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On Nudel'man's Problem and Indefinite Interpolation in the Generalized Schur and Nevanlinna Classes

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Abstract. This work is a revised and corrected version of the authors' joint paper [Trans. Amer. Math. Soc. **355** (2003), 813–836] with T. Constantinescu. Some of the theorems from the original paper that are withdrawn are recast as new open problems for indefinite interpolation. Partial results are obtained by other methods, including Kronecker's theorem for Hankel operators.

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1. Introduction and Preliminaries

Nudel′man's problem may be viewed as a method to implement the operator approach to interpolation due to Sarason [15]. Assume given a linear operator A on a complex vector space $\mathcal{V}$ into itself. Suppose there is a natural way to interpret $f(A)$, where f is an analytic function which is defined and bounded by one on the open unit disk of the complex plane. The problem is to find solutions f of the equation $b = f(A)c$ for given vectors b and c in $\mathcal{V}$. A general theorem based on the commutant lifting theorem of Sz.-Nagy and Foias [17] provides conditions for the existence of solutions. Simple choices of A, b, c yield classical interpolation theorems of the Nevanlinna-Pick, Carathéodory-Fejér, and Loewner types. See [13, Chapter 2].

Indefinite generalizations of operator methods for interpolation were pioneered by Ball and Helton [7]. Arocena, Azizov, Dijksma, and Marcantognini [6] use a theorem of Ball and Helton to prove an indefinite form of

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the commutant lifting theorem. This raised the possibility of finding an indefinite generalization of Nudel′man's problem, and such a generalization was proposed in our paper [3] with T. Constantinescu. However, the proof of the Main Theorem in [3] has gaps, which are identified in the Corrigendum [4]. Appendix B in this work describes the problems in the proof and includes an example showing what can go wrong. We have no counterexample to the original statement of the Main Theorem in [3], but we feel that its validity is seriously in doubt. Five open questions are identified in the present work, in Problems 3.3, 4.1, 5.1, 5.3, 5.5. Negative answers to any of them would provide a counterexample to the original form of the Main Theorem in [3].

This paper is a revised version of [3] that repairs the Main Theorem and shows the changes needed in the applications. Briefly, a stronger hypothesis fixes the problems in the proof of the Main Theorem. The applications in [3] to the classical interpolation problems of Pick-Nevanlinna, Carathéodory-Fejér, and Sarason survive with minor changes. The main losses are the theorems on boundary interpolation, for which the stronger hypothesis has not yet been proved or disproved. The boundary theorems in [3] are reformulated here as open problems, for which we obtain some partial results.

An effort has been made to make this paper self-contained, and thus we repeat unaffected results from the original paper, including some verbatim passages. However, although the statement and proof of the Main Theorem (Alternative Form) in [3, p. 834] could be inserted verbatim at the end of this paper, we shall not do so.

Throughout, κ denotes a nonnegative integer. By a **Hermitian form** or **Hermitian kernel** on a set Ω we mean a complex-valued function K on $\Omega \times \Omega$ such that $K(\zeta, z) = \overline{K(z,\zeta)}$ for all ζ, z in Ω. We say that K has κ **negative squares** and write

$$\mathrm{sq}_-\, K = \kappa$$

if the maximum number of negative eigenvalues (counting multiplicities) among all matrices $\big(K(\zeta_j,\zeta_i)\big)_{i,j=1}^n$, $\zeta_1,\dots,\zeta_n \in \Omega$, $n \geqslant 1$, is κ. Inner products are examples of Hermitian forms.

Proposition 1.1. *A linear and symmetric inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ on a complex vector space $\mathcal{H}$ has κ negative squares if and only if the maximum dimension of a strictly negative subspace of $\mathcal{H}$ is κ.*

For the purpose of this work, a **strictly negative subspace** of $\mathcal{H}$ is a subspace $\mathcal{N}$ such that $\langle f, f\rangle_{\mathcal{H}} < 0$ for every $f \neq 0$ in $\mathcal{N}$.

Proof of Proposition 1.1. By definition, $\langle\cdot,\cdot\rangle_{\mathcal{H}}$ has κ negative squares if and only if the maximum number of negative eigenvalues of every Gram matrix

$$\big(\langle g_j, g_i\rangle_{\mathcal{H}}\big)_{i,j=1}^n, \qquad g_1,\dots,g_n \in \mathcal{H}, \quad n \geqslant 1,$$

is κ. By [5, Lemma 1.1.1′], this occurs if and only if κ is the maximum dimension of a strictly negative subspace of $\mathcal{H}$. □

A bounded selfadjoint operator T on a Hilbert space $\mathcal{H}$ is said to have κ **negative squares** if the inner product

$$\langle f, g\rangle_T = \langle Tf, g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{H},$$

has κ negative squares. In this case, we write $\mathrm{sq}_- T = \kappa$.

Proposition 1.2. *Let $\mathcal{H}$ be a Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ a selfadjoint operator. Then $\mathrm{sq}_- T = \kappa$ if and only if the negative spectrum of T consists of a finite number of eigenvalues of total multiplicity κ.*

Proof. Write $\mathcal{H} = \mathcal{H}_- \oplus \mathcal{H}_+$, where $\mathcal{H}_\pm$ are the spectral subspaces for T for the intervals $(-\infty, 0)$ and $[0, \infty)$. We must show that the inner product $\langle\cdot,\cdot\rangle_T$ has κ negative squares if and only if $\dim \mathcal{H}_- = \kappa$.

Suppose $\dim \mathcal{H}_- = \kappa$. Then $\mathcal{H}_-$ is a strictly negative subspace of $(\mathcal{H}, \langle\cdot,\cdot\rangle_T)$ of dimension κ. Let $\mathcal{N}$ be an arbitrary strictly negative subspace of $(\mathcal{H}, \langle\cdot,\cdot\rangle_T)$. If P_- is the projection onto $\mathcal{H}_-$, then $P_-|\mathcal{N}$ is a one-to-one linear mapping from $\mathcal{N}$ into $\mathcal{H}_-$. For if $f \in \mathcal{N}$ and $P_- f = 0$, then $f \in \mathcal{H}_+$ and so $\langle f, f\rangle_T = \langle Tf, f\rangle_{\mathcal{H}} \geqslant 0$. Since $f \in \mathcal{N}$, $f = 0$. Therefore $\dim \mathcal{N} \leqslant \kappa$. By Proposition 1.1, the inner product $\langle\cdot,\cdot\rangle_T$ has κ negative squares.

Conversely, suppose $\langle\cdot,\cdot\rangle_T$ has κ negative squares. Again by Proposition 1.1, since $\mathcal{H}_-$ is a strictly negative subspace of $(\mathcal{H}, \langle\cdot,\cdot\rangle_T)$, $\mathcal{H}_-$ has dimension at most κ, say $\dim \mathcal{H}_- = \kappa'$. Then by what we just showed, $\langle\cdot,\cdot\rangle_T$ has κ' negative squares. Hence $\kappa' = \kappa$, and therefore $\dim \mathcal{H}_- = \kappa$. □

2. Main Theorem

A function $S(z)$ which is analytic on a subregion of the unit disc is in the **generalized Schur class** $\mathbf{S}_\kappa = \mathbf{S}_\kappa(\mathbb{D})$ if the Hermitian kernel $[1 - S(z)\overline{S(\zeta)}]/(1 - z\bar{\zeta})$ has κ negative squares. In this case, $S(z)$ has an analytic continuation to $\mathbb{D}$ except for at most κ poles. When $\kappa = 0$, $\mathbf{S}_0$ is the **Schur class** of analytic functions which are defined and bounded by one on $\mathbb{D}$. By the Kreĭn-Langer factorization [10, p. 382], every $S(z)$ in $\mathbf{S}_\kappa$ has the form

$$S(z) = B(z)^{-1} f(z), \tag{2.1}$$

where $f(z)$ belongs to $\mathbf{S}_0$, $B(z)$ is a Blaschke product of degree κ, and $f(z)$ does not vanish at the zeros of $B(z)$. Conversely, every such function belongs to $\mathbf{S}_\kappa$. Recall that a Blaschke product of degree κ is a function of the form

$$B(z) = c \prod_{j=1}^{\kappa} \frac{z - a_j}{1 - z\bar{a}_j}, \qquad |c| = 1, \quad a_1, \dots, a_\kappa \in \mathbb{D}.$$

Here the points $a_1, \dots, a_\kappa$ need not be distinct.

Assume given a complex vector space $\mathcal{V}$ with algebraic dual $\mathcal{V}'$. We write

$$(x, x') = x'(x)$$

for the action of a linear functional x' in $\mathcal{V}'$ on a vector x in $\mathcal{V}$. Every linear operator $A\colon \mathcal{V} \to \mathcal{V}$ has a dual $A'\colon \mathcal{V}' \to \mathcal{V}'$ defined by

$$(x, A'x') = (Ax, x'), \qquad x \in \mathcal{V}, \;\; x' \in \mathcal{V}'.$$

Nudel′man's Problem. *Given vectors b, c in $\mathcal{V}$ and a linear operator A on $\mathcal{V}$ into itself, find a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that*

$$f(A)c = B(A)b \tag{2.2}$$

in the sense described below. We call (A, b, c) the **data** *of the problem.*

By an **admissible set** for given data (A, b, c), we understand a subset $\mathcal{D}$ of $\mathcal{V}'$ such that[1]

(i) $\mathcal{D}$ is a linear subspace of $\mathcal{V}'$ which is invariant under A';
(ii) the sums $\sum_{j=0}^{\infty} |(A^j b, x')|^2$ and $\sum_{j=0}^{\infty} |(A^j c, x')|^2$ are finite for all x' in $\mathcal{D}$;
(iii) there is a constant $M > 0$ such that for all x' in $\mathcal{D}$,

$$\sum_{j=0}^{\infty} |(A^j b, x')|^2 \leqslant M \sum_{j=0}^{\infty} |(A^j c, x')|^2. \tag{2.3}$$

When an admissible set $\mathcal{D}$ has been chosen, we interpret (2.2) to mean that

$$\sum_{j=0}^{\infty} f_j (A^j c, x') = \sum_{j=0}^{\infty} B_j (A^j b, x'), \qquad x' \in \mathcal{D}, \tag{2.4}$$

where $f(z) = \sum_{j=0}^{\infty} f_j z^j$ and $B(z) = \sum_{j=0}^{\infty} B_j z^j$ are Taylor expansions.

Theorem 2.1 (Main Theorem). *Let (A, b, c) be given data, $\mathcal{D}$ an admissible set. Define a Hermitian form $\mathcal{K}$ on $\mathcal{D} \times \mathcal{D}$ by*

$$\mathcal{K}(x', y') = \sum_{j=0}^{\infty} \Big[(A^j c, x')\overline{(A^j c, y')} - (A^j b, x')\overline{(A^j b, y')} \Big] \tag{2.5}$$

for all $x', y' \in \mathcal{D}$. Let κ be a nonnegative integer.

(1) *If $\mathcal{K}$ has κ negative squares, there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that $f(A)c = B(A)b$.*

(2) *If there is a pair (f, B) as in (1), then $\mathcal{K}$ has κ' negative squares for some $\kappa' \leqslant \kappa$.*

This is proved for the case $\kappa = 0$ in [13, pp. 23–24] using the Sz.-Nagy and Foias commutant lifting theorem. The general case is derived using the Ball-Helton almost commutant lifting theorem in the following form.

[1] We emphasize that condition (iii) for an admissible family in the form used in this paper is stronger than that of [3]. See Appendix B for an explanation why the stronger form is needed.

Theorem 2.2. *For each $j = 1, 2$, let $T_j \in \mathcal{L}(\mathcal{H}_j)$ be a contraction on the Hilbert space $\mathcal{H}_j$, let $W_j \in \mathcal{L}(\mathcal{G}_j)$ be an isometric dilation of T_j on a Hilbert space $\mathcal{G}_j$, and let P_j be the projection of $\mathcal{G}_j$ onto $\mathcal{H}_j$. Let $C \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ be an operator such that $CT_1 = T_2C$.*

(1) *If* $\operatorname{sq}_-(1 - C^*C) = \kappa$, *there exists a pair $(\mathcal{E}, \widetilde{C})$ such that $\mathcal{E}$ is a closed W_1-invariant subspace of $\mathcal{G}_1$ of codimension κ and $\widetilde{C}$ is a contraction operator on $\mathcal{E}$ into $\mathcal{G}_2$ satisfying*

$$\widetilde{C}W_1|_{\mathcal{E}} = W_2\widetilde{C} \qquad \text{and} \qquad P_2\widetilde{C} = CP_1|_{\mathcal{E}}.$$

(2) *If there is a pair $(\mathcal{E}, \widetilde{C})$ as in* (1), *then* $\operatorname{sq}_-(1 - C^*C) \leqslant \kappa$.

