\mathbb{D}$, at $z = 0$ and $z = \frac{-i}{2}$. Thus, by factoring out $z$ and multiplying and dividing by $(1 + \overline{\frac{i}{2}}z)$, we get:*

$$F(z) = \left(-z \cdot \frac{\frac{1}{2}}{\frac{-i}{2}} \cdot \frac{\frac{i}{2} + z}{1 - \frac{i}{2}z}\right) \cdot \left(\frac{\frac{-i}{2}}{\frac{1}{2}}(1 - \frac{i}{2}z)(z-2)\right) = B \cdot G.$$

There is a further decomposition that is sometimes computed on Hardy spaces, called the *inner-outer factorization.* Essentially, the function $G$ can be further decomposed as

$$G = S \cdot O,$$

where $S$ is a singular inner function and $O$ is called an outer function. In this dissertation, the inner-outer factorization will not be used, as it will not affect our series representations. For relevant definitions and results about this topic, the reader is directed to [9].

### 2.1.3 Classical Weighted Hardy Spaces

In this section we provide the definitions of several common weighted Hardy spaces. All of the weighted spaces in this section are weighted $\mathcal{H}^2$ spaces, and norms of functions in these spaces can be defined using series expansions similar to (2.6). We assume that given a function $F \in \mathcal{H}^2$, we can express

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The *Dirichlet space*, denoted $\mathcal{D}$, is the space of $\mathcal{H}^2$ functions with squared derivatives that have finite integrals with respect to area measure. That is,

$$D(F) := \frac{1}{\pi} \iint_{\mathbb{D}} |F'(z)|^2 dA = \frac{1}{2\pi} \iint_{\mathbb{D}} |\partial_x F|^2 + |\partial_y F|^2 dxdy < \infty.$$

The typical Dirichlet norm is given by

$$\|F\|_{\mathcal{D}}^2 = \|F\|_{\mathcal{H}^2}^2 + D(F).$$

We can also equivalently express this norm as

$$\|F\|_{\mathcal{D}}^2 := \sum_{n=0}^{\infty} (n+1)|a_n|^2. \tag{2.8}$$

The analytic *Hardy-Sobolev spaces*, denoted $W^{s,2}$, where $s \in \mathbb{N}$, consist of $\mathcal{H}^2$ functions with $s$ derivatives in $\mathcal{H}^2$. Since we can express the $s$th derivative of $F$ with a Fourier series

$$F^{(s)}(z) = \sum_{n=s}^{\infty} n(n-1)\cdots(n-s+1)a_n z^{n-s},$$

we have that $F^{(s)} \in \mathcal{H}^2$ if and only if

$$\sum_{n=s}^{\infty} n^2(n-1)^2 \cdots (n-s+1)^2 |a_n|^2 < \infty.$$

Therefore, $F \in W^{s,2}$ if and only if for every $k = 0, 1, \ldots s$,

$$\sum_{n=k}^{\infty} n^{2k} |a_n|^2 < \infty.$$

There are many equivalent norms on these spaces, but we choose the commonly used norm that satisfies the identity

$$\|F\|_{W^{s,2}}^2 = \sum_{n=0}^{\infty} (n^2+1)^s |a_n|^2. \tag{2.9}$$

These first two examples of weighted spaces are contained in $\mathcal{H}^2$. This is because the associated weights are monotone increasing and nonnegative. If we consider weights that are positive, bounded above, and decrease to 0, we will have weighted spaces that contain $\mathcal{H}^2$. This brings us to the following spaces.

The *Bergman space*, denoted $\mathcal{A}$, is the space of functions that are absolutely square integrable on $\mathbb{D}$ with respect to area measure. That is,

$$\|F\|_{\mathcal{A}}^2 = \int_{\mathbb{D}} |F(z)|^2 dA < \infty. \tag{2.10}$$

This norm can also be expressed as

$$\|F\|_{\mathcal{A}}^2 := \sum_{n=0}^{\infty} \frac{1}{n+1} |a_n|^2. \tag{2.11}$$

Similar to the Bergman space are the *$\beta$-weighted Bergman spaces*, discussed at length in

[7] for $0 < \beta < \infty$, denoted $\mathcal{A}_\beta$, which have associated norms

$$\|F\|^2_{\mathcal{A}_\beta} := \sum_{n=0}^{\infty} \frac{1}{(n+1)^\beta} |a_n|^2. \tag{2.12}$$

There is much theory on these weighted spaces that will not be discussed, but we will summarize details that are of importance to this dissertation.

- As we saw earlier, the zero set of an $\mathcal{H}^2$ function must satisfy the Blaschke condition (2.1). On the other hand, the Dirichlet space and the Hardy-Sobolev spaces have a condition on the zero set that is more restrictive than (2.1). This is detailed in a paper by Shapiro and Shields [33]. Roughly speaking, the characterization in [33] shows that the faster the weights grow, the faster the zero set must accumulate to $\partial\mathbb{D}$.

- The Dirichlet space is highly studied. In particular, Blaschke decompositions were investigated on this space by Carleson in [2]. As a main result in that work, a formula for functions in $\mathcal{H}^\infty \cap \mathcal{D}$ was provided that relies on the comparison of the Dirichlet norms of $F$ and $G$ as in (2.7). We will discuss this further in Chapter 3.

- There are still many open questions regarding the spaces "between" $\mathcal{H}^2$ and $\mathcal{D}$ [1]. That is, weighted Hardy spaces with weights that grow sub linearly. Although we provide some results on these spaces in Chapter 3, we emphasize that there is currently new research by many authors in the area.

### 2.1.4 The Hilbert Transform and Analytic Projections

The motivation of the unwinding series comes from applications in signal analysis. We start with a real valued, $2\pi$ periodic signal, $s : [0, 2\pi) \to \mathbb{R}$, and find its unique analytic extension to a function, $F \in \mathcal{H}^2$, which we will expand using the unwinding series. The tools that we can use to accomplish this are the Hilbert transform and Poisson kernel.

To quote Krantz, *"The Hilbert transform is, without question, the most important operator in analysis. It arises in so many different contexts, and all these contexts are intertwined in profound and influential ways. What it all comes down to is that there is only one singular integral in dimension 1, and it is the Hilbert transform. The philosophy is that all significant analytic questions reduce to a singular integral; and in the first dimension there is just one choice."*[19]

In practice, if we are given a $2\pi$ periodic function $s : [0, 2\pi] \to \mathbb{R}$, where $s \in L^2([0, 2\pi])$ then we define the Hilbert transform of $s$ as

$$\mathbf{H}s(\theta) := \frac{1}{2\pi}\text{p.v.} \int_0^{2\pi} s(t) \cot\left(\frac{\theta - t}{2}\right) dt, \tag{2.13}$$

where p.v. stands for the principal value integral (see [9] for more details). The new function, $\mathbf{H}s$, will also be in $L^2([0, 2\pi])$, and will be the *harmonic conjugate* of $s$. Therefore, if we consider the function $f : \partial\mathbb{D} \to \mathbb{C}$, where

$$f(e^{i\theta}) := s(\theta) + i\mathbf{H}s(\theta),$$

then $f$ will have real component that agrees with $s$, and be in the space $L^2(\partial\mathbb{D})$.

From here, we can obtain the unique analytic extension, of $f$ to $\mathbb{D}$, denoted $F$, through *analytic projection*. By using the Poisson kernel

$$P_r(\theta) := \frac{1 - r^2}{1 - 2r\cos(\theta) + r^2},$$

the function, $F$, described by

$$F(re^{i\theta}) := \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(\theta - t) f(e^{it})dt, \quad 0 \leq r < 1,$$

will be analytic in $\mathbb{D}$ and agree with $f$ almost everywhere on $\partial\mathbb{D}$ with respect to the arc length measure. Further, since $f \in L^2(\partial\mathbb{D})$, we know that $F \in \mathcal{H}^2(\mathbb{D})$. This allows us to utilize the previously discussed theory throughout the remainder of this dissertation.

### 2.1.5 Rouché's Theorem

In our study of Blaschke decompositions and the unwinding series, it is of fundamental importance to have a tool to determine whether of not a function has roots in $\mathbb{D}$. If a function has no such root, then the Blaschke decomposition (2.7) will be trivial and we will not have to spend time computing it.

In the study of one complex variable, Rouché's theorem is a useful result in determining whether or not the sum of two analytic functions $f + g$ will have the same number of roots as the function $f$. The result is stated as follows:

**Theorem 2.1.3.** *Given $R > 0$, and two functions $f$ and $g$, analytic in $\mathbb{D}_R$, suppose that $|f(z)| > |g(z)|$ for all $|z| = r$, where $r < R$. If $f$ has $N \geq 0$ roots in $\mathbb{D}_r$, then the function $f + g$ also has $N$ roots in $\mathbb{D}_r$.*

The proof of this result can be found in Chapter 8 of [11], and makes use of the argument principal and logarithmic integrals, which are also discussed in that chapter. As was previously mentioned, this theorem will be useful to us throughout this dissertation. Since we are working with the unit disc, $\mathbb{D}$, we can apply this theorem in a few different ways.

Firstly, if we are given a function $F \in \mathcal{O}(\mathbb{D}_{1+\epsilon})$, for some $\epsilon > 0$, we can express its Fourier series

$$F(z) = \sum_{n=0}^{\infty} a_n z^n.$$

If there exists some $k \in \mathbb{N}$ such that

$$|a_k| > \sum_{n \neq k} |a_n|,$$

then $F$ will have precisely $k$ roots in $\mathbb{D}$. This is due to Rouché's theorem, with $R = 1 + \epsilon$, $r = 1$, $f(z) = a_k z^k$, and $g(z) = F(z) - f(z)$. In the situation when $k = 0$, then $F \in \mathcal{H}^2$ has no roots in $\mathbb{D}$, and the Blaschke decomposition will be trivial.

Rouché's theorem also provides a sufficient condition for functions, $F \in \mathcal{H}^2 \setminus \mathcal{O}(\mathbb{D}_{1+\epsilon})$, for all $\epsilon > 0$, to have trivial Blaschke decompositions. Essentially, by expressing

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

if

$$|a_0| \geq \sum_{n=1}^{\infty} |a_n|$$

then $F$ has no roots in $\mathbb{D}$. This proof is by contradiction, where if such a root existed, it would be at a point $z^*$ where $|z^*| = \rho < 1$. By hypothesis, for any $|z| = \frac{1+\rho}{2}$,

$$|a_0| > \sum_{n=1}^{\infty} |a_n||z^n| > \left| \sum_{n=1}^{\infty} a_n z^n \right| = |F(z) - a_0|.$$

By Rouché's theorem, this implies that by setting $r = \frac{1+\rho}{2}$, $R = 1$, $f(z) = a_0$ and $g(z) = F(z) - a_0$, we have that $F$ and $f$ have the same number of roots in $\mathbb{D}_{\frac{1+\rho}{2}}$. This is a contradiction, and tells us that $F$ has no roots in $\mathbb{D}$.

We will see these applications of Rouché's theorem again in both Chapter 4 and Chapter 5.

### 2.1.6 The Convergence Rate of $\mathcal{H}^2$ Functions

If we are given a sequence $\{x\}_n$ that converges to a point $x^*$ in a Banach space, $X$, we can also talk about the rate of convergence of $\{x\}_n$ to $x^*$. The general idea is that by denoting $e_n := x_n - x^*$, we want to know how quickly the terms $\|e_n\|_X$ approach 0. There are many definitions of convergence rate that are used in different areas of mathematics and we will use language similar to that of [5] for this dissertation.

On the space $\mathcal{H}^2$, we say that a family of functions, $F_k$, converge to $F$ with an *exponential convergence* rate if there exist $N \in \mathbb{N}$ and $0 < \epsilon < 1$ such that for any $n \geq N$,

$$\|F_{n+1} - F\|_{\mathcal{H}^2} \leq \epsilon \|F_n - F\|_{\mathcal{H}^2}. \tag{2.14}$$

**Remark 2.1.3.** *We note that in many settings, this is called linear convergence.*

In the case when no such $\epsilon$ exists, that is,

$$\limsup_{n\to\infty} \frac{\|F_{n+1} - F\|_{\mathcal{H}^2}}{\|F_n - F\|_{\mathcal{H}^2}} = 1, \tag{2.15}$$

we call the convergence rate *non-exponential*.

**Remark 2.1.4.** *In many settings, this is referred to as sub linear convergence.*

## 2.2 Review of Existing Literature

### 2.2.1 Nonlinear Phase Unwinding

In the 2000 PhD thesis of Nahon [26], the concept of performing an unwinding series was first introduced. The idea is that given a $2\pi$ periodic, real valued signal, $s \in L^2([0, 2\pi])$, we can use the Hilbert transform and Poisson kernel to create a function $F \in \mathcal{H}^2(\mathbb{D})$ whose boundary values have real part that agrees with $s$. That is,

$$F(e^{i\theta}) = s(\theta) + i\mathbf{H}s(\theta), \quad \theta \in \mathbb{R}$$

where $\mathbf{H}s$ is the Hilbert transform of $s$. From here, the unwinding series can be formally defined from the following iterative process:

We begin with the Blaschke decomposition,

$$F(z) = B_0(z) \cdot G_0(z), \quad z \in \mathbb{D}.$$

Since $G_0 \in \mathcal{H}^2$, by adding and subtracting the term $G_0(0)$, we can introduce a root at $z = 0$ for the function $G_0(z) - G_0(0)$. This implies that we can decompose

$$G_0(z) - G_0(0) = B_1(z) \cdot G_1(z).$$

This decomposition will either be trivial, which will occur if $G_0(z) - G_0(0) \equiv C$, for some $C \in \mathbb{C}$, or we will obtain a function $G_1(z) \neq G_0(z) - G_0(0)$. Similarly, for any $n \geq 0$, we can iteratively define

$$G_n(z) - G_n(0) = B_{n+1}(z) \cdot G_{n+1}(z).$$

With all of this, through adding and subtracting the terms $G_j(0)$, for $0 \leq j \leq n$, this gives us the following series expansion:

$$F(z) = G_0(0)B_0(z) + G_1(0)B_0(z)B_1(z) + \cdots + G_n(0)\prod_{j=1}^{n} B_j(z) + \prod_{j=1}^{n} B_j(z)(G_n(z) - G_n(0)). \quad (2.16)$$

This series allows us to approximate the original signal, $s$ in a meaningful way. In particular, by considering the unwinding series of the complexified signal, $F(e^{i\theta})$, we can express it as a summation where each term is the product of a constant and a function with unit magnitude. The function with unit magnitude, $B_k(z)$, can be viewed as an underlying *time varying frequency* of the complex signal, and the constant term, $G_k(0)$, can be viewed as the corresponding *amplitude* of that frequency. By considering the real component of these frequencies and amplitudes, we express the original signal $s$ as a summation of these time varying frequencies, which is meaningful in signal analysis. Further, early numerical results by Nahon, for specific choices of $F$, showed exponential convergence of this formal expansion to $F$.

Motivated by these early experiments, Coifman, Steinerberger, and Wu further investigated the unwinding series of functions in the two articles [5, 6].

In the first paper [5], the definitions of the weighted Hardy spaces (seen in the next chapter as Definition 3.1.1), $X_\gamma$ and $Y_\gamma$, were introduced and several results (including our reference result, Theorem 3.3.1) were proven. This article was the main inspiration of this work. It was also there that the term *unwinding series* was first coined. The reader is directed to [5] for more details, but we illustrate the main idea with the following example.

If we consider the analytic function, $F$, produced by the analytic projection of the boundary data

$$F(e^{i\theta}) = e^{-(\theta-\pi)^2} e^{i10\theta},$$

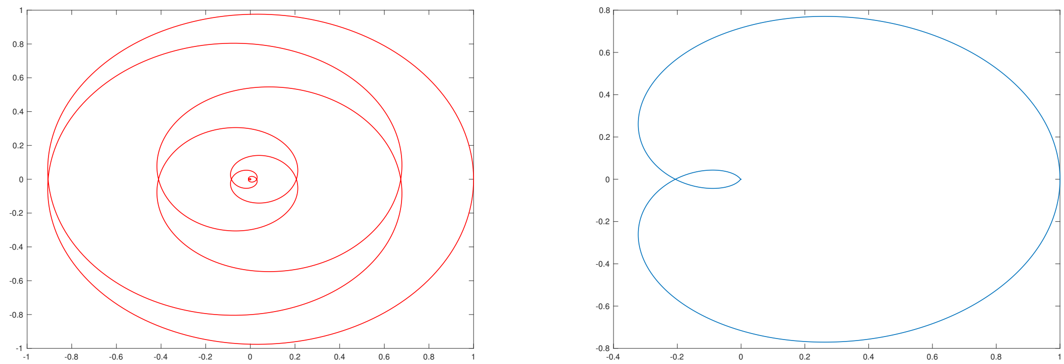

Figure 1: The complex plots of $F(e^{i\theta})$ (left) and $G(e^{i\theta})$ (right), $\theta \in [0, 2\pi]$, where $F(e^{i\theta}) = e^{-(\theta-\pi)^2} e^{i10\theta}$. We can see that the average winding of $G$ is much lower than the average winding of $F$.

then Figure 1, which plots $F(e^{i\theta})$ and $G(e^{i\theta})$ as in Equation (2.7) shows how the average winding (number of loops the closed curve creates around each point $\omega \in \mathbb{C}$) is much smaller for $G$ than it is for $F$. Since the two functions have the same magnitude for fixed $\theta \in [0, 2\pi]$, we can view $G$ as an "unwound" version of $F$.

In the second paper [6], the study of the unwinding series was extended beyond the $\mathcal{H}^2$ space to include non-analytic signals. More precisely, the authors studied functions with small (in the $L^2(\partial\mathbb{D})$ sense) antiholomorphic components. In addition to this, they proved an elementary result allowing for a direct computation of the unwinding series, restated in Chapter 4 as Theorem 4.3.1. This was based on term-wise equality between the Fourier series and unwinding series for a certain class of functions, and is further discussed in Chapter 4.

Since then, Coifman and Peyriere have shown the convergence of the unwinding series for any $\mathcal{H}^p$ function with $p \geq 1$ in [4] . While the convergence has been shown, the arguments are based on invariant spaces and do not tell us about the rate of convergence of the unwinding series.

Applications of the unwinding series to the Doppler effect have been studied privately by Healy [15], algorithmic stability was studied by Letelier and Saito [21], and the literature on this subject is growing rapidly. Time and space constraints compel us to limit this discussion to the most relevant works for this dissertation, but we direct the readers to the recent papers [23, 35, 36, 34, 7, 37, 24] for more content.

### 2.2.2 Adaptive Fourier Decomposition

For the sake of completeness, we briefly mention here that other research groups, in particular Qian et al. have investigated the related topic of Adaptive Fourier Decompositon (AFD). The main distinction between the iterations resulting from the unwinding series and AFD is that at each step of AFD, instead of adding and subtracting the term $G_j(0)$ (as was done in the unwinding series), an algorithm seeks the "optimal" point $a_j \in \mathbb{D}$ that will provide the best finite approximation if we add and subtract $G_j(a_j)$. The existence of the optimal points $a_j$, for each step $j$, has been shown in [29], however there are no closed formulas for the explicit computation of such points (See [12] for related results.). Therefore, there is a computational cost in approximating the optimal points that is avoided in the unwinding series. At the time of this writing, we are not aware of results that compare the convergence rate of the AFD algorithm with the convergence rate of the unwinding series.

# Chapter 3

# Blaschke Decompositions on Weighted Hardy Spaces

As we saw in the previous chapter, if we consider a function $F \in \mathcal{H}^2$, then it will have a decomposition, $F = B \cdot G$, where $B$ is a Blaschke product and

$$\|F\|_{\mathcal{H}^2} = \|G\|_{\mathcal{H}^2}.$$

In this chapter, we explore the relationship between the weighted Hardy norms of $F$ and $G$ and improve previous results [5] on the matter. To do this, we begin in Section 3.1 by defining two classes of weighted Hardy spaces, denoted $X_\gamma$ and $Y_\gamma$, using similar notation as [5]. It turns out that many important spaces in complex function theory can be identified with an $X_\gamma$ or $Y_\gamma$ space, so we provide some concrete examples of these spaces. From there, in Section 3.2 we prove several results on the containment relationships between $\mathcal{H}^2$, $X_\gamma$, and $Y_\gamma$.

After this, in Section 3.3 we state the main result of [5], which provides a bound on the

$X_\gamma$ norms of functions $F$ and $G$ as in (2.7). After providing some commentary, we ask two main questions for this chapter, both of which seek to improve upon the aforementioned result. This sets the ground for our own results which, as we will see, provide stronger and more comprehensive bounds than the original results in [5]. In Section 3.4, we prove our main results.

To end this chapter, in Section 3.5 we apply our new results to concrete instances of weighted Hardy spaces we previously defined to gain new insights and provide a worked example.

## 3.1 The $X_\gamma$ and $Y_\gamma$ Spaces: Definitions and Examples

We saw in Theorem 2.1.1 that a function's $\mathcal{H}^2$ norm can be expressed using its Fourier series as in Equation (2.6). One method of creating weighted $\mathcal{H}^2$ spaces is to weight each of the terms in such series. While the idea of weighting Fourier coefficients is not new, the idea in [5] was to consider the growth rate of the weights. It was in that work that the following definitions were first introduced.

**Definition 3.1.1.** *Let $\{\gamma_n\} \not\equiv 0$ be a monotone increasing sequence of real numbers that satisfies $\gamma_0 = 0$. Given a function $F \in \mathcal{H}^2$, we say that $F$ belongs to the space $X_\gamma$ if*

$$\|F(z)\|_{X_\gamma}^2 = \|\sum_{j\geq 0} a_j z^j\|_{X_\gamma}^2 := \sum_{j\geq 0} \gamma_j |a_j|^2 < \infty, \tag{3.1}$$

*where $a_j$ is the $j$th Fourier coefficient of $F$. Moreover, we say that $F$ belongs to the space $Y_\gamma$ if*

$$\|F(z)\|_{Y_\gamma}^2 = \|\sum_{j\geq 0} a_j z^j\|_{Y_\gamma}^2 := \sum_{j\geq 0}(\gamma_{j+1}-\gamma_j)|a_j|^2 < \infty. \tag{3.2}$$

The introduction of these weighted spaces provide a generalized framework to prove results on many classical function spaces. With these definitions in mind, we now consider the following subclasses of weighted spaces.

**Definition 3.1.2.** *Let* $\{\gamma_n\} \not\equiv 0$ *be a monotone increasing sequence of real numbers that satisfies* $\gamma_0 = 0$*.*

*(i) We say* $X_\gamma$ *is convex if for all* $n \geq 0$

$$\gamma_{n+2} - 2\gamma_{n+1} + \gamma_n \geq 0. \tag{3.3}$$

*(ii) We say* $X_\gamma$ *is concave if for all* $n \geq 0$

$$\gamma_{n+2} - 2\gamma_{n+1} + \gamma_n \leq 0. \tag{3.4}$$

Essentially, the space $X_\gamma$ is convex/concave if the sequence $\gamma_n$ is convex/concave. The notion of convex and concave sequences is not new and has been studied by many authors, for example in [25], where Mitrinovic and Vasic study analytic inequalities. For our purposes the notion is further simplified to: sequences that increase at an increasing rate are convex, and sequences that increase at a decreasing rate are concave.

We now provide some concrete examples for the $X_\gamma$ and associated $Y_\gamma$ spaces.

**Example 3.1.1.** *The space* $X_\gamma$*, where* $\gamma_n = n$*, can be identified with the Dirichlet space,* $\mathcal{D}$ *in the following sense: the spaces are equal as sets. This can be seen directly by Equation*

(2.8). *The associated* $Y_\gamma$ *space will be exactly* $\mathcal{H}^2$. *In this example,* $X_\gamma$ *is both convex and concave, as both Equations* (3.3) *and* (3.4) *hold.*

**Example 3.1.2.** *The space* $X_\gamma$, *where* $\gamma_n = n^{2s}, s \in \mathbb{N}$ *can be identified with the Hardy-Sobolev space* $W^{s,2}$, *regarding set equality. This can be seen by Equation* (2.9). *The associated* $Y_\gamma$ *space can be identified with the non-integer Hardy-Sobolev space* $W^{s-\frac{1}{2},2}$. *In this example,* $X_\gamma$ *is convex.*

**Example 3.1.3.** *Given the sequence*

$$\gamma_0 = 0, \quad \gamma_n = \sum_{j=0}^{n-1} \frac{1}{j+1} \quad \forall n > 0,$$

*the space* $X_\gamma$ *will contain all elements* $F \in \mathcal{H}^2$ *whose Fourier coefficients,* $a_n$, *satisfy*

$$\sum_{n=0}^{\infty} \log(n+1)|a_n|^2 < \infty. \tag{3.5}$$

*The associated* $Y_\gamma$ *space is equal to the Bergman space,* $\mathcal{A}$, *seen in Equation* (2.11). *In this example,* $X_\gamma$ *is concave.*

With these examples in mind, we now look into some of the relationships between $\mathcal{H}^2$, $X_\gamma$, and $Y_\gamma$.

