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A Multi-item Inventory Model

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A MULTI-ITEM INVENTORY MODEL

BY

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A MULTI-ITEM INVENTORY MODEL

This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser	Date
Head, Mechanical Engineering Department	Date

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I wish to express my deepest appreciation to Dr. Richard P. Covert for introducing me to this topic and guiding the progress of this work from its inception.

I thank Professor John F. Sandfort for his guidance throughout my graduate program.

I also wish to dedicate this thesis to my parents, Mr. and Mrs. S. Ramakrishnan, whose constant encouragement and endless sacrifices have made my higher education abroad possible.

RK

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CHAPTER I

INTRODUCTION

Brief Historical Sketch

During the last fifteen years there has been a rapid growth of interest in scientific inventory control; the use of mathematical models to obtain rules for operating inventory systems. The subject has attracted such wide interest that today every serious student in the Industrial Engineering and related fields is expected to have had some experience with inventory models. Originally, the development of inventory models had practical applications as an immediate objective. Today, inventory models are being developed at many different levels, ranging from the direct application in practical problems to the development of abstract mathematical models.

The earliest derivation of the simple lot size formula was obtained by Ford Harris of the Westinghouse Corporation in 1915 (1). Subsequently, it has been developed and applied independently by many individuals and is often referred to as the "Wilson Formula" after R. H. Wilson, who developed it as an integral part of inventory control procedure which he applied in many organizations.

Basic Structure of Inventory Systems

Inventory Control is concerned with the storage and release of physical items. An item may be stocked at a single location, or it may be stocked at many locations. For example, in the Air Force Supply

System, a spare part for a certain type of aircraft may be stocked at over 100 bases and repair facilities throughout the world. In a privately owned lumber yard, the entire stock may be stocked at a single yard.

When there is more than a single stocking point, there are many possibilities for interaction between the stocking points. The simplest form of interaction involves one stocking point which serves as a warehouse for one or more other stocking points. This leads to a multi-echelon, or multi-level, inventory system. This type of multi-echelon system is illustrated in Figure 1-1. The arrows indicate the normal pattern for the flow of goods through the system. This is referred to as a four echelon system. Customer demands occur only at stocking points in level 1. These stocking points have their stocks replenished by shipments from warehouses at level 2, which in turn receive replenishments for their stock from level 3, which in turn receive shipment from the fourth and final level. In other cases, customer demands might occur at all levels, or stocking points at any level might not only receive shipments from the next highest level but might also get replenishments from any higher level or from the source. Also, it might be allowable, in some occasions, to permit redistribution of stocks among various stocking points at a given level.

Figure 1-2 shows a single echelon inventory system. It consists of one stocking point with a single resupply source. Customer demands arrive at the single stocking point, and at appropriate times orders are placed with the source for replenishing the inventory.

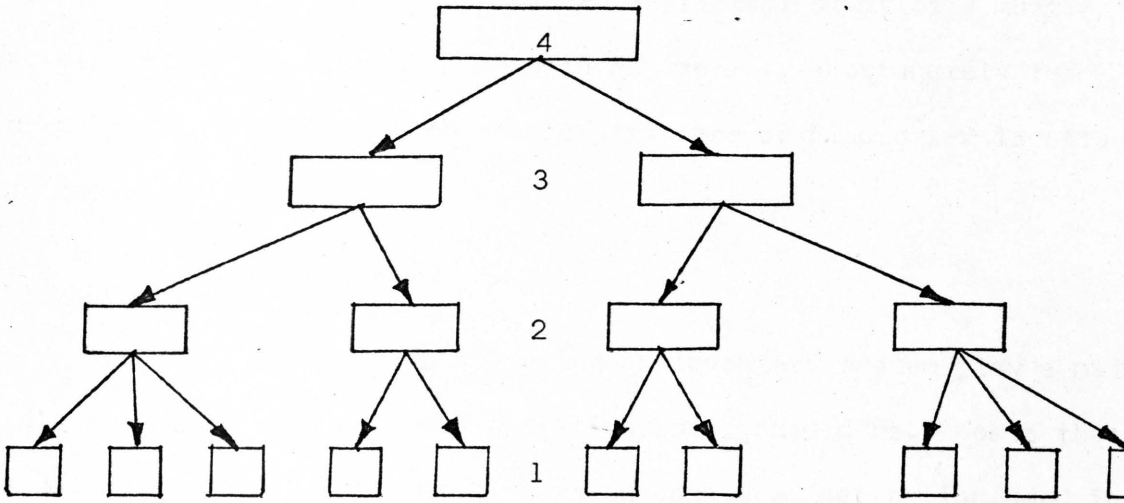


Figure 1-1. A Four Echelon Inventory System.

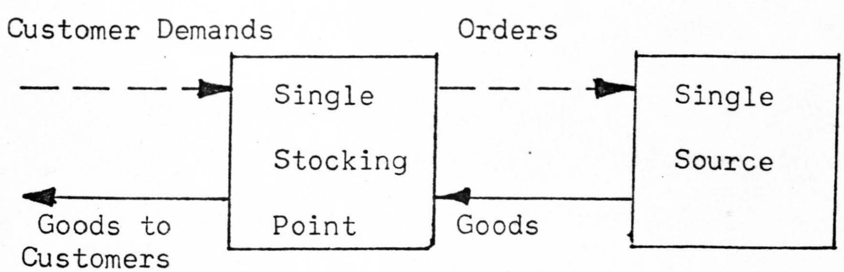


Figure 1-2. A Single Echelon Inventory System.

It is very difficult to make an analytical study of a multi-echelon system of the type shown in Figure 1-1. Fortunately for practical applications, the simple structure of Figure 1-2 is often adequate.

Relevant Costs

The costs incurred in operating an inventory system play a major role in determining what the operating rules should be. Costs that are independent of the operating rules used need not be included in the analysis.

Fundamentally, there are four types of costs which may be important.

1. Procurement costs.
2. Inventory carrying costs.
3. Costs of filling customer's orders.
4. Stockout costs.

Each of the above costs will be discussed in detail.

Procurement Costs

Procurement costs can be divided into two parts. First, there is the amount of money which must be paid to the source from which this procurement is made. The sum paid to this source represents the cost of the units procured. Secondly, there are the internal costs incurred by the inventory system in making a procurement. These are the ordering costs and are usually assumed to be directly proportional to the number of orders.

Inventory Carrying Costs

Inventory carrying costs are the out-of-pocket costs such as insurance, taxes, breakage and pilferage at the storage site and the costs of maintaining the warehouse. An additional expenditure, which is frequently the most important one, is not a direct cost but rather an opportunity cost. This is the outlay incurred by having capital tied up in inventory rather than having it invested elsewhere, and it is equal to the rate of return which the system could obtain from alternative investments. Obviously, when one has funds invested in inventory he foregoes this rate of return, and hence it represents a cost of carrying inventory.

Costs of Filling Customer's Orders

Costs of filling customer's orders normally consist of the expenses of the paperwork system involved in filling the customer's orders. There are other costs such as the salaries of those in the warehouse who are concerned with filling orders, the charges of packing and shipping fees when included in the cost of the item.

Stockout Costs

Stockout costs are the costs incurred by having demands occur when the system is out of stock. Perhaps the most important component of stockout costs is the somewhat intangible goodwill loss, whether the item can be backordered, or whether the sale is lost.

Costs of Operating the Information System

The cost of operating the information system usually includes obtaining the necessary information for decision making. It normally depends upon the type of operating rule used, and may include expenditures such as the fees associated with having a computer continuously update the inventory records, the cost of making an actual inventory count and that of making demand predictions.

Selection of an Operating Rule

One criterion for selecting an operating rule is that of profit maximization. In some cases it may be uneconomical to determine the optimal operating rule, and instead, one optimizes with respect to some subset of the constraints on inventory policy.

One useful tool employed in the solution of inventory problems is a computer simulation. About the best that could be done using simulation is to study a small number of sets of operating rules and to select the best one studied.

The methodology of aggregate plannings, including the application of mathematical organization models to the aggregate planning problem, was first developed as a part of the Post-World War II Management Science Movement (2). Models were developed in which a mathematical optimum solution was possible.