Theorem 2.2 is proved in a more general form in Theorem 1.1 in Arocena, Azizov, Dijksma, and Marcantognini [6].

Proof of Theorem 2.1. Let $\mathcal{H}_c$ be the set of functions on $\mathbb{D}$ of the form

$$h_{x'}(z) = \sum_{j=0}^{\infty} (A^j c, x')z^j, \qquad x' \in \mathcal{D}. \tag{2.6}$$

By conditions (i) and (ii) for an admissible set, $\mathcal{H}_c$ is a linear subspace of the Hardy space H^2 which is invariant under S^*, where S is multiplication by z in H^2. Condition (iii) assures that the formula

$$X_0\colon \sum_{j=0}^{\infty} (A^j c, x')z^j \to \sum_{j=0}^{\infty} (A^j b, x')z^j, \qquad x' \in \mathcal{D}, \tag{2.7}$$

defines a bounded operator on $\mathcal{H}_c$ into H^2. By the definition of X_0,

$$S^*X_0h = X_0S^*h, \qquad h \in \mathcal{H}_c. \tag{2.8}$$

Let $X \in \mathcal{L}(\bar{\mathcal{H}}_c, H^2)$ be the extension by continuity of X_0. By (2.5),

$$\mathcal{K}(x', y') = \langle h_{x'}, h_{y'}\rangle_{H^2} - \langle X_0h_{x'}, X_0h_{y'}\rangle_{H^2}, \qquad x', y' \in \mathcal{D}. \tag{2.9}$$

An approximation argument shows that the number of negative squares of $\mathcal{K}$ is the same as the number of negative squares of the Hermitian form

$$\langle (1 - X^*X)h, k\rangle_{H^2} = \langle h, k\rangle_{H^2} - \langle Xh, Xk\rangle_{H^2}, \qquad h, k \in \bar{\mathcal{H}}_c,$$

which is the same as $\operatorname{sq}_-(1 - X^*X)$. That is, if any one of these numbers is κ, all are equal to κ. By standard methods for Hilbert space operators, $\operatorname{sq}_-(1 - XX^*) = \operatorname{sq}_-(1 - X^*X)$.

Proof of (1). Assume that $\mathcal{K}$ has κ negative squares, so $\operatorname{sq}_-(1 - X^*X) = \kappa$. Set

$\mathcal{H}_1 = H^2$ and $T_1 = S$;
$\mathcal{H}_2 = \bar{\mathcal{H}}_c$ and $T_2 = E_2^*SE_2$, where $E_2\colon \mathcal{H}_2 \to H^2$ is inclusion;
$\mathcal{G}_1 = \mathcal{G}_2 = H^2$ and $W_1 = W_2 = S$;
$C = X^* \in \mathcal{L}(H^2, \mathcal{H}_2)$.

Then W_1, W_2 are isometric dilations of T_1, T_2. By (2.8), $CT_1 = T_2C$. Also $\mathrm{sq}_-(1 - C^*C) = \mathrm{sq}_-(1 - XX^*) = \mathrm{sq}_-(1 - X^*X) = \kappa$, and so the assumptions of Theorem 2.2(1) are met. By that result, there is a closed S-invariant subspace $\mathcal{E}$ of H^2 of codimension κ, and a contraction $\widetilde{C} \in \mathcal{L}(\mathcal{E}, H^2)$ such that $S\widetilde{C}f = \widetilde{C}Sf$ for all $f \in \mathcal{E}$, and

$$Cf = P_2\widetilde{C}f, \qquad f \in \mathcal{E}. \tag{2.10}$$

Here P_2 is the projection on H^2 with range $\mathcal{H}_2$. Write $\tilde{\varphi}(z) = \overline{\varphi(\bar{z})}$ for any complex-valued function φ on $\mathbb{D}$. Then $\mathcal{E} = \widetilde{B}H^2$ where B is a Blaschke product of degree κ. For any $\varphi \in H^\infty$, let M_φ be multiplication by φ on H^2. For every $h \in H^2$,

$$\widetilde{C}M_{\widetilde{B}}Sh = \widetilde{C}S(\widetilde{B}h) = S\widetilde{C}\widetilde{B}h = S\widetilde{C}M_{\widetilde{B}}h,$$

and therefore $\widetilde{C}M_{\widetilde{B}}$ commutes with S. Since $\widetilde{C}$ is a contraction, $\widetilde{C}M_{\widetilde{B}} = M_{\tilde{f}}$ for some $f \in \mathbf{S}_0$. To verify (2.4), consider any $x' \in \mathcal{D}$ and $h \in H^2$. Then by (2.10) and (2.7),

$$\begin{aligned}\left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, M_{\tilde{f}}h \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, \widetilde{C}M_{\widetilde{B}}h \right\rangle_{H^2} \\ = \left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, P_2\widetilde{C}M_{\widetilde{B}}h \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, CM_{\widetilde{B}}h \right\rangle_{H^2} \\ = \left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, X^*M_{\widetilde{B}}h \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^jb, x')z^j, \widetilde{B}h \right\rangle_{H^2}.\end{aligned}$$

When $h = 1$, this reduces to (2.4).

Proof of (2). Assume (2.4) holds for some f and B as in (1). For all $x' \in \mathcal{D}$,

$$\left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, \tilde{f}(z) \right\rangle_{H^2} = \left\langle \sum_{j=0}^{\infty}(A^jb, x')z^j, \widetilde{B}(z) \right\rangle_{H^2}. \tag{2.11}$$

Hence for all $x' \in \mathcal{D}$ and $n \geqslant 0$,

$$\begin{aligned}\left\langle \sum_{j=0}^{\infty}(A^jc, x')z^j, z^n\tilde{f}(z) \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^{j+n}c, x')z^j, \tilde{f}(z) \right\rangle_{H^2} \\ = \left\langle \sum_{j=0}^{\infty}(A^jc, (A')^nx')z^j, \tilde{f}(z) \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^jb, (A')^nx'))z^j, \widetilde{B}(z) \right\rangle_{H^2} \\ = \left\langle \sum_{j=0}^{\infty}(A^{j+n}b, x')z^j, \widetilde{B}(z) \right\rangle_{H^2} &= \left\langle \sum_{j=0}^{\infty}(A^jb, x')z^j, z^n\widetilde{B}(z) \right\rangle_{H^2}.\end{aligned}$$

Let $h_{x'}$ be as in (2.6). Recalling the definition (2.7) of X_0, we deduce that

$$\left\langle h_{x'}, \tilde{f}g \right\rangle_{H^2} = \left\langle X_0h_{x'}, \widetilde{B}g \right\rangle_{H^2}$$

first for $g(z) = z^n$ and then for any g in H^2. By the arbitrariness of x',

$$\left\langle h, \tilde{f} g \right\rangle_{H^2} = \left\langle Xh, \widetilde{B} g \right\rangle_{H^2} = \left\langle h, X^* \widetilde{B} g \right\rangle_{H^2}, \quad h \in \bar{\mathcal{H}}_c, \ \ g \in H^2.$$

Therefore the restriction of X^* to $\mathcal{E} = \widetilde{B} H^2$ is a contraction. Write

$$X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}, \qquad X_1 \in \mathcal{L}(\bar{\mathcal{H}}_c, \mathcal{E}), \quad X_2 \in \mathcal{L}(\bar{\mathcal{H}}_c, \mathcal{E}^\perp).$$

Then X_1 is a contraction because $X_1^* = X^*|_{\mathcal{E}}$ is a contraction. Thus

$$1 - X^* X = 1 - X_1^* X_1 - X_2^* X_2,$$

where $1 - X_1^* X_1 \geqslant 0$. Since $\widetilde{B}$ is a Blaschke product of degree κ, $\dim \mathcal{E}^\perp = \kappa$. Thus $-X_2^* X_2$ has rank at most κ. It follows that $\operatorname{sq}_- (1 - X^* X) \leqslant \kappa$, and hence the kernel $\mathcal{K}$ has $\kappa' \leqslant \kappa$ negative squares. □

3. Classical interpolation problems on the disc

The classical interpolation problem of Pick-Nevanlinna falls within the scope of Theorem 2.1.

Theorem 3.1 (Cf. [3, Theorem 3.1])**.** *Let $z_1, \dots, z_n$ be distinct points in the unit disc $\mathbb{D}$, $w_1, \dots, w_n$ any complex numbers, and let κ be a nonnegative integer. Set*

$$P = \left(\frac{1 - w_j \bar{w}_i}{1 - z_j \bar{z}_i} \right)_{i,j=1}^n . \tag{3.1}$$

(1) *If P has κ negative eigenvalues, then there is a pair (f, B) with $f \in \mathbf{S}_0$ and B a Blaschke product of degree κ such that $f(z_j) = B(z_j) w_j$ for all $j = 1, \dots, n$.*
(2) *If there is a pair (f, B) as in* (1)*, then P has $\kappa' \leqslant \kappa$ negative eigenvalues.*

Proof. We apply Theorem 2.1 with $\mathcal{V} = \mathbb{C}^n$. Identify $\mathcal{V}'$ with $\mathbb{C}^n$ with the pairing $(x, y) = x_1 y_1 + \cdots + x_n y_n$, where x_j, y_j are the entries of x, y. For data, choose

$$A = \operatorname{diag}\{z_1, \dots, z_n\}, \qquad b = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

We show that $\mathcal{D} = \mathcal{V}'$ is an admissible set for the data. Conditions (i) and (ii) are easily verified. We check (iii). For any x in $\mathcal{D}$,

$$\sum_{j=0}^\infty (A^j b, x) z^j = \sum_{j=0}^\infty \sum_{k=1}^n z_k^j w_k x_k z^j = \sum_{k=1}^n \frac{w_k x_k}{1 - z_k z},$$
$$\sum_{j=0}^\infty (A^j c, x) z^j = \sum_{j=0}^\infty \sum_{k=1}^n z_k^j x_k z^j = \sum_{k=1}^n \frac{x_k}{1 - z_k z}.$$

Thus (iii) requires that for some $M > 0$,

$$\left\| \sum_{k=1}^{n} \frac{w_k x_k}{1 - z_k z} \right\|^2 \leqslant M \left\| \sum_{k=1}^{n} \frac{x_k}{1 - z_k z} \right\|^2, \qquad x \in \mathbb{C}^n, \tag{3.2}$$

where $\| \cdot \|$ is the norm in H^2. The inequality (3.2) is easily brought to the form of a matrix inequality $WCW^* \leqslant MC$, where

$$C = \left(\frac{1}{1 - z_i \bar{z}_j} \right)_{i,j=1}^{n}, \qquad W = \operatorname{diag}\{w_1, \ldots, w_n\}.$$

Here

$$C = \left(\langle g_i, g_j \rangle \right)_{i,j=1}^{n},$$

where $g_k(z) = 1/(1 - z_k z)$, $k = 1, \ldots, n$. Since $z_1, \ldots, z_n$ are distinct, the functions $g_1, \ldots, g_n$ are linearly independent. Therefore C is nonnegative and invertible [9, p. 407]. Thus $\delta I_n \leqslant C \leqslant \mu I_n$ for some $\delta, \mu > 0$. If $\eta = \max\{|w_k| \colon k = 1, \ldots, n\}$, then

$$WCW^* \leqslant \mu \eta^2 I_n \leqslant \delta^{-1} \mu \eta^2 C,$$

which implies (iii) with $M = \delta^{-1}\mu\eta^2$. Thus $\mathcal{D}$ is an admissible set.