## 3.2 Containment Relationships Between $\mathcal{H}^2$, $X_\gamma$ and $Y_\gamma$

Recalling Definition 3.1.1 on the $X_\gamma$ and $Y_\gamma$ spaces, we now seek a better understanding of the relationship between the spaces. By studying this relationship, we are able to see how the sequence $\{\gamma_n\}$ affects properties of both $X_\gamma$ and $Y_\gamma$.

We first see that for any choice of $\{\gamma_n\}$ satisfying the conditions of Definition 3.1.1, we will always yield the set inclusion $X_\gamma \subset \mathcal{H}^2$. We now ask what further conditions can be imposed on $\{\gamma_n\}$ so that we guarantee the set equality $X_\gamma = \mathcal{H}^2$. We also ask whether such conditions can be proved to be necessary and sufficient. This brings us to the following result, where we constructively prove that every unbounded sequence $\{\gamma_n\}$ will have an associated space $X_\gamma$ that is a proper subspace of $\mathcal{H}^2$.

**Proposition 3.2.1.** *For any unbounded, monotone increasing sequence $\{\gamma_n\}$, with $\gamma_0 = 0$, there exists a function $F \in \mathcal{H}^2$ such that $F \notin X_\gamma$. Thus, $X_\gamma \subsetneq \mathcal{H}^2$.*

*Proof.* Let the monotone increasing sequence $\{\gamma_n\}$ be given. For each $n$, we define the sets $K_0 = \emptyset$, and $K_n := \{j : j \in (\gamma_{n-1}, \gamma_n] \cap \mathbb{N}\}$ for every $n \geq 1$.

Note: for each $m \neq n$, $K_n \cap K_m = \emptyset$ and $\cup_{n=0}^{\infty} K_n = \mathbb{N}$.

Then, we define

$$a_n := \sqrt{\sum_{j \in K_n} \frac{1}{j^2}}.$$

Note, if $K_n = \emptyset$, then $a_n = 0$. As a direct consequence of their construction,

$$\sum_{n=0}^{\infty} |a_n|^2 = \sum_{n=0}^{\infty} \left| \sqrt{\sum_{j \in K_n} \frac{1}{j^2}} \right|^2 = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

Therefore, letting $F(z) = \sum a_n z^n$, we see that $F \in \mathcal{H}^2$ from Equation (2.6). However, we obtain

$$\|F\|_{X_\gamma}^2 = \sum_{n=0}^{\infty} \gamma_n |a_n|^2 = \sum_{n=0}^{\infty} \gamma_n \left| \sqrt{\sum_{j \in K_n} \frac{1}{j^2}} \right|^2 \geq \sum_{n=0}^{\infty} \max_{j \in K_n}(j) \sum_{j \in K_n} \frac{1}{j^2} \geq \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

Here we have used the fact that $\gamma_n \geq \max_{j \in K_n}(j)$ since $K_n \subset (\gamma_{n-1}, \gamma_n]$. Therefore, $F \notin X_\gamma$.

$\square$

With this result, we now move on to a necessary and sufficient condition for the set of functions in $X_\gamma$ to be equal to the set of $\mathcal{H}^2$ functions.

**Corollary 3.2.1.** *Let $\{\gamma_n\}$ be a sequence as in Definition 3.1.1. The set equality $X_\gamma = \mathcal{H}^2$ if and only if $\{\gamma_n\}$ is bounded.*

*Proof.* ( $\implies$ ) This is a direct result of the contrapositive of Proposition 3.2.1.

($\Longleftarrow$) If $\{\gamma_n\}$ is bounded by $M$, then for any $F \in \mathcal{H}^2$,

$$0 \leq \|F\|_{X_\gamma}^2 \leq M\|F\|_{\mathcal{H}^2}^2 < \infty.$$

Thus, $F \in X_\gamma$, and it follows that $\mathcal{H}^2 \subseteq X_\gamma$ . By definition, $X_\gamma \subseteq \mathcal{H}^2$, so $X_\gamma = \mathcal{H}^2$.

$\square$

Next, we investigate the containment relationships between $X_\gamma$ and $Y_\gamma$. Looking back to Example 3.1.1, we saw that for $\gamma_n = n$, we have the set equality $X_\gamma = \mathcal{D}$ and $Y_\gamma = \mathcal{H}^2$. Therefore, $X_\gamma \subset Y_\gamma$, and $Y_\gamma \not\subset X_\gamma$. This property, while true for some weighted Hardy Spaces, will not always hold. We can see this in the following result.

**Proposition 3.2.2.** *Let $\{\gamma_n\}$ be a sequence as in Definition 3.1.1. The set inclusion, $X_\gamma \subseteq Y_\gamma$ holds if and only if*

$$\limsup_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} < \infty. \tag{3.6}$$

*Proof.* ( $\implies$ ) We prove this direction by contrapositive. Let $\{\gamma_n\}$ be a sequence such that $\limsup_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = \infty$. We will construct a function $F \in X_\gamma$ such that $F \notin Y_\gamma$.

By hypothesis, there exists a subsequence $\{\gamma_k\}$ of $\{\gamma_n\}$ such that $\frac{\gamma_{k+1}}{\gamma_k} \to \infty$. Then we can find a strictly increasing sequence of natural numbers $K_1, K_2, \ldots$ such that for any $j$, if

$k \geq K_j$,

$$\gamma_{k+1} \geq (j+1)\gamma_k > 0.$$

From here, we define the Fourier coefficients of the function $F$. Let

$$F(z) = \sum_{n=0}^{\infty} a_n z^n$$

be a function where

$$a_n = \begin{cases} \sqrt{\frac{1}{\gamma_n j^2}} & \text{if } n = K_j; \\ 0 & \text{if } n \neq K_j. \end{cases}$$

Then, we have that

$$\|F\|_{X_\gamma}^2 = \sum_{n=0}^{\infty} \gamma_n |a_n|^2 = \sum_{j=1}^{\infty} \gamma_{K_j} \frac{1}{\gamma_{K_j} j^2} = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

However,

$$\|F\|_{Y_\gamma}^2 = \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n)|a_n|^2 \geq \sum_{j=1}^{\infty} \left[(j+1)\gamma_{K_j} - \gamma_{K_j}\right] \frac{1}{\gamma_{K_j} j^2} = \sum_{j=1}^{\infty} \frac{1}{j} = \infty.$$

Therefore, $X_\gamma \not\subseteq Y_\gamma$. Thus, by contrapositive, the proof is complete.

($\Longleftarrow$) Suppose that

$$\limsup_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = M_1 < \infty.$$

This implies that there exists some $M_2 < \infty$ and some $N \in \mathbb{N}$ such that if $n \geq N$,

$$\gamma_{n+1} - \gamma_n \leq M_2 \gamma_n.$$

Let $F \in X_\gamma$ be given, where $F(z) = \sum a_n z^n$. Then

$$\sum_{n=0}^{\infty} \gamma_n |a_n|^2 < \infty.$$

Therefore we obtain

$$\|F\|_{Y_\gamma}^2 = \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n)|a_n|^2 < \sum_{n=0}^{N-1} (\gamma_{n+1} - \gamma_n)|a_n|^2 + M_2 \sum_{n=N}^{\infty} \gamma_n |a_n|^2 < \infty.$$

Therefore, $F \in Y_\gamma$. This completes the proof. □

Our next example shows why spaces $X_\gamma$ for which $X_\gamma \not\subseteq Y_\gamma$ are rarely of practical use, as they contain too few elements.

**Example 3.2.1.** *Consider the sequence $\gamma_0 = 0$, $\gamma_n = (n!)^2$. Letting $F(z) = e^z$, then $F \notin X_\gamma$.*

*To see this, notice that*

$$\|F\|_{X_\gamma}^2 = \sum_{n=1}^{\infty} \gamma_n \left(\frac{1}{n!}\right)^2 = \sum_{n=1}^{\infty} 1 = \infty.$$

*We also have*

$$\lim_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = \lim_{n\to\infty} \frac{(n+1)!^2}{n!^2} = \lim_{n\to\infty} n + 1 = \infty.$$

*Therefore, by Proposition 3.2.2, $X_\gamma \not\subseteq Y_\gamma$.*

From here, we look at conditions that allow the reverse containment $Y_\gamma \subseteq X_\gamma$ to hold. This brings us to the following result.

**Proposition 3.2.3.** *Let $\{\gamma_n\}$ be a sequence as in Definition 3.1.1. The set inclusion, $Y_\gamma \subseteq X_\gamma$, if and only if*

$$\liminf_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} > 1. \tag{3.7}$$

*Proof.* ( $\Longrightarrow$ ) We prove this direction by contrapositive. Since $\{\gamma_n\}$ is a monotone increasing sequence,

$$\liminf_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} \geq 1,$$

thus the contrapositive is reduced to

$$\liminf_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} = 1.$$

By picking the proper subsequence of $\{\gamma_n\}$, we have

$$s_k := \frac{\gamma_{k+1} - \gamma_k}{\gamma_k},$$

satisfies $s_k \searrow 0$ and $\frac{1}{s_k} \nearrow \infty$. Therefore, by applying the construction of Proposition 3.2.1, we can find Fourier coefficients, $b_k$, such that

$$\sum_{k\geq 0} |b_k|^2 < \infty \quad \text{and} \quad \sum_{k\geq 0} \frac{1}{s_k}|b_k|^2 = \infty.$$

Let

$$F(z) = \sum_{k\geq 0} a_k z^k,$$

where

$$a_k = \frac{|b_k|}{\sqrt{\gamma_{k+1} - \gamma_k}}.$$

Then we have

$$\|F\|^2_{Y_\gamma} = \sum_{k\geq 0} (\gamma_{k+1} - \gamma_k) \frac{|b_k|^2}{(\gamma_{k+1} - \gamma_k)} = \sum_{n\geq 0} |b_k|^2 < \infty.$$

However,

$$\|F\|_{X_\gamma}^2 = \sum_{k\geq 0} \frac{\gamma_k}{\gamma_{k+1}-\gamma_k}|b_k|^2 = \sum_{k\geq 0} \frac{1}{s_k}|b_k|^2 = \infty.$$

Therefore we have found a function $F \in Y_\gamma$ such that $F \notin X_\gamma$, which proves the result by contrapositive.

($\Longleftarrow$) Suppose that

$$\liminf_{n\to\infty} \frac{\gamma_{n+1}}{\gamma_n} > 1.$$

Then for some $N \in \mathbb{N}$, if $n \geq N$,

$$\gamma_{n+1} \geq M\gamma_n,$$

where $M > 1$. Then for any function $F \in Y_\gamma$,

$$\|F\|_{Y_\gamma}^2 = \sum_{n\geq 0}(\gamma_{n+1}-\gamma_n)|a_n|^2 \geq \sum_{n=0}^{N-1}(\gamma_{n+1}-\gamma_n)|a_n|^2 + (M-1)\sum_{n=N}^{\infty}\gamma_n|a_n|^2.$$

Therefore, since

$$\sum_{n=0}^{N-1}\gamma_n|a_n|^2 < \infty,$$

and the tail of the summation is bounded, $F \in X_\gamma$.

□

By combining Proposition 3.2.3 and Proposition 3.2.3, we arrive at the following Corollary which characterizes when the set equality $X_\gamma = Y_\gamma$ holds.

**Corollary 3.2.2.** *Let $\{\gamma_n\}$ be a sequence as in Definition 3.1.1. The set equality, $X_\gamma = Y_\gamma$ if and only if Equation* (3.6) *and Equation* (3.7) *hold.*

## 3.3 Main Results

Recall that the decomposition theorem states that any nonzero function, $F \in \mathcal{H}^2$, can be expressed

$$F = B \cdot G$$

where $B$ is a Blaschke product, $G$ vanishes nowhere in $\mathbb{D}$, and the identity

$$\|F\|_{\mathcal{H}^2} = \|G\|_{\mathcal{H}^2}$$

is satisfied. The main reason we care about the weighted Hardy spaces, $X_\gamma$, is that this identity no longer holds.

### 3.3.1 Reference Result

As was pointed out in [5], the $X_\gamma$ and $Y_\gamma$ spaces are well suited to study Blaschke decompositions compared to $\mathcal{H}^2$. In fact, Coifman and Steinerberger were able to prove the following result on these spaces.

**Theorem 3.3.1.** *(Main Result of [5]) Given a sequence $\{\gamma_n\}$ as in Definition 3.1.1, suppose $F \in X_\gamma$ is analytic in $\mathbb{D}_{1+\epsilon}$, for some $\epsilon > 0$. Then*

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2. \tag{3.8}$$

*Suppose further that $F$ has a root at $\alpha \in \mathbb{D}$. Then we have the improved inequality*

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - (1 - |\alpha|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha} e^{i\cdot}} \right\|_{Y_\gamma}^2. \tag{3.9}$$

Moreover, from the proof of this inequality, the authors of [5] were able to provide an enhanced version of (3.9) for the special case $\gamma_n = n$, for which the set equality $X_\gamma = \mathcal{D}$ holds. The result is stated as Corollary 2 in [5], and tells us that for functions $F \in \mathcal{D} \cap \mathcal{O}(\mathbb{D}_{1+\epsilon})$, with $m$ roots [1], labeled $\alpha_1, \dots \alpha_m$, we have the identity

$$\|G\|_{\mathcal{D}}^2 = \|F\|_{\mathcal{D}}^2 - \sum_{j=1}^{m} (1 - |\alpha_j|^2) \left\| \frac{F(\cdot)}{\cdot - \alpha_j} \right\|_{\mathcal{H}^2}^2 . \tag{3.10}$$

This formula is the same as a special case of a famous result by Carleson in [2]. In that work, Carleson uses properties of the Dirichlet space $\mathcal{D}$ to show that a function $F$ in $\mathcal{H}^\infty \cap \mathcal{D}$ with roots at $\alpha_j \in \mathbb{D}$ for $j \in J$ satisfies the same identity, regardless of the cardinality of $J$.

With these results in hand, we considered relevant questions. The first comes from the assumption made on the functions to which this theorem applies. When considering the set of $\mathcal{H}^2$ functions versus the set of $\mathcal{H}^2$ functions that are analytic on $\mathbb{D}_{1+\epsilon}$, we can see that this assumption is quite restrictive. Clearly no nonzero function with an infinite number of roots in $\mathbb{D}$ satisfies this property, along with functions that have discontinuities on $\partial\mathbb{D}$. If a function has an infinite number of roots in $\mathbb{D}$, but satisfies the Blaschke condition (2.1), then this result may still hold for spaces $X_\gamma$, where $\{\gamma_n\}$ is bounded. Similarly, if we consider $\mathcal{H}^2$ functions such as

$$F(z) = \log(1 - z),$$

we would like to know if (3.9) still holds even though we cannot extend $F$ continuously to $\partial\mathbb{D}$. Because the requirement $F \in \mathcal{O}(\mathbb{D}_{1+\epsilon})$ is so restrictive, we ask the question:

[1] We note that the assumption in Theorem 3.3.1 that $F$ is analytic on $\mathbb{D}_{1+\epsilon}$ automatically precludes the case of infinitely many roots.

**Can we relax the assumptions in Theorem 3.3.1 so that we have a result for all $F \in \mathcal{H}^2$?**

We point out another shortcoming of Theorem 3.3.1: when considering the applicability of unwinding series versus Fourier series, an important point in the comparison process is that in each step we can factor out multiple roots in $\mathbb{D}$ to improve our finite approximation. In Theorem 3.3.1, we only factor out a *single* root of $F$ in $\mathbb{D}$. This is improved upon in Equation (3.10), but can only applied to the Dirichlet space. Therefore, we ask the question:

**Can we improve the bounds in Theorem 3.3.1 so that we include *all* roots of $F$ in $\mathbb{D}$?**

Our main results for this chapter are dedicated to answering these two questions in the affirmative.

### 3.3.2 Statement of Results

It is at this stage that we focus on two different types of weighted Hardy spaces: convex and concave. These classes are not restrictive, as we have seen many practical examples of this.

The first main result of this dissertation, Theorem 3.3.2, investigates the case where the sequence $\{\gamma_n\}$ is convex; that is, the sequence satisfies (3.3). As was previously mentioned, if $\gamma_n = n$, then the space $X_\gamma$ is equivalent to the Dirichlet space, denoted $\mathcal{D}$, and $Y_\gamma$ is $\mathcal{H}^2$. Moreover, if $\gamma_n = n^2$, then $X_\gamma$ is equivalent to the Hardy-Sobolev space, denoted $W^{1,2}$, and $Y_\gamma$ is equivalent to $\mathcal{D}$.

We point out that functions in these types of weighted Hardy spaces may have a finite or an infinite number of roots in $\mathbb{D}$. In the latter case, a sufficient condition was given by Shapiro and Shields in [33] for an infinite set of points, $a_n \in \mathbb{D}$, to be the zero set of a function in weighted Hardy spaces. Essentially, given $X_\gamma$, the growth rate of $\gamma_n$ dictates the minimal convergence rate the sequence of interior zeros of $F$ must go to $\partial\mathbb{D}$ for $F$ to be in the space. Due to this dependence, we limit our first theorem to functions with a finite number of roots in $\mathbb{D}$ and leave the convergence in the case when $F$ has an infinite number of roots as an open question.

This gives us the following result.

**Theorem 3.3.2.** *Suppose that $\{\gamma_n\}$ is monotone increasing, satisfying $\gamma_0 = 0$ and the convexity condition,* (3.3)*. For functions $F \in X_\gamma$ with a finite number of zeros inside the unit disc labeled $\alpha_1, \alpha_2, \ldots, \alpha_m$, we have*

$$\|G(e^{i\cdot})\|^2_{X_\gamma} \leq \|F(e^{i\cdot})\|^2_{X_\gamma} - \sum_{j=1}^{m} \left( (1-|\alpha_j|^2) \left\| \frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}} \right\|^2_{Y_\gamma} \right). \tag{3.11}$$

From here, we investigate the case when $\{\gamma_n\}$ is concave. That is, for any $n \geq 0$, we require $\{\gamma_n\}$ to satisfy (3.4).

We begin our study of spaces $X_\gamma$, where $\{\gamma_n\}$ satisfies (3.4) with a result regarding functions with a finite number of roots in $\mathbb{D}$.

**Theorem 3.3.3.** *Suppose that $\{\gamma_n\}$ is a monotone increasing sequence satisfying $\gamma_0 = 0$ and the concavity condition,* (3.4)*. For functions $F \in X_\gamma$ with a finite number of zeros inside $\mathbb{D}$ labeled $\alpha_1, \alpha_2, \ldots, \alpha_m$, we have*

$$\|G(e^{i\cdot})\|^2_{X_\gamma} \leq \|F(e^{i\cdot})\|^2_{X_\gamma} - \sum_{j=1}^{m} \left( (1-|\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j} \right\|^2_{Y_\gamma} \right). \tag{3.12}$$

Similar to our first Theorem, this result connects the $X_\gamma$ norm of $F$ and $G$ by using *all* of the roots of $F$ in $\mathbb{D}$. However, we no longer have an expression involving $G$ on the right hand side of the inequality.

Our next goal is to extend this result to arbitrary functions in $\mathcal{H}^2$, which may have an infinite number of roots in $\mathbb{D}$. To prove such a result, we need to impose two additional conditions to the sequence $\{\gamma_n\}$: boundedness and a prescribed convergence rate of the sequence to its limit (recalling that bounded monotone sequences converge). This gives us the following result.

**Theorem 3.3.4.** *Suppose that $\{\gamma_n\} \nearrow M$ is a bounded monotone increasing sequence satisfying $\gamma_0 = 0$, the concavity condition (3.4) and*

$$\sum_{n\geq 0} M - \gamma_n < \infty.$$

*For any function $F \in \mathcal{H}^2$ with zeros inside the unit disc, $\alpha_j$ for $j \in J$, we have*

1. $$\sum_{j\in J}(1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2 < \infty$$

   .

2. $$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j\in J}\left((1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2\right). \tag{3.13}$$

These last two theorems, while general, help us to better understand the spaces, $X_\gamma$, that are situated between $\mathcal{H}^2$ and the Dirichlet space $\mathcal{D}$. Many open questions still exist on these spaces, as many of the proofs used for the Dirichlet space do not translate to these "in between" spaces. Examples of such problems can be found in the monograph [1] by Arcozzi, Rochberg, Sawyer, and Wick.

## 3.4 Proof of Main Results

In this section, we develop the theory necessary to prove Theorems 3.3.2, 3.3.3, and 3.3.4. This section will be broken into three parts for clarity.

In the first part, see Section 3.4.1, we expand upon the relationship between the functions $F$ and $G$ in the decomposition theorem to see that reflecting the roots of $F$ across $\partial\mathbb{D}$ provides a method of producing $G$. With this knowledge, we study how the act of reflecting a root in $\mathbb{D}$ across the unit circle affects the $X_\gamma$ norm of a function, seen in Proposition 3.4.1. To end this part, we state Corollary 3.4.1 which provides an identity for the case when $F$ has a single root in $\mathbb{D}$.

In the second part, see Section 3.4.2, we investigate the case when $F$ has finitely many roots in $\mathbb{D}$. We begin by defining intermediate functions (seen in Definition 3.4.2) that can be viewed as partial decompositions. Using these functions, we invoke Proposition 3.4.1 to obtain an identity seen in Lemma 3.4.2, which connects the $X_\gamma$ norms of $F$ and $G$. From there, we state and prove Lemma 3.4.4, which provides bounds on the $Y_\gamma$ norm of functions based on the growth rate of the sequence $\{\gamma_n\}$. With this, we have the tools necessary to handle all functions with a finite number of zeros in $\mathbb{D}$, and prove Theorem 3.3.2 and Theorem 3.3.3.

In the final part, see Section 3.4.3, we prove the convergence of the inequality in Theorem 3.3.3 for functions with infinitely many roots in $\mathbb{D}$ at the cost of imposing further conditions on $\{\gamma_n\}$ (see Lemma 3.4.5 and Lemma 3.4.6). From there, we directly prove Theorem 3.3.4.

In every part, we remove the assumption that the function $F$ is analytic in $\mathbb{D}_{1+\epsilon}$ for some $\epsilon > 0$, and prove the results accordingly.

### 3.4.1 Part 1: Reflecting a Root Across $\partial\mathbb{D}$

To begin, suppose that a function $F$ has a finite number of roots in $\mathbb{D}$ labeled in increasing order of magnitudes $\alpha_1, \alpha_2, \ldots, \alpha_m$, where the roots need not be distinct. Then for $z \in \mathbb{D}$, we can express

$$F(z) = \left( \prod_{j=1}^{m} \frac{z - \alpha_j}{1 - \overline{\alpha_j} z} \right) \cdot G(z).$$

**Remark 3.4.1.** *Since $m$ is finite, we let $G$ absorb the term $e^{i\phi}$ for some $\phi \in [0, 2\pi)$ so that we can express $F$ in this way. We also do not preclude $\alpha_j = 0$ for ease of notation.*

One way of generating the function $G$ from $F$ is by replacing each term in the Blaschke product $z - \alpha_j$ with the term $1 - \overline{\alpha_j} z$. This can be viewed as the reflection of each zero of $F$ in $\mathbb{D}$ across the complex unit circle (see Figure 2). When all zeros are reflected, we will have $m$ removable singularities, all outside $\mathbb{D}$, so we will have a function equivalent to $G$ on $\mathbb{D}$.

In the case where $F$ has infinitely many roots in $\mathbb{D}$, we use more careful notation and rely on existing theory to show that this process will induce a sequence of functions that converge to $G$ uniformly on compact subsets of $\mathbb{D}$. This will be further discussed in the third part of this section.