Denzler (3) presented a model which seeks to schedule the production of "m" products over "n" production planning periods to minimize the total annual costs, subject to the constraint imposed by the capacities of the production resources. He used a modified version of the Simplex Method to determine the optimal solutions.

Iglehart (4) developed three models. The first model involved n products and m classes of demand without any constraints. The second one involved a two-product model subject to a constraint on the number of pieces that can be ordered. The third model involved a n-period multi-product problem without constraints.

Parsons (5) presented both the classical inventory models and "production lead time" models. The basic difference between the two models is that the latter considers the real production time, while the former assumes zero production time. In addition the following restrictions were considered independently:

1. Limited hours available for annual set up.
2. An upper limit on the total inventory investment in all products.
3. An upper limit on the total number of production runs which can be made in a year.
4. Available warehouse floor space.

Dzielinski and Manne (7) made a simulation study of a hypothetical multi-item production and inventory system. They used computer techniques to simulate the behaviour of an idealized multi-item production and inventory control system, in which a producer manufactured standardized items for stock. No practical constraints were considered.

Banks (8) applied the lagrangian multiplier technique to determine the economic lot size for a multi-item, single source deterministic model subject to a warehouse space restriction.

Homer (9) presented two models. In the first, a phased delivery model was used to determine the phasing of deliveries which yield the minimum storage space requirements when optimal quantities are ordered. In the second, a constrained phased delivery model was used to determine the optimal ordering quantities under the constraint of a smaller storage capacity than the minimum space requirements indicated by the first case. His second model is a single constraint model, whereas the first one is not subject to any constraints.

Plossl and Wight (10) presented the LIMIT technique which is a procedure to apply economic order quantities to realistic situations where constraints are almost always present. They considered the following limitations independently:

a. A limit on the number of setups.

b. A limit on the inventory investments.

Morse (11) suggested that 'sometimes operational criteria influence our choice of the reorder quantity'. For example, warehouse space limitations or shipping requirements may make it cheaper to order in integral multiples of a lot size. In other words, the maximum inventory is an integral multiple of the economic order quantity which has already been determined by some other operational criterion. He developed a model to evaluate the optimum value of the integral multiple.

Starr and Miller (12) developed two models for aggregate inventory management. The first model considers a constraint on inventory investment. The second model considers a constraint on the total number of orders to be produced in a period of time. Based on these two models, they developed procedures to show how the inventory carrying charge can be used as a management policy variable.

Eilon (13) suggested a linear programming model for multi-product analysis. In a plant "n" products are manufactured by use of "m" parallel processes. Of the many alternatives (i.e., different combinations of product quantities) that are possible, one may want to maximize the profit. The process capacity presents a limitation on the amount produced of each product.

Solutions to this class of problems are obtained through linear programming techniques.

Churchman, Ackoff and Arnoff (14) developed four multi-item inventory models. The first model is without any restrictions. The second model is subject to a restriction on warehouse capacity. The third model is subject to a restriction on setup times. The fourth model is subject to both the restrictions on warehouse capacity and setup time.

Problem Formulation

The literature survey indicates there is a need for further research in the area of aggregate inventory management. As many authors have pointed out, several techniques have been developed for the management of specific lot size inventory when not constrained, or singularly constrained.

It is desirable to develop a multi-item inventory model subject to the constraints on the inventory investment and on the number of orders or setups per year.

Each of the possible solution types will be explored and the information obtained used to suggest a course of action.

CHAPTER 2

DEVELOPMENT OF THE MODEL

The previous discussion has indicated a relatively large number of single constraint models, but relatively few models with two or more constraints. It is therefore proposed to develop a model with two constraints. The constraints to be considered are:

1. An upper limit on the number of orders per year.

2. An upper limit on the amount in dollars that can be invested for inventory.

For ease of understanding, two single constraint models will be developed first. Then, the two constraint model will be developed.

The following definitions and assumptions are used throughout the developments:

n = number of products or items

λ_j = demand for item j per year

C_j = cost of item j, dollars per unit

Q_j = lot size of item j

A_j = cost of placing an order for item of type j, dollars/order

I_j = carrying charge for item j, per cent/year

h = maximum allowable number of orders/year

K = annual cost, dollars

D = maximum allowable amount in dollars for inventory investment

h_j = number of orders/year for item j

D_j = inventory investment in dollars for item j

Assumptinns

a. demand is at a fixed, known rate

b. lead time is zero (or is known exactly)

c. production is instantaneous

d. no shortages are permitted

The annual cost, K, can be obtained using the following reasoning: since $\sum_{j=1}^{n} \lambda_j$ units per year are demanded, $\sum_{j=1}^{n} \lambda_j$ units per year must be procured at a cost of $\sum_{j=1}^{n} C_j \times \lambda_j$. The total ordering cost per year is $\sum_{j=1}^{n} \frac{\lambda_j}{Q_j} \times A_j$. Since the usage rate was assumed constant, the total of the average inventories is $\sum_{j=1}^{n} Q_j/2$. Hence, the total inventory carrying cost per year is $\sum_{j=1}^{n} C_j \times \frac{Q_j}{2} \times I$.

For the purpose of this thesis, the unit cost of the item is independent of the quantity ordered. Therefore, $\lambda_j \times C_j$ is independent of Q_j and need not be included in these discussions.

Summing the cost terms defined above,

$$K = \sum_{j=1}^{n} \frac{\lambda_j}{Q_j} \times A_j + \frac{Q_j}{2} \times I_j \times C_j \qquad \text{Equation 1}$$

It may now be useful to determine the lot sizes Q_j's in such a manner as to minimize the annual cost, K. This can be easily done by differentiating K with respect to Q_j and setting the derivative equal to zero. This results in

$$Q_j = \sqrt{\frac{2 \times \lambda_j \times A_j}{C_j \times I_j}} \quad j = 1,2.....n \qquad \text{Equation 2}$$

This considers no constraints, and is therefore the most specialized, but simplest, model. A similar situation occurs if the constraint is such that the unrestricted optimum values also satisfy the constraint. In this case, the constraint is said to be inactive. Obviously, if any constraint prevents the unrestricted optimum, it is known as an active constraint.

a. A Multi-Item (Economic Order Quantity) Model Subject to the Limitation on the Number of Orders Per Year

The multi-item Economic Order Quantity model may be subject to a restriction on the number of orders per year. As indicated in the previous section, the total number of orders per year is

$$\sum_{j=1}^{n} h_j = \sum_{j=1}^{n} \lambda_j/Q_j$$

Since 'h' is the maximum allowable number of orders per year,

$$\sum_{j=1}^{n} \lambda_j/Q_j \leq h \qquad \text{Equation 3}$$

and $h - \sum_{j=1}^{n} (\lambda_j/Q_j)$ must be either zero or positive. If $h - \sum_{j=1}^{n} \lambda_j/Q_j$ is positive, then the constraint is inactive. If it is zero, then the solution is optimum for that constraint. Obviously, if $h - \sum_{j=1}^{n} \lambda_j/Q_j$ is negative, no solution is possible, since the constraint is violated. Therefore, η , a lagrangian multiplier, may be defined such that

$$\eta = 0 \quad \text{when} \quad h - \sum_{j=1}^{n} \lambda_j/Q_j > 0$$

$$\eta < 0 \quad \text{when} \quad h - \sum_{j=1}^{n} \lambda_j/Q_j = 0$$

Under these conditions,

$\eta \left(h - \sum_{j=1}^{n} \lambda_j/Q_j\right)$ is identically equal to zero. Hence, it can be added to Equation 1 without changing the relationships in the equation.

The discussion above may be generalized to any other case, such as a constraint on warehouse floor space or allowed capital expenditures.

The Lagrangian multiplier can be interpreted as the imputed cost or shadow price for the constraint considered. The absolute value of the Lagrangian multiplier, when optimized, would give the decrease in the minimum cost if the constraint were relaxed by one unit (15).