A short calculation shows that the Hermitian form (2.5) is given by

$$\mathcal{K}(x, y) = \sum_{i,j=1}^{n} \frac{1 - w_j \bar{w}_i}{1 - z_j \bar{z}_i} x_j \bar{y}_i = \langle Px, y \rangle_{\mathbb{C}^n},$$

for all $x, y \in \mathbb{C}^n$. Therefore $\operatorname{sq}_- \mathcal{K} = \operatorname{sq}_- P$, which by Proposition 1.2 is the number of negative eigenvalues of P (counting multiplicity). The condition $f(A)c = B(A)b$ is also easily seen to be equivalent to the relations $f(z_j) = B(z_j)w_j$, $j = 1, \ldots, n$. Thus Theorem 3.1 is a special case of Theorem 2.1. □

In [3, Theorem 3.1], the matrix (3.1) is replaced by its transpose P^t. This does not change anything, because P and P^t have the same eigenvalues and multiplicities. In fact, for any selfadjoint $n \times n$ matrix M, the transpose and conjugate of M coincide: $M^t = \overline{M}$. If λ is an eigenvalue for M with eigenvector x, then $Mx = \lambda x$, and so $\overline{M}\bar{x} = \lambda \bar{x}$. A complete orthonormal system of eigenvectors $x_1, \ldots, x_n$ for M with eigenvalues $\lambda_1, \ldots, \lambda_n$ thus induces a complete orthonormal system of eigenvectors $\bar{x}_1, \ldots, \bar{x}_n$ for $\overline{M}$ with the same eigenvalues.

Theorem 3.2 of [3] is withdrawn (see Appendix B), but the special case for countable sets can be derived from Theorem 3.1 using a normal families argument.

Theorem 3.2 (Cf. [3, Theorem 3.2]). *Let Ω be a countable subset of $\mathbb{D}$. Let $S_0 : \Omega \to \mathbb{C}$ be a given function, and let κ be a nonnegative integer. Set*

$$K_0(\zeta, z) = \frac{1 - S_0(z)\overline{S_0(\zeta)}}{1 - z\bar{\zeta}}, \qquad \zeta, z \in \Omega.$$

(1) *If $\operatorname{sq}_- K_0 = \kappa$, then there is a pair (f, B) with $f \in \mathbf{S}_0$ and B a Blaschke product of degree κ such that $f(z) = B(z)S_0(z)$ for all $z \in \Omega$.*

(2) *If there is a pair* (f, B) *as in* (1), *then* $\operatorname{sq}_- K_0 = \kappa' \leqslant \kappa$.

It is an open question what happens when Ω is uncountable.

Problem 3.3. *Is Theorem* 3.2 *true for an arbitrary subset* Ω *of* $\mathbb{D}$*?*

The answer is affirmative when $\kappa = 0$ according to the following known result: *For any subset* Ω *of* $\mathbb{D}$*, a function* $S_0 : \Omega \to \mathbb{C}$ *is the restriction of a Schur function* $f \in \mathbf{S}_0$ *if and only if the kernel* K_0 *is nonnegative.* This result is due to Krein and Rekhtman [11]; see also Akhiezer [2, p. 104]. It also follows from our Main Theorem (Theorem 2.1) in the definite case, as shown in [13, p. 25].

Our proof of Theorem 3.2 (the countable case) uses a compactness property of Blaschke products.

Lemma 3.4. *Let* $B_1, B_2, \dots$ *be a sequence of Blaschke products, each of degree at most* κ*. Then there exist positive integers* $n_1 < n_2 < \cdots$ *and a Blaschke product* B *of degree* $\kappa' \leqslant \kappa$ *such that* $B_{n_k} \to B$ *uniformly on all compact subsets of* $\mathbb{D}$*.*

Proof. Write each $B_j(z)$ as a product of κ factors (in any order),

$$B_j(z) = B_{j1}(z) \cdots B_{j\kappa}(z),$$

where each $B_{jk}(z)$ is either a constant of modulus one or a simple Blaschke factor

$$\gamma\,\frac{z-\alpha}{1-\bar{\alpha}z}, \qquad |\gamma| = 1 \quad \text{and} \quad |\alpha| < 1.$$

Consider the sequence of first factors: $B_{11}(z), B_{21}(z), B_{31}(z), \dots$. If infinitely many terms in this sequence are constants γ_k of modulus one, we can find a subsequence that converges to a constant γ of modulus one as scalars, and hence as functions uniformly on compact sets. Otherwise infinitely many terms have the form

$$\gamma_k\,\frac{z-\alpha_k}{1-\bar{\alpha}_k z}, \qquad |\gamma_k| = 1 \quad \text{and} \quad |\alpha_k| < 1.$$

By passing to a subsequence we can arrange that $\gamma_k \to \gamma$ and $\alpha_k \to \alpha$ as scalars, where $|\gamma| = 1$ and $|\alpha| \leqslant 1$. When $|\alpha| < 1$, it is easy to see that

$$\gamma_k\,\frac{z-\alpha_k}{1-\bar{\alpha}_k z} \to \gamma\,\frac{z-\alpha}{1-\bar{\alpha}z}$$

uniformly on compact subsets of $\mathbb{D}$. When $|\alpha| = 1$, one can show that

$$\gamma_k\,\frac{z-\alpha_k}{1-\bar{\alpha}_k z} \to \eta$$

uniformly on compact subsets of $\mathbb{D}$, where $\eta = -\gamma\alpha$ is a constant of absolute value one. When $\kappa = 1$, we have produced the required subsequence. For $\kappa > 1$, we need only repeat the process for the second factors, third factors, and so on. At each stage we choose the next subsequence from the previous one. The final subsequence has the required properties. $\square$

Proof of Theorem 3.2. (1) Assume $\mathrm{sq}_- K_0 = \kappa$. Suppose first that Ω is a finite set consisting of the points $z_1, \dots, z_n$. If $w_j = S(z_j)$, $j = 1, \dots, n$, then

$$\big(K_0(z_j, z_i)\big)_{i,j=1}^n = \left(\frac{1 - w_i \bar{w}_j}{1 - z_i \bar{z}_j}\right)_{i,j=1}^n$$

is the transpose P^t of the matrix (3.1). As noted above, P and P^t have the same number of negative eigenvalues and multiplicities, and therefore (1) follows from Theorem 3.1(1) in this case.

Suppose Ω is countably infinite. Choose finite subsets $\Omega_1 \subseteq \Omega_2 \subseteq \cdots$ such that $\Omega = \bigcup_1^\infty \Omega_n$. We can assume that K_0 has κ negative squares on each of the finite sets. For each $n \geqslant 1$, by what we just showed there exist $f_n \in \mathbf{S}_0$ and B_n a Blaschke product of degree at most κ such that

$$f_n(z) = B_n(z) S_0(z), \qquad z \in \Omega_n.$$

By passing to a subsequence, without loss of generality we can assume that $f_n \to f$ uniformly on compact subsets of $\mathbb{D}$ for some $f \in \mathbf{S}_0$ (see the theory of normal families in e.g. Ahlfors [1, Chapter IV]). By passing to another subsequence using Lemma 3.4, we can also assume that $B_n \to B$ uniformly on compact subsets of $\mathbb{D}$, where B is a Blaschke product of degree κ' for some $\kappa' \leqslant \kappa$.

Consider an arbitrary $z \in \Omega$. Then $z \in \Omega_{n_0}$ for some n_0. Hence $z \in \Omega_n$ for every $n \geqslant n_0$, and so

$$f_n(z) = B_n(z) S_0(z), \qquad n \geqslant n_0.$$

Letting $n \to \infty$, we obtain $f(z) = B(z) S_0(z)$. If $\kappa' = \kappa$, we are done. If $\kappa' < \kappa$, we can multiply both f and B by $\kappa - \kappa'$ simple Blaschke factors to obtain a pair (f, B) having the required properties. This proves (1).

(2) This is an immediate consequence of Theorem 3.1(2). □

Another choice of data in Theorem 2.1 yields a result of Carathéodory-Fejér type.

Theorem 3.5 (Cf. [3, Theorem 3.4]). *Let $w(z) = w_0 + w_1 z + \cdots + w_n z^n$ be a polynomial with complex coefficients, and set*

$$T = \begin{pmatrix} w_0 & w_1 & \cdots & w_n \\ 0 & w_0 & \cdots & w_{n-1} \\ & & \cdots & \\ 0 & 0 & \cdots & w_0 \end{pmatrix}.$$

Let κ be a nonnegative integer such that $\kappa \leqslant n+1$.

(1) *If $1 - T^*T$ has κ negative eigenvalues, then there is a pair (f, B) with $f \in \mathbf{S}_0$ and B a Blaschke product of degree κ such that $B(z)w(z) = f(z) + \mathcal{O}(z^{n+1})$.*

(2) *If there is a pair (f, B) as in (1), then $1 - T^*T$ has $\kappa' \leqslant \kappa$ negative eigenvalues.*

Corollary 3.6 (Cf. [3, Corollary 3.5]). *Let $w(z) = w_0 + w_1 z + \cdots + w_n z^n$ and T be as in Theorem 3.5, and let κ be a nonnegative integer such that $\kappa \leqslant n+1$. If $1 - T^*T$ has κ negative eigenvalues, there is a $\kappa' \leqslant \kappa$ and a function $S(z)$ in $\mathbf{S}_{\kappa'}$ which is holomorphic at the origin and such that $w(z) = S(z) + \mathcal{O}(z^{n+1-\kappa})$.*

Proof of Theorem 3.5. Let $\mathcal{V}$ and $\mathcal{V}'$ be as in the proof of Theorem 3.1 but with $\mathbb{C}^n$ replaced by $\mathbb{C}^{n+1}$. For the data (A, b, c), choose

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ & & \cdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad b = \begin{pmatrix} w_0 \\ w_1 \\ \vdots \\ w_n \end{pmatrix}, \qquad c = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

We check that the set $\mathcal{D} = \mathcal{V}'$ is admissible. Condition (i) in the definition of admissibility is clear, and (ii) is trivial because $A^j = 0$ for $j > n$. The sums in (2.3) can be evaluated, reducing (iii) to the assertion that T is bounded as an operator on $\mathbb{C}^{n+1}$ in the Euclidean metric. Thus $\mathcal{D} = \mathcal{V}'$ is admissible.

The Hermitian form (2.5) is given by $\mathcal{K}(x, y) = \langle (1 - T^*T)x, y \rangle_{\mathbb{C}^{n+1}}$ for all $x, y \in \mathbb{C}^{n+1}$. In fact, for any $x \in \mathbb{C}^{n+1}$,

$$\begin{aligned} \mathcal{K}(x,x) &= \big(|x_0|^2 + \cdots + |x_n|^2\big) \\ &\qquad - \big(|w_0 x_0 + \cdots + w_n x_n|^2 + |w_0 x_1 \cdots + w_{n-1} x_n|^2 + \cdots + |w_0 x_n|^2\big) \\ &= \|x\|^2_{\mathbb{C}^{n+1}} - \|Tx\|^2_{\mathbb{C}^{n+1}} \\ &= \langle (1 - T^*T)x, x \rangle_{\mathbb{C}^{n+1}}. \end{aligned}$$

By Proposition 1.2, $\mathrm{sq}_-\,\mathcal{K}$ is equal to the number of negative eigenvalues of $1 - T^*T$. The equation $f(A)c = B(A)b$ with $f(z) = \sum_{j=0}^\infty f_j z^j$ and $B(z) = \sum_{j=0}^\infty B_j z^j$ is equivalent to the identities

$$f_0 = w_0 B_0,\ f_1 = w_1 B_0 + w_0 B_1,\ \ldots,\ f_n = w_n B_0 + w_{n-1} B_1 + \cdots + w_0 B_n,$$

or $B(z)w(z) = f(z) + \mathcal{O}(z^{n+1})$. The result thus follows from the Theorem 2.1, □

Proof of Corollary 3.6. Let (f, B) be a pair as in part (1) of Theorem 3.5. If $B(z)$ has a zero of order r at the origin, $f(z)$ has a zero of order at least r at the origin. Hence $S(z) = f(z)/B(z)$ belongs to $\mathbf{S}_{\kappa'}$ for some $\kappa' \leqslant \kappa$ and is holomorphic at the origin, and $w(z) = S(z) + \mathcal{O}(z^{n-r+1}) = S(z) + \mathcal{O}(z^{n+1-\kappa})$. □

A simultaneous generalization of the Pick-Nevanlinna and Carathéodory-Fejér problems can be treated in the same way by choosing A in Jordan form. The calculations are straightforward but somewhat lengthy, and we shall not pursue this direction. For the definite case, see [8] and [13, §2.6].