With this knowledge, we want to investigate how the act of reflecting a single root across

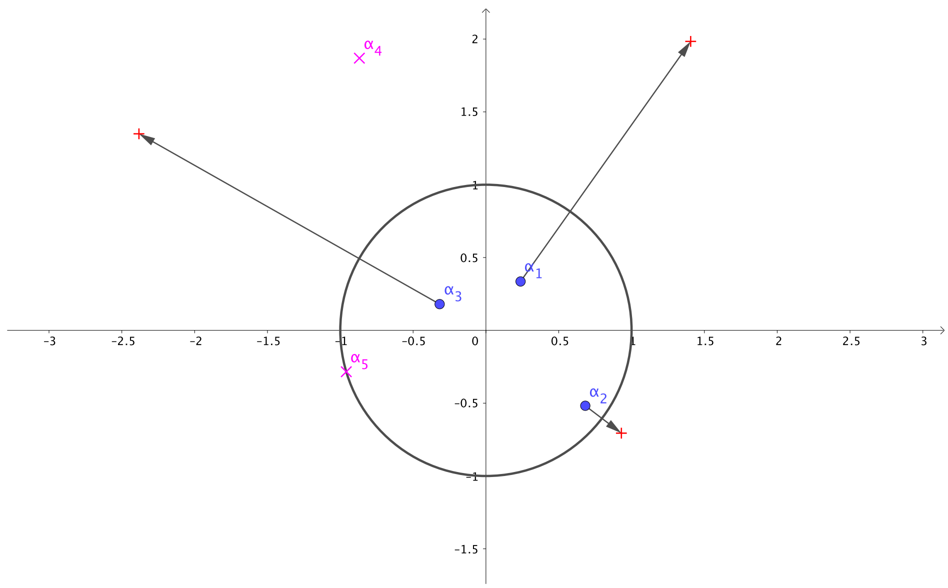


Figure 2: Roots of a function $F$ in the complex plane, labeled $\alpha_j$, and the corresponding roots of the factor $G$. Roots $\alpha_j \in \mathbb{D}$ that are reflected to the roots of $G$, namely $\frac{1}{\overline{\alpha_j}}$, are labeled with "+" ticks. Roots $\alpha_j \notin \mathbb{D}$, are also roots of $G$ and are labeled with a "×" tick.

the unit circle changes the $X_\gamma$ norm of a function. To understand how a single reflection works, we first define the following operator.

**Definition 3.4.1.** *Suppose that $F \in \mathcal{H}^2$ has a root at $\alpha \in \mathbb{D}$, that is, $F(\alpha) = 0$ and for all $z \in \mathbb{D}$, we can express $F(z) = (z - \alpha)H_\alpha(z)$. We define $\phi_\alpha$ be the operator that acts on functions in $\mathcal{H}^2$ with roots at $\alpha$ and satisfies*

$$\phi_\alpha(F(\cdot)) = \phi_\alpha((\cdot - \alpha)H_\alpha(\cdot)) := (1 - \overline{\alpha}\cdot)H_\alpha(\cdot). \tag{3.14}$$

This definition tells us that the operator $\phi_\alpha$ only affects a single root of a function. In the case where a function has a root at $\alpha$ of higher multiplicity, this operator will reduce the multiplicity of the root by one. With this definition, we have the following result on the $\mathcal{H}^2$

norm of $H_\alpha$.

**Lemma 3.4.1.** *If $F \in \mathcal{H}^2$ satisfies $F(\alpha) = 0$ where $|\alpha| < 1$, then $H_\alpha \in \mathcal{H}^2$. Further, we have the inequality*

$$\|H_\alpha\|_{\mathcal{H}^2} \leq \frac{2}{1-|\alpha|}\|F\|_{\mathcal{H}^2}.$$

*Proof.* By definition,

$$\|F\|_{\mathcal{H}^2} = \limsup_{0\leq r<1}\left(\frac{1}{2\pi}\int_0^{2\pi}|F(re^{i\theta})|^2 d\theta\right)^{\frac{1}{2}} = M < \infty. \tag{3.15}$$

Since $\alpha$ is a root of $F$, $\frac{F(\cdot)}{(\cdot-\alpha)} = H_\alpha$ has a removable singularity at $\alpha$, so because $H_\alpha$ can be treated as an analytic function in $\mathbb{D}$,

$$\limsup_{0\leq r<1}\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{re^{i\theta}-\alpha}\right|^2 d\theta = \lim_{r\to 1^-}\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{re^{i\theta}-\alpha}\right|^2 d\theta.$$

Since $|\alpha| < 1$, if $r \geq \frac{1+|\alpha|}{2}$, for any $\theta \in [0, 2\pi]$,

$$\left|re^{i\theta}-\alpha\right| \geq \frac{1-|\alpha|}{2}.$$

Therefore,

$$\begin{aligned}
2\pi\|H_\alpha\|^2_{\mathcal{H}^2} &= \lim_{r\to 1^-}\int_0^{2\pi}\left|\frac{F(re^{i\theta})}{re^{i\theta}-\alpha}\right|^2 d\theta \\
&\leq \left(\frac{2}{1-|\alpha|}\right)^2 \lim_{r\to 1^-}\int_0^{2\pi}|F(re^{i\theta})|^2 d\theta \\
&\leq \left(\frac{2}{1-|\alpha|}\right)^2 2\pi\|F\|^2_{\mathcal{H}^2} < \infty.
\end{aligned}$$

Dividing each side by $2\pi$ and taking square roots gives us the result. □

With this lemma proved, the function $H_\alpha$ can be represented with a Fourier series satisfying (2.6). This allows us to properly study how the operator, $\phi_\alpha$, affects the $X_\gamma$ norm of functions and leads us to the following proposition, which is a generalization of a result in [5].

**Proposition 3.4.1.** *Let $F \in X_\gamma$ satisfy $F(\alpha) = 0$ for some $\alpha \in \mathbb{D}$, so that for all $z \in \mathbb{D}$, $F(z) = (z - \alpha)H_\alpha(z)$. With $\phi_\alpha$ defined in (3.14), we have the following results:*

1. *$\phi_\alpha(F) \in X_\gamma$ and $H_\alpha \in Y_\gamma$*

2. *$\|\phi_\alpha(F)\|^2_{X_\gamma} = \|F\|^2_{X_\gamma} - (1 - |\alpha|^2)\|H_\alpha\|^2_{Y_\gamma}$.*

*Proof.* Given $F \in X_\gamma$, $X_\gamma \subseteq \mathcal{H}^2$, so it follows from Lemma 3.4.1 that $H_\alpha \in \mathcal{H}^2$. Therefore, for all $z \in \mathbb{D}$ we may represent

$$H_\alpha(z) = \sum_{j=0}^{\infty} a_j z^j, \quad \text{where } \sum_{j=0}^{\infty} |a_j|^2 < \infty.$$

With this notation,

$$F(z) = (z - \alpha)H_\alpha(z) \quad \text{and} \quad \phi_\alpha(F(z)) = (1 - \overline{\alpha}z)H_\alpha(z),$$

so we can express

$$F(z) = (z - \alpha)\sum_{j=0}^{\infty} a_j z^j = -\alpha a_0 + \sum_{j=1}^{\infty}(a_{j-1} - \alpha a_j)z^j,$$

and

$$\phi_\alpha(F(z)) = (1 - \overline{\alpha}z)\sum_{j=0}^{\infty} a_j z^j = a_0 + \sum_{j=1}^{\infty}(a_j - \overline{\alpha}a_{j-1})z^j.$$

With these expressions, we can compute the $X_\gamma$ norm of each function, and obtain

$$\|F\|_{X_\gamma}^2 = \gamma_0|\alpha a_0|^2 + \sum_{j=1}^{\infty} \gamma_j |a_{j-1} - \alpha a_j|^2, \tag{3.16}$$

$$\|\phi_\alpha(F)\|_{X_\gamma}^2 = \gamma_0|a_0|^2 + \sum_{j=1}^{\infty} \gamma_j |a_j - \overline{\alpha} a_{j-1}|^2. \tag{3.17}$$

From here, if we can show that $\|F\|_{X_\gamma}^2 - \|\phi_\alpha(F)\|_{X_\gamma}^2 \geq 0$, then this will imply that $\phi_\alpha(F) \in X_\gamma$. Towards this end, we consider the following finite difference:

$$\gamma_0|\alpha a_0|^2 + \sum_{j=1}^{N} \gamma_j |a_{j-1} - \alpha a_j|^2 - \left(\gamma_0|a_0|^2 - \sum_{j=1}^{N} \gamma_j |a_j - \overline{\alpha} a_{j-1}|^2\right).$$

For each $j \in \{1, 2, \dots, N\}$,

$$|a_{j-1} - \alpha a_j|^2 - |a_j - \overline{\alpha} a_{j-1}|^2 = |a_{j-1}|^2 + |\alpha|^2|a_j|^2 - |a_j|^2 - |\alpha|^2||a_{j-1}|^2 = (1 - |\alpha|^2)\left(|a_{j-1}|^2 - |a_j|^2\right),$$

because $\mathcal{R}e(a_{j-1}\overline{\alpha a_j}) = \mathcal{R}e(a_j \alpha \overline{a_{j-1}})$. Therefore,

$$\begin{aligned}
&\gamma_0|\alpha a_0|^2 + \sum_{j=1}^{N} \gamma_j |a_{j-1} - \alpha a_j|^2 - \gamma_0|a_0|^2 - \sum_{j=1}^{N} \gamma_j |a_j - \overline{\alpha} a_{j-1}|^2 \\
&= -\gamma_0(1 - |\alpha|^2)|a_0|^2 + (1 - |\alpha|^2)\sum_{j=1}^{N} \gamma_j \left(|a_{j-1}|^2 - |a_j|^2\right) \\
&= (1 - |\alpha|^2)\left(\left[\sum_{j=0}^{N-1} (\gamma_{j+1} - \gamma_j)|a_j|^2\right] + \gamma_N |a_N|^2\right).
\end{aligned}$$

Thus, by passing limits into the summation, we have

$$\|F\|_{X_\gamma}^2 - \|\phi_\alpha(F)\|_{X_\gamma}^2 = \lim_{N\to\infty}(1-|\alpha|^2)\left(\left[\sum_{j=0}^{N-1}(\gamma_{j+1}-\gamma_j)|a_j|^2\right] + \gamma_N|a_N|^2\right). \tag{3.18}$$

Since $\|F\|_{X_\gamma}^2 < \infty$,

$$\lim_{n\to\infty}\gamma_n^{\frac{1}{2}}|a_{n-1}-\alpha a_n| = 0.$$

Because

$$\gamma_n^{\frac{1}{2}}|a_{n-1}-\alpha a_n| \geq \gamma_n^{\frac{1}{2}}\left||a_{n-1}|-|\alpha||a_n|\right|,$$

$$\lim_{n\to\infty}\gamma_n^{\frac{1}{2}}\left||a_{n-1}|-|\alpha||a_n|\right| = 0.$$

Since the sequence $\{\gamma_n\}$ is monotone increasing, implies that given any $\epsilon > 0$, there exists some $M$ such that if $n \geq M$,

$$\gamma_{n-1}^{\frac{1}{2}}|a_{n-1}| \leq \gamma_n^{\frac{1}{2}}|a_{n-1}| < \epsilon + |\alpha|\gamma_n^{\frac{1}{2}}|a_n|.$$

Letting $L = \limsup \gamma_n|a_n|^2$, this gives,

$$L \leq \epsilon + |\alpha|L \implies L \leq \frac{\epsilon}{1-|\alpha|}.$$

Since $|\alpha| < 1$, $L = 0$ so we have

$$\lim_{N\to\infty}\gamma_N|a_N|^2 = 0. \tag{3.19}$$

By definition,

$$\lim_{N\to\infty}\sum_{j=0}^{N-1}(\gamma_{j+1}-\gamma_j)|a_j|^2 = \|H_\alpha\|_{Y_\gamma}^2, \tag{3.20}$$

so, by substituting (3.19) and (3.20) into (3.18), we have the identity

$$\|F\|_{X_\gamma}^2 - \|\phi_\alpha(F)\|_{X_\gamma}^2 = (1-|\alpha|^2)\|H_\alpha\|_{Y_\gamma}^2.$$

Since $(1-|\alpha|^2)\|H_\alpha\|_{Y_\gamma}^2 \geq 0$, this immediately tells us that $\|\phi_\alpha(F)\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 < \infty$, and that $\|H_\alpha\|_{Y_\gamma}^2 < \infty$, which proves (1).

Lastly, by rearranging the terms, we have

$$\|\phi_\alpha(F)\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - (1-|\alpha|^2)\|H_\alpha\|_{Y_\gamma}^2,$$

which completes the proof.

□

This result shows us that the reflection of a single root about the complex unit circle will alter the $X_\gamma$ norm of a function in a predictable way, and will always decrease the $X_\gamma$ norm. We end this subsection with a Corollary involving functions with a single root in $\mathbb{D}$.

**Corollary 3.4.1.** *Let $\{\gamma_n\}$ be a monotone increasing sequence with $\gamma_0 = 0$. If $F \in X_\gamma$ has a single root, $\alpha$, in $\mathbb{D}$, then*

$$\|G\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - (1-|\alpha|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha}e^{i\cdot}}\right\|_{Y_\gamma}^2. \tag{3.21}$$

*Proof.* We begin by noting that for all $z \in \mathbb{D}$,

$$F(z) = \frac{z-\alpha}{1-\overline{\alpha}z}G(z).$$

Therefore, $\phi_\alpha(F) = G$, and $H_\alpha(z) = \frac{G(z)}{1-\overline{\alpha}z}$. Therefore, by applying Proposition 3.4.1, we have the result. $\square$

With this result proved, we have completed this part of the section, and move on to functions with finitely many roots in $\mathbb{D}$.

### 3.4.2 Part 2: Performing a Finite Number of Reflections

In the previous section we identified the relationship between $F$ and $G$, and studied how the $X_\gamma$ norm is affected by reflecting a single root across $\partial\mathbb{D}$ with Proposition 3.4.1. Unfortunately, for functions with multiple zeros in $\mathbb{D}$, a single reflection will not produce the function $G$. Further, after we have performed a reflection, if we reflect a second root, we will be acting upon a new function. Therefore, to utilize the full potential of Proposition 3.4.1, we require the following definition.

**Definition 3.4.2.** *Let $F \in X_\gamma$ have $m$ roots in $\mathbb{D}$, enumerated $\alpha_1, \alpha_2 \ldots \alpha_m$, in increasing order of magnitude. Then expressing*

$$F_0(z) := F(z) = \left(\prod_{j=1}^{m} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}\right) \cdot G(z),$$

*where $G$ has no zeros in $\mathbb{D}$, we define*

$$\begin{aligned} F_k(z) &:= \left(\prod_{j=k+1}^{m} \frac{z-\alpha_j}{1-\overline{\alpha_j}z}\right) \cdot G(z) \quad \textit{where } 1 \le k < m \\ F_m(z) &:= G(z). \end{aligned}$$

With this definition and using the same notation as Definition 3.4.1 , we notice that for each $1 \leq k \leq m$,

$$\phi_{\alpha_k}(F_{k-1}) = F_k, \quad H_{\alpha_k}(z) := \frac{1}{1-\overline{\alpha_k}z} F_k(z). \tag{3.22}$$

With all of this, we can prove a simple, yet useful lemma.

**Lemma 3.4.2.** *Suppose that $F$ has $m$ roots in $\mathbb{D}$, labeled in increasing order of magnitude $\alpha_1, \ldots \alpha_m$. Then for any $1 \leq n \leq m$, we have the identity*

$$\|F_n\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n} \left( (1-|\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right). \tag{3.23}$$

*Proof.* We begin by observing that for $0 \leq k \leq n-1$, we have $\phi_{\alpha_{k+1}}(F_k) = F_{k+1}$. Then by applying Proposition 3.4.1 to $F_k$, for $0 \leq k \leq m-1$, we have the identity

$$\|F_{k+1}\|_{X_\gamma}^2 = \|F_k\|_{X_\gamma}^2 - (1-|\alpha_k|^2)\|H_{\alpha_k}\|_{Y_\gamma}^2. \tag{3.24}$$

Now, by applying Equation (3.24) to each $F_k$, we have

$$\|F_1\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - (1-|\alpha_1|^2)\|H_{\alpha_1}\|_{Y_\gamma}^2 \tag{3.25}$$

$$\|F_2\|_{X_\gamma}^2 = \left( \|F\|_{X_\gamma}^2 - (1-|\alpha_1|^2)\|H_{\alpha_1}\|_{Y_\gamma}^2 \right) - (1-|\alpha_2|^2)\|H_{\alpha_2}\|_{Y_\gamma}^2 \tag{3.26}$$

$$\vdots$$

$$\|F_{n-1}\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n-1} \left( (1-|\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right) \tag{3.27}$$

$$\|F_n\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n} \left( (1-|\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right). \tag{3.28}$$

□

As a direct consequence of this Lemma, we now have an identity for any function $F \in X_\gamma$

with $m$ roots in $\mathbb{D}$:

$$\|G\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{m} \left( (1 - |\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right). \tag{3.29}$$

With this identity established, we now look to restrict our choices of $\{\gamma_n\}$ to create a bound on the terms $\|H_{\alpha_j}\|_{Y_\gamma}^2$ involving the functions $F$ and $G$.

We begin by noticing that the $Y_\gamma$ semi-norm will carry the same desired properties that the $X_\gamma$ semi-norm has if the sequence

$$\Gamma_n := \gamma_{n+1} - \gamma_n$$

is monotone increasing. In other words, if $\{\gamma_n\}$ satisfies the convexity condition, (3.3), we can treat $Y_\gamma$ as $X_\Gamma$ (relaxing the assumption to $\Gamma_0 \geq 0$). This means that any inequalities that can be applied to the $X_\Gamma$ norm can be applied to the $Y_\gamma$ norm. Namely, for any function $F \in Y_\gamma$, we can apply the results of Proposition 3.4.1 to see

$$\|F\|_{Y_\gamma}^2 - \|\phi_\alpha(F)\|_{Y_\gamma}^2 \geq 0. \tag{3.30}$$

If the bounded sequence $\{\gamma_n\}$ is monotone decreasing, then we want to obtain the reverse of the inequality (3.30). To do this, we need to verify that the results from the last section still hold. This brings us to the following Lemma.

**Lemma 3.4.3.** *Let $\{\gamma_n\}$ be a monotone increasing sequence satisfying $\gamma_0 = 0$ and the concavity condition,* (3.4). *If $F \in Y_\gamma \cap \mathcal{H}^2$, then*

$$\|\phi_\alpha(F)\|_{Y_\gamma}^2 - \|F\|_{Y_\gamma}^2 \geq 0. \tag{3.31}$$

*Proof.* Since $F \in \mathcal{H}^2$, we can express the series expansion of $H_\alpha$, where $F(z) = (z-\alpha)H_\alpha(z)$ for any $z \in \mathbb{D}$ as

$$H_\alpha(z) = \sum_{n=0}^{\infty} a_n z^n, \quad \text{where} \quad \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

As was done in the proof of Proposition 3.4.1,

$$F(z) = (z-\alpha)\sum_{j=0}^{\infty} a_j z^j = -\alpha a_0 + \sum_{j=1}^{\infty}(a_{j-1} - \alpha a_j)z^j,$$

and

$$\phi_\alpha(F(z)) = (1-\overline{\alpha}z)\sum_{j=0}^{\infty} a_j z^j = a_0 + \sum_{j=1}^{\infty}(a_j - \overline{\alpha}a_{j-1})z^j.$$

With these expressions, we can look at the $Y_\gamma$ norm of each function, and we see that

$$\|F\|_{Y_\gamma}^2 = \Gamma_0|\alpha a_0|^2 + \sum_{j=1}^{\infty}\Gamma_j|a_{j-1} - \alpha a_j|^2,$$

$$\|\phi_\alpha(F)\|_{Y_\gamma}^2 = \Gamma_0|a_0|^2 + \sum_{j=1}^{\infty}\Gamma_j|a_j - \overline{\alpha}a_{j-1}|^2.$$

From here, if we can show that $\|F\|_{Y_\gamma}^2 - \|\phi_\alpha(F)\|_{Y_\gamma}^2 \leq 0$, then we are finished.

$$\|F\|_{Y_\gamma}^2 - \|\phi_\alpha(F)\|_{Y_\gamma}^2 = \Gamma_0|\alpha a_0|^2 + \sum_{j=1}^{\infty}\Gamma_j|a_{j-1} - \alpha a_j|^2 - \Gamma_0|a_0|^2 - \sum_{j=1}^{\infty}\Gamma_j|a_j - \overline{\alpha}a_{j-1}|^2.$$

For each $j$,

$$|a_{j-1} - \alpha a_j|^2 - |a_j - \overline{\alpha}a_{j-1}|^2 = (1-|\alpha|^2)\left(|a_{j-1}|^2 - |a_j|^2\right).$$

Therefore,

$$\begin{aligned}
\|F\|_{Y_\gamma}^2 - \|\phi_\alpha(F)\|_{Y_\gamma}^2 &= \Gamma_0|\alpha a_0|^2 + \sum_{j=1}^{\infty}\Gamma_j|a_{j-1} - \alpha a_j|^2 - \Gamma_0|a_0|^2 - \sum_{j=1}^{\infty}\Gamma_j|a_j - \overline{\alpha}a_{j-1}|^2 \\
&= -\Gamma_0(1-|\alpha|^2)|a_0|^2 + (1-|\alpha|^2)\sum_{j=1}^{\infty}\Gamma_j\left(|a_{j-1}|^2 - |a_j|^2\right) \\
&= (1-|\alpha|^2)\left(\left[\sum_{j=0}^{\infty}(\Gamma_{j+1}-\Gamma_j)|a_j|^2\right]\right).
\end{aligned}$$

Since $\{\gamma_n\}$ is a nonnegative monotone decreasing sequence, for any $j$, we have $-\Gamma_0 \leq \Gamma_{j+1} - \Gamma_j \leq 0$, implying

$$-\infty < -\Gamma_0(1-|\alpha|^2)\left(\left[\sum_{j=0}^{\infty}|a_j|^2\right]\right) \leq (1-|\alpha|^2)\left(\left[\sum_{j=0}^{\infty}(\Gamma_{j+1}-\Gamma_j)|a_j|^2\right]\right) \leq 0,$$

so

$$-\infty < \|F\|_{Y_\gamma}^2 - \|\phi_\alpha(F)\|_{Y_\gamma}^2 \leq 0.$$

This proves the result.

□

These results directly lead us to the following lemma.

**Lemma 3.4.4.** *Let $\{\gamma_n\}$ be a monotone increasing sequence satisfying $\gamma_0 = 0$, and let $F \in X_\gamma$ have $m \leq \infty$ roots in $\mathbb{D}$ labeled in increasing order of magnitude, $\alpha_j$.*

1. *If $\{\gamma_n\}$ satisfies the convexity condition* (3.3)*, then for any* $1 \leq k < m$*,*

$$\left\| \frac{G(e^{i\cdot})}{1-\overline{\alpha_k}e^{i\cdot}} \right\|_{Y_\gamma}^2 \leq \|H_{\alpha_k}\|_{Y_\gamma}^2 \leq \left\| \frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_k} \right\|_{Y_\gamma}^2 .$$

2. *If $\{\gamma_n\}$ satisfies the concavity condition* (3.4)*, then for any* $1 \leq k < m$*,*

$$\left\| \frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_k} \right\|_{Y_\gamma}^2 \leq \|H_{\alpha_k}\|_{Y_\gamma}^2 \leq \left\| \frac{G(e^{i\cdot})}{1-\overline{\alpha_k}e^{i\cdot}} \right\|_{Y_\gamma}^2 .$$

*Proof.* To begin, let

$$F(z) = \prod_{j=0}^{m} \frac{\alpha_j - z}{1-\overline{\alpha_j}z} G(z).$$

For any $1 \leq k < m$, we can express

$$H_{\alpha_k} = \left( \prod_{j=k+1}^{m} \frac{\alpha_j - z}{1-\overline{\alpha_j}z} \right) \cdot \left( \frac{1}{1-\overline{\alpha_k}z} G(z) \right).$$

Further, for any $k$, $H_{\alpha_k} \in \mathcal{H}^2$. Similarly, by rearranging,

$$\frac{F(z)}{\alpha_k - z} = \left( \prod_{\substack{j=0 \\ j\neq k}}^{m} \frac{\alpha_j - z}{1-\overline{\alpha_j}z} \right) \cdot \frac{1}{1-\overline{\alpha_k}z} G(z).$$

Thus, by reflecting the first $k-1$ roots of $\frac{F(z)}{z-\alpha_k}$ across the unit circle, we get $H_{\alpha_k}$. By reflecting the remaining roots, we get $\frac{1}{1-\overline{\alpha_k}\cdot}G(\cdot)$.