A function J, similar to K, is selected such that

$$J = \sum_{j=1}^{n} \left(\frac{I_j \times C_j \times Q_j}{2} + \frac{\lambda_j \times A_j}{Q_j}\right) + \eta \left(h - \sum_{j=1}^{n} \lambda_j/Q_j\right) \qquad \text{Equation 4}$$

To obtain optimal lot sizes, J is partially differentiated with respect to Q_j and η and the partial derivatives are set equal to zero. Thus,

$$\frac{\partial J}{\partial Q_j} = \frac{I_j \times C_j}{2} - \frac{\lambda_j \times A_j}{Q_j^2} + \eta \times \frac{A_j}{Q_j^2} = 0$$

$$j = 1,2.....n \qquad \text{Equation 5}$$

and

$$\frac{\partial J}{\partial \eta} = h - \sum_{j=1}^{n} \frac{\lambda_j}{Q_j} = 0 \qquad \text{Equation 6}$$

From Equation 5,

$$\overset{*}{Q}_j = \sqrt{\frac{2 \times \lambda_j \times (A_j - \overset{*}{\eta})}{I_j \times C_j}} \qquad j = 1,2.....n \qquad \text{Equation 7}$$

where

$\overset{*}{Q}_j$ = optimal lot size for product j

$\overset{*}{\eta}$ = optimal value of η

Substituting Equation 7 into Equation 6 gives

$$h = \sum_{j=1}^{n} \frac{1}{\sqrt{2}} \sqrt{\frac{I_j \times C_j \times \lambda_j}{A_j - \eta^*}} \qquad \text{Equation 8}$$

The solution of Equation 8 for $\overset{*}{\eta}$ must be completed numerically. The absolute value of $\overset{*}{\eta}$ can be interpreted as the marginal cost of placing an additional order, and it is not economical to increase the limitation on the number of orders per year unless the additional ordering cost is less than the absolute value of $\overset{*}{\eta}$.

b. A Multi-Item (Economic Order Quantity) Model Subject to the Limitation on Inventory Investment

It would be expected that the derivation of the model with a constraint on inventory investment would be similar to that of the model with the limitation on number of orders per year. This is, in fact, true.

The dollar investment in inventory for item j is D_j or $C_j \times Q_j$. Here one may be tempted to equate D_j to $\frac{C_j \times Q_j}{2}$, since $Q_j/2$ is the average inventory of item j in the long run. Nevertheless, the instantaneous inventory investment at some point would have one or more items at a maximum. Particularly if the cost of the items are wide-spread, the maximum investment would approach the maximum inventory multiplied by the cost of the items. Because no smaller figure can be justified in a general model, the maximum value is used, while recognizing that it may be above the actual maximum that is reached.

Since the total inventory investment for all items should be less than the maximum allowable inventory investment D,

$$\sum_{j=1}^{n} C_j \times Q_j \leq D \qquad \text{Equation 9}$$

For clarity in the two constraint model, the lagrangian multiplier, for inventory investment, will be called $\emptyset$ such that

$$\emptyset < 0, \quad \text{when} \quad D - \sum_{j=1}^{n} C_j \times Q_j = 0$$

$$\emptyset = 0, \quad \text{when} \quad D - \sum_{j=1}^{n} C_j \times Q_j > 0$$

if $D - \sum_{j=1}^{n} C_j \times Q_j$ is negative, no solution is possible since it would violate the constraint represented by Equation 9.

Again, a new function J is found by adding the zero-valued term to the cost function K.

$$J = \sum_{j=1}^{n} \left(\frac{\lambda_j}{Q_j} \times A_j + \frac{I_j \times C_j \times Q_j}{2} \right) + \emptyset \left(D - \sum_{j=1}^{n} C_j \times Q_j \right) \qquad \text{Equation 10}$$

The optimal values of Q_j and $\emptyset$ can be evaluated by taking partial derivatives of J with respect to Q_j and $\emptyset$ respectively and setting the partial derivatives equal to zero. This results in

$$\overset{*}{Q}_j = \sqrt{\frac{2 \times \lambda_j \times A_j}{C_j \times (I_j - 2\overset{*}{\emptyset})}} \qquad j = 1,2.....n \qquad \text{Equation 11}$$

$$D = \sum_{j=1}^{n} C_j \times \overset{*}{Q}_j \qquad \text{Equation 12}$$

where,

$\overset{*}{Q}_j$ = optimal lot size for item j

$\overset{*}{\emptyset}$ = optimal value of $\emptyset$

$\emptyset$ is the marginal cost of capital investment.

c. Development of a Multi-Item (Economic Order Quantity) Model With Two Constraints

The previous discussions showed the developments of two single constraint models. Following the same reasoning used, the model with two constraints can be developed.

In a multi-item inventory problem subject to two constraints, the following cases may arise.

a. Both constraints may be inactive. In other words, the unrestricted optimum lot sizes satisfy both the constraints.

b. One of the constraints may be active, while the other one may be inactive. This essentially reduces to a single constraint problem.

c. Both constraints may be active, with a feasible solution. In this case, the lot sizes may be determined which satisfy both the constraints.

d. Both constraints may be active, with no possible solution. In this case, it is only possible to determine the minimum value of the product of the constraints, which could be satisfied.

The annual cost equation can be constructed with two "zero-valued" terms added.

$$J = \sum_{j=1}^{n} \left(\frac{\lambda_j \times A_j}{Q_j} + \frac{I_j \times C_j \times Q_j}{2} \right) + \eta \left(h - \sum_{j=1}^{n} \frac{\lambda_j}{Q_j} \right)$$

$$+ \phi \left(D - \sum_{j=1}^{n} C_j \times Q_j \right) \qquad \text{Equation 13}$$

Optimal Q_j's, ϕ and η can be obtained by taking the partial derivatives of J with respect to Q_j, ϕ and η respectively and setting the partial derivatives equal to zero. Thus,

$$\overset{*}{Q}_j = \sqrt{\frac{2 \times \lambda_j \times (A_j - \overset{*}{\eta})}{C_j \times (I_j - 2\overset{*}{\phi})}} \qquad j = 1,2.....n$$

$\overset{*}{\eta}$ = optimal value of η

$\overset{*}{\phi}$ = optimal value of ϕ Equation 14

A simple example can be solved to show the numerical solutions methods. A manufacturing company makes two products X_1 and X_2. Certain information concerning these products is available in Table 2-1.

In addition, management has made decisions resulting in the following information:

Cost of Inventory, I = \$0.20/unit year

Maximum Number of Orders/Year h = 200

Maximum Inventory Investment D = \$12,000.

TABLE 2-1

Product	Annual Demand λ_j Units	Cost of Setup A_j, Dollars	Unit Cost C_j, Dollars
X_1	10,000	\$10.00	\$100.00
X_2	7,200	\$ 2.00	\$ 40.00

Sample computations as necessary for the various solutions are in Appendix A.

Lot sizes have been plotted against the annual costs for both products X_1 and X_2 as shown in Figure 2-1.

Because there are only two products, the quantities of each may also be plotted against the other, with iso-cost curves as shown in Figure 2-2.

It can be seen that the annual cost, K, has the minimum value of \$2,480, when lot sizes Q_1 and Q_2 correspond to optimum lot sizes $\overset{*}{Q}_1$ and $\overset{*}{Q}_2$ respectively. The iso-cost curve for this case is a unique point represented by $(\overset{*}{Q}_1, \overset{*}{Q}_2)$.