The Main Theorem also yields a result on generalized interpolation in the sense of Sarason [15]. Let C be an inner function on $\mathbb{D}$, and let

$$\mathcal{H}(C) = H^2 \ominus CH^2$$

in the inner product of H^2. The reproducing kernel for $\mathcal{H}(C)$ is given by

$$K_C(w,z) = \frac{1 - C(z)\overline{C(w)}}{1 - z\bar{w}}, \qquad z, w \in \mathbb{D}.$$

Let S be the shift operator $S\colon h(z) \to zh(z)$ on H^2, and let T be the compression of S to $\mathcal{H}(C)$, that is,

$$T = P_{\mathcal{H}(C)} S|_{\mathcal{H}(C)},$$

where $P_{\mathcal{H}(C)}$ is the projection operator on H^2 with range $\mathcal{H}(C)$. The space $\mathcal{H}(C)$ is invariant under S^* and $T^* = S^*|_{\mathcal{H}(C)}$. Since T is completely nonunitary, for any $\varphi \in H^\infty$ an operator $\varphi(T)$ on $\mathcal{H}(C)$ is defined by the H^∞-functional calculus (see [16] and [17, p. 114]):

$$\varphi(T) = s\text{-}\lim_{r \uparrow 1} \varphi(rT).$$

Equivalently, for this particular situation, $\varphi(T) = P_{\mathcal{H}(C)} M_\varphi|_{\mathcal{H}(C)}$, where M_φ is multiplication by φ on H^2. For every $\varphi \in H^\infty$, $\varphi(T)$ commutes with T, and $\varphi(T)$ is a contraction if φ is a Schur function.

Theorem 3.7 (Cf. [3, Theorem 3.6])**.** *Let C be an inner function on the unit disc, and define T on $\mathcal{H}(C)$ as above. Let R be a bounded linear operator on $\mathcal{H}(C)$ such that $TR = RT$.*

(1) *If $1 - RR^*$ has κ negative squares, then there is a pair (f, B), where $f \in \mathbf{S}_0$ and B is a Blaschke product of degree κ, such that*

$$B(T)R = f(T).$$

(2) *If there is a pair (f, B) as in* (1)*, $1 - RR^*$ has κ' negative squares for some $\kappa' \leqslant \kappa$.*

If R is a contraction, the condition in (1) is satisfied with $\kappa = 0$, and in this case the result reduces to the original theorem of Sarason [15, Theorem 1].

Proof. In the Main Theorem, let $\mathcal{V} = \mathcal{H}(C)$, $A = T$, $c = K_C(0,\cdot)$, and $b = Rc = RK_C(0,\cdot)$. Let $\mathcal{D}$ be the set of continuous linear functionals on $\mathcal{V} = \mathcal{H}(C)$; thus $\mathcal{D} = \{x'_k \colon k \in \mathcal{H}(C)\}$ where for any $k \in \mathcal{H}(C)$,

$$(h, x'_k) = \langle h, k \rangle_{\mathcal{H}(C)}, \qquad h \in \mathcal{H}(C).$$

Then $A'x'_k = x'_{T^*k}$ for any $k \in \mathcal{H}(C)$, and so condition (i) holds in the definition of admissibility. To verify (ii), notice that for any $k(z) = \sum_0^\infty a_j z^j$ in $\mathcal{H}(C)$,

$$\sum_{j=0}^\infty |(A^j c, x'_k)|^2 = \sum_{j=0}^\infty |\big\langle K_C(0,\cdot), T^{*j}k \big\rangle_{\mathcal{H}(C)}|^2 = \sum_{j=0}^\infty |a_j|^2 = \|k\|^2_{H^2} < \infty. \tag{3.3}$$

If we replace c by b and use the identity $RT = TR$, we obtain

$$\sum_{j=0}^\infty |(A^j b, x'_k)|^2 = \sum_{j=0}^\infty |\big\langle K_C(0,\cdot), T^{*j}R^*k \big\rangle_{\mathcal{H}(C)}|^2 = \|R^*k\|^2_{H^2} < \infty, \tag{3.4}$$

and thus (ii) holds. Condition (iii) is immediate from (3.3) and (3.4) because R is bounded in the norm of $\mathcal{H}(C)$, which is the norm of H^2.

The form (2.5) is given by

$$\begin{aligned}\mathcal{K}(x'_h, x'_k) &= \sum_{j=0}^{\infty} \Big[\big\langle T^j K_C(0,\cdot), h \big\rangle_{\mathcal{H}(C)} \big\langle k, T^j K_C(0,\cdot) \big\rangle_{\mathcal{H}(C)} \\ &\qquad\qquad - \big\langle T^j R K_C(0,\cdot), h \big\rangle_{\mathcal{H}(C)} \big\langle k, T^j R K_C(0,\cdot) \big\rangle_{\mathcal{H}(C)} \Big] \\ &= \langle k, h \rangle_{H^2} - \langle R^* k, R^* h \rangle_{H^2} \\ &= \langle k, h \rangle_{\mathcal{H}(C)} - \langle R^* k, R^* h \rangle_{\mathcal{H}(C)} \\ &= \langle (1 - RR^*) k, h \rangle_{\mathcal{H}(C)} \end{aligned} \tag{3.5}$$

for any h and k in $\mathcal{H}(C)$.

(1) Assume that $1-RR^*$ has κ negative squares. By (3.5), the Hermitian form (2.5) has κ negative squares. Hence by part (1) of the Main Theorem, there is a function $f \in \mathbf{S}_0$ and a Blaschke product B of degree κ such that $B(A)b = f(A)c$, that is, if $B(z) = \sum_0^\infty B_j z^j$ and $f(z) = \sum_0^\infty f_j z^j$, then for every $h \in \mathcal{H}(C)$,

$$\sum_{j=0}^{\infty} B_j \big\langle T^j R K_C(0,\cdot), h \big\rangle_{\mathcal{H}(C)} = \sum_{j=0}^{\infty} f_j \big\langle T^j K_C(0,\cdot), h \big\rangle_{\mathcal{H}(C)}.$$

Using Abel summation of these series, we see that

$$B(T) R K_C(0,\cdot) = f(T) K_C(0,\cdot). \tag{3.6}$$

Since R commutes with T, it commutes with $B(T)$ and $f(T)$. Hence $B(T)R$ and $f(T)$ agree on the smallest invariant subspace of T containing $K_C(0,\cdot)$. The latter subspace is all of $\mathcal{H}(C)$, and thus we obtain $B(T)R = f(T)$.

(2) Assume that a pair (f, B) exists as in (1). Reversing the preceding steps, we see that $B(A)b = f(A)c$, hence by part (2) of the Main Theorem the form (2.5) has κ negative squares. Therefore by (3.5), $1 - RR^*$ has κ' negative squares for some $\kappa' \leqslant \kappa$. □

4. Boundary problems, disc case

Theorem 3.8 of [3] is withdrawn (see Appendix B), but in its place we can formulate an open problem. We make a minor change in the hypotheses of [3, Theorem 3.8] by assuming there that $|b/c| \leqslant 1$ a.e.; this is a necessary condition for the desired representation, and so nothing is lost. This change appears in Problem 4.1 in the hypothesis that $|S_0(u)| \leqslant 1$ a.e. It is essential for the application of Kronecker's theorem in Theorem 4.2(3).

Let σ be normalized Lebesgue measure on $\partial\mathbb{D} = \{u \colon |u| = 1\}$, and write L^2, L^∞ for $L^2(\partial\mathbb{D}), L^\infty(\partial\mathbb{D})$. We identify H^2 with a subspace of L^2 in the usual way. The class of boundary functions for $\mathbf{S}_\kappa(\mathbb{D})$ is denoted $\mathbf{S}_\kappa(\partial\mathbb{D})$. If Δ is a Borel subset of $\partial\mathbb{D}$, $L^2(\Delta)$ is the subspace of functions in L^2 supported on Δ. Let $\mathbf{S}_\kappa(\Delta)$ be the space of restrictions to Δ of functions in $\mathbf{S}_\kappa(\partial\mathbb{D})$. The characteristic function of Δ is denoted 1_Δ; if φ is a function on Δ, we view $\varphi 1_\Delta$ as a function defined on $\partial\mathbb{D}$ which is equal to φ a.e. on Δ and equal to zero on the complement of Δ. In what follows, we exclude the degenerate case that Δ is a Lebesgue null set.

Problem 4.1 (Cf. [3, Theorem 3.8])**.** *Let S_0 be a measurable complex-valued function on a Borel subset Δ of $\partial\mathbb{D}$ such that $|S_0(u)| \leqslant 1$ a.e. on Δ, and let κ be a nonnegative integer. Define a Hermitian form on $L^2(\Delta) \times L^2(\Delta)$ by*

$$L(\varphi,\psi) = \lim_{r\uparrow 1} \int_\Delta \int_\Delta \frac{1 - S_0(u)\overline{S_0(v)}}{1 - r^2 u\bar{v}}\, \varphi(u)\overline{\psi(v)}\, d\sigma(u)d\sigma(v), \qquad \varphi,\psi \in L^2(\Delta).$$

Does it follow that $S_0 \in \mathbf{S}_\kappa(\Delta)$ if and only if $\operatorname{sq}_- L = \kappa$*?*

The limit defining the Hermitian form L always exists [13, Theorem A, p. 30]. In fact, for all $\varphi,\psi \in L^2(\Delta)$,

$$\begin{aligned} &\lim_{r\uparrow 1} \int_\Delta \int_\Delta \frac{\varphi(u)\overline{\psi(v)}}{1 - r^2 u\bar{v}}\, d\sigma(u)d\sigma(v) \\ &\quad = \sum_{j=0}^\infty \Big(\int_\Delta u^j \varphi(u)\, d\sigma(u)\Big)\overline{\Big(\int_\Delta v^j \psi(v)\, d\sigma(v)\Big)} = \langle Q_-(\varphi 1_\Delta), \psi 1_\Delta\rangle_{L^2}, \end{aligned} \tag{4.1}$$

where Q_- is the orthogonal projection on L^2 whose range is the closed span of all functions u^j, $j \leqslant 0$.

Theorem 4.2. *In Problem* 4.1*:*

1. *The answer is affirmative for $\kappa = 0$.*
2. *If $S_0 \in \mathbf{S}_\kappa(\Delta)$, then* $\operatorname{sq}_- L \leqslant \kappa$ *with equality for $\kappa = 0$ and $\kappa = 1$. Moreover,* $\operatorname{sq}_- L \neq 0$ *for all $\kappa \geqslant 1$.*
3. *If $\Delta = \partial\mathbb{D}$, then $S_0 \in \mathbf{S}_\kappa(\partial\mathbb{D})$ if and only if* $\operatorname{sq}_- L = \kappa$*.*

Proof. (1) This follows from [13, Theorem A, p. 30].

(2) The case $\kappa = 0$ is covered in (1). Assume $\kappa \geqslant 1$ and $S_0 \in \mathbf{S}_\kappa(\Delta)$. The Hermitian form L is an inner product on $L^2(\Delta)$. Hence by Proposition 1.1, to prove that $\operatorname{sq}_- L \leqslant \kappa$, it is sufficient to show that any subspace of $L^2(\Delta)$ which is strictly negative with respect to L has dimension at most κ.