If $\{\gamma_n\}$ is monotone increasing, then by (3.30),

$$\left\| \frac{F(z)}{\alpha_k - z} \right\|_{Y_\gamma}^2 \geq \left\| \phi_{\alpha_1} \left( \frac{F(z)}{\alpha_k - z} \right) \right\|_{Y_\gamma}^2 \geq \cdots \geq \|H_{\alpha_k}\|_{Y_\gamma}^2 \geq \|\phi_{\alpha_{k+1}}(H_{\alpha_k})\|_{Y_\gamma}^2 \geq \cdots .$$

Clearly, if $m$ is finite the result holds. If $m = \infty$, by the monotonicity of the sequence of $Y_\gamma$ norms, this implies that

$$\|H_{\alpha_k}\|_{Y_\gamma}^2 \geq \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_k} e^{i\cdot}} \right\|_{Y_\gamma}^2 ,$$

which proves the first part of the inequality.

Proving the second inequality follows verbatim with all inequalities flipped, due to (3.31). This gives the result. □

With this lemma, we can see that by restricting $\{\gamma_n\}$ to be a sequence that is either increasing at a non-increasing or non-decreasing rate, we can replace the intermediary $H_{\alpha_j}$ terms in Equation (3.29). With all of this, we are now able to prove Theorem 3.3.2 and Theorem 3.3.3.

### Proofs of Theorem 3.3.2 and Theorem 3.3.3

With the previous results, we now restate and prove Theorem 3.3.2 and Theorem 3.3.3

We begin with the proof of Theorem 3.3.2 .

**Theorem 3.3.2.** *Suppose that $\{\gamma_n\}$ is a monotone increasing satisfying $\gamma_0 = 0$ and* (3.3). *For functions $F \in X_\gamma$ with a finite number of zeros inside the unit disc labeled $\alpha_1, \alpha_2, \ldots, \alpha_m$, we have*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j=1}^{m} \left( (1 - |\alpha_j|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2 \right) .$$

*Proof.* Since $F$ has a finite number of roots in $\mathbb{D}$, by Lemma 3.4.2 we have the identity

$$\|G\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n} \left( (1 - |\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right).$$

By Lemma 3.4.4 we have that for all $1 \le j \le m$,

$$\left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \ge \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2 .$$

Therefore, by replacing each $H_{\alpha_j}$ with $\frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}}$, we preserve the inequality and have the result.

□

**Theorem 3.3.3.** *Suppose that $\{\gamma_n\}$ is a monotone increasing sequence satisfying $\gamma_0 = 0$ and* (3.4)*. For functions $F \in X_\gamma$ with a finite number of zeros inside $\mathbb{D}$ labeled $\alpha_1, \alpha_2, \ldots, \alpha_m$, we have*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \le \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j=1}^{m} \left( (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 \right).$$

*Proof.* Since $F$ has a finite number of roots in $\mathbb{D}$, by Lemma 3.4.2 we have the identity

$$\|G\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n} \left( (1 - |\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right).$$

By Lemma 3.4.4 we have that for all $1 \le j \le m$,

$$\left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \ge \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 .$$

Therefore, by replacing each $H_{\alpha_j}$ with $\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}$, we preserve the inequality and have the result.

□

With all results proved in the case when the function $F$ has finite number of roots in $\mathbb{D}$, we now consider the case when $F$ has an infinite number of roots in $\mathbb{D}$.

### 3.4.3 Part 3: Functions with Infinitely Many Roots in $\mathbb{D}$

Similar to the previous section, this section will study the relationship between the $X_\gamma$ norm of $F$ and $G$, with the exception that $F$ will now be assumed to have infinitely many zeros in $\mathbb{D}$. In this case, we rely on some well known literature about the convergence of the partial decompositions.

To begin, $F$ has Blaschke decomposition $F = B \cdot G$, where $B$ is an infinite Blaschke Product, and $G \in \mathcal{H}^2$. Therefore, if we enumerate the nonzero roots of $F$ in increasing order of magnitude as $\alpha_1, \alpha_2, \ldots$, we can express

$$F(z) = z^m \prod_{j=1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z} G(z). \tag{3.32}$$

We divide $F$ by $z^m$ (if $m > 0$) and define:

$$F_k(z) = \left( \prod_{j=k+1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z} \right) \cdot G(z),$$

$$B_k(z) = \prod_{j=1}^{k} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z}, \quad \text{and} \quad B(z) = \prod_{j=1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z},$$

where $F_k$ are the partial decomposition of $F$. As a fact that is proven in Section 7 of [31], $B_n \to B$ uniformly on compact subsets of $\mathbb{D}$. We will now show that this fact immediately implies that $F_n \to G$ uniformly on compact subsets of $\mathbb{D}$.

Let $0 < r < 1$ be given. We begin by choosing $M$ large enough so that $|\alpha_M| > r$ which can be done since $|\alpha_j| \to 1$. Consider the Blaschke product

$$\tilde{B}(z) = \prod_{j=M}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z},$$

which has no roots in $cl(\mathbb{D}_r)$, as $\{\alpha\}_n$ is expressed in increasing order of magnitude. Since $\tilde{B}$ is a Blashke product, we know that

$$\tilde{B}_k \to \tilde{B}$$

uniformly on compact subsets of $\mathbb{D}$. Therefore, for any $\epsilon > 0$, and $z \in cl(\mathbb{D}_r)$, we can choose $L$ large enough so that if $k \geq L$,

$$|\tilde{B}_k(z) - \tilde{B}(z)| = |\tilde{B}_k(z)| \left| \prod_{j=M+k+1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z} - 1 \right| < \epsilon.$$

Since $\tilde{B}_k(z) \neq 0$ for any $z \in cl(\mathbb{D}_r)$, there is some $s > 0$, such that $|\tilde{B}_k(z)| \geq s > 0$ when $|z| \leq r$. Therefore, setting $N = M + L$, we see that for any $z \in cl(\mathbb{D}_r)$, if $k \geq N$,

$$|F_k(z) - G(z)| = \left| \prod_{j=k+1}^{\infty} \frac{|\alpha_j|}{\alpha_j} \frac{\alpha_j - z}{1 - \overline{\alpha_j} z} - 1 \right| |G(z)| < \frac{\epsilon}{s} |G(z)|.$$

Since $G$ is analytic in $\mathbb{D}$, it is bounded on compact subsets, $cl(\mathbb{D}_r)$, which implies that

$$F_k \to G$$

uniformly on compact subsets of $\mathbb{D}$.

Further, by Proposition 3.4.2, we have that for any finite $n$, we have the identity

$$\|F_n\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n}(1-|\alpha_j|^2)\|H_{\alpha_j}\|_{Y_\gamma}^2.$$

Since $\sum_{j=1}^{\infty}(1-|\alpha_j|^2) < \infty$ by the Blaschke condition (2.1), by showing that

$$\lim_{j\to\infty} \|F_j\|_{X_\gamma}^2 = \|G\|_{X_\gamma}^2 \quad \text{and} \tag{3.33}$$

$$\sup_j \|H_{\alpha_j}\|_{Y_\gamma}^2 < \infty, \tag{3.34}$$

we will have a meaningful analogue to Equation (3.29) for functions with infinitely many zeros in $\mathbb{D}$, and can prove Theorem 3.3.4.

We begin by proving a sufficient condition on $\{\gamma_n\}$ to ensure Equation (3.33) holds.

**Lemma 3.4.5.** *Suppose that* $\{\gamma_n\} \nearrow M < \infty$. *Then*

$$\lim_{m\to\infty} \|F_m\|_{X_\gamma}^2 = \|G\|_{X_\gamma}^2.$$

*Proof.* Let

$$F_m(z) = \sum_{j=0}^{\infty} a_j^{(m)} z^j \quad \text{and} \qquad G(z) = \sum_{j=0}^{\infty} b_j z^j.$$

Since

$$F_m \to G,$$

uniformly on compact subsets of $\mathbb{D}$, for any $\epsilon > 0$, and for each $k \in \mathbb{N}$, we can find $N(k,\epsilon) \in \mathbb{N}$

such that if $m \geq N$, for any $0 \leq j \leq k$,

$$|a_j^{(m)} - b_j|^2 < \frac{\epsilon}{2M(k+1)}.$$

Since the magnitude of the Fourier coefficients are shifted to earlier terms via reflection of roots (Immediate result after applying Equation (3.29) with $\gamma_n = 0$ when $n < k+1$ and $\gamma_n = 1$ otherwise), for every $m \geq 0$ and every $k > 0$,

$$\sum_{j=k+1}^{\infty} |a_j^{(0)}|^2 \geq \sum_{j=k+1}^{\infty} |a_j^{(m)}|^2 \geq \sum_{j=k+1}^{\infty} |b_j|^2.$$

If we choose $k$ such that

$$\sum_{j=k+1}^{\infty} |a_j^{(0)}|^2 < \frac{\epsilon}{8M},$$

then

$$\sum_{j=k+1}^{\infty} |a_j^{(m)} - b_j|^2 < \frac{\epsilon}{2M}.$$

Choosing appropriate $k$ and $N$, if $m \geq N$, we therefore satisfy

$$\|F_m - G\|_{X_\gamma}^2 = \sum_{j=0}^{\infty} \gamma_j |a_j^{(m)} - b_j|^2 < M \sum_{j=0}^{k} |a_j^{(m)} - b_j|^2 + M \sum_{j=k+1}^{\infty} |a_j^{(m)} - b_j|^2 < \epsilon.$$

Therefore, by the reverse triangle inequality

$$\|F_m\|_{X_\gamma}^2 \to \|G\|_{X_\gamma}^2.$$

□

With this condition, we now have to restrict our choice of $\{\gamma_n\}$ to bounded sequences.

Clearly no bounded sequence can increase at an increasing rate, so for the remainder of this section we assume that the concavity condition, (3.4), holds.

From here, we need to find a condition on $\{\gamma_n\}$ so that (3.34) is true on the space $Y_\gamma$. By Lemma 3.4.4 and (3.4), for any $j \in \mathbb{N}$,

$$\left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 \leq \|H_{\alpha_j}\|_{Y_\gamma}^2 \leq \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2 .$$

Therefore, if we can find a condition on $\{\gamma_n\}$ such that

$$\sup_j \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2 < \infty,$$

we will also satisfy (3.34).

From Lemma 3.4.1, we obtained the inequality

$$\|H_\alpha\|_{\mathcal{H}^2}^2 \leq \frac{2}{1 - |\alpha|} \|F\|_{\mathcal{H}^2}.$$

While this bound works well if $F$ has a finite number of roots in $\mathbb{D}$, it is less useful in the case of infinite roots, as the roots must accumulate to $\partial\mathbb{D}$.

For example, if we consider the zero set $\alpha_j = \sqrt{1 - \frac{1}{2^j}}$, then

$$\frac{B(z)}{\alpha_k - z} = \left( \prod_{\substack{j=0 \\ j \neq k}}^{\infty} \frac{\alpha_j - z}{1 - \alpha_j z} \right) \frac{1}{1 - \alpha_k z}.$$

Since we are dealing with unweighted $\mathcal{H}^2$, the Blaschke product will not alter the norm so we obtain

$$\left\| \frac{B(z)}{\alpha_k - z} \right\|_{\mathcal{H}^2}^2 = \left\| \frac{1}{1 - \alpha_k z} \right\|_{\mathcal{H}^2}^2 .$$

From here, we use geometric series to get the series expansion:

$$\left\|\frac{1}{1-\alpha_k z}\right\|_{\mathcal{H}^2}^2 = \left\|\sum_{j=0}^{\infty}\alpha_k^j z^j\right\|_{\mathcal{H}^2}^2 = \sum_{j=0}^{\infty}|\alpha_k|^{2j} = \frac{1}{1-|\alpha_k|^2} = \frac{1}{1-(1-\frac{1}{2^k})}.$$

So we get

$$\left\|\frac{B(z)}{\alpha_k - z}\right\|_{\mathcal{H}^2}^2 = 2^k.$$

Since this would not provide us a meaningful bound, we look to impose a condition on the weights $\{\gamma_n\}$ to provide a finite bound on our desired $Y_\gamma$ norm.

**Lemma 3.4.6.** *Let $\{\gamma_n\} \nearrow M$ be a monotone increasing sequence satisfying* (3.4)*. If*

$$\sum_{j=0}^{\infty} M - \gamma_j < \infty, \tag{3.35}$$

*then, for every function $F \in \mathcal{H}^2(\mathbb{D})$ with infinitely many roots labeled in increasing order of magnitude $\alpha_1, \alpha_2, \ldots$, with Blaschke decomposition $F = B \cdot G$, we have*

$$\sup_j \left\|\frac{G(z)}{1-\overline{\alpha_j} z}\right\|_{Y_\gamma}^2 < \infty. \tag{3.36}$$

*Proof.* We break this proof into two cases, $0 \leq |\alpha| < \frac{1}{2}$ and $\frac{1}{2} \leq |\alpha| < 1$. In both cases we will provide a bound that is independent of the choice of $\alpha$, giving us the result.

To begin, let $\alpha$ be a root of $F$ with $|\alpha| < 1$. By the decomposition theorem,

$$\left\|\frac{F(z)}{z-\alpha}\right\|_{\mathcal{H}^2} = \left\|\frac{G(z)}{1-\overline{\alpha} z}\right\|_{\mathcal{H}^2},$$

since

$$F(z) = \frac{z-\alpha}{1-\overline{\alpha}z} \cdot \tilde{B}(z) \cdot G(z).$$

If $0 \le |\alpha| < \frac{1}{2}$, by Lemma 3.4.1 and the fact that $\gamma_1 \ge \gamma_{n+1} - \gamma_n$, for any $n \ge 0$,

$$\left\| \frac{G(z)}{1-\overline{\alpha}z} \right\|_{Y_\gamma}^2 \le \gamma_1 \left\| \frac{G(z)}{1-\overline{\alpha}z} \right\|_{\mathcal{H}^2}^2 = \gamma_1 \left\| \frac{F(z)}{z-\alpha} \right\|_{\mathcal{H}^2}^2 < 4\gamma_1 \|F(z)\|_{\mathcal{H}^2}^2.$$

Since this bound is independent of $\alpha$, we are finished with this case.

If $\frac{1}{2} \le |\alpha| < 1$, then we can rewrite

$$\frac{G(z)}{1-\overline{\alpha}z} = \frac{-1}{\overline{\alpha}} \left( \frac{G(z)}{z-\frac{1}{\overline{\alpha}}} \right).$$

By our bound on $\alpha$,

$$\left\| \frac{-1}{\overline{\alpha}} \left( \frac{G(z)}{z-\frac{1}{\overline{\alpha}}} \right) \right\|_{Y_\gamma}^2 \le 4 \left\| \frac{G(z)}{z-\frac{1}{\overline{\alpha}}} \right\|_{Y_\gamma}^2.$$

Therefore, if we bound the term

$$\left\| \frac{G(z)}{z-\frac{1}{\overline{\alpha}}} \right\|_{Y_\gamma}^2,$$

we will have completed the proof.

For simplicity, we will denote $\beta = \frac{1}{\overline{\alpha}}$ where $1 < |\beta| \le 2$. Since $G \in \mathcal{H}^2$, we may express $G(z) = \sum_{n=0}^{\infty} c_n z^n$, for all $z \in \mathbb{D}$. Since $|\beta| > 1$, we have that for all $z \in \mathbb{D}$, we can rewrite

$$G_\beta(z) := \frac{G(z)}{z-\beta} = \sum_{n=0}^{\infty} d_n z^n.$$

By Cauchy's formula for derivatives, for each $n$,

$$d_n = \frac{G_\beta^{(n)}(0)}{n!}.$$

By the generalized Leibniz rule, we have that for any $|z| < 1$,

$$G_\beta^{(n)}(z) = \frac{d^n}{dz^n}\left[G(z) \cdot \frac{1}{z-\beta}\right] = \sum_{k=0}^{n} \binom{n}{k} \frac{d^{n-k}}{dz^{n-k}}[G(z)] \cdot \frac{d^k}{dz^k}\left[\frac{1}{z-\beta}\right].$$

Since $G^{(n-k)}(0) = c_{n-k}(n-k)!$, and

$$\frac{d^k}{dz^k}\left[\frac{1}{z-\beta}\right] = (k!)(-1)^k(z-\beta)^{-(k+1)},$$

we obtain

$$G_\beta^{(n)}(0) = \sum_{k=0}^{n} \frac{n!}{(n-k)!k!} c_{n-k}(n-k)! \cdot (k!) \frac{1}{\beta^{k+1}} = \frac{n!}{\beta^{n+1}} \sum_{k=0}^{n} c_k \beta^k.$$

Therefore, for each $0 \leq n < \infty$,

$$d_n = \frac{-1}{\beta^{n+1}} \sum_{k=0}^{n} c_k \beta^k.$$

From here, we consider the $Y_\gamma$ norm of $\frac{G(z)}{z-\beta}$.

$$\left\| \frac{G(z)}{z-\beta} \right\|_{Y_\gamma}^2 = \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n)|d_n|^2 = \sum_{n=0}^{\infty} \frac{(\gamma_{n+1} - \gamma_n)}{|\beta|^{2n+2}} \left| \sum_{k=0}^{n} c_k \beta^k \right|^2.$$

By the Cauchy-Schwarz inequality we have

$$\left| \sum_{k=0}^{n} c_k \beta^k \right|^2 \leq \sum_{k=0}^{n} |c_k|^2 \sum_{k=0}^{n} |\beta^k|^2.$$

Therefore

$$\left\|\frac{G(z)}{z-\beta}\right\|_{Y_\gamma}^2 \le \sum_{n=0}^{\infty}(\gamma_{n+1}-\gamma_n)\left(\sum_{k=0}^{n}|c_k|^2\right)\frac{\sum_{k=0}^{n}|\beta^k|^2}{|\beta|^{2n+2}}.$$

Clearly, $\sum_{k=0}^{n}|c_k|^2 \le \|G\|_{\mathcal{H}^2}^2$. By finite geometric series and the bound $1<|\beta|\le 2$,

$$\frac{1}{3} \le \frac{\sum_{k=0}^{n}|\beta^k|^2}{|\beta|^{2n+2}} < (n+1).$$

Lastly, since $\{\gamma_n\}$ satisfies

$$\sum_{n=0}^{\infty} M-\gamma_n < \infty,$$

we see that

$$\sum_{n=0}^{\infty}(\gamma_{n+1}-\gamma_n)(n+1) = \sum_{n=0}^{\infty}(M-\gamma_n-(M-\gamma_{n+1}))(n+1) = \sum_{n=0}^{\infty} M-\gamma_n < \infty,$$

where the last equality is due to telescoping series. Therefore, we have

$$\left\|\frac{G(z)}{z-\beta}\right\|_{Y_\gamma}^2 \le \|G\|_{\mathcal{H}^2}^2 \sum_{n=0}^{\infty}(M-\gamma_n) < \infty.$$

With this, we have found a bound for $\left\|\frac{G(z)}{z-\beta}\right\|_{Y_\gamma}^2$ that is independent of $\beta$, so the result is proven.

Therefore we have shown

$$\sup_j \left\|\frac{G(z)}{1-\overline{\alpha_j}z}\right\|_{Y_\gamma}^2 < \infty.$$

□

We now have a sufficient condition to show (3.34). An interesting observation about this Lemma is the fact that it imposes a similar condition to the Blaschke condition on the sequence $\{\gamma_n\}$. That is, both

$$\sum_{j=0}^{\infty} M - \gamma_j < \infty \quad \text{and} \quad \sum_{j=1}^{\infty} 1 - |\alpha_j| < \infty$$

must hold true for our results.

With all of this, we can now prove Theorem 3.3.4.

**Proof of Theorem 3.3.4.**

**Theorem 3.3.4.** *Suppose that $\{\gamma_n\} \nearrow M$ is a bounded monotone increasing sequence satisfying $\gamma_0 = 0$, (3.4) and*

$$\sum_{n \geq 0} M - \gamma_n < \infty.$$

*For any function $F \in \mathcal{H}^2$ with zeros inside the unit disc labeled in increasing order of magnitude, $\alpha_j$ for $j \in J$, we have*

1.

$$\sum_{j \in J} (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 < \infty$$

.

2.

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j \in J} \left( (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 \right).$$

*Proof.* By hypothesis, $F \in X_\gamma$, and the roots $\alpha_j$, for $j \in J$ satisfy the Blaschke condition, (2.1). By Proposition 3.4.2, we see that for any finite $n$, we have the identity

$$\|F_n\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{n} \left( (1 - |\alpha_j|^2) \|H_{\alpha_j}\|_{Y_\gamma}^2 \right).$$

By Lemma 3.4.4 and by Lemma 3.4.6,

$$\sup_{j\in J}\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2 \leq \sup_{j\in J}\|H_{\alpha_j}\|_{Y_\gamma}^2 \leq \sup_{j\in J}\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}}\right\|_{Y_\gamma}^2 < \infty.$$

Further, since for every $j \in \mathbb{N}$, $|\alpha_j| < 1$,

$$\sum_{j=1}^{\infty} 1-|\alpha_j|^2 < 2\sum_{j=1}^{\infty} 1-|\alpha_j| < \infty.$$

With this, we have

$$\sum_{j=1}^{\infty}(1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2 < \infty.$$

Since the right hand summation converges, we obtain

$$\lim_{n\to\infty}\|F_n\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j=1}^{\infty}\left((1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2\right).$$

Since $\{\gamma_n\}$ is bounded, by Lemma 3.4.5, we may pass the limit through the left hand side and have the inequality

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j=1}^{\infty}\left((1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2\right),$$

which proves the result.

□

## 3.5 Applications of Main Results to Classical Spaces

We now look at applying the main results, proved in the previous section, to several important examples of weighted Hardy spaces. We begin by looking at Corollaries to our main results. From there, we provide a worked example demonstrating the improvement of our bounds versus that of the main result of [5].

### 3.5.1 Corollaries to Main Results

To begin, as an immediate application of Theorem 3.3.2, we obtain an alternate method of proving Corollary (3.10) found in [5].

**Corollary 3.5.1.** *Suppose that $\{\gamma_n\}$ is monotone increasing such that for any $n \geq 0$, $\gamma_{n+1} - \gamma_n \equiv C$, for some constant $C > 0$. For functions $F \in X_\gamma$ with a finite number of zeros inside the unit disc labeled $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, we have the identity*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 = \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j=1}^{m} \left( (1 - |\alpha_j|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2 \right). \tag{3.37}$$

*Proof.* Let $\{\gamma_n\} \equiv C$. Then both the convexity condition, (3.3), and the concavity condition, (3.3), hold true. Therefore, by Lemma 3.4.4, for all $1 \leq j \leq m$

$$\left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2 = \|H_{\alpha_j}\|_{Y_\gamma}^2 = \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2. \tag{3.38}$$

Therefore, by replacing each term in (3.29), we have the result. □

**Remark 3.5.1.** *For each $j = 1, 2, \ldots m$, by replacing*

$$\left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{Y_\gamma}^2$$

*with*

$$\left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{Y_\gamma}^2,$$

*we can obtain a similar identity to Equation* (3.37)*.*

Our second Corollary provides a new inequality on the Hardy-Sobolev norm of the function $G$ involving both the Dirichlet and Hardy norms of $G$. In this Corollary, we provide the bound for the space $W^{1,2}$, but note that the same techniques can be used to create bounds on the spaces $W^{s,2}$, where $s \in \mathbb{N}$.