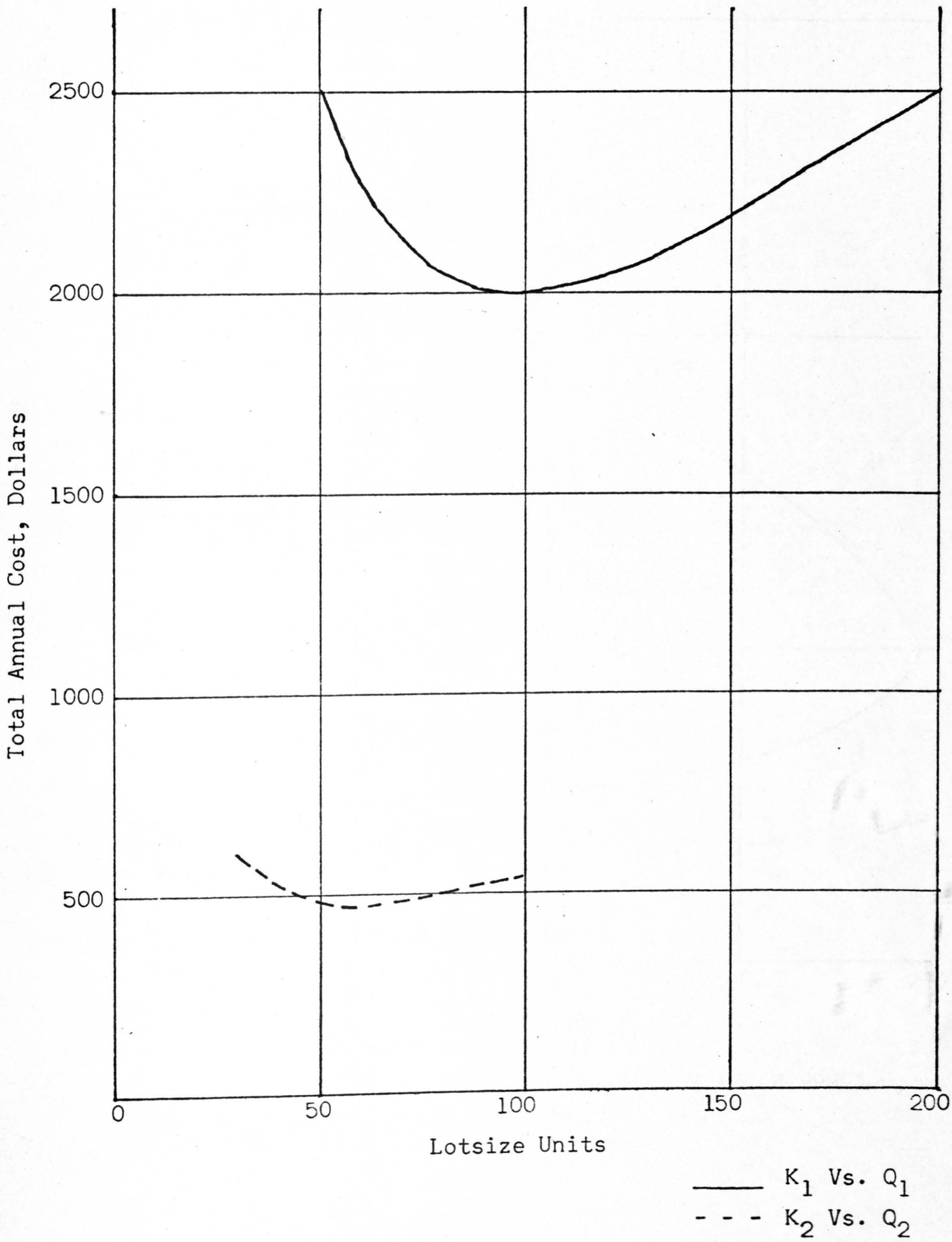


Figure 2-1. Lot sizes vs. annual costs for each of products X_1 and X_2.

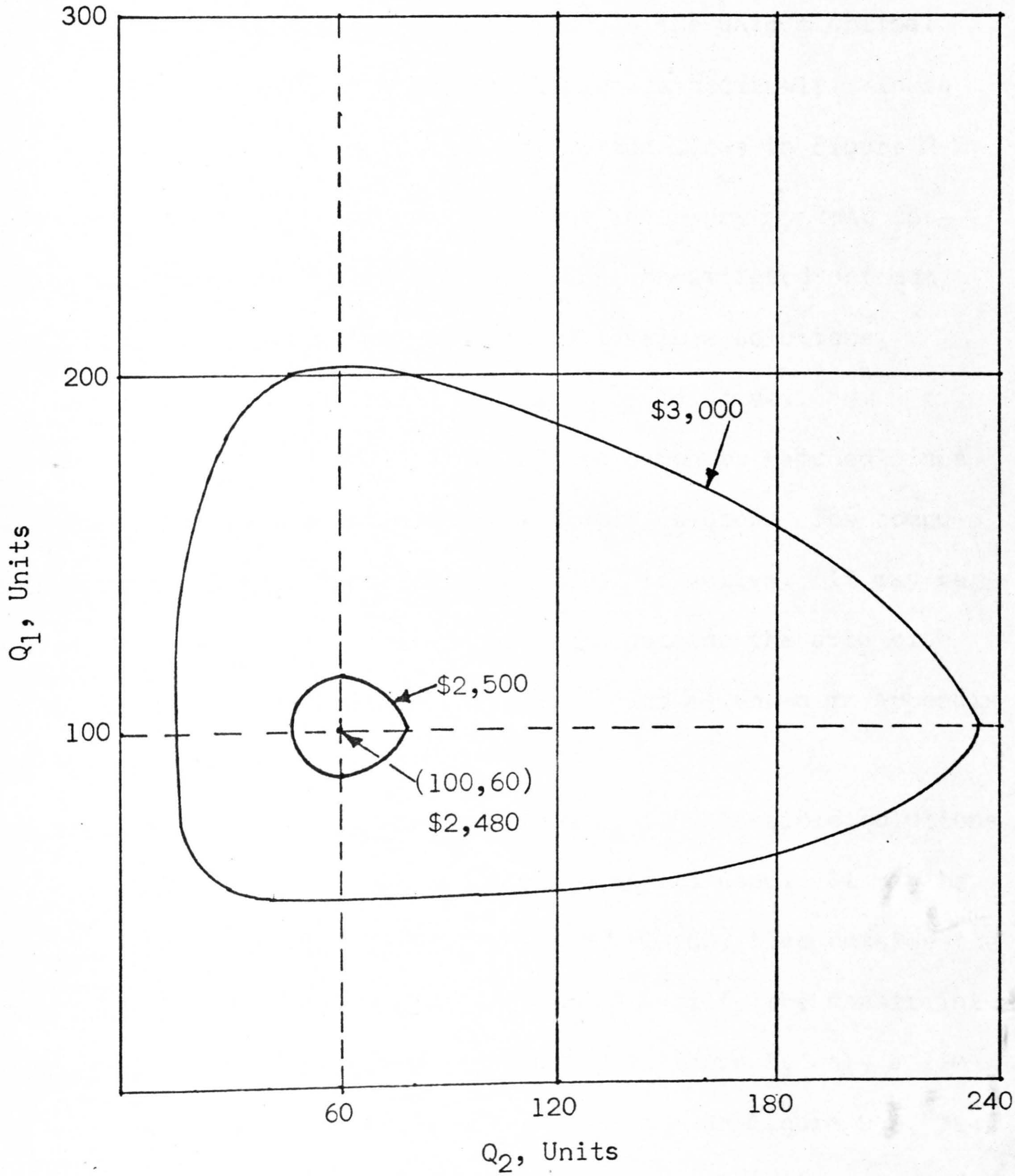


Figure 2-2. Lot size for product X_1 Vs. lot size for product X_2.

It may be observed that the maxima and minima of all iso-cost curves lie on the straight lines passing through the unique optimal point ($\overset{*}{Q}_1$, $\overset{*}{Q}_2$) and parallel to the Y and X axes respectively. These lines $X = \overset{*}{Q}_2$ and $Y = \overset{*}{Q}_1$ are represented by dotted lines in Figure 2-2.

When there is a constraint acting on the inventory system, the area of feasible solutions is restricted. The unrestricted optimum point would be outside or inside the area of feasible solutions, depending on whether the constraint is active or inactive.

Figure 2-3a shows the area of feasible solution by hatched lines when only the constraint on the number of orders is used. The computations in Appendix A show that this constraint is active. It may be noticed that the unrestricted optimum point is outside the area of feasible solutions. A new optimum point was found as shown in Appendix A to suit the constraint.

In a similar way, Figure 2-3b shows the area of feasible solutions when only the constraint on inventory investment was used. It may be noticed that the unrestricted optimum point P (100,60) lies outside the area. New optimum lot sizes were determined to satisfy the constraint.