By the Kreĭn-Langer factorization, $S_0 = B^{-1} f$ a.e. on Δ, where B is a Blaschke product of degree κ and $f \in \mathbf{S}_0(\partial\mathbb{D})$. By the definite case applied to f, the Hermitian form

$$L_0(\varphi,\psi) = \lim_{r\uparrow 1} \int_\Delta \int_\Delta \frac{1 - f(u)\overline{f(v)}}{1 - r^2 u\bar{v}}\, \varphi(u)\overline{\psi(v)}\, d\sigma(u)d\sigma(v), \qquad \varphi,\psi \in L^2(\Delta),$$

is nonnegative. Write

$$\frac{1 - S_0(u)\overline{S_0(v)}}{1 - r^2 u\bar{v}} = \frac{1 - f(u)\overline{f(v)}}{1 - r^2 u\bar{v}} - S_0(u)\, \frac{1 - B(u)\overline{B(v)}}{1 - r^2 u\bar{v}}\, \overline{S_0(v)}.$$

An induction argument shows that

$$1 - B(u)\overline{B(v)} = (1 - u\bar{v}) \sum_{j=1}^{\kappa} e_j(u)\overline{e_j(v)},$$

where $e_1, \dots, e_\kappa$ are rational functions which are bounded on $\bar{\mathbb{D}}$. Thus

$$\frac{1 - S_0(u)\overline{S_0(v)}}{1 - r^2 u\bar{v}} = \frac{1 - f(u)\overline{f(v)}}{1 - r^2 u\bar{v}} - \frac{1 - u\bar{v}}{1 - r^2 u\bar{v}} \sum_{j=1}^{\kappa} S_0(u)e_j(u)\overline{e_j(v)S_0(v)}.$$

Therefore for all $\varphi, \psi \in L^2(\Delta)$,

$$L(\varphi, \psi) = L_0(\varphi, \psi) \\ - \lim_{r\uparrow 1} \int_\Delta \int_\Delta \frac{1 - u\bar{v}}{1 - r^2 u\bar{v}} \sum_{j=1}^{\kappa} \varphi(u)S_0(u)e_j(u)\overline{e_j(v)S_0(v)\psi(v)}\, d\sigma(u)\, d\sigma(v).$$

For all $u, v \in \Delta$ and r in $(0,1)$,

$$\left| \frac{1 - u\bar{v}}{1 - r^2 u\bar{v}} \right| = \left| 1 - \frac{(1 - r^2)u\bar{v}}{1 - r^2 u\bar{v}} \right| \leqslant 2.$$

Hence

$$L(\varphi, \psi) = L_0(\varphi, \psi) \\ - \sum_{j=1}^{\kappa} \int_\Delta \varphi(u)S_0(u)e_j(u)\, d\sigma(u) \int_\Delta \overline{e_j(v)S_0(v)\psi(v)}\, d\sigma(v). \tag{4.2}$$

Consider now any subspace $\mathcal{N}$ of $L^2(\Delta)$ such that

$$L(\varphi, \varphi) < 0, \qquad 0 \neq \varphi \in \mathcal{N}. \tag{4.3}$$

We show that any $\kappa + 1$ elements $\varphi_1, \dots, \varphi_{\kappa+1}$ of $\mathcal{N}$ are linearly dependent. Set

$$\varphi_* = \eta_1\varphi_1 + \cdots + \eta_{\kappa+1}\varphi_{\kappa+1},$$

where $\eta_1, \dots, \eta_{\kappa+1}$ are scalars to be determined. No matter how $\eta_1, \dots, \eta_{\kappa+1}$ are chosen, φ_* belongs to $\mathcal{N}$ because $\mathcal{N}$ is a subspace, and so $L(\varphi_*, \varphi_*) \leqslant 0$. We choose $\eta_1, \dots, \eta_{\kappa+1}$, not all zero, such that

$$\int_\Delta \varphi_*(u)S_0(u)e_j(u)\, d\sigma(u) = 0, \qquad j = 1, \dots, \kappa. \tag{4.4}$$

Such a choice is possible because (4.4) is a system of κ equations in $\kappa + 1$ unknowns. Then by (4.2) and (4.4),

$$L(\varphi_*, \varphi_*) = L_0(\varphi_*, \varphi_*) \geqslant 0.$$

Therefore $L(\varphi_*, \varphi_*) = 0$ and so $\varphi_* = 0$ by (4.3). This yields a nontrivial dependence relation for $\varphi_1, \dots, \varphi_{\kappa+1}$. It follows that $\dim \mathcal{N} \leqslant \kappa$, and hence $\mathrm{sq}_-\, L \leqslant \kappa$.

It remains to show that $\mathrm{sq}_-\, L \neq 0$. If $\mathrm{sq}_-\, L = 0$, then $S_0 \in \mathbf{S}_0(\Delta)$ by [13, Theorem A, p. 30]. This is impossible since we assume $S_0 \in \mathbf{S}_\kappa(\Delta)$ with $\kappa \geqslant 1$. Therefore $\mathrm{sq}_-\, L \neq 0$. This completes the proof of (2).

(3) Set

$$P_- = \text{projection on } L^2_- = [u^{-1}, u^{-2}, \dots],$$
$$Q_- = \text{projection on } uL^2_- = [1, u^{-1}, u^{-2}, \dots],$$

where $[\cdot]$ indicates closed span in L^2. Then $P_- = u^{-1}Q_-u$.

Assume $\mathrm{sq}_-\, L = \kappa$. Define a Hermitian form $L_1(h,k)$ on $H^2 \times H^2$ by

$$L_1(h,k) = L(uh, uk), \qquad h,k \in H^2.$$

Since L_1 is essentially a restriction of L, $\mathrm{sq}_-\, L_1 = \kappa' \leqslant \kappa$. By (4.1), for all $\varphi, \psi \in L^2$,

$$L(\varphi,\psi) = \langle Q_-\varphi, \psi\rangle_{L^2} - \langle Q_- S_0\varphi, S_0\psi\rangle_{L^2}.$$

If $\varphi, \psi \in uH^2$, then $Q_-\varphi = 0$, and so

$$L(\varphi,\psi) = -\langle Q_- S_0\varphi, S_0\psi\rangle_{L^2}.$$

For any $h,k \in H^2$,

$$\begin{aligned} L_1(h,k) = L(uh,uk) &= -\langle Q_- S_0 uh, S_0 uk\rangle_{L^2} = -\langle P_- S_0 h, S_0 k\rangle_{L^2} \\ &= -\langle P_- S_0 h, P_- S_0 k\rangle_{L^2} = -\langle H_{S_0} h, H_{S_0} k\rangle_{L^2_-} = -\big\langle H_{S_0}^* H_{S_0} h, k\big\rangle_{H^2}, \end{aligned}$$

where $H_{S_0} \colon H^2 \to L^2_-$ is the Hankel operator with symbol S_0 (see Appendix A). By Proposition 1.2,

$$\kappa' = \mathrm{sq}_-\, L_1 = \mathrm{sq}_-\, (-H_{S_0}^* H_{S_0}) = \operatorname{rank} H_{S_0}^* H_{S_0} = \operatorname{rank} H_{S_0}.$$

By Kronecker's Theorem (Theorem A.1), there is a Blaschke product B_0 of degree κ' such that $B_0 S_0 = f_0 \in H^\infty$. Since we assume $|S_0(u)| \leqslant 1$ a.e. on $\partial\mathbb{D}$, $|f_0(u)| \leqslant 1$ a.e. on $\partial\mathbb{D}$, and so $f_0 \in \mathbf{S}_0(\partial\mathbb{D})$. If B_0 and f_0 have common zeros in $\mathbb{D}$, we can remove them by dividing B_0 and f_0 by appropriate Blaschke factors. Then we obtain a Blaschke product B of degree $\kappa'' \leqslant \kappa'$ and $f \in \mathbf{S}_0(\partial\mathbb{D})$ such that

$$S_0 = B^{-1} f \in \mathbf{S}_{\kappa''}(\partial\mathbb{D}).$$

By part (2) of the theorem proved above,

$$\kappa = \mathrm{sq}_-\, L \leqslant \kappa''.$$

By construction, $\kappa'' \leqslant \kappa' \leqslant \kappa$, so $\kappa'' = \kappa' = \kappa$. Thus $S_0 \in \mathbf{S}_\kappa(\partial\mathbb{D})$, and the sufficiency part of (3) follows.

Conversely, assume $S_0 \in \mathbf{S}_\kappa(\partial\mathbb{D})$. Again by part (2),

$$\mathrm{sq}_-\, L = \kappa_1 \leqslant \kappa.$$

Hence by what we just proved, $S_0 \in \mathbf{S}_{\kappa_1}(\partial\mathbb{D})$. Since $S_0 \in \mathbf{S}_\kappa(\partial\mathbb{D})$ by assumption, this is possible only if $\kappa_1 = \kappa$. Thus $\mathrm{sq}_-\, L = \kappa$, and the necessity part of (3) follows. □

5. Boundary problems on a half-plane

The half-plane boundary theorems in [3, Section 4] are also withdrawn (Appendix B). We shall similarly reformulate them here as open problems and give partial results analogous to the disc case. We omit [3, Theorem 4.5] in the interest of brevity.

The **generalized Schur class** $\mathbf{S}_\kappa(\mathbb{C}_+)$ on the upper half-plane is the set of functions

$$S(z) = S_0\Big(\frac{z-i}{z+i}\Big),$$

where S_0 belongs to $\mathbf{S}_\kappa(\mathbb{D})$. The **generalized Nevanlinna class** $\mathbf{N}_\kappa(\mathbb{C}_+)$ is the set of functions $f(z)$ which are analytic on a subregion Ω of $\mathbb{C}_+$ such that the Hermitian kernel $[f(z) - \overline{f(\zeta)}]/(z-\bar\zeta)$ has κ negative squares on $\Omega \times \Omega$. If $S(z)$ belongs to $\mathbf{S}_\kappa(\mathbb{C}_+)$, then

$$f(z) = i\,\frac{1+S(z)}{1-S(z)} \tag{5.1}$$

defines a function in $\mathbf{N}_\kappa(\mathbb{C}_+)$, and every function in $\mathbf{N}_\kappa(\mathbb{C}_+)$ is obtained in this way; when $\kappa = 0$ we exclude $S(z) \equiv 1$ from this correspondence. The associated boundary classes are denoted $\mathbf{S}_\kappa(\mathbb{R})$ and $\mathbf{N}_\kappa(\mathbb{R})$. Given a Borel subset Δ of $\mathbb{R}$, $\mathbf{S}_\kappa(\Delta)$ and $\mathbf{N}_\kappa(\Delta)$ are the spaces of restrictions to Δ. Let $H^2(\mathbb{C}_\pm)$ be the Hardy classes for the upper and lower half-planes, $H^2_\pm(\mathbb{R})$ their boundary classes. We note that $L^2(\mathbb{R})$ is the orthogonal direct sum of $H^2_-(\mathbb{R})$ and $H^2_+(\mathbb{R})$.

Problem 5.1 (Cf. [3, Theorem 4.1]). *Let S_0 be a measurable complex-valued function on a Borel subset Δ of $\mathbb{R}$ such that $|S_0(u)| \leqslant 1$ a.e. on Δ, and let κ be a nonnegative integer. Define a Hermitian form on $L^2(\Delta) \times L^2(\Delta)$ by*

$$L(\varphi,\psi) = \lim_{\epsilon\downarrow 0} \frac{i}{2}\int_\Delta\int_\Delta \frac{1 - S_0(s)\overline{S_0(t)}}{s-t+i\varepsilon}\,\varphi(s)\overline{\psi(t)}\,ds\,dt, \qquad \varphi,\psi \in L^2(\Delta).$$

Does it follow that $S_0 \in \mathbf{S}_\kappa(\Delta)$ if and only if $\operatorname{sq}_- L = \kappa$?

By [13, pp. 33–34], for all $\varphi,\psi \in L^2(\Delta)$,

$$\begin{aligned}
&\lim_{\epsilon\downarrow 0} \frac{i}{2}\int_\Delta\int_\Delta \frac{\varphi(s)\overline{\psi(t)}}{s-t+i\epsilon}\,ds\,dt \\
&\qquad = \sum_{j=0}^\infty \left(\int_\Delta \Big(\frac{t-i}{t+i}\Big)^j \frac{1}{t+i}\,\varphi(t)\,dt\right)\overline{\left(\int_\Delta \Big(\frac{t-i}{t+i}\Big)^j \frac{1}{t+i}\,\psi(t)\,dt\right)} \\
&\qquad = \pi\,\langle \mathbb{P}_-(\varphi 1_\Delta), \psi 1_\Delta\rangle_{L^2(\mathbb{R})}\,,
\end{aligned} \tag{5.2}$$

where $\mathbb{P}_-$ is the projection from $L^2(\mathbb{R})$ onto $H^2_-(\mathbb{R})$. Hence the Hermitian form L is well defined.