**Corollary 3.5.2.** *Let $F \in W^{1,2}$ have Blaschke decomposition $F = B \cdot G$. Suppose $F$ has a finite number of roots in $\mathbb{D}$ labeled $\alpha_1, \alpha_2, \ldots, \alpha_m$. Then*

$$\|G\|_{W^{1,2}}^2 \leq \|F\|_{W^{1,2}}^2 - \sum_{j=1}^{m} (1 - |\alpha_j|^2) \left[ 2 \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{\mathcal{D}}^2 - \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{\mathcal{H}^2}^2 \right].$$

*Proof.* By Equation (2.9), given $F \in W^{1,2}$, we can represent

$$\|F\|_{W^{1,2}}^2 = \sum_{n=0}^{\infty} (1 + n^2)|a_n|^2 = \sum_{n=0}^{\infty} n^2 |a_n|^2 + \sum_{n=0}^{\infty} |a_n|^2.$$

By letting $\gamma_n = n^2$, we have the identity

$$\| \cdot \|_{W^{1,2}}^2 = \| \cdot \|_{X_\gamma}^2 + \| \cdot \|_{\mathcal{H}^2}^2. \tag{3.39}$$

Since $\gamma_{n+1} - \gamma_n = (n+1)^2 - n^2 = 2n + 1 = 2(n+1) - 1$, we have the identity

$$\| \cdot \|_{Y_\gamma}^2 = 2\| \cdot \|_{\mathcal{D}}^2 - \| \cdot \|_{\mathcal{H}^2}^2 \tag{3.40}$$

Therefore, by invoking Theorem 3.3.2 on $\gamma_n$, we have

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j=1}^{m} (1 - |\alpha_j|^2) \left[ 2 \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{\mathcal{D}}^2 - \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha_j} e^{i\cdot}} \right\|_{\mathcal{H}^2}^2 \right]. \tag{3.41}$$

Finally, by the fact that $\|G\|_{\mathcal{H}^2}^2 = \|F\|_{\mathcal{H}^2}^2$ and (3.39), we have the result. □

In studying the proof of this Corollary, we can see that similar techniques can also be used to create bounds for the spaces $W^{s,2}$ where $s \in \mathbb{N}$. By expanding the terms $(1 + n^2)^s$, and applying the same techniques, we can obtain corresponding results.

Next, we apply Theorem 3.3.4 to special choices of $\{\gamma_n\}$ to obtain a new result that holds for all $\mathcal{H}^2$ functions.

**Corollary 3.5.3.** *Let $\beta > 2$ be a real number and sequence, $\{\gamma_n\}$ be given, where*

$$\gamma_0 = 0 \quad \textit{and for all } n \geq 1, \quad \gamma_n = \sum_{j=1}^{n} \frac{1}{(j)^\beta}. \tag{3.42}$$

*Then for every function $F \in \mathcal{H}^2$ with decomposition $F = B \cdot G$ and roots $\alpha_j \in \mathbb{D}$ for $j \in J$, we have*

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j \in J} (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \alpha_j} \right\|_{\mathcal{A}^\beta}^2. \tag{3.43}$$

*Proof.* Since $\beta > 2$, the concavity condition, (3.4), clearly holds. By setting

$$M := \lim_{n \to \infty} \gamma_n = \lim_{n \to \infty} \sum_{j=1}^{n} \frac{1}{j^\beta},$$

we have $M < \infty$ and

$$\sum_{n=0}^{\infty} M - \gamma_n = \sum_{n=0}^{\infty} (\gamma_{n+1} - \gamma_n)(n+1) = \sum_{n=0}^{\infty} \left( \frac{1}{n+1} \right)^{\beta - 1} < \infty.$$

Where the first equality is shown in Lemma 3.4.6. Therefore, by Theorem 3.3.4, the result holds. $\square$

This result is important because the $X_\gamma$ space associated with the sequence $\{\gamma_n\}$ as in Equation (3.42) is equivalent to $\mathcal{H}^2$ (norm equivalence). Also, with this choice of weights we can connect $\beta$-weighted Bergman spaces, $\mathcal{A}^\beta$, defined in (2.12), to $\mathcal{H}^2$ spaces in a new way.

Within the proofs of Theorem 3.3.2, Theorem 3.3.3, and Theorem 3.3.4, we also obtain Equation (3.29):

$$\|G\|_{X_\gamma}^2 = \|F\|_{X_\gamma}^2 - \sum_{j=1}^{m} \left( (1 - |\alpha_j|^2) \left\| H_{\alpha_j} \right\|_{Y_\gamma}^2 \right).$$

This identity can be used to show how a Blaschke decomposition redistributes the magnitude of the Fourier coefficients of a function $F \in \mathcal{H}^2$. Simply put, if $F = B \cdot G$, with $m$ roots in $\mathbb{D}$ labeled $\alpha_1, \ldots \alpha_m$, where

$$F(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{and} \quad G(z) = \sum_{n=0}^{\infty} b_n z^n,$$

then for any integer $k > 0$, a result by Qian in [30] states

$$\sum_{n=k}^{\infty} |b_n|^2 \leq \sum_{n=k}^{\infty} |a_n|^2. \tag{3.44}$$

In our work, by selecting the sequence

$$\gamma_n = \begin{cases} 0 & n < k \\ 1 & n \geq k \end{cases} \tag{3.45}$$

and applying Equation (3.29) to such a choice of $\{\gamma_n\}$, we obtain the following identity, which is an improvement of the inequality (3.44).

$$\sum_{n=k}^{\infty} |b_n|^2 = \left(\sum_{n=k}^{\infty} |a_n|^2\right) - \frac{1}{(k-1)!} \sum_{j=1}^{m} \left( (1-|\alpha_j|^2) \left| \frac{d}{d^{k-1}} \left[ \frac{F(\cdot) \prod_{\ell=1}^{j-1}(1-\overline{\alpha_\ell}\cdot)}{\prod_{\ell=1}^{j}(\cdot - \alpha_\ell)} \right](0) \right|^2 \right). \tag{3.46}$$

To obtain this identity, we note that

$$H_{\alpha_j}(z) = \frac{F(z) \prod_{\ell=1}^{j-1}(1-\overline{\alpha_\ell}z)}{\prod_{\ell=1}^{j}(z-\alpha_\ell)}$$

and express the $(k-1)$st Fourier coefficient of this function using derivatives.

With this identity, we have a better understanding of the distribution of the magnitude of the Fourier coefficients of $F$ and $G$. This also implies that finite Fourier approximations of $G$ will have less error than the corresponding finite Fourier approximation of $F$, as long as $F$ has a root in $\mathbb{D}$.

### 3.5.2 Worked Examples

To end this chapter, we provide worked examples to demonstrate the improvements that our bounds have in several commonly studied weighted Hardy spaces.

For our examples, we test the function

$$\begin{aligned} F(z) &= \frac{3i}{8}z - \left(\frac{i}{2}+\frac{3}{4}\right)z^2 + z^3 && (3.47)\\ &= z\left(z-\frac{i}{2}\right)\left(z-\frac{3}{4}\right) && (3.48)\\ &= \left(z\cdot\frac{z-\frac{i}{2}}{1+\frac{i}{2}z}\cdot\frac{z-\frac{3}{4}}{1-\frac{3}{4}z}\right)\cdot\left((1+\frac{i}{2}z)(1-\frac{3}{4}z)\right). && (3.49)\\ & && (3.50) \end{aligned}$$

and the two sequences:

$$\gamma_n = n^2, \quad \text{and} \quad \gamma_n = \sum_{j=1}^{n}\frac{1}{j},$$

which are examples of a convex and concave sequence, respectively.

**Example 3.5.1.** *Using Theorem 3.3.2, and the assumption $\gamma_n = n^2$, Find an upper bound on $\|G\|^2_{X_\gamma}$ in the factorization $F = B\cdot G$, where $F$ is given in Equation (3.47).*

***Solution:*** *Since $\gamma_n = n^2$, we may invoke Theorem 3.3.2,*

$$\|G(e^{i\cdot})\|^2_{X_\gamma} \leq \|F(e^{i\cdot})\|^2_{X_\gamma} - \sum_{j=1}^{m}\left((1-|\alpha_j|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}}\right\|^2_{Y_\gamma}\right).$$

*Firstly,*

$$\|F(e^{i\cdot})\|^2_{X_\gamma} = \sum_{n=0}^{3} n^2|a_n|^2 = 0^2|0|^2 + 1^2\left|\frac{3i}{8}\right|^2 + 2^2\left|\frac{i}{2}+34\right|^2 + 3^2|1|^2 = 12.39.$$

*From here, $F$ has 3 roots in $\mathbb{D}$, namely at $0, \frac{i}{2}$, and $\frac{3}{4}$. For each root, we compute*

$$(1-|\alpha_j|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}}\right\|^2_{Y_\gamma}.$$

- *For* $\alpha = 0$ :

$$(1-|0|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{0}e^{i\cdot}}\right\|_{Y_\gamma}^2 = \left\|\left((1+\frac{i}{2}z)(1-\frac{3}{4}z)\right)\right\|_{Y_\gamma}^2 =$$

$$= (1^2-0^2)|1|^2 + (2^2-1^2)\left|\frac{i}{2}+\frac{3}{4}\right|^2 + (3^2-2^2)\left|\frac{3i}{8}\right|^2 = 4.140625.$$

- *For* $\alpha = \frac{i}{2}$ :

$$\left(1-\left|\frac{i}{2}\right|^2\right)\left\|\frac{G(e^{i\cdot})}{1+\frac{i}{2}e^{i\cdot}}\right\|_{Y_\gamma}^2 = \frac{3}{4}\left\|\left(1-\frac{3}{4}z\right)\right\|_{Y_\gamma}^2 =$$

$$= \frac{3}{4}\left((1^2-0^2)|1|^2 + (2^2-1^2)\left|\frac{3}{4}\right|^2\right) = 2.015625.$$

- *For* $\alpha = \frac{3}{4}$ :

$$\left(1-\left|\frac{3}{4}\right|^2\right)\left\|\frac{G(e^{i\cdot})}{1-\frac{3}{4}e^{i\cdot}}\right\|_{Y_\gamma}^2 = \frac{7}{16}\left\|\left(1+\frac{i}{2}z\right)\right\|_{Y_\gamma}^2 =$$

$$= \frac{7}{16}\left((1^2-0^2)|1|^2 + (2^2-1^2)\left|\frac{i}{2}\right|^2\right) = 0.765625.$$

*Therefore, we have the bound*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq 12.39 - 4.140625 - 2.015625 - 0.765625 = 5.468125. \tag{3.51}$$

In this example, we notice a few important details. Firstly, we notice that the smaller the magnitude of the root, the larger the term

$$(1-|\alpha_j|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha_j}e^{i\cdot}}\right\|_{Y_\gamma}^2$$

will be. Therefore, since the main result of [5], Theorem 3.3.1, only takes into account a single root of $F$, the most accurate bound it can provide is

$$\|G\|_{X_\gamma}^2 \leq 12.39 - 4.140625 = 8.249375.$$

When comparing this bound to our bound in Equation 3.51, we can see the advantage of our result. This is because we consider all roots of $F$ in $\mathbb{D}$, each of which further tightens the bound on $G$.

On the other hand, by direct computation, we also have that in this example, where $\gamma_n = n^2$,

$$\|G\|_{X_\gamma}^2 = 1.375.$$

This implies that while the bound provided in Theorem 3.3.2 is more accurate than that of the bound provided in [5], it will still not be optimal.

We now move on to our next example, where we study the same function in a different weighted space.

**Example 3.5.2.** *Using Theorem 3.3.3, and the assumption $\gamma_0 = 0, \gamma_n = \sum_{j=1}^{n}\frac{1}{j}$, Find an upper bound on $\|G\|_{X_\gamma}^2$ in the factorization $F = B\cdot G$, where $F$ is given in Equation* (3.47)*.*

***Solution:*** *Since $\gamma_n = \sum_{j=1}^{n}\frac{1}{j}$, we may invoke Theorem 3.3.3, so*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq \|F(e^{i\cdot})\|_{X_\gamma}^2 - \sum_{j=1}^{m}\left((1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|_{Y_\gamma}^2\right).$$

*Similar to the previous example, we begin by finding*

$$\|F(e^{i\cdot})\|^2_{X_\gamma} = \sum_{n=0}^{3} \gamma_n |a_n|^2 = (1)\left|\frac{3i}{8}\right|^2 + (1+\frac{1}{2})\left|\frac{i}{2}+\frac{3}{4}\right|^2 + (1+\frac{1}{2}+\frac{1}{3})|1|^2 =$$

$$= \frac{613}{192} \approx 3.1927.$$

*From here, $F$ has 3 roots in $\mathbb{D}$, namely at $0, \frac{i}{2}$, and $\frac{3}{4}$. For each root, we compute*

$$(1-|\alpha_j|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\alpha_j}\right\|^2_{Y_\gamma}.$$

- *For $\alpha = 0$ :*

$$(1-|0|^2)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-0}\right\|^2_{Y_\gamma} = \left\|\frac{3i}{8} - (\frac{i}{2}+\frac{3}{4})z + z^2\right\|^2_{Y_\gamma} =$$

$$= (1)\left|\frac{3i}{8}\right|^2 + \frac{1}{2}\left|\frac{i}{2}+\frac{3}{4}\right|^2 + \frac{1}{3}|1|^2 = \frac{169}{192} \approx .8802.$$

- *For $\alpha = \frac{i}{2}$ :*

$$\left(1-\left|\frac{i}{2}\right|^2\right)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\frac{i}{2}}\right\|^2_{Y_\gamma} = \frac{3}{4}\left\|z^2 - \frac{3}{4}z\right\|^2_{Y_\gamma} =$$

$$= \frac{3}{4}\left(1|0|^2 + \frac{1}{2}\left|\frac{3}{4}\right|^2 + \frac{1}{3}|1|^2\right) = \frac{177}{384} \approx .4609$$

- *For $\alpha = \frac{3}{4}$ :*

$$\left(1-\left|\frac{3}{4}\right|^2\right)\left\|\frac{F(e^{i\cdot})}{e^{i\cdot}-\frac{3}{4}}\right\|^2_{Y_\gamma} = \frac{7}{16}\left\|z^2 - \frac{i}{2}z\right\|^2_{Y_\gamma} =$$

$$= \frac{7}{16}\left(1|0|^2 + \frac{1}{2}\left|\frac{i}{2}\right| + \frac{1}{3}|1|^2\right) = \frac{77}{384} \approx .2005.$$

*Therefore, we have the bound*

$$\|G(e^{i\cdot})\|_{X_\gamma}^2 \leq \frac{613}{192} - \frac{169}{192} - \frac{177}{384} - \frac{77}{384} = \frac{634}{384} \approx 1.651.$$

Again, the root at 0 will have the largest effect on this inequality. Using this information, we can see that the most accurate bound Theorem 3.3.1 can provide is

$$\|G\|_{X_\gamma}^2 \leq \frac{613}{192} - \frac{279}{192} = \frac{334}{192} \approx 1.7396.$$

Through direct computation, we get that for this example,

$$\|G\|_{X_\gamma}^2 = \frac{131}{128} \approx 1.023,$$

which again validates that our bound is tighter than that of [5].

In looking at these examples, we can see that the bounds provided in Theorem 3.3.2 and Theorem 3.3.3 are dependent on the relationship between the $X_\gamma$ and $Y_\gamma$ spaces. When $\gamma_n = n$, we obtain an identity for the $X_\gamma$ norm of $G$. As the growth rate becomes either more convex or more concave, we expect that the inequalities will become less sharp. The reasoning for this comes from the proof of Lemma 3.4.4. We further investigate this topic in Chapter 5.

# Chapter 4

# Comparing the Unwinding Series and the Fourier Series

In the previous chapter, we investigated a single Blaschke decomposition. More specifically, the relationship between the weighted Hardy norms of functions $F$ and $G$, defined in Equation (2.7). We showed that the weighted Hardy norm of $G$ is always bounded above by the weighted Hardy norm of $F$, and the bound involves the roots of $F$ that sit inside the unit disc, $\mathbb{D}$. In this chapter, we perform a formal iteration of Blaschke decompositions and investigate the convergence rate of the corresponding unwinding series, expressed at step $n$ as in Equation (2.16) as

$$F = G_0(0)B_0 + G_1(0)B_0B_1 + \cdots + G_n(0)\prod_{j=1}^{n} B_j + \prod_{j=1}^{n} B_j(G_n(z) - G_n(0)).$$

It was proven by Qian et al. [12] and again by Coifman and Steinerberger [5] that the unwinding series converges for $F \in \mathcal{H}^2$. In fact, the proof of convergence for $\mathcal{H}^2$ functions shows that the unwinding series will converge at least as fast as the corresponding Fourier series. Even with this result, finding the exact convergence rate of the unwinding series is a difficult problem. After proving the convergence of the unwinding series, the authors Coifman and Steinerberger wrote:

> *The proof shows that convergence will happen at least as quickly as Fourier series but potentially much faster* [...]. *It would be interesting to quantifying how precisely this happens.*[5]

Many numerical examples in the PhD thesis of Nahon [26] showed that the unwinding series of $F \in \mathcal{H}^2$ will converge to $F$ at an exponential rate. With this in mind, our main goal of this chapter is to develop tools necessary to address this question:

**Are there functions, $F \in \mathcal{H}^2$, for which the unwinding series converges to $F$ non-exponentially?**

By finding an example of a family of functions for which the unwinding series and Fourier series are term-wise equal, and showing that the latter has non-exponential convergence, we answer this question in the affirmative. Our family comes from the collection of the polylogarithm functions, denoted $Li_s$, and defined as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \tag{4.1}$$

where $s \in \mathbb{C}$. A subset of these functions, namely $s \in \mathbb{R}, s > \frac{1}{2}$, will be in $\mathcal{H}^2$, have Fourier series that is term-wise equal to the unwinding series, and have Fourier series that converges non-exponentially. In particular, we show that the function

$$Li_1(z) = -\log(1 - z)$$

has a Fourier series that converges non-exponentially.

To arrive at this result, in this chapter we investigate the similarities and differences between the unwinding series and Fourier series. We begin in Section 4.1 by considering the situation when $F$ is a Blaschke product; the unwinding series of $F$ can be expressed using a single term and the Fourier series consists of infinitely many terms, implying that the unwinding series is much more efficient than the Fourier series.

From there, in Section 4.2 we look into the unwinding series of polynomial functions. In this case, both the unwinding series and Fourier series consist of finitely many terms. By viewing Blaschke decompositions in a similar way to the previous chapter, we derive a formula to compute the $\mathcal{H}^2$ error between polynomials and their penultimate partial unwinding series. We provide a necessary and sufficient condition for this particular partial unwinding series to have the same error as the corresponding Fourier series, and this leads us to the notion of term-wise equality between the two series.

To end the chapter, in Section 4.3 we look for functions for which the unwinding series will be term-wise equal to the Fourier series. This brings us to two families of functions: those with exponentially decaying Fourier coefficients, and the polylogarithm functions (4.1). The former will have unwinding series that converges exponentially, which does not help us answer our main question for this chapter. The unwinding series of the latter, however, will not converge exponentially and therefore answers this chapter's main question in the affirmative.

## 4.1 Unwinding vs. Fourier: Blaschke Products

We begin our comparison of the unwinding series and the Fourier series by looking at Blaschke products. We will see that while nontrivial Blaschke products ($B(z) \neq z^n$, where $n \in \mathbb{N} \cup \{0\}$) have infinitely many nonzero Fourier coefficients, the unwinding series can trivially be expressed with a single term.

Suppose we have a Blaschke factor

$$B(z) = \frac{z - \alpha}{1 - \overline{\alpha} z},$$

where $0 < |\alpha| < 1$. When we look at the Fourier coefficients of $B$, we see that by geometric series,

$$B(z) = (z - \alpha) \sum_{n=0}^{\infty} (\overline{\alpha} z)^n = \sum_{n=0}^{\infty} \overline{\alpha}^n z^{n+1} - \alpha \overline{\alpha}^n z^n =$$

$$= -\alpha + (1 - |\alpha|) \sum_{n=1}^{\infty} \overline{\alpha}^{n-1} z^n.$$

Clearly this Blaschke factor will have an infinite number of nonzero Fourier coefficients, so the unwinding series is much more efficient in representing these functions than the Fourier series.

When we move to finite and infinite Blaschke products, there will still be an infinite number of nonzero Fourier coefficients. In this case, the goal of expressing the product as a Fourier series becomes much more challenging. An article by Kim [18] looked into finding the derivatives of Blaschke products, and studied whether or not Fourier coefficients could represent a finite or infinite Blaschke product. However, we are unaware of any explicit formula to find the Fourier coefficients of a Blaschke product with $N$ roots in $\mathbb{D} \setminus \{0\}$. While the explicit Fourier coefficients of a Blaschke product, $a_n$, are unknown, we will always have the identity

$$\sum_{n=0}^{\infty} |a_n|^2 = 1.$$

This is due to the characterization $\|B\|_{\mathcal{H}^2} = 1$.

Due to the difficult nature of finding the Fourier coefficients of both finite and infinite Blaschke products, we can see why the unwinding series can be an efficient method of expressing functions. From here, we investigate whether or not the unwinding series will always be more efficient than the Fourier series for polynomials.

## 4.2 Unwinding vs. Fourier: Polynomials

To begin our study of polynomial unwinding, we look at an example of producing the unwinding series for a given polynomial to get a concrete sense of how the series is produced.

**Example 4.2.1.** *Write the unwinding series for*

$$F(z) = z(z - \frac{1}{2})(z + 3) = -\frac{3}{2}z + \frac{5}{2}z^2 + z^3.$$

***Solution:*** *Since $F$ has two roots in $\mathbb{D}$, namely at $z = 0$ and $z = \frac{1}{2}$,*

$$F(z) = \left(z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) \cdot \left((1 - \frac{1}{2}z)(z + 3)\right) = B_0(z) \cdot G_0(z).$$

*$G_0(0) = 3$, so the first partial unwinding series can be expressed*

$$F(z) = B_0(z)(G_0(z) - G_0(0) + G_0(0)) = G_0(0)B_0(z) + B_0(z)(G_0(z) - G_0(0)) =$$

$$= 3\left(z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) + \left(z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right)\left(\frac{1}{2}z - \frac{1}{2}z^2\right).$$

*From here, we define*

$$G_0(z) - G_0(0) = B_1(z) \cdot G_1(z),$$

*and see*

$$G_0(z) - G_0(0) = \frac{1}{2}z - \frac{1}{2}z^2 = z \cdot \left(\frac{1}{2} - \frac{1}{2}z\right) = B_1(z) \cdot G_1(z),$$

*and* $G_1(0) = \frac{1}{2}$. *Therefore, the second partial unwinding series can be expressed*

$$F(z) = G_0(0)B_0(0) + G_1(0)B_0(z)B_1(z) + B_0(z)B_1(z)(G_1(z) - G_1(0)) =$$

$$= 3\left(z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) + \frac{1}{2}\left(z^2\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) + \left(z^2\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right)\left(\frac{1}{2}z\right).$$

*At this point, we have the full unwinding series. Since* $G_1(z) - G_1(0) = \frac{1}{2}z$, *we can define*

$$B_2(z) = z \quad \textit{and} \quad G_2(z) = \frac{1}{2},$$

*and write* $F$ *as*

$$F(z) = 3\left(z\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) + \frac{1}{2}\left(z^2\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right) + \frac{1}{2}\left(z^3\frac{z-\frac{1}{2}}{1-\frac{1}{2}z}\right).$$

This simple example helps to illustrate a few previously known (and further discussed in [5]) points about the unwinding series for polynomials. Firstly, the unwinding series for a degree $m$ polynomial will always converge using at most $m$ terms. Essentially, since each polynomial $G_n(z) - G_n(0)$ has a root at $z = 0$, the function $B_{n+1}$ must contain a $z^{s_{n+1}}$ term were $s_{n+1} \geq 1$. This means that

$$deg(G_{n+1}) = deg(G_n) - s_{n+1}.$$

Since the degree of $G_0$ is at most $m$, we have convergence in at most $m$ terms. Further, the penultimate term, indexed $k-1 < m$, in the unwinding series will always satisfy

$$G_{k-1}(z) - G_{k-1}(0) = bz^{s_k},$$

where $s_k \geq 1$. That is, the unwinding series will either terminate when $G_{k-1}(z) - G_{k-1}(0)$ is a degree one polynomial, or when the only roots of $G_{k-1}(z) - G_{k-1}(0)$ are at $z = 0$.

Looking back to the previous example, we have $k = 2$, and $G_1(z) - G_1(0) = \frac{1}{2}z$. As was the case in this example, the value of $k$ may not equal the degree of the polynomial, $p$. With a smaller value of $k$, we require fewer terms in the unwinding series to represent $p$.

We now investigate the coefficient, $b$, that shows up in the final term of the unwinding series.