If these two are combined, as in Figure 2-4, there is only a limited area in which both constraints are satisfied. In Figure 2-5, the values for η and $\emptyset$ are also shown. As expected, the lot sizes $\overset{*}{Q}_1$ and $\overset{*}{Q}_2$, which optimize the annual cost, K, subject to the two constraints, form one of the intersections of the two curves representing the constraints. It may also be noticed that the optimum point (87,85) lies between $\eta = -8$ and $\eta = -12$ and $\emptyset = 0$ and $\emptyset = -2$. This agrees with the calculated values of $\eta = -9.297$ and $\emptyset = -0.167$.

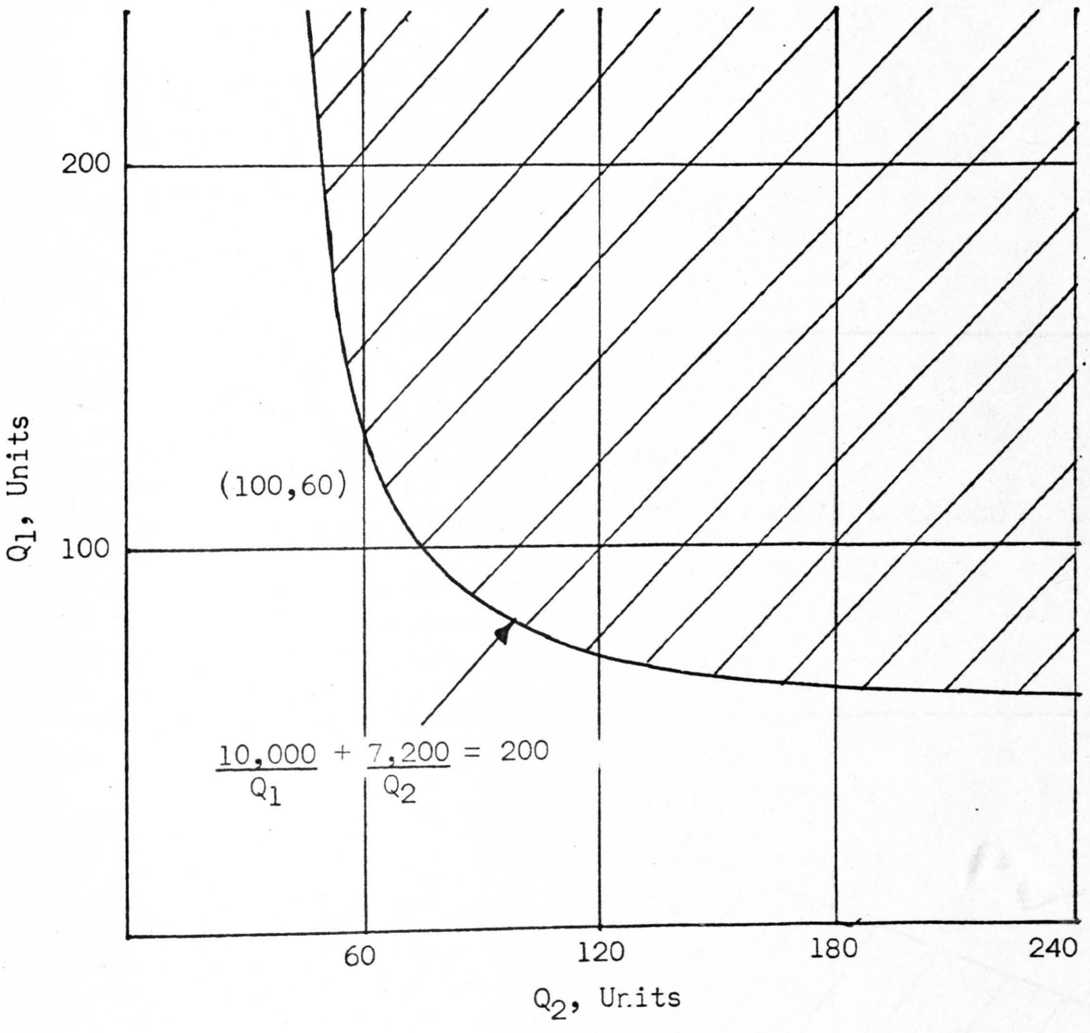


Figure 2-3a. Area of feasible solutions for the two product sample problem subject to the constraint on the number of orders.

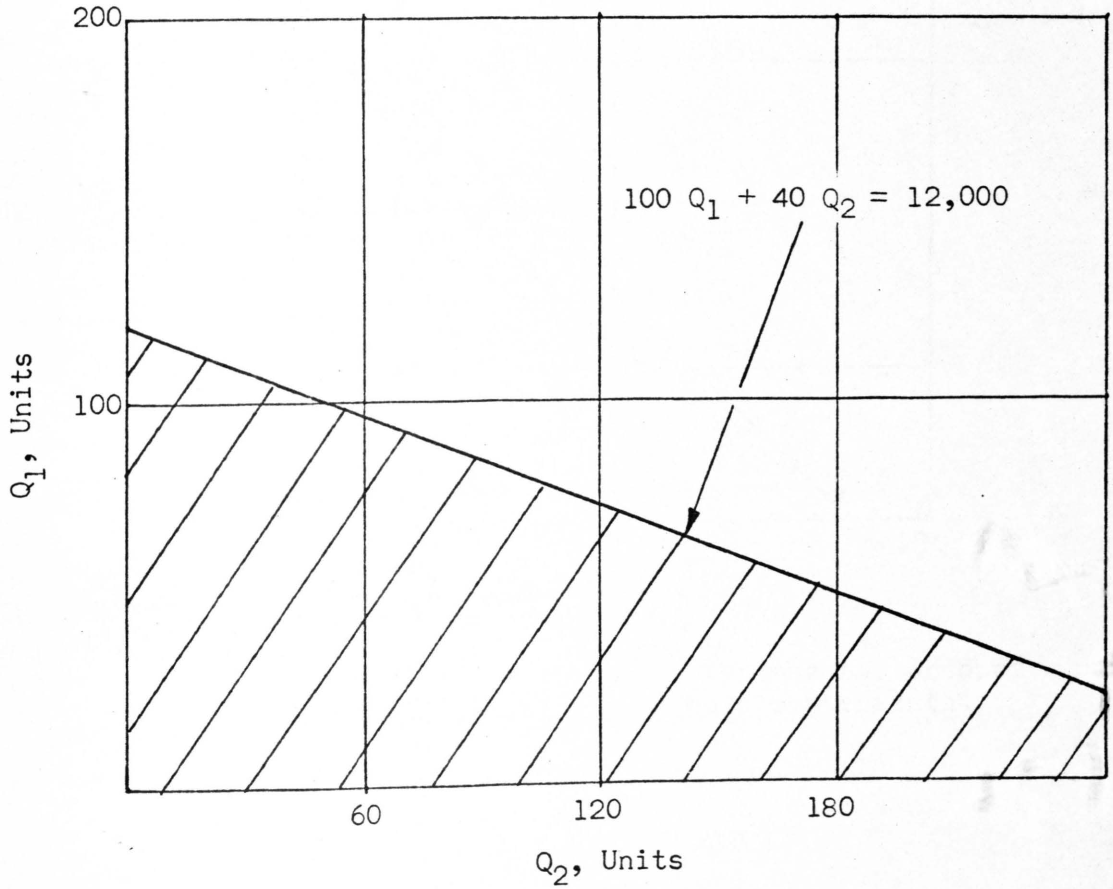


Figure 2-3b. Area of feasible solutions for the two product sample problem subject to the constraint on the inventory investment.