Theorem 5.2. *In Problem* 5.1*:*

(1) *The answer is affirmative for* $\kappa = 0$.

(2) *If* $S_0 \in \mathbf{S}_\kappa(\Delta)$, *then* $\mathrm{sq}_- L \leqslant \kappa$ *with equality for* $\kappa = 0$ *and* $\kappa = 1$. *Moreover,* $\mathrm{sq}_- L \neq 0$ *for all* $\kappa \geqslant 1$.

(3) *If* $\Delta = \mathbb{R}$, *then* $S_0 \in \mathbf{S}_\kappa(\mathbb{R})$ *if and only if* $\mathrm{sq}_- L = \kappa$.

Proof. (1) This follows from [13, Theorem B, p. 31].

(2) By (1) it is sufficient to treat the case $\kappa \geqslant 1$. Since $S_0 \in \mathbf{S}_\kappa(\mathbb{C}_+)$, we can write $S_0(x) = B(x)^{-1} f(x)$ a.e. on $\mathbb{R}$, where B is a Blaschke product on $\mathbb{C}_+$ of degree κ and $f \in \mathbf{S}_0(\mathbb{C}_+)$. The Hermitian form

$$L_0(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \frac{i}{2} \int_\Delta \int_\Delta \frac{1 - f(s)\overline{f(t)}}{s - t + i\varepsilon} \varphi(s)\overline{\psi(t)}\, ds\, dt, \qquad \varphi, \psi \in L^2(\Delta),$$

is nonnegative by the known case $\kappa = 0$. Write

$$\frac{1 - S_0(s)\overline{S_0(t)}}{s - t + i\varepsilon} = \frac{1 - f(s)\overline{f(t)}}{s - t + i\varepsilon} - S_0(s)\, \frac{1 - B(s)\overline{B(t)}}{s - t + i\varepsilon}\, \overline{S_0(t)}.$$

By induction,

$$1 - B(z)\overline{B(w)} = \frac{2}{i}\,(z - \bar{w}) \sum_{j=1}^{\kappa} e_j(z)\overline{e_j(w)},$$

where each $e_1(z), \dots, e_\kappa(z)$ is rational and belongs to $H^2(\mathbb{C}_+)$. Thus

$$\frac{i}{2}\, \frac{1 - S_0(s)\overline{S_0(t)}}{s - t + i\varepsilon} = \frac{i}{2}\, \frac{1 - f(s)\overline{f(t)}}{s - t + i\varepsilon} - \frac{s - t}{s - t + i\varepsilon} \sum_{j=1}^{\kappa} S_0(s) e_j(s) \overline{e_j(t) S_0(t)}.$$

Then for all φ, ψ in $L^2(\Delta)$,

$$\begin{aligned} L(\varphi, \psi) &= L_0(\varphi, \psi) \\ &\qquad - \lim_{\varepsilon \downarrow 0} \int_\Delta \int_\Delta \frac{s - t}{s - t + i\varepsilon} \sum_{j=1}^{\kappa} \varphi(s) S_0(s) e_j(s) \overline{e_j(t) S_0(t) \psi(t)}\, ds\, dt. \end{aligned}$$

Here $\varphi S_0 e_j$ and $\psi S_0 e_j$ are in $L^1(\Delta)$ for all $j = 1, \dots, \kappa$, and

$$\left| \frac{s - t}{s - t + i\varepsilon} \right|^2 = \frac{(s - t)^2}{(s - t)^2 + \varepsilon^2} \leqslant 1$$

for all $s, t \in \Delta$ and $\varepsilon > 0$. Hence

$$L(\varphi, \psi) = L_0(\varphi, \psi) - \sum_{j=1}^{\kappa} \int_\Delta \varphi(s) S_0(s) e_j(s)\, ds \int_\Delta \overline{e_j(t) S_0(t) \psi(t)}\, dt.$$

This identity is parallel to (4.2) in the proof of Theorem 4.2(2). We use it in the same way to show that any subspace $\mathcal{N}$ of $L^2(\Delta)$ which is strictly negative with respect to L has dimension at most κ. Hence $\mathrm{sq}_- L \leqslant \kappa$ by Proposition 1.1. The last statement in (2) also follows as in Theorem 4.2(2).

(3) Assume $\mathrm{sq}_- L = \kappa$. By (5.2), for all φ, ψ in $L^2(\mathbb{R})$,

$$\lim_{\varepsilon \downarrow 0} \frac{i}{2} \int_\mathbb{R} \int_\mathbb{R} \frac{\varphi(s)\overline{\psi(t)}}{s - t + i\varepsilon}\, ds\, dt = \pi \langle \mathbb{P}_- \varphi, \psi \rangle_{L^2(\mathbb{R})},$$

and hence

$$L(\varphi,\psi) = \pi\langle \mathbb{P}_-\varphi, \psi\rangle_{L^2(\mathbb{R})} - \pi\langle \mathbb{P}_- S_0\varphi, S_0\psi\rangle_{L^2(\mathbb{R})}.$$

Let L_1 be the restriction of L to $H^2_+(\mathbb{R}) \times H^2_+(\mathbb{R})$. For φ, ψ in $H^2_+(\mathbb{R})$,

$$L_1(\varphi,\psi) = -\pi\langle \mathbb{P}_- S_0\varphi, S_0\psi\rangle_{L^2(\mathbb{R})},$$

because $\mathbb{P}_-\varphi = 0$. Then

$$\begin{aligned} L_1(\varphi,\psi) &= -\pi\langle \mathbb{P}_- S_0\varphi, \mathbb{P}_- S_0\psi\rangle_{L^2(\mathbb{R})} \\ &= -\pi\langle \mathcal{H}_{S_0}\varphi, \mathcal{H}_{S_0}\psi\rangle_{H^2_-(\mathbb{R})} \\ &= -\pi\big\langle \mathcal{H}^*_{S_0}\mathcal{H}_{S_0}\varphi, \psi\big\rangle_{H^2_+(\mathbb{R})}, \end{aligned}$$

where $\mathcal{H}_{S_0}\colon H^2_+(\mathbb{R}) \to H^2_-(\mathbb{R})$ is the Hankel operator with symbol S_0 (see Appendix A). Since L_1 is a restriction of L,

$$\operatorname{sq}_- L_1 = \kappa' \leqslant \kappa.$$

By Proposition 1.2,

$$\kappa' = \operatorname{sq}_- L_1 = \operatorname{sq}_- (-\mathcal{H}^*_{S_0}\mathcal{H}_{S_0}) = \operatorname{rank} \mathcal{H}^*_{S_0}\mathcal{H}_{S_0} = \operatorname{rank} \mathcal{H}_{S_0}.$$

By Kronecker's Theorem for the half-plane (Theorem A.3), there is a Blaschke product B_0 on the half-plane of degree κ' such that the function

$$f_0 = B_0 S_0$$

belongs to $H^\infty(\mathbb{R})$. Since we assume that $|S_0(x)| \leqslant 1$ a.e., f_0 is bounded by one a.e. on $\mathbb{R}$, and hence $f_0 \in \mathbf{S}_0(\mathbb{R})$. Viewed as functions on $\mathbb{C}_+$, f_0 and B_0 may have common zeros. These can be removed by cancelling appropriate Blaschke factors. We thus obtain a Blaschke product B of degree $\kappa'' \leqslant \kappa'$ and an $f \in \mathbf{S}_0(\mathbb{R})$ having no common zeros, such that

$$S_0 = B^{-1} f \in \mathbf{S}_{\kappa''}(\mathbb{R}).$$

By part (2) of the theorem,

$$\kappa = \operatorname{sq}_- L \leqslant \kappa''.$$

Since $\kappa'' \leqslant \kappa' \leqslant \kappa$ by construction, $\kappa'' = \kappa' = \kappa$. Therefore $S_0 \in \mathbf{S}_\kappa(\mathbb{R})$, as was to be shown.

For the converse direction, suppose $S_0 \in \mathbf{S}_\kappa(\mathbb{R})$. Apply part (2) again to conclude that $\operatorname{sq}_- L \leqslant \kappa$. By what we just proved, it follows that $S_0 \in \mathbf{S}_{\kappa_1}(\mathbb{R})$, where $\kappa_1 = \operatorname{sq}_- L$. Then $S_0 \in \mathbf{S}_{\kappa_1}(\mathbb{R}) \cap \mathbf{S}_\kappa(\mathbb{R})$, and this is possible only if $\kappa_1 = \kappa$. □

Problem 5.3 (Cf. [3, Theorem 4.2]). *Let f_0 be a measurable complex-valued function on a Borel subset Δ of $\mathbb{R}$ such that $\operatorname{Im} f_0(x) \geqslant 0$ a.e. on Δ, and let κ be a nonnegative integer. Let $\mathcal{D}$ be the linear space of measurable functions φ on Δ such that φ and $f_0\varphi$ belong to $L^2(\Delta)$. Define a Hermitian form on $\mathcal{D} \times \mathcal{D}$ by*

$$L(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{f_0(s) - \overline{f_0(t)}}{s - t + i\varepsilon} \varphi(s)\overline{\psi(t)}\, ds\, dt, \qquad \varphi, \psi \in \mathcal{D}. \tag{5.3}$$

Does it follow that $f_0 \in \mathbf{N}_\kappa(\Delta)$ if and only if $\operatorname{sq}_- L = \kappa$?

Theorem 5.4. *In Problem 5.3:*

(1) *The answer is affirmative for $\kappa = 0$.*

(2) *If $f_0 \in \mathbf{N}_\kappa(\Delta)$, then $\operatorname{sq}_- L \leqslant \kappa$, with equality when $\kappa = 0$ or $\kappa = 1$. Moreover, $\operatorname{sq}_- L \neq 0$ for all $\kappa \geqslant 1$.*

(3) *When $\Delta = \mathbb{R}$, then $f_0 \in \mathbf{N}_\kappa(\mathbb{R})$ if and only if $\operatorname{sq}_- L = \kappa$.*

Proof. Part (1) follows from the theorem in [13, p. 34].

For parts (2) and (3), set $S_0(x) = (f_0(x) - i)/(f_0(x) + i)$. Then

$$1 - S_0(s)\overline{S_0(t)} = \frac{2i\big[\overline{f_0(t)} - f_0(s)\big]}{\big[f_0(s) + i\big]\big[\overline{f_0(t)} - i\big]}\,. \tag{5.4}$$

Hence $|S_0(x)| \leqslant 1$ a.e. on Δ because $\operatorname{Im} f_0(x) \geqslant 0$ a.e. on Δ. Define a Hermitian form on $L^2(\Delta) \times L^2(\Delta)$ by

$$K(\varphi, \psi) = \lim_{\varepsilon \downarrow 0} \frac{i}{2} \int_{\Delta} \int_{\Delta} \frac{1 - S_0(s)\overline{S_0(t)}}{s - t + i\varepsilon} \varphi(s)\overline{\psi(t)}\, ds\, dt$$

for all $\varphi, \psi \in L^2(\Delta)$. We show that

$$\operatorname{sq}_- K = \operatorname{sq}_- L\,. \tag{5.5}$$

By (5.4),

$$\begin{aligned} K(\varphi, \psi) &= \lim_{\varepsilon \downarrow 0} \frac{i}{2} \int_{\Delta} \int_{\Delta} \frac{2i\big[\overline{f_0(t)} - f_0(s)\big]}{\big[f_0(s) + i\big]\big[\overline{f_0(t)} - i\big]} \frac{\varphi(s)\overline{\psi(t)}}{s - t + i\varepsilon}\, ds\, dt \\ &= \lim_{\varepsilon \downarrow 0} \int_{\Delta} \int_{\Delta} \frac{f_0(s) - \overline{f_0(t)}}{s - t + i\varepsilon} \frac{\varphi(s)}{f_0(s) + i} \frac{\overline{\psi(t)}}{\overline{f_0(t)} - i}\, ds\, dt \\ &= \lim_{\varepsilon \downarrow 0} \int_{\Delta} \int_{\Delta} \frac{f_0(s) - \overline{f_0(t)}}{s - t + i\varepsilon} \tilde{\varphi}(s)\, \overline{\tilde{\psi}(t)}\, ds\, dt \\ &= L(\tilde{\varphi}, \tilde{\psi}), \end{aligned} \tag{5.6}$$

where

$$\tilde{\varphi}(s) = \frac{\varphi(s)}{f_0(s) + i}, \qquad \tilde{\psi}(t) = \frac{\psi(t)}{f_0(t) + i}\,.$$

To deduce (5.5), we need to show that $V\colon \varphi(x) \to \varphi(x)/\big[f_0(x) + i\big]$ is a one-to-one mapping from $L^2(\Delta)$ onto $\mathcal{D}$. Let $\varphi(x) \in L^2(\Delta)$, and set $\tilde{\varphi}(x) =$

$\varphi(x)/\big[f_0(x)+i\big]$. Decompose $f_0(x)$ into its real and imaginary parts, $f_0(x) = u(x) + iv(x)$. Then $v(x) \geqslant 0$ a.e., and so

$$\left|\frac{1}{f_0(x)+i}\right|^2 = \frac{1}{u(x)^2 + \big[v(x)+1\big]^2} \leqslant 1,$$
$$\left|\frac{f_0(x)}{f_0(x)+i}\right|^2 = \frac{u(x)^2 + v(x)^2}{u(x)^2 + \big[v(x)+1\big]^2} \leqslant 1.$$

Therefore $\tilde\varphi, f_0\tilde\varphi$ belong to $L^2(\Delta)$, and hence $\tilde\varphi$ is in $\mathcal{D}$. Conversely, if $\tilde\varphi \in \mathcal{D}$, then $\tilde\varphi, f_0\tilde\varphi$ belong to $L^2(\Delta)$, and hence $\varphi(x) = \big[f_0(x)+i\big]\tilde\varphi(x)$ is in $L^2(\Delta)$. Thus V is one-to-one and onto, and hence (5.5) follows from (5.6).