**Theorem 4.2.1.** *Suppose that $p$ is a monic degree $m$ polynomial and we are given the integers $k \leq m$ and $s_k > 0$, along with the complex number $b$ such that the unwinding series of $p$ is given by*

$$p(z) = G_0(0)B_0(z) + G_1(0)\prod_{j=0}^{1} B_j(z) + \cdots + G_{k-1}(0)\prod_{j=0}^{k-1} B_j(z) + \prod_{j=0}^{k-1} B_j(z)\left(bz^{s_k}\right). \quad (4.2)$$

*If we define the set*

$$A_j = \{\alpha \in \mathbb{D} \setminus \{0\} : G_j(\alpha) - G_j(0) = 0\}, \quad (4.3)$$

*then*

$$b = \prod_{j=0}^{k-1} \prod_{\alpha_j \in A_j} -\overline{\alpha}_j. \tag{4.4}$$

*Proof.* Since $p$ is a monic degree $m$ polynomial, we can express

$$p(z) = z^{s_0}(z-\alpha_1)(z-\alpha_2)\dots(z-\alpha_{t_0})(z-\beta_1)(z-\beta_2)\dots(z-\beta_{m-s_0-t_0}),$$

where $\alpha_j \in \mathbb{D}\setminus\{0\}$ and $\beta_j \notin \mathbb{D}$. In this factorization, $A_0 = \{\alpha_j : j \in \{1, 2, \dots t_0\}\}$.

Therefore, we can express the Blaschke decomposition of $p$ as

$$p(z) = z^{s_0}\left(\prod_{j=1}^{t_0} \frac{\alpha_j - z}{1-\overline{\alpha}_j z}\right)\cdot\left(\prod_{j=1}^{t_0}(1-\overline{\alpha}_j z)\cdot\prod_{j=1}^{m-s_0-t_0}(z-\beta_j)\right).$$

From this, we have

$$G_0(z) = \prod_{j=1}^{t_0}(1-\overline{\alpha}_j z)\cdot\prod_{j=1}^{m-s_0-t_0}(z-\beta_j),$$

which is a degree $m_0 := m - s_0$ polynomial with lead coefficient

$$b_0 := \prod_{j=1}^{t_0} -\alpha_j.$$

The function $\frac{G_0 - G_0(0)}{b_0}$ will also be a monic degree $m_0$ polynomial, so decomposing this function and denoting

$$A_1 = \{\alpha_j \in \mathbb{D}\setminus\{0\} : G_0(\alpha_j) - G_0(0) = 0\}, \quad \text{where } |A_1| = t_1,$$

we have

$$\frac{G_0(z) - G_0(0)}{b_0} = z^{s_1} \left( \prod_{\alpha_j \in A_1} \frac{\alpha_j - z}{1 - \overline{\alpha}_j z} \right) \cdot \left( \prod_{\alpha_j \in A_1} (1 - \overline{\alpha}_j z) \cdot \prod_{j=1}^{m_0 - s_1 - t_1} (z - \beta_j) \right). \qquad (4.5)$$

Note: $s_1 \geq 1$ since we are introducing a root at the origin. Thus

$$\frac{1}{b_0} G_1(z) = \prod_{\alpha_j \in A_1} (1 - \overline{\alpha}_j z) \cdot \prod_{j=1}^{m_0 - s_1 - |A_1|} (z - \beta_j),$$

which is a degree $m_1 := m_0 - s_1$ polynomial with lead coefficient

$$b_1 := \prod_{\alpha \in A_1} -\alpha_j.$$

Similarly, the polynomial

$$\frac{G_1(z) - G_1(0)}{b_0 b_1}$$

is monic, so we can factor this function and obtain the function $\frac{1}{b_0 b_1} G_2$, which can be expressed similarly to (4.5) with subscript 2.

As long as $m_\ell > 0$, the same iterative process can be used to obtain formulas for $G_2, G_3, \ldots G_\ell$ that are similar to (4.5), and at some step $k - 1 < m$, we obtain

$$\frac{G_{k-1}(z) - G_{k-1}(0)}{\prod_{j=1}^{k-1} b_j} = z^{s_k}$$

where for each $n \leq k - 1$,

$$b_n = \prod_{\alpha_j \in A_n} -\overline{\alpha}_j.$$

Therefore, we have the result. $\square$

Essentially, this result allows us to describe the $\mathcal{H}^2$ error between $p$ and its second to last partial unwinding series. It also allows us to compare this error to the corresponding error between $p$ and its partial Fourier series. That is, given the unwinding series as in Equation we have

$$\left\| p(z) - (G_0(0)B_0(z) + \cdots + G_k(0)\prod_{j=0}^{k} B_j(z)) \right\|_{\mathcal{H}^2} = |b|.$$

On the other hand, if we compare this to the Fourier series the $(m-1)$st Fourier approximation to $p$ will have error

$$\|p - (a_0 + a_1 z + \cdots + a_{m-1}z^{m-1})\|_{\mathcal{H}^2} = 1.$$

This implies that as long as $|b| < 1$, the final partial unwinding series of monic polynomials will have less error than the partial Fourier series.

We now use the proof of Theorem 4.2.1 to investigate a necessary and sufficient on the polynomial $p$ as in (4.2) so that $|b| = 1$. This gives us the following result.

**Corollary 4.2.1.** *Let $p$ be a monic degree $m$ polynomial, expressed*

$$p(z) = a_0 + a_1 z + \cdots + a_{m-1}z^{m-1} + z^m,$$

*with unwinding series*

$$p(z) = G_0(0)B_0(z) + \cdots + G_{k-1}(0)\prod_{j=0}^{k-1} B_j(z) + \prod_{j=0}^{k-1} B_j(z)(bz^s).$$

*Then* $|b| = 1$ *if and only if* $b = 1$ *and for each* $0 \leq j \leq k-1$*,*

$$B_j(z) = z^{n_j},$$

*where* $n_0 \geq 0$ *and* $n_j \geq 1$ *are integers.*

*Proof.* By Theorem 4.2.1, we know that

$$b = \prod_{j=0}^{k-1} \prod_{\alpha_j \in A_j} -\overline{\alpha}_j,$$

where

$$A_j = \{\alpha \in \mathbb{D} \setminus \{0\} : G_j(\alpha) - G_j(0) = 0\}.$$

Therefore, $|b| = 1$ if and only if for each $0 \leq j \leq k-1$, $A_j = \emptyset$ implying $b = 1$. The sets $A_j$ will be empty if and only if the only roots of the polynomial $p$, and the polynomials

$$G_j(z) - G_j(0),$$

where $0 \leq j \leq k-2$, are at $z = 0$. This is equivalent to the condition for each $0 \leq j \leq k-1$,

$$B_j(z) = z^{n_j},$$

where $n_0 \geq 0$ and $n_j \geq 1$ are integers. □

By our corollary, if **none** of the functions $G_n(z) - G_n(0)$ have nonzero roots in $\mathbb{D}$, then $b = 1$. In this case, each function $B_n(z)$ will be of the form

$$B_n(z) = z^{s_n},$$

and we will essentially recover the Fourier series. This is the motivation for our next section.

## 4.3 Unwinding vs. Fourier: Term-wise Equality

In this section, we study classes of functions in $\mathcal{H}^2$ for which the unwinding series will be term-wise equal to the Fourier series. As we have already seen, there is a large class of functions for which this will not be the case. For polynomials, we saw in Corollary 4.2.1 that the final partial unwinding series will have error that is strictly less than that of the corresponding Fourier series, as long as one of the functions $G_n$, as in Equation (2.16), has a nonzero root in $\mathbb{D}$. We therefore investigate conditions on $F$ to ensure that every function $G_n$ in the unwinding series will only have roots at the origin. We then look for functions, $F$, for which the Fourier series will be term-wise equal to the unwinding series, under the assumption that the Fourier series of $F$ does not converge exponentially. This will allow us to answer the main question of this chapter.

To begin, we must formally define what it means for the Fourier series of $F$ to be term-wise equal to the unwinding series. This idea comes from the following iterative process that will produce the Fourier series of $F$, described in [5]:

Given $F \in \mathcal{H}^2$, let $F_0 = F$ and we iteratively define

$$z^{s_n} F_{n+1}(z) := F_n(z) - F_n(0), \tag{4.6}$$

where $s_n \geq 1$. Then we have for any $n \geq 0$,

$$F(z) = F_0(0) + z^{s_1} F_1(0) + \cdots + z^{s_1+\cdots+s_n} F_n(0) + z^{s_1+\cdots+s_n}(F_n(z) - F_n(0)). \tag{4.7}$$

In this iterative method, if we express $F$ as

$$F(z) = \sum_{k=0}^{\infty} a_k z^k,$$

then $F_0(0) = F(0)$, and for each $n$, $F_n(0)$ is the $n$th nonzero Fourier coefficient of $F$. Simply put, this method expresses the Fourier series of $F$ by reindexing the nonzero Fourier coefficients. In the case where $a_n \neq 0$ for all $n$, then we have $F_n(0) = a_n$.

By removing the Fourier coefficients of magnitude 0, we can now compare the Fourier series and unwinding series in a more standardized way.

**Definition 4.3.1.** *Given a function $F \in \mathcal{H}^2$ with unwinding series as in Equation* (2.16) *and Fourier series as in Equation* (4.7)*, the two series are term-wise equal if for every $n \geq 0$, and for all $z \in \mathbb{D}$,*

1. $G_n(z) = F_n(z)$
2. $B_n(z) = z^{n_s}$

### 4.3.1 Term-wise Equality and Exponential Convergence

With Definition 4.3.1, we now review an existing result by Coifman et al. in [6] on the term-wise equality of the Fourier and unwinding series.

**Theorem 4.3.1** (Proposition 3.4 of [6]). *Let $0 \leq n_0 < n_1 < n_2 < \dots$ be a strictly increasing sequence of integers and*

$$f(z) = \sum_{k=0}^{\infty} a_k z^{n_k} \quad \textit{where, for all } n \quad |a_n| > \sum_{k=n+1}^{\infty} |a_k|.$$

*Then the $N$-th term of the unwinding series is given by*

$$f(0) + a_1 B_1 + a_2 B_1 B_2 + \cdots + a_N B_1 \cdots B_n = \sum_{k=0}^{N} a_k z^{n_k}.$$

In this result, the assumptions imply that the nonzero Fourier coefficients, $a_k$, of the function, $F$, decay at a rate faster than that of $\frac{1}{2^k}$. The proof of this follows by an inductive application of Rouché's Theorem, where at each step, we can argue that the terms in the unwinding series

$$G_n(z) - G_n(0)$$

will only have roots at the point $z = 0$.

Unfortunately, whenever we make the assumption that the Fourier coefficients of a function decay exponentially, the Fourier series, and hence the unwinding series, will converge at an exponential rate.

Further, this result does not encompass all functions for which the unwinding series and Fourier series are term-wise equal, which we can see in the next example.

**Example 4.3.1.** *Show that the unwinding series of $F(z) = \frac{1}{1-\frac{z}{2}}$ is term-wise equal to its Fourier series.*

***Solution:*** *By geometric series that for any* $|z| < 2$

$$F(z) = \frac{1}{1 - \frac{z}{2}} = \sum_{j=0}^{\infty} \frac{z^j}{2^j}.$$

*Clearly* $F$ *has no roots in* $\mathbb{D}$*, So if we consider*

$$F(z) - F(0) = \sum_{j=1}^{\infty} \frac{z^j}{2^j} = \frac{z}{2} \sum_{j=0}^{\infty} \frac{z^j}{2^j} = \frac{z}{2} F(z).$$

*With this,* $B_1(z) = z$ *and* $G_1(z) = \frac{F(z)}{2}$*, so* $G_1(0) = \frac{F(0)}{2} = \frac{1}{2}$*. Since* $G_1(z) - G_1(0) = \frac{1}{2}(F(z) - F(0))$*, we can see* $B_2(z) = z$ *and* $G_2(z) = \frac{1}{2^2} F(z)$*. By the same argument, we can see that for any* $n$*,*

$$G_n(z) = \frac{F(z)}{2^n}, \quad B_n(z) = z.$$

*Therefore the unwinding series will be term-wise equal to the Fourier series.*

In this example, the hypotheses of Theorem 4.3.1 are not satisfied and yet the conclusion still holds. Again, the convergence rate of the Fourier coefficients in this example decay exponentially, which implies that the unwinding series will converge at an exponential rate.

### 4.3.2 Term-wise Equality and Non-exponential Convergence

From here, we ask the question of whether or not there exists nonpolynomial functions, $F \in \mathcal{H}^2$, that have Fourier series that do not converge exponentially and are term-wise equal to the unwinding series.

We begin by noting that if such a function, $F$, exists, it will not be in the class of functions that are analytic in $\mathbb{D}_{1+\epsilon}$ for some $\epsilon > 0$. This is because for every $|z| \leq 1 + C\epsilon$, where $0 < C < 1$, we can express

$$F(z) = \sum_{n=0}^{\infty} a_n z^n,$$

and this summation converges absolutely. That implies that

$$\sum_{n=0}^{\infty} |a_n|(1 + C\epsilon)^n < \infty,$$

which indicates that the Fourier coefficients, $a_n$, must decay at a rate faster than $(1+C\epsilon)^{-n}$. This immediately implies that there is exponential convergence.

With this in mind, we investigate functions $F \in \mathcal{H}^2 \setminus \mathcal{O}(\mathbb{D}_{1+\epsilon})$, and have the following result.

**Proposition 4.3.1.** *There exist functions $F \in \mathcal{H}^2$ such that*

1. *The Fourier series of $F$ converges to $F$ non-exponentially in $\mathcal{H}^2$.*

2. *The unwinding series is term-wise equal to the Fourier series.*

*Proof.* Consider the function $F(z) = -\log(1 - z)$, where log is chosen to be the principal branch of the logarithm function. The Fourier series of $F$ is expressed for $z \in \mathbb{D}$ as

$$F(z) = \sum_{n=1}^{\infty} \frac{z^n}{n}.$$

We point out that $F \in \mathcal{H}^2$ since

$$\|F\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty.$$

We first show that this Fourier series will converge to $F$ non exponentially (sub linearly) in $\mathcal{H}^2$, using the criteria (2.15). To begin,

$$\lim_{k\to\infty} \frac{\|F_{k+1} - F\|_{\mathcal{H}^2}}{\|F_k - F\|_{\mathcal{H}^2}} = \sqrt{\lim_{k\to\infty} \frac{\sum_{n=k+2}^{\infty} \frac{1}{n^2}}{\sum_{n=k+1}^{\infty} \frac{1}{n^2}}} =$$

$$= \sqrt{\lim_{k\to\infty} 1 - \frac{1}{\sum_{n=k+1}^{\infty} \left(\frac{k+1}{n}\right)^2}}.$$

For fixed $k > 0$,

$$\sum_{n=k+1}^{\infty} \left(\frac{k+1}{n}\right)^2 > \frac{k}{4},$$

so

$$\sqrt{\lim_{k\to\infty} 1 - \frac{1}{\sum_{n=k+1}^{\infty} \left(\frac{k+1}{n}\right)^2}} = 1.$$

Therefore,

$$\lim_{k\to\infty} \frac{\|F_{k+1} - F\|_{\mathcal{H}^2}}{\|F_k - F\|_{\mathcal{H}^2}} = 1,$$

which shows the convergence rate is non exponential.

With this, we now prove that the unwinding series of $F$ will be term-wise equal to its Fourier series.

To begin, we notice that the function $F(z) = -\log(1-z)$ will only have one root in $\mathbb{D}$, namely at 0. This comes from the identity

$$\log(z) = \ln|z| + i\text{Arg}(z),$$

where $Arg(z) \in [0, 2\pi)$ is the principle argument of $z$. This means that

$$-\log(1-z) = z \cdot \sum_{n=0}^{\infty} \frac{z^n}{n+1} = B_1(z) \cdot G_1(z),$$

and that $G_1(0) = 1$, the first Fourier coefficient of $F$. Therefore the first term of the unwinding series will equal the first term of the Fourier series.

From here, we proceed by strong induction. Suppose that the first $k$ terms of the unwinding series of $F$ are equal to the Fourier terms as in Definition 4.3.1. In other words, $G_n(0) = \frac{1}{n}$ and $B_n(z) = z \ \forall n \leq k$. Then the $k$th step in the unwinding series expansion of $F$ is

$$F(z) = \sum_{n=1}^{k} \left(\frac{z^n}{n}\right) + z^k(G_k(z) - G_k(0)),$$

where

$$G_k(z) - G_k(0) = \sum_{j=1}^{\infty} \frac{z^j}{j+k} = z \cdot \sum_{j=0}^{\infty} \frac{z^j}{j+k+1}.$$

From here, if we can show that

$$\sum_{j=0}^{\infty} \frac{z^j}{j+k+1} \neq 0 \quad \forall z \in \mathbb{D}, \tag{4.8}$$

then we will have

$$G_k(z) - G_k(0) = zG_{k+1}(z),$$

implying that the $k+1$ term of the unwinding series will be the same as the Fourier series.

We now prove (4.8). Suppose for the sake of contradiction that

$$\sum_{j=0}^{\infty} \frac{z_0^j}{j+k+1} = 0$$

for some $z_0 \in \mathbb{D}$. Then we can find an $r < 1$ such that $|z_0| < r$.

Next, consider the function

$$H(z) = (z-1)\sum_{j=0}^{\infty} \frac{z^j}{j+k+1} = \sum_{j=0}^{\infty} \frac{z^{j+1}}{j+k+1} - \frac{z^j}{j+k+1} =$$

$$= \frac{-1}{k+1} + \sum_{j=1}^{\infty} \frac{z^j}{k+j} - \frac{z^j}{k+j+1} = \frac{-1}{k+1} + \sum_{j=1}^{\infty} \frac{z^j}{(k+j)(k+j+1)}.$$

If $|z| \leq r < 1$, then

$$|\sum_{j=1}^{\infty} \frac{z^j}{(k+j)(k+j+1)}| < \sum_{j=1}^{\infty} \frac{1}{(k+j)(k+j+1)} = \frac{1}{k+1}.$$

Therefore, since

$$|\frac{-1}{k+1}| > |\sum_{j=1}^{\infty} \frac{z^j}{(k+j)(k+j+1)}| \quad \text{for } |z| = r,$$

By Rouché's Theorem, $H$ will have no roots in $\mathbb{D}_r$. Since $H$ has the same roots as

$$\sum_{j=0}^{\infty} \frac{z^j}{j+k+1},$$

inside $\mathbb{D}_r$, we arrive at a contradiction. Therefore,

$$\sum_{j=0}^{\infty} \frac{z^j}{j+k+1} \neq 0 \quad \forall z \in \mathbb{D},$$

which proves the result by induction.

□

With this proof, we have a concrete example of a function $F \in \mathcal{H}^2$ with an unwinding series that converges linearly. Further, $F(z) = -\log(1-z)$ has no analytic extension to $\mathbb{D}_{1+\epsilon}$ for any $\epsilon > 0$.

The proof of Proposition 4.3.1 has also led to an important discovery. The exact same proof can be extended to the *polylogarithm* functions, defined for $z \in \mathbb{D}$ and $s \in \mathbb{C}$ as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}.$$

Notice that when $s = 1$, $Li_s(z) = -\log(1-z)$. Polylogarithm functions have been well studied in many areas of mathematics [22], and are connected to the Riemann zeta function, which can be expressed as a function of $s \in \mathbb{C}$ as $\zeta(s) = Li_s(1)$.

For these functions,

$$\|Li_s\|_{\mathcal{H}^2}^2 = \sum_{n=1}^{\infty} \frac{1}{|n^{2s}|}.$$

In particular, $Li_s \in \mathcal{H}^2$ as long as $s \in \mathbb{R}$ and $s > \frac{1}{2}$. We plot the image of $\partial\mathbb{D}$ of degree 1000 polynomial approximations to $Li_s$ for several real valued $s > \frac{1}{2}$ in Figure 4.3.2 for reference.

For these functions, we have the following result.

**Theorem 4.3.2.** *For any $s > \frac{1}{2}$, function $Li_s \in \mathcal{H}^2$ has an unwinding series that is term-wise equal to its Fourier series.*

*Proof.* To begin, we notice that the function $F(z) = Li_s(z)$ will only have one root in $\mathbb{D}$, namely at 0. This means that

$$Li_s(z) = z \cdot \sum_{n=0}^{\infty} \frac{z^n}{(n+1)^s} = B_1(z) \cdot G_1(z),$$

and that $G_1(0) = 1$, the first Fourier coefficient of $F$. Therefore the first term of the unwinding series will equal the first term of the Fourier series.

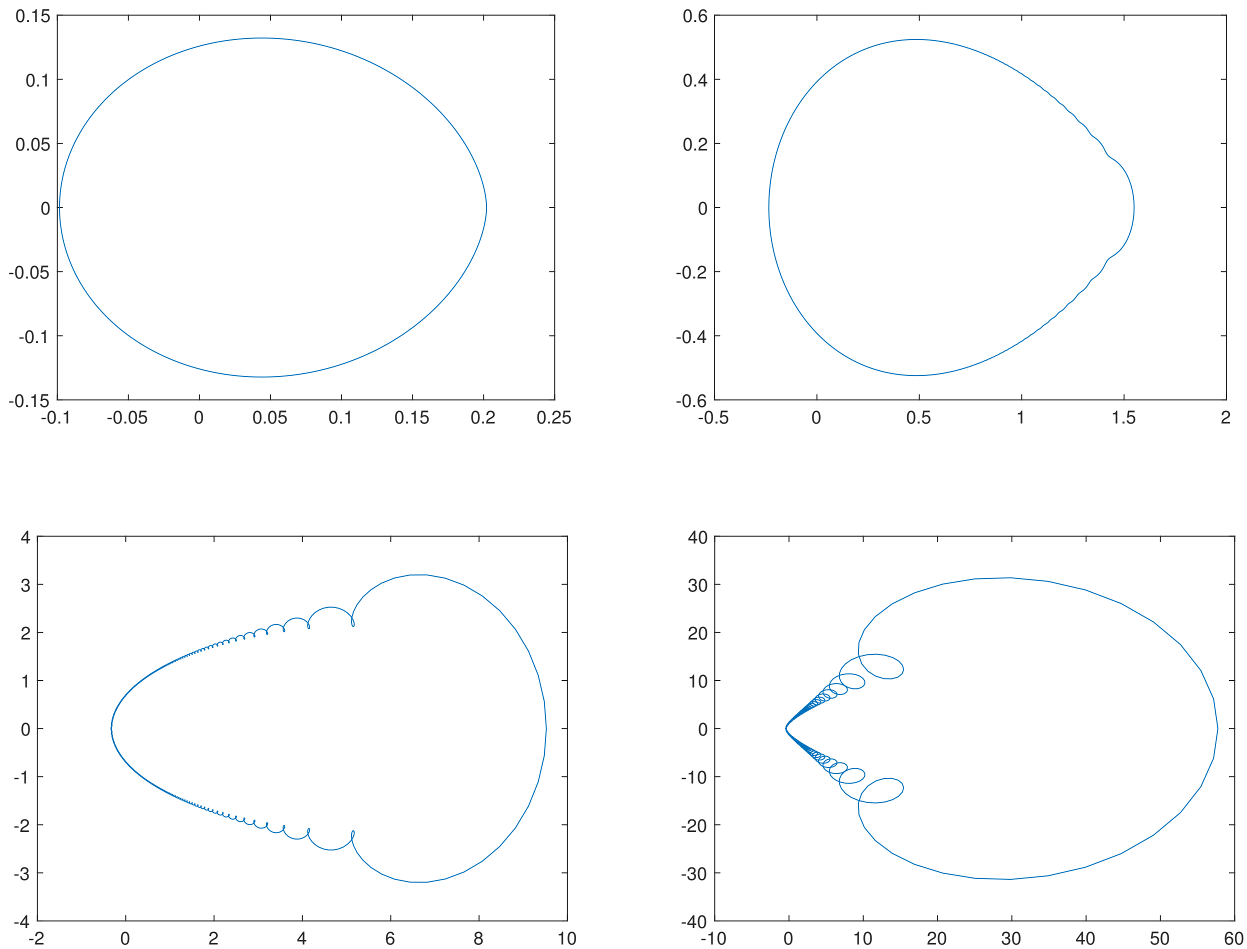


Figure 3: The image of $\partial\mathbb{D}$ under polynomial approximations of degree 1000 to: $Li_3$ (top left), $Li_{1.5}$ (top right), $Li_{.9}$ (bottom left) and $Li_{.51}$ (bottom right).