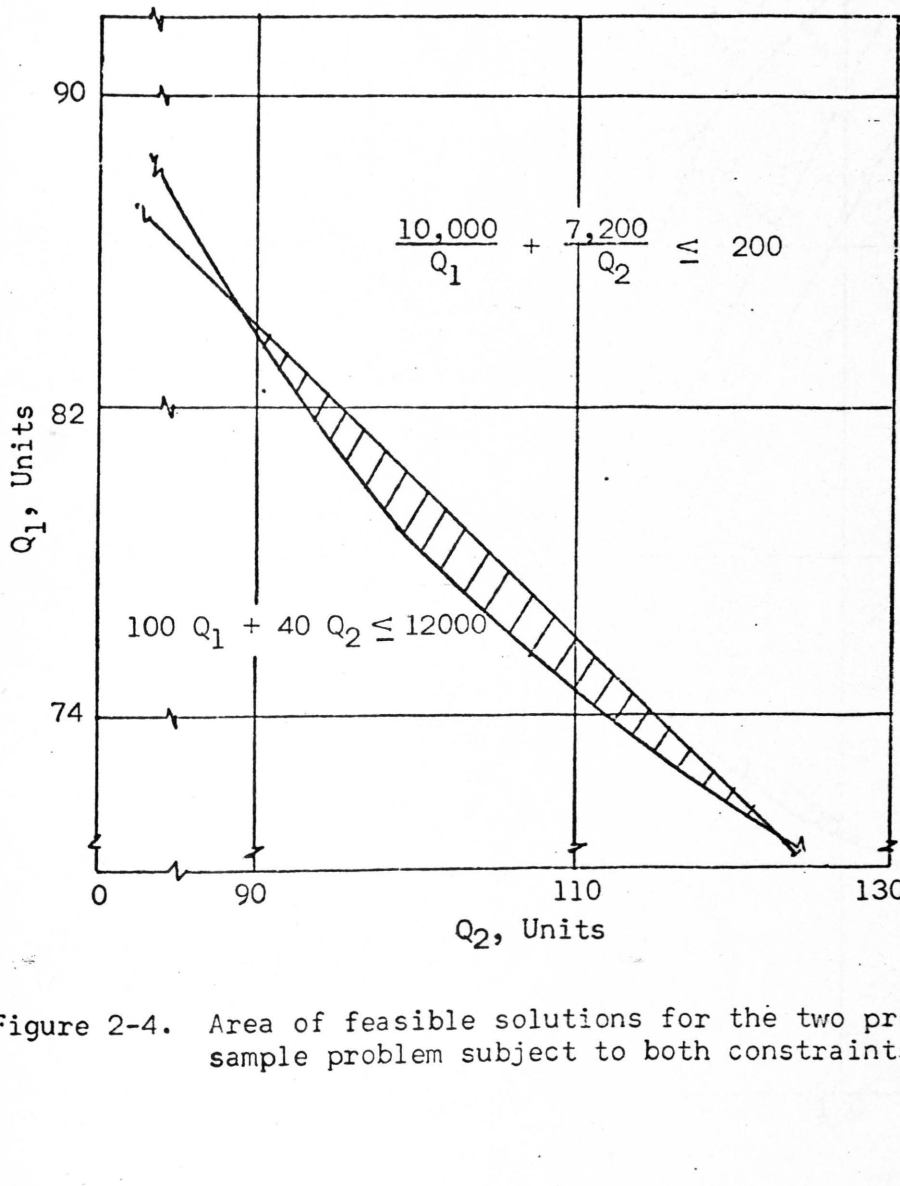


Figure 2-4. Area of feasible solutions for the two product sample problem subject to both constraints.

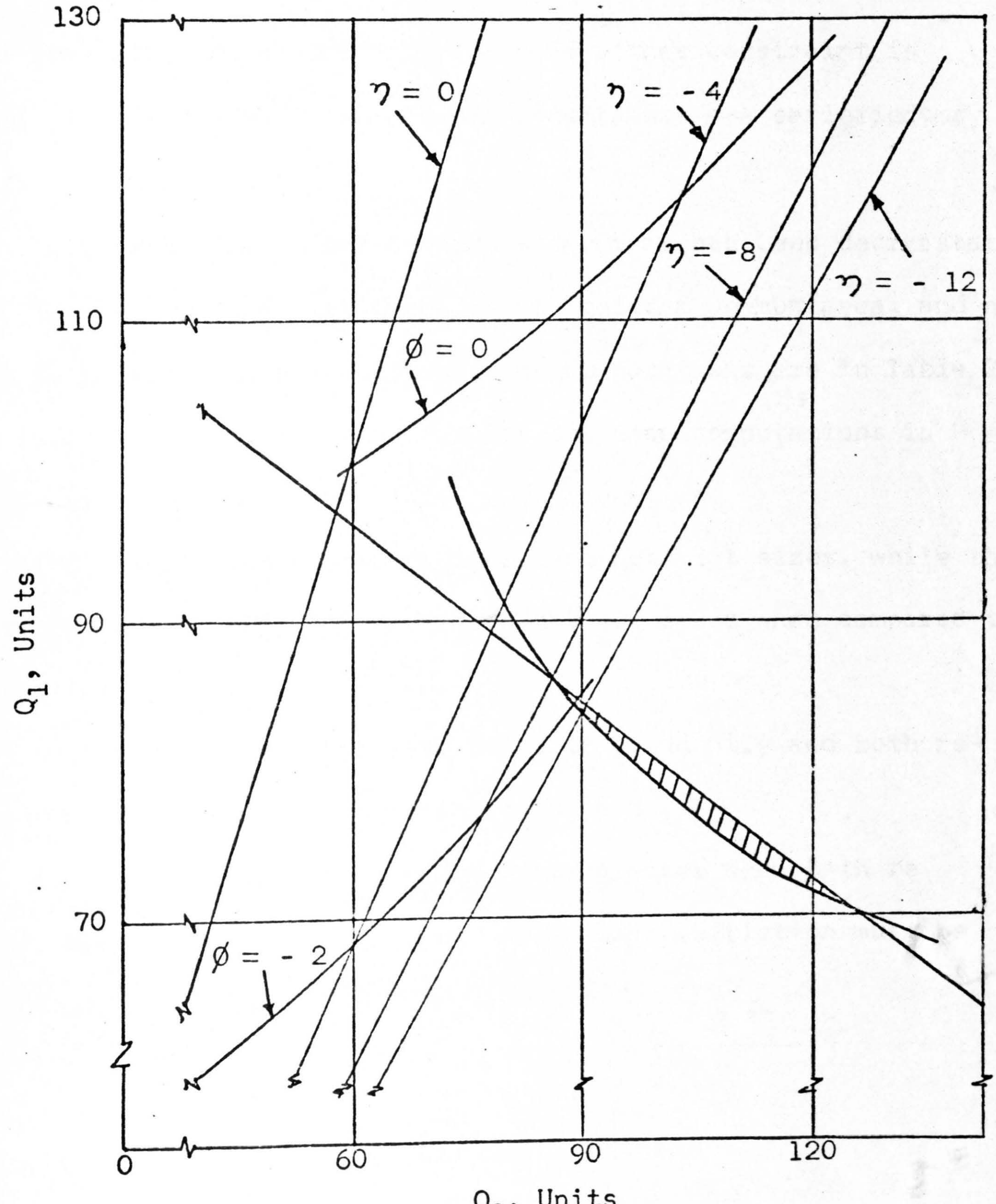


Figure 2-5. Lot size for product X_1 Vs. lot size for product X_2 for different values of and ϕ with constraints shown.

If the constraints are relaxed, then the area in which both constraints are satisfied should increase. If either constraint is tightened, then the area in which both constraints are satisfied may disappear.

In Figure 2-6, the number of orders per year has been decreased to 100, and it may be noted that there is no longer a common area, and no solution is possible. The results of the computations are in Table 2-2.

It is desirable to note the results of these computations in greater detail.

1. The restriction on orders increased both lot sizes, while the inventory investment restriction lowers the lot sizes when compared to the unrestricted optimum condition.

2. Each restriction increases costs independently and both restrictions taken together further increase costs.

3. The absolute values of $\overset{*}{\eta}$ and $\overset{*}{\phi}$ are greater when both restrictions must be satisfied than when only one restriction must be satisfied.