By (5.1), $f_0 \in \mathbf{N}_\kappa(\Delta)$ if and only if $S_0 \in \mathbf{S}_\kappa(\Delta)$. Thus by (5.5), parts (2) and (3) of the theorem follow from the corresponding parts of Theorem 5.2. □

The classical Loewner Theorem uses difference-quotient kernels and applies to real-valued functions [3, p. 38]. In Problem 5.3, the Hermitian form (5.3) can be written in an analogous form when f_0 is real valued.

Problem 5.5 (Cf. [3, Theorem 4.4]). *Let f_0 be a measurable real-valued function on a Borel subset Δ of $\mathbb{R}$, and let κ be a nonnegative integer. Let $\mathcal{D}$ be the linear space of measurable functions φ on Δ such that φ and $f_0\varphi$ belong to $L^2(\Delta)$. Define a Hermitian form on $\mathcal{D}\times\mathcal{D}$ by*

$$L(\varphi,\psi) = \lim_{\varepsilon\downarrow 0} \iint\limits_{|t-s|>\varepsilon} \frac{f_0(s)-f_0(t)}{s-t}\,\varphi(s)\overline{\psi(t)}\,ds\,dt, \qquad \varphi,\psi\in\mathcal{D}. \tag{5.7}$$

Does it follow that $f_0 \in \mathbf{N}_\kappa(\Delta)$ if and only if $\operatorname{sq}_- L = \kappa$*?*

The double integral in (5.7) is taken over $\{(s,t)\in\Delta\times\Delta\colon |t-s|>\varepsilon\}$, but we write simply $|t-s|>\varepsilon$ when no confusion can arise.

Theorem 5.6. *In Problem* 5.5*:*

(1) *The answer is affirmative for $\kappa = 0$.*

(2) *If $f_0 \in \mathbf{N}_\kappa(\Delta)$, then* $\operatorname{sq}_- L \leqslant \kappa$*, with equality when $\kappa = 0$ or $\kappa = 1$. Moreover,* $\operatorname{sq}_- L \neq 0$ *for all $\kappa \geqslant 1$.*

(3) *When $\Delta = \mathbb{R}$, then $f_0 \in \mathbf{N}_\kappa(\mathbb{R})$ if and only if* $\operatorname{sq}_- L = \kappa$.

As before, let $\mathbb{P}_-$ be the projection from $L^2(\mathbb{R})$ onto $H^2_-(\mathbb{R})$. Then [14, p. 113]

$$\mathbb{P}_- = \frac{1}{2}(1+iH), \tag{5.8}$$

where $H\colon L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is the Hilbert transform. This is defined by

$$(H\varphi)(x) = PV\frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(t)}{t-x}\,dt = \lim_{\varepsilon\downarrow 0}\frac{1}{\pi}\int_{|t-x|>\varepsilon} \frac{\varphi(t)}{t-x}\,dt$$

for all φ in $L^2(\mathbb{R})$. The limit exists pointwise a.e. on $\mathbb{R}$ and in the norm of $L^2(\mathbb{R})$. The operator iH is selfadjoint and unitary. For any Borel subset Δ of $\mathbb{R}$, let H_Δ be the compression of H to $L^2(\Delta)$. Then for all φ in $L^2(\Delta)$,

$$(H_\Delta\varphi)(x) = PV\frac{1}{\pi}\int_\Delta \frac{\varphi(t)}{t-x}\,dt$$

a.e. on Δ. The operator iH_Δ is selfadjoint, and therefore $H_\Delta^* = -H_\Delta$.

Proof of Theorem 5.6. By (5.8), we can write (5.2) in the form

$$\begin{aligned}\lim_{\epsilon\downarrow 0}\int_\Delta\int_\Delta \frac{\varphi(s)\overline{\psi(t)}}{s-t+i\epsilon}\,ds\,dt &= \frac{2\pi}{i}\,\langle \mathbb{P}_-(\varphi 1_\Delta), \psi 1_\Delta\rangle_{L^2(\mathbb{R})}\\ &= \frac{\pi}{i}\,\langle (1+iH)(\varphi 1_\Delta), \psi 1_\Delta\rangle_{L^2(\mathbb{R})}\\ &= \pi\,\langle (H_\Delta - i)\varphi, \psi\rangle_{L^2(\Delta)}.\end{aligned}$$

Hence if $\varphi, f_0\varphi$ and $\psi, f_0\psi$ belong to $L^2(\Delta)$, the Hermitian form (5.3) is given by

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\int_{\mathbb{R}}\int_{\mathbb{R}} &\frac{f_0(s)-\overline{f_0(t)}}{s-t+i\varepsilon}\,\varphi(s)\overline{\psi(t)}\,ds\,dt\\ &= \pi\,\langle (H_\Delta - i)(f_0\varphi), \psi\rangle_{L^2(\Delta)}\\ &\qquad - \pi\,\langle (H_\Delta - i)\varphi, f_0\psi\rangle_{L^2(\Delta)}\\ &= \pi\,\langle H_\Delta(f_0\varphi), \psi\rangle_{L^2(\Delta)} + \pi\,\langle \varphi, H_\Delta(f_0\psi)\rangle_{L^2(\Delta)}\\ &\qquad + 2\pi\,\langle (\operatorname{Im} f_0)\varphi, \psi\rangle_{L^2(\Delta)}.\end{aligned}$$

The last term on the right side is zero because we assume that f_0 is real valued. For (5.7) we obtain

$$\begin{aligned}\lim_{\varepsilon\downarrow 0}\iint\limits_{|t-s|>\epsilon} &\frac{f_0(s)-f_0(t)}{s-t}\,\varphi(s)\overline{\psi(t)}\,ds\,dt\\ &= \lim_{\varepsilon\downarrow 0}\int_\Delta\left(\int_{|t-s|>\epsilon}\frac{f_0(s)\varphi(s)}{s-t}\,ds\right)\overline{\psi(t)}\,dt\\ &\qquad - \lim_{\varepsilon\downarrow 0}\int_\Delta\left(\int_{|t-s|>\epsilon}\frac{\varphi(s)}{s-t}\,ds\right)f_0(t)\overline{\psi(t)}\,dt\\ &= \pi\,\langle H_\Delta(f_0\varphi), \psi\rangle_{L^2(\Delta)} - \pi\,\langle H_\Delta\varphi, f_0\psi\rangle_{L^2(\Delta)}\\ &= \pi\,\langle H_\Delta(f_0\varphi), \psi\rangle_{L^2(\Delta)} + \pi\,\langle \varphi, H_\Delta(f_0\psi)\rangle_{L^2(\Delta)}.\end{aligned}$$

Thus (5.3) and (5.7) coincide when f_0 is real valued, and so the result follows from Theorem 5.4. □

Appendix A. Hankel operators

In this appendix, we review Kronecker's theorem for the disc and half-plane. The standard source for Hankel operators is Peller [12].

Let P_- be the projection from $L^2(\partial\mathbb{D})$ onto the closed span L^2_- of $u^{-1}, u^{-2}, \dots$. The Hardy space H^2 is identified with the associated space of boundary functions in $L^2(\partial\mathbb{D})$. Given $\varphi \in L^\infty(\partial\mathbb{D})$, the Hankel operator $H_\varphi\colon H^2 \to L^2_-$ is defined by

$$H_\varphi f = P_- \varphi f, \qquad f \in H^2.$$

The identity

$$P_- S H_\varphi = H_\varphi S \tag{A.1}$$

is verified by checking the action of each side on u^n for all $n \geqslant 0$. By (A.1), the kernel of H_φ is invariant under the shift operator S.

Theorem A.1 (Kronecker's Theorem for the Unit Circle). *Let $\varphi \in L^\infty(\partial\mathbb{D})$.*

(1) *If* $\operatorname{rank} H_\varphi = \kappa < \infty$, *there is a Blaschke product B of degree κ such that $B\varphi \in H^\infty$.*
(2) *If $B\varphi \in H^\infty$ for some finite Blaschke product B, then* $\operatorname{rank} H_\varphi \leqslant \deg B$.

Proof. (1) Suppose $\operatorname{rank} H_\varphi = \kappa$. Since H_φ is one-to-one on the orthogonal complement of its kernel,

$$\dim \left(H^2 \ominus \ker H_\varphi\right) = \kappa. \tag{A.2}$$

Since $\ker H_\varphi$ is invariant under S, $\ker H_\varphi = BH^2$ for some inner function B, by Beurling's theorem. By (A.2), B is a Blaschke product of degree κ. Since H_φ is zero on BH^2,

$$P_- \varphi B = H_\varphi B = 0.$$

Therefore $B\varphi \in H^2 \cap L^\infty(\partial\mathbb{D}) = H^\infty$.

(2) Assume B is Blaschke and $B\varphi \in H^\infty$. Then $B\varphi \in H^2$ and so $H_\varphi B = P_-\varphi B = 0$. Since $\ker H_\varphi$ is invariant under S, $BH^2 \subseteq \ker H_\varphi$. Therefore

$$\operatorname{rank} H_\varphi = \dim\left(H^2 \ominus \ker H_\varphi\right) \leqslant \dim\left(H^2 \ominus BH^2\right) = \deg B,$$

as was to be shown. □

To derive a version of Kronecker's Theorem for the real line, we define mappings α and $\beta = \alpha^{-1}$ connecting the unit disc and upper half-plane by

$$\alpha(w) = i\,\frac{1+w}{1-w}, \qquad w \in \bar{\mathcal{D}}\backslash\{1\},$$
$$\beta(z) = \frac{z-i}{z+i}, \qquad z \in \bar{\mathbb{C}}_+\backslash\{\infty\}.$$

We use a natural unitary operator $U\colon L^2(\partial\mathbb{D}) \to L^2(\mathbb{R})$, which is defined by

$$Uf = F,$$

where $f \in L^2(\partial\mathbb{D})$ and $F \in L^2(\mathbb{R})$ are connected by

$$F(t) = \frac{1}{\sqrt{\pi}}\,\frac{1}{t+i}\,f(\beta(t)), \qquad t \in \mathbb{R}\backslash\{\infty\}, \tag{A.3}$$

$$f(e^{i\theta}) = \frac{2i\sqrt{\pi}}{1-e^{i\theta}}\,F(\alpha(e^{i\theta})), \qquad e^{i\theta} \in \partial\mathbb{D}\backslash\{1\}. \tag{A.4}$$