From here, we proceed by strong induction. Suppose that the first $k$ terms of the unwinding series of $F$ are equal to the Fourier terms. In other words, $G_n(0) = \frac{1}{n^s}$ and $B_n(z) = z$ $\forall n \leq k$. Then the $k$th partial unwinding series of $F$ is

$$F(z) = \sum_{n=1}^{k} \left( \frac{z^n}{n^s} \right) + z^k (G_k(z) - G_k(0)),$$

where

$$G_k(z) - G_k(0) = \sum_{j=1}^{\infty} \frac{z^j}{(j+k)^s} = z \cdot \sum_{j=0}^{\infty} \frac{z^j}{(j+k+1)^s}.$$

From here, if we can show that

$$\sum_{j=0}^{\infty} \frac{z^j}{(j+k+1)^s} \neq 0 \quad \forall z \in \mathbb{D},$$

then we will have

$$G_k(z) - G_k(0) = z G_{k+1}(z),$$

implying that the $k+1$ term of the unwinding series will be the same as the Fourier series.

Suppose for the sake of contradiction that

$$\sum_{j=0}^{\infty} \frac{z_0^j}{(j+k+1)^s} = 0$$

for some $z_0 \in \mathbb{D}$. Then we can find an $r < 1$ such that $|z_0| < r$.

Next, consider the function

$$H(z) = (z-1) \sum_{j=0}^{\infty} \frac{z^j}{(j+k+1)^s} = \sum_{j=0}^{\infty} \frac{z^{j+1}}{(j+k+1)^s} - \frac{z^j}{(j+k+1)^s} =$$

$$= \frac{-1}{(k+1)^s} + \sum_{j=1}^{\infty} \frac{z^j}{(k+j)^s} - \frac{z^j}{(k+j+1)^s}.$$

If $|z| < r < 1$, then

$$\left| \sum_{j=1}^{\infty} \frac{z^j}{(k+j)^s} - \frac{z^j}{(k+j+1)^s} \right| < \frac{1}{(k+1)^s},$$

since

$$\sum_{j=1}^{\infty} \frac{1}{(k+j)^s} - \frac{1}{(k+j+1)^s} = \frac{1}{(k+1)^s}.$$

Therefore, since

$$\left| \frac{-1}{(k+1)^s} \right| > \left| \sum_{j=1}^{\infty} \frac{z^j}{(k+j)^s} - \frac{z^j}{(k+j+1)^s} \right| \quad \text{for } |z| = r,$$

by Rouché's Theorem, $H$ will have no roots in $\mathbb{D}_r$. Since $H$ has the same roots as

$$\sum_{j=0}^{\infty} \frac{z^j}{(j+k+1)^s},$$

inside $\mathbb{D}_r$, we arrive at a contradiction. Therefore,

$$\sum_{j=0}^{\infty} \frac{z^j}{(j+k+1)^s} \neq 0 \quad \forall z \in \mathbb{D},$$

which proves the result by induction.

□

With this result, we now have a family of functions in $\mathcal{H}^2$ that have identical unwinding series and Fourier series. Further, we know that the functions $Li_s(z)$, where $s = \frac{1}{2} + \epsilon$ will have unbounded $\mathcal{H}^2$ norms as $\epsilon \to 0$. Thus, we have functions whose unwinding series have non exponential (sub linear) convergence, along with significant initial error, namely

$\|F(z) - F(0)\|_{\mathcal{H}^2}$. For these functions, a very large number of terms in the unwinding series are required to provide a suitable approximation, which demonstrates the theoretical limitations of the unwinding series.

# Chapter 5

# Algorithms and Numerical Examples

In this chapter, we develop a method (implemented using Matlab) that will create the finite unwinding series of a real valued, $2\pi$ periodic signal, $s(\theta)$, and then perform numerical tests.

We begin, in Section 5.1, by discussing an applicable method to find the analytic extension of the signal $s(\theta)$, namely $F(z)$, so that we can apply previously mentioned results in complex analysis. In practice, we may not have a closed form of $s$, so we discuss the process of approximating the function $F$ given a discrete set of data describing $s$.

From there, by using the main idea from the 1962 Guido and Mary Weiss algorithm [39], in Section 5.2 we provide a method for approximating the Blaschke decomposition of $F$ to obtain $B$ and $G$ as in Equation (2.7). This algorithm is extremely useful, and does not require the knowledge of the location of the roots of $F$ in $\mathbb{D}$ a priori. With this in hand, we then provide an algorithm to produce the partial unwinding series of $F$ by using addition and subtraction, along with the iterative definition:

$$B_{k+1}(z)G_{k+1}(z) = G_k(z) - G_k(0).$$

After providing the pseudocode for our algorithms, in Section 5.3 we move on to numerical experiments. These experiments help to verify some of the theory provided in the previous sections, and also look to provide insights into the mechanics of the unwinding series. We define a special family of functions, which we call *Rouché dominated functions*. With this family, we are able to observe some of the mechanics that allow the unwinding series to approximate many signals using only a few terms.

## 5.1 The Complexification of a Real Valued Signal

To begin, we assume that we are given a real valued, $2\pi$ periodic signal, $s : [0, 2\pi) \to \mathbb{R}$. If we treat the signal, $s$, as the real component of the boundary values of the uniquely determined $\mathcal{H}^2$ function $F$, then we have for every $\theta \in [0, 2\pi)$,

$$s(\theta) = \mathcal{R}e(F(e^{i\theta})).$$

The problem of finding the analytic function, $F$, was described in Chapter 2. We briefly recall the argument here. By using the Hilbert transform of $s$, denoted $\mathbf{H}s$, we may express the boundary values of the analytic $F$ as

$$F(e^{i\theta}) = s(\theta) + i\mathbf{H}s(\theta). \tag{5.1}$$

This method of complexification is extremely useful in signal analysis as it preserves the frequency and amplitude of the original signal under certain conditions [38]. Essentially, if we can express our signal as a *phase signal*,

$$s(\theta) = a(\theta)\cos(\phi(\theta)),$$

where $a$ represents the instantaneous amplitude of $s$ and $\phi$ expresses the instantaneous phase of $s$, then Equation (5.1) is equivalent to expressing

$$F(e^{i\theta}) = a(\theta)e^{i\phi(\theta)}.$$

We note that not all signals are phase signals, and a summary of the properties of phase signals can be found in [28]. When we are given a phase signal, neither the phase nor the amplitude of the signal are affected by the complexification process.

In general, computing the Hilbert transform of a function is numerically unstable as it involves the principle value integral of a singular kernel [8]. Singular integral operators, while mathematically sound, prove to be challenging for many computations. Luckily, there is a method to obtain $F$ from $s$ that does not require singular integrals: the Fourier series.

If we are given the Fourier series of $s$, namely

$$s(\theta) = \frac{a_0}{2} + \sum_{k\geq 1} a_k\cos(k\theta) + b_k\sin(k\theta),$$

then

$$\mathbf{H}s(\theta) = \sum_{k\geq 1} a_k\sin(k\theta) - b_k\cos(k\theta).$$

This is discussed in the first chapter of the first volume, and elaborated upon in the second volume of [41]. Therefore, by the identity

$$e^{ik\theta} = \cos(k\theta) + i\sin(k\theta),$$

we know the Fourier series of $F$:

$$F(e^{i\theta}) = s(\theta) + i\mathbf{H}s(\theta) = \frac{a_0}{2} + \sum_{k\geq 1}(a_k - ib_k)e^{ik\theta}.$$

This being the case, if we know the Fourier series of $s$, we can easily obtain the Fourier coefficients of $F$. If $s$ is integrable and has a closed form, we can compute for each $k$,

$$a_k = \frac{1}{\pi}\int_0^{2\pi} s(x)\cdot\cos(kx)dx,$$

$$b_k = \frac{1}{\pi}\int_0^{2\pi} s(x)\cdot\sin(kx)dx.$$

Therefore, in this case, computing $F$ becomes simple. Unfortunately, in many instances we do not have a closed form for $s$, and are often only given a discrete set of points, $s(x_i)$. In this case, we must use a different approach.

### 5.1.1 The Discrete and Fast Fourier Transform

A popular method to approximate unknown Fourier coefficients of real valued, $2\pi$ periodic data, $s(x_i)$ is the Discrete Fourier Transform (DFT). In many applications, we do not have a closed form of the signal with which we are working. Instead, we only assume that we can obtain a discrete set of $N$ points, taken at equally spaced intervals of $[0, 2\pi]$. That is, for $j \in \{0, 1, \ldots, N-1\}$, we are given $s(\frac{2\pi j}{N})$. Given these points, we can compute a DFT. In this process, we define the first $N$ approximate Fourier coefficients in the following way:

$$\hat{c}_k = \hat{a}_k + i\hat{b}_k := \sum_{j=0}^{N-1} e^{-i\frac{2\pi j}{N}k} s\left(\frac{2\pi j}{N}\right),$$

where $k \in \{\frac{-N}{2}+1, \frac{-N}{2}+2, \ldots, -1, 0, 1, \ldots, \frac{N}{2}-1, N\}$. For each $k$, we can view each term in the summation as the product of a root of unity with the known value of the signal. Using this definition, it is clear that finding each coefficient requires $N$ multiplications, thus the overall cost in computing the DFT is of the order $N^2$ multiplications. This has been improved greatly in recent years with the creation of the Fast Fourier Transform (FFT). The main idea of the FFT algorithm is to use the symmetry involved in computation that exists when we can express $N = 2^n$. In this case, we can reduce the cost of computing FFT to order $N\log(N)$ multiplications [16]. Due to its efficiency, we will be using the FFT algorithm to create approximations of the first non-negatively indexed $\frac{N}{2}+1$ Fourier coefficients of $s$: $\hat{c}_k$, where $0 \leq k \leq \frac{N}{2}$,

With this, we now have a method of approximating the first $\frac{N}{2}+1$ nonnegative Fourier coefficients of $F$ from a discrete set of $N$ values of the signal $s$.

## 5.2 The Guido and Mary Weiss Algorithm

Since we now have a method of approximating the $\mathcal{H}^2$ function $F$ from either $s$ or from a discrete set of values $s(x_i)$, we now look at producing the Blaschke decomposition

$$F = B \cdot G.$$

Given $F \in \mathcal{H}^2$, if we assume that for any $\theta \in [0, 2\pi)$, $F(e^{i\theta}) \neq 0$, then we can use the algorithm first proposed by Guido and Mary Weiss in [39].

The algorithm can be summarized as follows:

1. Define the real valued function $\ell$ as

$$\ell(\theta) := \log |F(e^{i\theta})|.$$

2. Complexify $\ell$ into $L$ via

$$L(e^{i\theta}) = \ell(\theta) + i\mathbf{H}\ell(\theta).$$

3. Compute:

$$G(e^{i\theta}) = e^{L(e^{i\theta})} \quad \text{and} \quad B(e^{i\theta}) = \frac{F(e^{i\theta})}{G(e^{i\theta})}.$$

It is important to note that while this algorithm will produce the Blaschke decomposition of $F$, the functions $G$ and $B$ may be off by a factor of a rotation term, $e^{i\phi}$. This is because the algorithm always guarantees $G(0) > 0$. Further, if the function $F$ has a root on $\partial\mathbb{D}$, or has a very small magnitude on $\partial\mathbb{D}$, this algorithm will become unstable due to the logarithm in the first step. Several authors have looked at improving the stability of this step, including Nahon in [26] (whose work was revisited by Coifman and Steinerberger in [5]), who looked at perturbing the data by adding a small constant, as well as Letelier and Saito in [21], who chose to add a sinusoid with small amplitude to the signal. An independent rediscovery of this algorithm was also provided by Qian in [30].

### 5.2.1 Implementation

With the idea behind the algorithm stated, we now provide pseudocode for finding the Blaschke Decomposition $F = B \cdot G$ in the case when we are given a discrete set of data on the signal $s$, seen as Algorithm 1. The goal of this algorithm is to input a discrete set of $N$ (assumed to be even) points, $(\theta_k, s(\theta_k))$, where $\theta_k = \frac{k}{N2\pi}$ for $k = 0, 1, \dots N-1$, and output the approximation to the functions $F$, $B$, and $G$ at each point $\theta_k$. In this pseudocode, for vectors $v \in \mathbb{C}^n$, we denote $v_k$ as the $k$th coordinate of $v$.

---

**Algorithm 1:** $[F, B, G, Z]$ =`DiscreteBlaschkeDecomposition`$(s)$

---

**Input :** $s \in \mathbb{R}^N$, where $N$ is even and $s_k = s(\theta_k)$, where $\theta_k = \frac{2\pi(k-1)}{N}, 1 \leq k \leq N$.
**Output:** $F, B, G \in \mathbb{C}^N, Z \in \mathbb{R}^+$, where $F_k$, $B_k$, and $G_k$ are the approximation of $F$, $B$ and $G$, at the points $e^{i\theta_k}$ respectively and $Z = G(0)$.

$\hat{c} \leftarrow$ first $\frac{N}{2} + 1$ complex Fourier coefficients of $s$
**for** $k = 1$ ***to*** $N$ **do**
  $F_k = \sum_{j=0}^{\frac{N}{2}} \overline{\hat{c}_j} \cdot e^{i \cdot j \cdot \theta_k}$
$\ell \leftarrow \log |F|$
$\hat{d} \leftarrow$ first $\frac{N}{2} + 1$ complex Fourier coefficients of $\ell$
**for** $k = 1$ ***to*** $N$ **do**
  $L_k = \sum_{j=0}^{\frac{N}{2}} \overline{\hat{d}_j} \cdot e^{i \cdot j \cdot \theta_k}$
$G \leftarrow e^L$
$Z \leftarrow e^{\hat{d}_0}$
$B \leftarrow F/G$ (coordinate-wise division)
**return** $F, B, G, Z$

---

The first step of this algorithm approximates the coefficients $\frac{a_0}{2}, (a_1 + ib_1), \dots, (a_{\frac{N}{2}} + ib_{\frac{N}{2}})$ for the Fourier series

$$s(x) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx).$$

We note that the $(j+1)$st entry in this vector corresponds to the $j$th Fourier coefficient. From there, the first $\frac{N}{2} + 1$ Fourier coefficients of the function $F$ are approximated by conjugation, as are the values $F(e^{i\theta_k})$. Next, the Weiss algorithm is applied: we set $\ell = \log |F|$, and using the same process as before, find the complex extension of $\ell$, namely $L$. Lastly, we obtain the values of $G(e^{i\theta_k})$ and $B(e^{i\theta_k})$, using their definitions, the value $G(0)$ by the identity

$G(0) = e^{L(0)}$. With this algorithm, we can now discus a method to produce arbitrarily many terms of the unwinding series.

### 5.2.2 Algorithm for Unwinding Series

By using Algorithm 1, we can approximate the analytic extension, $F$ of a signal, $s$, by using a discrete set of points, and then approximate the Blaschke decomposition

$$F(e^{i\theta_k}) = B(e^{i\theta_k}) \cdot G(e^{i\theta_k}).$$

From here, we want to create an approximation to the partial unwinding series of $F$; that is, to obtain each term in the series

$$F(e^{i\theta_k}) = G_0(0)B_0(e^{i\theta_k}) + \cdots + G_n(0)\prod_{j=1}^{n} B_j(e^{i\theta_k}) + \prod_{j=1}^{n} B_j(e^{i\theta_k})(G_n(e^{i\theta_k}) - G_n(0)). \quad (5.2)$$

In this section we provide an algorithm, seen as Algorithm 2, to produce an approximation to $G_j(0)$ and $B_j(e^{i\theta_k})$, for $0 \leq j \leq n$, as well as the values of $G_n(e^{i\theta_k})$. Here $\theta_k = \frac{2\pi(k-1)}{N}$, where $1 \leq k \leq N$.

In this algorithm, we begin by invoking Algorithm 1 to produce, all terms in the Blaschke decomposition $F(e^{i\theta_k}) = B(e^{i\theta_k}) \cdot G(e^{i\theta_k})$ and the term $G_0(0)$ . The initial vector $B$ will approximate $B_0(z)$ in the unwinding series, so it is saved as the first column of $\tilde{B}$. After that, we use the iterative definition

$$G_{j+1}(e^{i\theta_k}) \cdot B_{j+1}(e^{i\theta_k}) = G_j(e^{i\theta_k}) - G_j(0),$$

to produce the unwinding series. We begin by taking the difference $G_j(e^{i\theta_k}) - G_j(0)$, and use

**Algorithm 2:** $[G(0), \tilde{B}, G]$ =`DiscreteUnwindingSeries`$(s, n)$

**Input :** $s \in \mathbb{R}^N$, where $N$ is even and $s_k = s(\theta_k)$, where $\theta_k = \frac{2\pi(k-1)}{N}, 1 \leq k \leq N$. $n \in \mathbb{N}$, the number of terms in the partial unwinding series.

**Output:** $G(0) \in \mathbb{C}^{n+1}$, $\tilde{B} \in \mathbb{C}^{N,n+1}$, where $\tilde{B}_{(k,j)} = B_{j-1}(e^{i\theta_k})$, and $G \in \mathbb{C}^N$, where $G_k = G_n(e^{i\theta_k})$.

```
[F, B̃_(·,1), G, G(0)_1] ← DiscreteBlaschkeDecomposition(s)
for j = 2 to n + 1 do
    G̃ ← G − Z
    g ← Re(G̃)
    [·, B̃_(·,j), G, G(0)_j] ← DiscreteBlaschkeDecomposition(g)
return G(0), B̃, G
```

the real part of this vector as the input of Algorithm 1. This produces $B_{j+1}(e^{i\theta_k})$, $G_{j+1}(e^{i\theta_k})$ and $G_{j+1}(0)$. When we terminate this algorithm at step $n$, we have an approximation to all terms in Equation (5.2).

## 5.3 Numerical Examples

In this section we look at applying Algorithm 1 and Algorithm 2 to verify some of our theoretical results and to gain an insight into the mechanics of the unwinding series. We begin by testing the bounds produced in Theorem 3.3.2 and Theorem 3.3.3 for a randomly generated polynomial of degree 50 using our implemented Blaschke decomposition algorithm, Algorithm 1. From there, we compute the unwinding series, via Algorithm 2, for approximations to the polylogarithm function, $Li_1(z)$, and verify that the terms $G_j(0)$ do not decay exponentially. From there, to produce a wider variety of examples, we study the polylogarithm, $Li_2$, which we perturb by adding a suitably selected polynomial to introduce multiple roots in $\mathbb{D}$. By studying the magnitude of relevant Fourier coefficients, this experiment provides insights into the mechanics of the unwinding series.

### 5.3.1 Numerical Tests of the Chapter 3 Bounds

We begin our numerical experiments by testing the sharpness of the bounds produced in Theorem 3.3.2 and Theorem 3.3.3 for polynomial functions with large degree. To construct these polynomials, we begin by specifying the degree, $n$ of the polynomial, $F$. From there, we generate $n$ roots inside the unit disc, denoted $\alpha_j \in \mathbb{D}$. These roots are created by generating an ordered pair $(r_j, \theta_j)$ chosen from the uniform distributions $r \in [0, 1]$, $\theta \in [0, 2\pi]$, so that

$$\alpha_j = r_j e^{i\theta_j}.$$

With these roots defined, we have the polynomial

$$F(z) = \prod_{j=1}^{n} (z - \alpha_j). \tag{5.3}$$

We constructed a polynomial of degree 50 with complex roots, $\alpha_j \in \mathbb{D}$, that can be seen in Figure 4.

We also provide the image of $\partial\mathbb{D}$ under $F$ along with the real component $\mathcal{R}e$, which we associate with the signal, $s(\theta)$, in Figure 5 as a reference to the behavior of this function.

From here, given the Blaschke decomposition $F = B \cdot G$, we can represent

$$G(z) = \prod_{j=1}^{n} (1 - \overline{\alpha_j} z),$$

and again plot the images of $G(e^{i\theta})$ along with $\mathcal{R}e(G(e^{i\theta})$ in Figure 6.

With these functions defined, we now look to test the sharpness of the bounds provided in Theorem 3.3.2 and Theorem 3.3.3. As was done in the final section of Chapter 3, we begin by choosing the two sequences

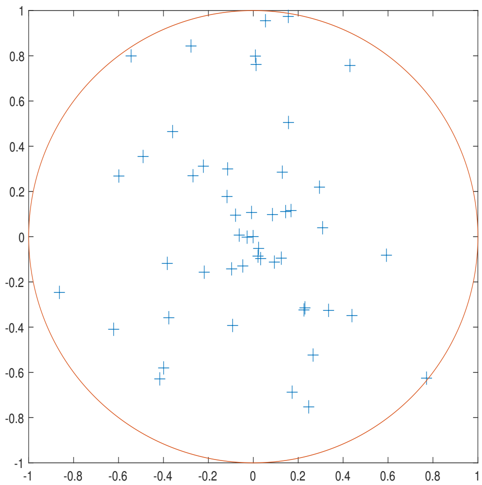


Figure 4: The location of the roots, $\alpha_j \in \mathbb{D}$, shown as "+" ticks, of the randomly generated test function, $F$, as in Equation (5.3).

$$\gamma_n = n^2, \; \forall n \geq 0 \quad \text{and} \quad \gamma_0 = 0, \; \gamma_n = \sum_{j=1}^{n} \frac{1}{j}, \; \forall n \geq 1.$$

to test these results.

For the space $X_\gamma$, where $\gamma_n = n^2$, for any $n \geq 0$ we compute

$$\|F\|_{X_\gamma}^2 \approx 271843.989.$$

From here, we compute

$$\sum_{j=1}^{50} (1 - |\alpha_j|^2) \left\| \frac{G(e^{i \cdot})}{1 - \overline{\alpha}_j e^{i \cdot}} \right\|_{Y_\gamma}^2 \approx 69972.660.$$

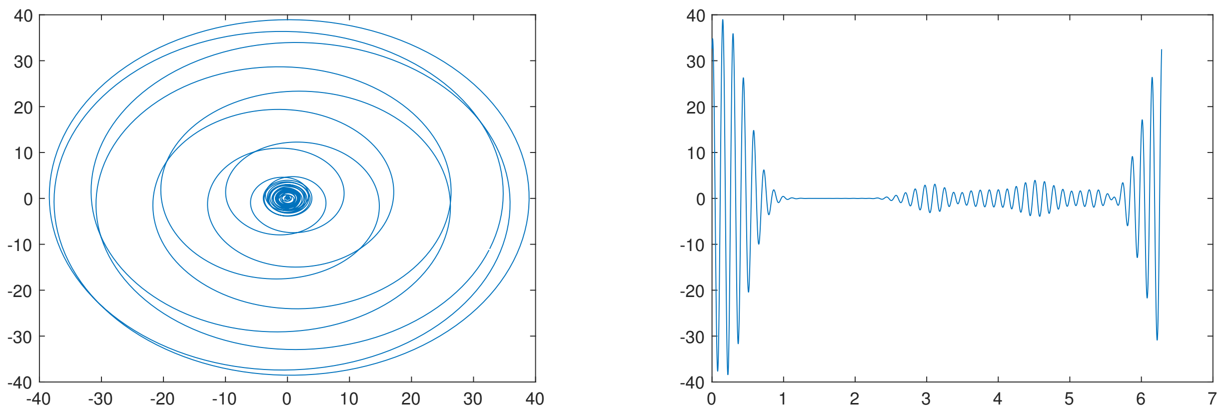


Figure 5: The images of $\partial\mathbb{D}$ under $F$(left) and $\mathcal{R}e(F)$ (right), for the test function $F$ as in Equation (5.3).