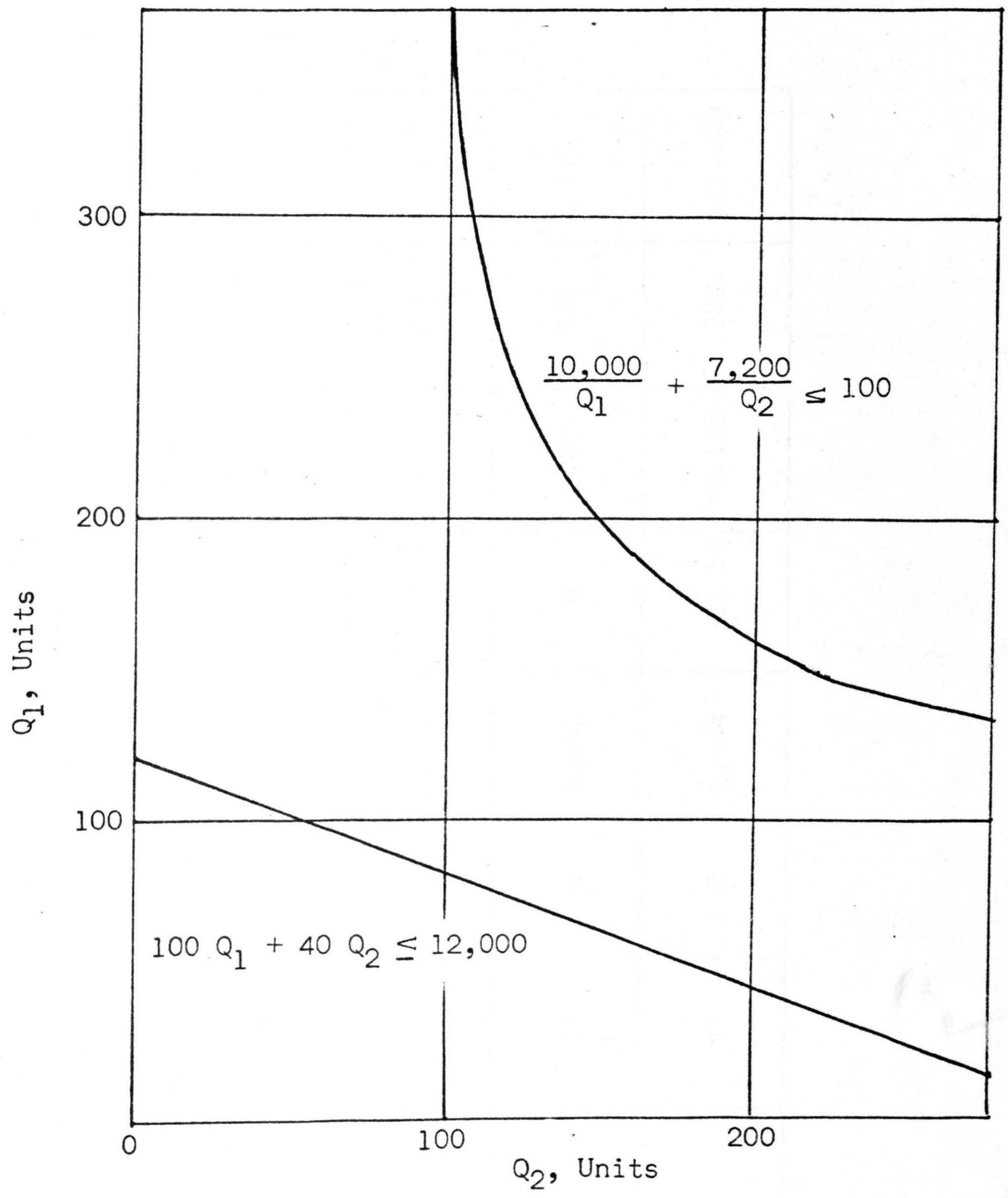


Figure 2-6. Lot size for product X_1 Vs. lot size for product X_2 with constraints shown.

TABLE 2-2

SUMMARY OF THE TWO-PRODUCT EXAMPLE PROBLEM

SUBJECT TO TWO CONSTRAINTS

Conditions	$\overset{*}{Q}_1$	$\overset{*}{Q}_2$	K	η	ϕ	$\sum_{j=1}^{2} \frac{\lambda_j}{Q_j}$	$\sum_{j=1}^{2} C_j Q_j$
Unrestricted	100	60	2,480	0	0	220	$12,400
Orders Restriction	103	70	2,486	-0.69	0	200	$13,100
Capital Investment Restriction	96	58	2,482	0	-0.007	232	$11,920
Both Restrictions	84	88	2,544	-10.58	-0.195	200	$11,920

CHAPTER 3

COMPUTERIZED MODEL

Inventory problems with either one or two constraints are normally solved by an iterative procedure and the amount of calculations involved rapidly becomes quite large. As a matter of fact, if there are more than two products, only iterative solutions are possible. Although a specific program could have been written, it was decided that a generalized computer program for the two-constraint problem would be a valuable addition to the thesis.

The completed computer program is listed in Appendix B. The flow diagram is shown in Figure 3-1. For ease of comparison of the explanation of the program with the flow diagram, Figure 3-1 is found on Page 41.

Briefly, the program starts with the unrestricted solution and by systematically selecting values for one of the Lagrangian multipliers, computes new optimum values which can be examined for being in the feasible solution area. If this occurs, or if the solution bypasses the least cost point, the program returns to the last previous solution and reduces the size of the interval for the multiplier. Thus the solution is always approached from the direction of the unrestricted solution. After a finite number of steps, the solution is found to be feasible optimum or non-feasible with a known minimum for the product of constraints.

A more detailed description of the program is of value.

Blocks A, B and C compute and print values for the unrestricted optimum lot sizes, corresponding annual cost, the total number of orders per year and the inventory investments. These are needed for comparison purposes, as well as a starting point for the restricted solutions.

Blocks D and E are decision blocks to determine if the constraints are active. If both constraints are inactive, the program drops into the block F and prints a final message. If either or both of the constraints are active, the program proceeds to block G.

The optimization procedure is started in block G, where the value of η is initialized. Block H computes a solution for the initialized value of η.

In block J the decision is made whether

1. the solution is feasible and optimum, or
2. the solution is feasible but not optimum, or
3. the solution is not feasible.

In the first case, the program proceeds directly to block R for final computations and output.

In the second case, the computed product of the number of orders per year and the inventory investment is less than the maximum allowable value. Therefore, this solution represents an "overshoot" and the computer program must correct in the other direction. The program proceeds to block L. The main function of block L is to correct the overshoot by returning to the non-feasible area. This is done by decreasing the absolute value of η. A review of Figure 2-5 indicates that as the absolute value of η is increased from the unrestricted solution, the optimum condition is being approached. If over-shoot has occurred, the absolute value of η should be changed to a smaller value to make the current solution non-feasible.

In the third case, the computed product of the number of orders per year and the inventory investment is more than the maximum allowable value. The program proceeds to decision block K. In block K, the value of the product of the number of orders per year and the inventory investment is compared with the previous computation to test for the existence of a feasible solution. If the value is decreasing, it shows that the solution is approaching minimum but within the non-feasible region. If the value is increasing, it shows that the solution is becoming less feasible assuming the increment for η is a small value. Hence, there is no solution. If the increment for η is larger than an arbitrarily chosen value as determined in block P, the program proceeds to block L. Here increment for η is reduced. This computation proceeds around loop LHJKP until a feasible and optimum solution is obtained or it is determined that there is no solution. In the latter case, the minimum possible value of the product of the number of orders per year and the inventory investment is printed out.

If the current solution is less than the previous solution, the program proceeds to block N. Here, value of η is algebraically decreased (remember that η is negative) by one increment and hence the product of the number of orders per year and the inventory investment is reduced. The logic directs the computation around the loop JKNH until either it is determined that there is no possible solution or until a feasible non-optimum solution is obtained. In that case, it follows loop JLH before returning to loop JKNH. At some point, the change in η becomes small enough that the solution is assumed to be feasible and optimum or non-feasible and minimum.

A numerical example will illustrate the iteration method.

Suppose that a hypothetical two-item inventory problem that is subject to two constraints, is as shown in Figure 3-2.