To check that U is unitary, we show that both mappings $f \to F$ and $F \to f$ are isometric. The following formulas to change variables are given in (5-4) and (5-5) in [14]. For $\varphi \in L^1(\partial\mathbb{D})$,

$$\frac{1}{2\pi}\int_0^{2\pi} \varphi(e^{i\theta})\,d\theta = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(\beta(t))}{t^2+1}\,dt. \tag{A.5}$$

For $\psi \in L^1(\mathbb{R})$,

$$\int_{\mathbb{R}} \psi(t)\,dt = \int_0^{2\pi} \frac{-2e^{i\theta}}{(1-e^{i\theta})^2}\,\psi(\alpha(e^{i\theta}))\,d\theta. \tag{A.6}$$

First suppose $f \in L^2(\partial\mathbb{D})$ and F is defined by (A.3). By (A.6),

$$\begin{aligned}
\int_{\mathbb{R}} |F(t)|^2\,dt &= \int_{\mathbb{R}} \overbrace{\frac{1/\pi}{|t+i|^2}\,|f(\beta(t))|^2}^{\psi(t)}\,dt \\
&\overset{(A.6)}{=} \int_0^{2\pi} \frac{-2e^{i\theta}}{(1-e^{i\theta})^2}\,\psi(\alpha(e^{i\theta}))\,d\theta \\
&= \int_0^{2\pi} \frac{-2e^{i\theta}}{(1-e^{i\theta})^2}\,\frac{1/\pi}{|\alpha(e^{i\theta})+i|^2}\,|f(\beta(\alpha(e^{i\theta})))|^2\,d\theta \\
&= \int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta.
\end{aligned}$$

Suppose $F \in L^2(\mathbb{R})$ is given and f is defined by (A.4). Then

$$\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta})|^2\,d\theta &= \frac{1}{2\pi}\int_0^{2\pi} \overbrace{\left|\frac{2i\sqrt{\pi}}{1-e^{i\theta}}\,F(\alpha(e^{i\theta}))\right|^2}^{\varphi(e^{i\theta})}\,d\theta \\
&\overset{(A.5)}{=} \frac{1}{\pi}\int_{\mathbb{R}} \frac{\varphi(\beta(t))}{t^2+1}\,dt \\
&= \frac{1}{\pi}\int_{\mathbb{R}} \frac{1}{t^2+1}\left|\frac{2i\sqrt{\pi}}{1-\beta(t)}\,F(\alpha(\beta(t)))\right|^2\,dt \\
&= \int_{\mathbb{R}} |F(t)|^2\,dt.
\end{aligned}$$

The unitarity of U follows.

Define $H^2_\pm(\mathbb{R})$ and $H^\infty(\mathbb{R})$ as the spaces of boundary functions of the Hardy classes $H^2(\mathbb{C}_\pm)$ and $H^\infty(\mathbb{C}_+)$. Write $L^2(\partial\mathbb{D}) = H^2 \oplus H^2_-$, where

$$H^2 = [1, u, u^2, \dots], \qquad H^2_- = [u^{-1}, u^{-2}, \dots],$$

and let $P_\pm$ be the projections on H^2 and H^2_-. One can show that

$$UH^2 = H^2_+(\mathbb{R}), \qquad UH^2_- = H^2_-(\mathbb{R}),$$

$$L^2(\mathbb{R}) = H^2_+(\mathbb{R}) \oplus H^2_-(\mathbb{R}).$$

Let $\mathbb{P}_\pm$ be the projections from $L^2(\mathbb{R})$ onto $H^2_\pm(\mathbb{R})$. By the preceding relations,

$$\mathbb{P}_\pm = UP_\pm U^{-1}. \tag{A.7}$$

We define the Hankel operator $\mathcal{H}_\psi\colon H^2_+(\mathbb{R}) \to H^2_-(\mathbb{R})$ for any $\psi \in L^\infty(\mathbb{R})$ as in Peller [12, p. 51]:

$$\mathcal{H}_\psi F = \mathbb{P}_- \psi F, \qquad F \in H^2_+(\mathbb{R}).$$

The following result is given in Peller [12, Lemma 8.3 on p. 51].

Theorem A.2. *If $\varphi \in L^\infty(\partial\mathbb{D})$ and $\psi \in L^\infty(\mathbb{R})$ are connected by $\varphi = \psi \circ \alpha$, then*

$$H_\varphi = U^{-1}\mathcal{H}_\psi U.$$

Proof. For all $f \in L^2(\partial\mathbb{D})$,

$$U\varphi f = \frac{1}{\sqrt{\pi}}\,\frac{1}{t+i}\,\varphi(\beta(t))f(\beta(t)) = \psi(t)\,\frac{1}{\sqrt{\pi}}\,\frac{1}{t+i}\,f(\beta(t)) = \psi U f.$$

Therefore

$$\mathbb{P}_- U\varphi f = \mathbb{P}_- \psi U f = \mathcal{H}_\psi U f.$$

By (A.7), $\mathbb{P}_- U = UP_-$ and hence

$$\mathbb{P}_- U\varphi f = UP_-\varphi f = UH_\varphi f.$$

Thus $\mathcal{H}_\psi U f = UH_\varphi f$. □

Theorem A.3 (Kronecker's Theorem for the Real Line). *Let $\psi \in L^\infty(\mathbb{R})$.*

(1) *If* $\operatorname{rank}\mathcal{H}_\psi = \kappa < \infty$*, there is a Blaschke product B for the upper half-plane of degree κ such that $B\psi \in H^\infty(\mathbb{R})$.*

(2) *If $B\psi \in H^\infty(\mathbb{R})$ for some finite Blaschke product B for the upper half-plane, then* $\operatorname{rank}\mathcal{H}_\psi \leqslant \deg B$.

Proof. Set $\varphi = \psi \circ \alpha$. A function B on $\mathbb{C}_+$ is a Blaschke product of degree κ if and only if the function $B \circ \alpha$ is a Blaschke product of degree κ on $\mathbb{D}$.

(1) If $\operatorname{rank}\mathcal{H}_\psi = \kappa$, then $\operatorname{rank} H_\varphi = \kappa$ by the Theorem A.2. By Theorem A.1(1), there is a Blaschke product $\widetilde{B}$ on $\mathbb{D}$ of degree κ such that $\widetilde{B}\varphi$ belongs to H^∞ on $\mathbb{D}$. Then $B = \widetilde{B} \circ \beta$ is a Blaschke product on $\mathbb{C}_+$ of degree κ, and $B\psi = (\widetilde{B}\varphi) \circ \alpha$ belongs to $H^\infty(\mathbb{R})$.

(2) Suppose $B\psi \in H^\infty(\mathbb{C}_+)$ for some finite Blaschke product B. Then $\widetilde{B} = B \circ \alpha$ is a finite Blaschke product on $\mathbb{D}$ such that $\widetilde{B}\varphi = (B\psi) \circ \alpha$ is in $H^\infty(\mathbb{D})$. By Theorem A.1(2), $\operatorname{rank} H_\varphi \leqslant \deg \widetilde{B}$. Therefore $\operatorname{rank}\mathcal{H}_\psi \leqslant \deg B$. □

Appendix B. Brief corrigendum

The first and third paragraphs that follow are quoted verbatim from the Corrigendum [4] *to the original paper* [3]*; the second is a paraphrase from* [4]*. The example at the end is new.*

The Main Theorem in [3, p. 816] has gaps. In the proof of Part (1), the statement on p. 832, line 6, that the operator $X_0 = Y_0 + K$ is bounded is an error, because no reason is given why the finite-rank summand K is bounded. A similar error occurs in Part (2) on p. 833, line 15, where again it is asserted without justification that X_0 is bounded. A correct version of the Main Theorem is obtained by replacing the condition (iii) in ([3, Definition 2.1]) with a stronger version:

(iii′) there is an $M > 0$ such that $\sum_{j=0}^{\infty} |(A^j b, x')|^2 \leqslant M \sum_{j=0}^{\infty} |(A^j c, x')|^2$ for all x' in $\mathcal{D}$.

Condition (iii′) makes X_0 bounded from the start, and then the proof of the Main Theorem goes through as written. The Main Theorem (Alternative Form) on p. 834 is correct as written provided that condition (iii′) is adopted.

A number of applications survive this change. The following results in [3] are true as stated: Theorem 3.1 on Pick-Nevanlinna interpolation, Theorem 3.4 and Corollary 3.5 on Carathéodory-Fejér interpolation, and Theorem 3.6 on Sarason generalized interpolation. (Proofs are given in Section 3 above.)

Other applications do not survive because (iii′) is not satisfied or cannot readily be verified. Theorem 3.2, its corollary, and Theorem 3.7 are in this category and are withdrawn. The boundary results in Theorem 3.8, its corollary, and the results in Section 4 are withdrawn for the same reason. An exception here is the Alternative form of Corollary 3.9 on p. 824, which does not use the Main Theorem and is correct as written. The Example on p. 825 remains valid when (iii) is replaced by (iii′).

Example. Let $\mathcal{M}$ be a linear subspace of a Hilbert space $\mathcal{H}$, X_0 a linear operator on $\mathcal{M}$ into $\mathcal{H}$. We show that the inner product

$$\langle f, g\rangle_{\mathcal{M}} = \langle f, g\rangle_{\mathcal{H}} - \langle X_0 f, X_0 g\rangle_{\mathcal{H}}, \qquad f, g \in \mathcal{M},$$

on $\mathcal{M}$ may have a finite number of negative squares with X_0 unbounded in the norm of $\mathcal{H}$. In fact, let $\mathcal{M}$ be the subspace of polynomials in $\mathcal{H} = L^2(-1,1)$. Let X_0 be the operator

$$X_0 \colon p(x) \to p(0)e(x), \qquad p \in \mathcal{M},$$

where $e(x) \equiv 1$ on $(-1,1)$. Then for all $p, q \in \mathcal{M}$,

$$\langle p, q\rangle_{\mathcal{M}} = \langle p, q\rangle_{\mathcal{H}} - \langle X_0 p, X_0 q\rangle_{\mathcal{H}} = \langle p, q\rangle_{\mathcal{H}} - 2p(0)\overline{q(0)}. \tag{B.1}$$

The operator X_0 is unbounded in the norm of $\mathcal{H}$ because we can make $|p(0)|$ arbitrarily large for p in $\mathcal{M}$ such that $\|p\|_{\mathcal{H}} \leqslant 1$. Such a polynomial can be chosen of the form $p(x) = C(1-x^2)^n$; with C any positive constant, by various means we can choose n large enough that $\|p\|_{\mathcal{H}} \leqslant 1$.

We show that the maximum dimension of a strictly negative subspace of $(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$ is one. If $p(x) = 1 - x^2$, then

$$\langle p, p\rangle_{\mathcal{M}} = \langle p, p\rangle_{\mathcal{H}} - 2|p(0)|^2 = \int_{-1}^{1} (1 - x^2)^2\, dx - 2 = -\frac{14}{15} < 0.$$

Hence there is a one-dimensional strictly negative subspace of $(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$. Let $\mathcal{N}$ be any strictly negative subspace of $(\mathcal{M}, \langle\cdot,\cdot\rangle_{\mathcal{M}})$, so for any p in $\mathcal{N}$, $\langle p, p\rangle_{\mathcal{M}} \leqslant 0$ with equality only for $p \equiv 0$. We show that any two elements p, q of $\mathcal{N}$ are linearly dependent.

Case 1: $p(0) = 0$ or $q(0) = 0$. If e.g. $p(0) = 0$, then $\langle p, p\rangle_{\mathcal{M}} = \langle p, p\rangle_{\mathcal{H}} \geqslant 0$ by (B.1). Since $p \in \mathcal{N}$ and $\mathcal{N}$ is strictly negative, $\langle p, p\rangle_{\mathcal{M}} \leqslant 0$ with equality only for $p \equiv 0$. Therefore $p \equiv 0$, and trivially p and q are linearly dependent.

Case 2: $p(0)q(0) \neq 0$. Set $r = q(0)p - p(0)q$. Then $r \in \mathcal{N}$ and $r(0) = q(0)p(0) - p(0)q(0) = 0$. As above, this implies $r \equiv 0$, and hence p and q are linearly dependent.

A particular instance of this situation is when X_0 is the operator defined in [3, p. 831, line 16]. The example shows that the hypothesis of a finite number of negative squares, by itself, is not sufficient to conclude that X_0 is bounded.

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