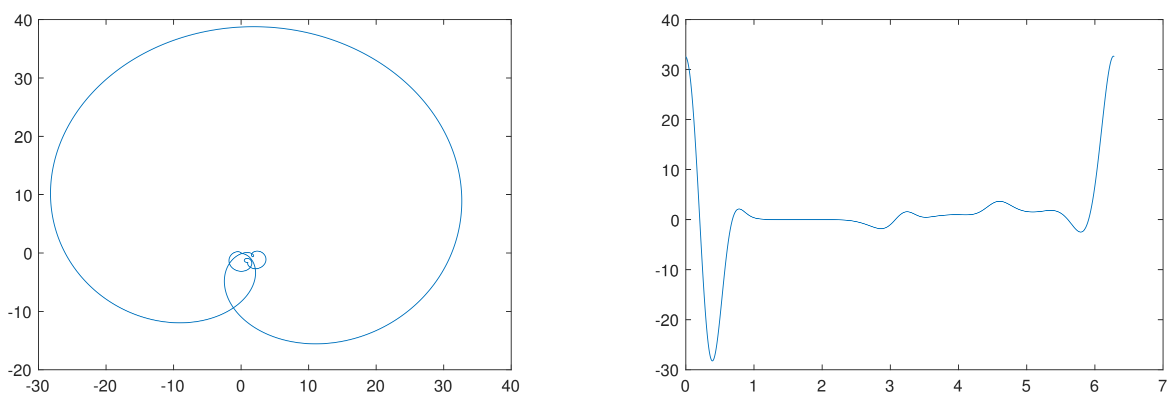


Figure 6: The image of the unit circle under $G$ (left) and $\mathcal{R}e(G)$ (right), where $F = B \cdot G$ and for $F$ as in Equation (5.3). Comparing to Figure 4, we see that $G$ has far less winding than $F$, and $\mathcal{R}e(G)$ has less oscillations than $\mathcal{R}e(F)$.

Therefore, by our bound in Theorem 3.3.2, we have that

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j=1}^{50} (1 - |\alpha_j|^2) \left\| \frac{G(e^{i\cdot})}{1 - \overline{\alpha}_j e^{i\cdot}} \right\|_{Y_\gamma}^2 \approx 201871.329.$$

By direct computation, we see that

$$\|G\|_{X_\gamma}^2 \approx 5734.418,$$

which verifies the bound, but shows that it is not sharp.

For the space $X_\gamma$, where $\gamma_0 = 0$ and $\gamma_n = \sum_{j=1}^{n} \frac{1}{j}$ for $n \geq 1$, we compute

$$\|F\|_{X_\gamma}^2 \approx 612.977.$$

From here, we compute

$$\sum_{j=1}^{50} (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \overline{\alpha}_j} \right\|_{Y_\gamma}^2 \approx 121.830.$$

Therefore, by our bound in Theorem 3.3.3, we have that

$$\|G\|_{X_\gamma}^2 \leq \|F\|_{X_\gamma}^2 - \sum_{j=1}^{50} (1 - |\alpha_j|^2) \left\| \frac{F(e^{i\cdot})}{e^{i\cdot} - \overline{\alpha}_j} \right\|_{Y_\gamma}^2 \approx 491.147.$$

By direct computation, we see that

$$\|G\|_{X_\gamma}^2 \approx 328.922,$$

which again verifies the bound.

| $t$ | $\|F\|_{X_\gamma}^2$ | $\sum_{j=1}^{50}(1-\|\alpha_j\|^2)\left\|\frac{G(e^{i\cdot})}{1-\overline{\alpha}_j e^{i\cdot}}\right\|_{Y_\gamma}^2$ | Bound on $\|G\|_{X_\gamma}^2$ | $\|G\|_{X_\gamma}^2$ | Error |
|---|---|---|---|---|---|
| 1.5 | 40925 | 20054 | 20871 | 2151.6 | 18719 |
| 1.1 | 9002.2 | 7016.1 | 1986.1 | 1002.3 | 983.8 |
| 1.01 | 6403.4 | 5479.9 | 923.5 | 846.4 | 76.1 |
| 1.001 | 6188.9 | 5344.5 | 844.4 | 832.3 | 12.1 |
| 1.0001 | 6167.9 | 5331.1 | 836.8 | 830.9 | 5.9 |

Table 1: Tests of the bound of $\|G\|_{X_\gamma}^2$ in Theorem 3.3.2 for varying $X_\gamma$, where $\gamma_n = n^t$. When $t = 1$, we get $X_\gamma = \mathcal{D}$,

For these two choices of weights, we see that the error between our bounds and the actual norm $\|G\|_{X_\gamma}^2$ is quite large. In Corollary 3.5.1, we saw that equality holds when the weights $\gamma_n$ grow at a linear rate. Therefore we want to verify our claim that the bounds we provide in Theorem 3.3.2 for weights that grow "near" linearly will be more accurate. To do this, we next consider the choice of weights

$$\gamma_n = n^t,$$

for different $t > 1$. We expect that as $t \to 1$, the bound will approach $\|G\|_{X_\gamma}^2$. Using the same function $F$ described above, Table 1 summarizes this phenomenon.

As expected, as the weights $\gamma_n$ approach the linear terms, $n$, the error between our bound and the $X_\gamma$ norm of $G$ decreases.

### 5.3.2 Non-exponential Convergence

In this section, we verify that the polylogarithm function, $Li_1$, defined in Chapter 4 will in fact have an unwinding series that does not converge exponentially.

We begin by defining our approximations to the polylogarithm function $Li_s(z)$, where

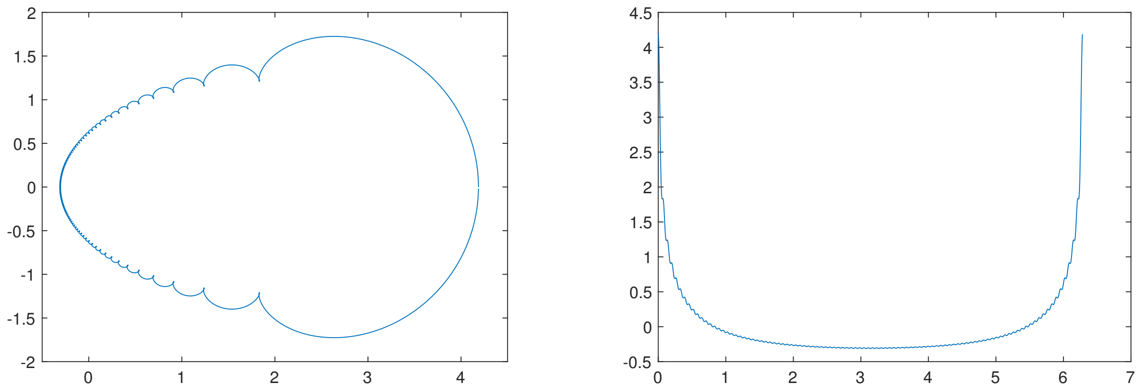


Figure 7: The images of $\partial\mathbb{D}$ under $F$ (left) and $\mathcal{R}e(F)$ (right), for $F$ as in Equation (5.4).

$s > \frac{1}{2}$. Since these functions can be represented as

$$Li_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s},$$

we will use the first $m$ Fourier coefficients in our approximations, that is

$$F_{(s,m)}(z) = \sum_{n=1}^{m} \frac{z^n}{n^s}.$$

In our experiment, we set $s = 1$ and $m = 100$ so that

$$F(z) := F_{(1,100)}(z) = \sum_{n=1}^{100} \frac{z^n}{n}. \tag{5.4}$$

Figure 7 plots $F(e^{i\theta})$ and $\mathcal{R}e(F(e^{i\theta})$, where $0 \leq \theta < 2\pi$ for reference.

After applying Algorithm 2 to $\mathcal{R}e(F(z)$ for 60 terms in the unwinding series, we looked at how the terms $G_j(0)$ decayed to 0. Figure 8 plots the terms $G_j(0)$ logarithmically, and demonstrates that the coefficients do not decay exponentially, as the decay would necessarily

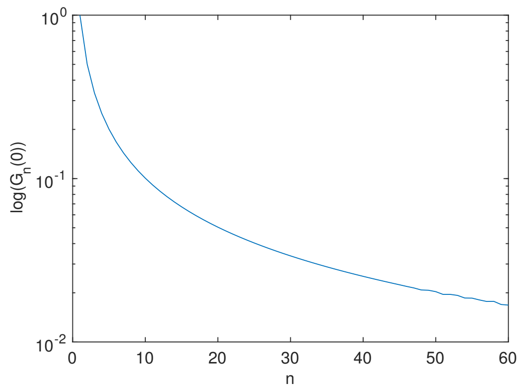


Figure 8: The magnitudes of $G_n(0)$ plotted logarithmically against $n$, for the function $F$ as in Equation (5.4). The decay rate of these terms is non-exponential.

occur linearly in the logarithmic scale. This verifies that there are signals $s$, and associated functions $F$ for which the unwinding series does not converge exponentially.

### 5.3.3 Rouché Dominated Functions

In the previous section, we saw numerical evidence that there are functions for which the unwinding series did not converge exponentially. We want to know what happens to these functions and their unwinding series if we introduce roots in $\mathbb{D}$. In particular, using the same notation as before, if we begin with

$$F_{(2,m)}(z) = \sum_{n=1}^{m} \frac{z^n}{n^2},$$

then for any $|z| = 1$, and $m < \infty$,

$$|F_{(2,m)}(z)| < \frac{\pi^2}{6}.$$

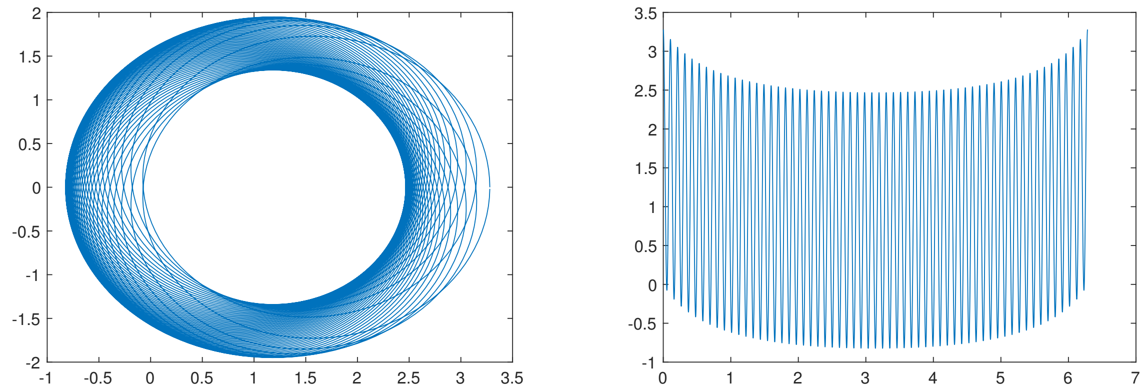


Figure 9: The images of $\partial\mathbb{D}$ under $F$ (left) and $\mathcal{R}e(F)$ (right), for $F$ as in Equation (5.6).

By Rouché's theorem, if we consider the function

$$A_k(z) = \frac{\pi^2}{6} z^k, \tag{5.5}$$

for some $k \in \mathbb{N}$, then for any $m$, the function

$$F(z) = F_{(2,m)}(z) + A_k(z), \tag{5.6}$$

will have exactly $k$ roots in $\mathbb{D}$. We refer to these types of functions as *Rouché dominated functions*, as a single term will determine the number of roots a function will have in $\mathbb{D}$. We want to know if the introduction of roots via this perturbation fundamentally changes the convergence rate of the unwinding series of $F$.

To begin, we set $m = 100$, and $k = 60$, and show the images of $F(e^{i\theta})$ and $\mathcal{R}e(F(e^{i\theta}))$ for $0 \leq \theta < 2\pi$ in Figure 9 as reference.

In our experiment, we looked at the effect the unwinding series has to the distribution of Fourier coefficients. After computing $G_0$ and $G_1$ using Algorithm 2, we then plot the magnitude of the Fourier coefficients in Figure 10

From this experiment, it appears as if the convergence rate of the unwinding series for Rouché dominated functions will be much faster than that of the unperturbed original function. The original function, $F_{(2,100)}$ will have an identical Fourier series and unwinding series, and the two will converge non-exponentially. After introducing roots inside $\mathbb{D}$ via the addition of a dominating function, we see that the Fourier coefficients, $a_n$, of $G_1$ begin to decay exponentially when $n > 60$. Since the unwinding series of $G_1$ will converge at least as quickly as its Fourier series, this immediately implies that the overall convergence rate of the unwinding series of the Rouché dominated function will be exponential. This phenomenon is quite interesting, and it would be interesting to continue to research this in future works, as the Fourier series convergence rate of a function will not change with the addition of a dominating function.

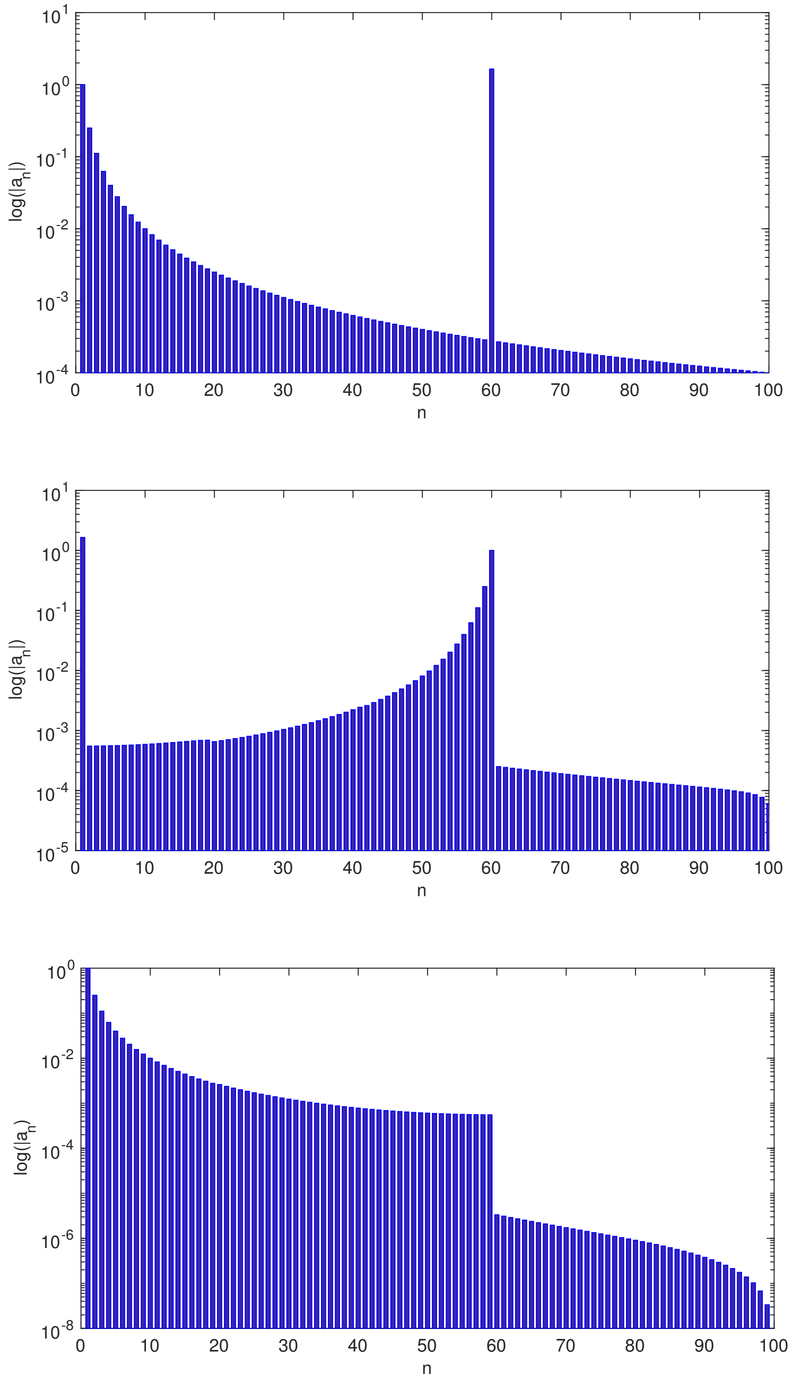


Figure 10: The magnitude of the Fourier coefficients for $F$ (top), $G_0$ (middle) and $G_1$ (bottom) in the unwinding series, for $F$ as in Equation (5.6) with $m = 100$ and $k = 60$. The Fourier coefficients decay non-exponentially for $F$, but for $G_1$, the decay rate of the coefficients $a_n$ switches from non-exponential to exponential at $k = 60$.

# Chapter 6

# Conclusion

In this dissertation, we have continued the exploration of Blaschke decompositions on weighted Hardy spaces and the investigation into both the limitations and advantages of the unwinding series.

In our study of Blaschke decompositions, $F = B \cdot G$ as in Equation (2.7), we have created sharper bounds on the weighted Hardy norms of $G$, by imposing nonrestrictive conditions on the growth rate of the weights. Further, we have extended the space of functions for which our bounds apply. In particular, we have shown that under certain conditions, our results can be applied to functions with an infinite number of zeros in $\mathbb{D}$.

While a significant step, we feel that there is still unexplored research on this topic. In Theorem 3.3.4, we proved that the convergence of an infinite series will occur if the weights, $\gamma_n$, are bounded by $M$ and

$$\sum_{n \geq 0} M - \gamma_n < \infty.$$

However, in the less restrictive statement of Corollary 3.5.1 by Carleson [2], we saw that for the Dirichlet space (for which the weights are unbounded) convergence of the same series still holds. This leads us to believe that further extensions of our results exist. This being the case, we now summarize the future steps we would like to take.

- We would first like to extend the result of Theorem 3.3.4 to any bounded, monotone increasing weights, for which the space $X_\gamma$ will be equal to $\mathcal{H}^2$ as sets. We believe this is possible since we utilized several inequalities in our proofs that may have imposed unnecessarily restrictive conditions.

- We would then like to explore convergence of our series in Theorem 3.3.4 for spaces $X_\gamma$ associated with unbounded weights. As was mentioned in Chapter 3, we have not yet taken advantage of a result by Shapiro and Shields [33] on the convergence rate of roots $\alpha_j$ to $\partial\mathbb{D}$ for these spaces. Since convergence partially depends on the term

$$\sum_{j\in J} 1 - |\alpha_j|^2,$$

  the additional knowledge of the rate of convergence of $|\alpha_j|$ to 1 may play an integral role in proving such extensions.

In our study of the unwinding series, we showed that there are limitations to the idea that the series will always have exponential convergence. In Chapter 5, our final numerical test showed that by perturbing functions with a slowly converging unwinding series, we may be able to regain the highly sought after exponential convergence. In future work, we would like to continue investigating this phenomenon and develop a theoretical approach to verify this result.

# Chapter 7

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# Vita

# Stephen Farnham

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## Education

- **Ph.D. Mathematics**, *Expected December 2019*
  *Syracuse University; Syracuse, New York*
  *Thesis Advisors: Dr. Loredana Lanzani and Dr. Lixin Shen*
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- **M.S. Mathematics**, *May 2016*
  *Syracuse University; Syracuse, New York*

- **B.S. Mathematics**, *May 2014*
  *Western New England University; Springfield, Massachusetts*

### Research Interests

- **Complex Analysis**
  Blaschke Decompositions, Interpolation Spaces, Weighted Hardy Spaces, Dirichlet Space, Unwinding Series, Adaptive Fourier Decomposition, Series Expansions, Signal Analysis, Harmonic Analysis
- **Numerical Linear Algebra**
  Principal Component Projection, Principal Component Analysis, Feature Selection, Scalability in Algorithmic Design
- **Other Interests**
  Machine Learning, Deep Learning

### Research Articles

- S. Farnham, *Blaschke Decompositions on Weighted Hardy Spaces* (preprint), Submitted, https://arxiv.org/pdf/1908.04665.pdf

- S. Farnham, L. Shen, B. Suter, *Principal Component Projection with Low Degree Polynomials* (preprint), Submitted, https://arxiv.org/pdf/1902.08656.pdf

## Employment

- **Syracuse University, Syracuse, NY**
  Graduate Assistant, Fall 2014 – Present

▶ **Griffiss Institute, Rome, NY**
Graduate Student Intern, May 2016 – May 2019
Internship at the United States Air Force Research Laboratory in Rome, NY. Produced research in applied mathematics.

## Teaching Experience

### Instructor of Record

▶ **Syracuse University**
- MAT 121: Probability and Statistics for Liberal Arts I, *Fall 2018*
- MAT 122: Probability and Statistics for Liberal Arts II, *Summer 2015, Spring 2018*
- MAT 183: Elements of Modern Mathematics, *Fall 2016*
- MAT 194: Precalculus, *Spring 2019, Fall 2019*
- MAT 295: Calculus I, *Fall 2015, Spring 2017*
- MAT 296: Calculus II, *Fall 2017*

### Teaching Assistant

▶ **Syracuse University**
- MAT 122: Probability and Statistics for Liberal Arts II, *Spring 2016*
- MAT 183: Elements of Modern Mathematics, *Fall 2014, Spring 2015*

### Mathematics Clinics & Tutoring

▶ **Syracuse University**
- Mathematics Help Room, *September 2015 – December 2015*

▶ **Western New England University**
- Mathematics Clinic Tutor, *September 2011 – May 2014*

## Awards

◇ 2017-2018 Syracuse University Outstanding Teaching Assistant Award

◇ 2014 Western New England University Allen E. Anderson Award for Excellence in Mathematics

## Departmental & University Service

### Departmental Service

▶ **Syracuse University**

◇ Conference Co-organizer, Syracuse University Graduate Conference *Spring 2016*

◇ Mathematics Graduate Organization Treasurer, *2016-2017*

◇ Preliminary and Qualifying Exam Help Session Instructor, *2017-2019*

### University Service

#### ▶ Syracuse University

- ◇ Graduate Student Organization Departmental Senator, *elected 2017-2018*
- ◇ Graduate Student Organization Finance Committee Member, *2017-2019*
- ◇ Graduate Student Organization At Large Senator, *elected 2018-2019*

## Advising

#### ▶ Undergraduate Directed Reading Program

- Fall 2019, *Project: Introduction to Advanced Complex Analysis*
- Fall 2018, *Project: Linear Optimization and the Simplex Method*

## Presentations

### Research Talks

- ◇ *Blaschke Decompositions on Weighted Hardy Spaces* — *September 2019*
  Interpolation in Spaces of Analytic Functions Conference, International Center of Mathematical Meetings, Marseille, France
- ◇ *Blaschke Decompositions on Weighted Hardy Spaces* — *September 2019*
  AMS Sectional Meeting, University of Wisconsin-Madison
- ◇ *Blaschke Expansion Convergence for Hardy Functions* — *April 2019*
  Geometric and Harmonic Analysis, University of Connecticut
- ◇ *The Convergence of Blaschke Expansions* — *April 2019*
  SU MGO Conference, Syracuse University
- ◇ *The Convergence of Blaschke Expansions* — *September 2018*
  Northeast Analysis Network, University of Albany
- ◇ *Fast and Stable Principal Component Projection* — *April 2018*
  SU MGO Conference, Syracuse University
- ◇ *A New Approach to Principal Component Projection* — *February 2018*
  Air Force Research Laboratory, Rome, NY
- ◇ *A Bound on the Rank of Elliptic Curves* — *April 2014*
  Hudson River Undergraduate Math Conference, Marist College

### University & Department Talks

- ◇ *Principal Component Projection with Low Degree Polynomials* — *September 2019*
  Syracuse University Applied Math Seminar

◇ *Blaschke Decompositions and the Unwinding Series* — *March 2019*
Syracuse University Math Graduate Organization Colloquium

◇ *An Introduction to Blaschke Products* — *April 2018*
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◇ *Principal Component Projection via Matrix Functions* — *April 2017*
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## Conferences and Workshops

### Conferences

- Interpolation in Spaces of Analytic Functions Conference, International Center of Mathematical Meetings, Marseille, France — *November 2019*
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- Syracuse University Mathematics Graduate Organization Conference — *Spring 2016, 2017, 2018, 2019*
- Geometric and Harmonic Analysis, University of Connecticut — *April 2019*
- Northeast Analysis Network, University of Albany — *September 2018*
- MAA Seaway Sectional Conference, Colgate University — *April 2015*
- Joint Mathematics Meeting, Boston, MA — *January 2012*

### Workshops

- Complex Analysis in Mathematical Physics and Applications, Isaac Newton Institute, Cambridge, UK — *October 2019*
- Introduction to Machine Learning Techniques, Griffiss Institute, Rome, NY — *July 2017*

## Skills

- **Mathematics Programs:** Sage, Mathematica, MATLAB, Minitab, Geogebra, Maple, (La)TeX
- **Presentational Programs:** Beamer, Microsoft Word, Microsoft Powerpoint, Microsoft Excel, Pages, Keynote
- **Programming:** Python (intermediate), SQL (intermediate)