Assume that $\eta = -301.5$ (Point F in Figure 3-2) and initially delta (the increment for η in block L) = 100. The program logic will reduce delta to 1/10th of its value in block L, each time a feasible solution is found.

Since the initial value of η is zero, block J would decide that the solution is not feasible (Point I). Therefore, the program would proceed in the JKNH loop.

$$\begin{aligned} \text{New } \eta &= \eta - \text{delta} \\ &= 0 - 100 \\ &= -100 \end{aligned}$$

The above loop would continue until η is set equal to -400 (Point a).

At this point, block J would find that the solution is feasible but not optimum. Then loop JLH is entered.

$$\begin{aligned} \text{New } \eta &= \eta + \text{delta} \\ &= -400 + 100 = -300 \end{aligned}$$

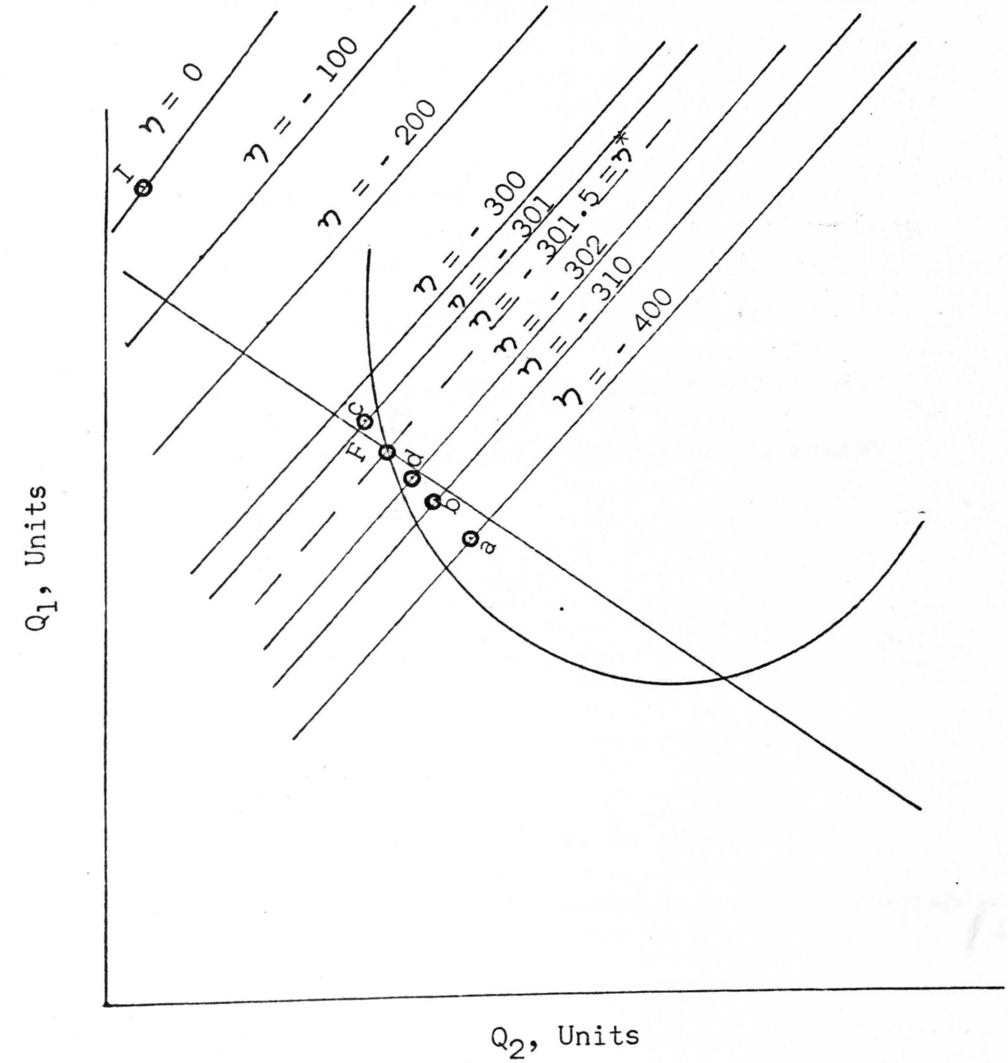


Figure 3-2. Shows the iterative procedure used to evaluate the lagrangian multiplier η.

η = -300 has already resulted in a non-feasible solution. So η needs to be updated to a value between -300 and -400. To accomplish this, the size of delta is decreased.

$$\text{delta} = \text{delta} \ /10$$

A new η can then be computed.

$$\text{New } \eta = \eta - \text{delta}$$
$$= -300 - 10 = -310. \text{ (Point b)}$$

This would result in a feasible but optimum solution.

Again loop JLH would be used and a new delta computed.

$$\text{New } \eta = \eta + \text{delta}$$
$$= -310 + 10 = -300$$
$$\text{delta} = \text{delta}/\ 10$$
$$= 10/10 = 1$$
$$\text{New } \eta = \eta - \text{delta}$$
$$= -300 - 1 = -301. \quad \text{(Point c)}$$

This would result in a non-feasible solution. The program would utilize loop JKNH.

$$\text{New } \eta = \eta - \text{delta}$$
$$= -301 - 1 = -\ 302. \quad \text{(Point d)}$$

This would result in a feasible but not optimum solution. Again loop JLH would be used and a new delta and η computed.

$$\eta = \eta + \text{delta}$$
$$= -301 + 1 = -301$$
$$\text{delta} = \text{delta}/10 = 1/10 = 0.1$$
$$\eta = \eta - \text{delta}$$
$$= -301 - 0.1 = -301.1$$

This would result in a non-feasible solution. Again loop JKNH will be utilized.

$$\eta = \eta - \text{delta}$$

$$= -301.1 - 0.1 = -301.2$$

This computation continues until

$$\eta = -301.5 = \eta^*$$

This would be the value of η corresponding to the feasible optimum solution, because delta is not allowed to reach the 0.01 level.

Upon reaching the optimum feasible solution, the program drops into block R. ϕ^* is the value of ϕ corresponding to the value of η^*. Further, optimal lot sizes subject to constraints and the corresponding annual cost are computed and the results are printed.

An example can be solved to show the numerical solution methods. A manufacturing company makes five products. Certain information concerning these products is available in Table 3-1.

The example problem has been solved using the computer for three different sets of values of the two constraints. These are also presented in Table 3-1.

The computer results for the three cases are shown in Figure 3-3.

The three cases are examples of the three of the different cases of solutions hypothesized.

TABLE 3-1

Data For An Example Prcblem Subject to Contraints On

Number of Orders/Year and On Capital Inventment

Item	1	2	3	4	5
Demand Rate Units/Year λ_j	1000	500	2000	3000	840
Unit Cost Dollars/Unit C_j	20	100	50	60	40
Setup Cost Dollars/Setup A_j	50	75	100	180	210

```
MAX VALUE OF CAPITAL INVESTMENT FOR INVENTORY = $100000.000

MAX NUMBER OF ORDERS PER YEAR =      100.000

AVERAGE ANNUAL COST (UNRESTRICTED OPTIMUM ) = $    9137.188

UNRESTRICTED ECONOMIC LOTSIZES Q
Q( 1)=      158.1
Q( 2)=       61.2
Q( 3)=      200.0
Q( 4)=      300.0
Q( 5)=      210.0
  BOTH CONSTRAINTS ARE INACTIVE
```

Figure 3-3a. Computer output for the five product sample problem subject to two constraints, Case I.

```
MAX VALUE OF CAPITAL INVESTMENT FOR INVENTORY = $ 45500.000

MAX NUMBER OF ORDERS PER YEAR =        37.000

AVERAGE ANNUAL COST (UNRESTRICTED OPTIMUM ) = $   9137.188

UNRESTRICTED ECONOMIC LOTSIZES Q
Q( 1)=       158.1
Q( 2)=        61.2
Q( 3)=       200.0
Q( 4)=       300.0
Q( 5)=       210.0

LAGRANGE MULTIPLIER ETA =    -117.513

LAGRANGE MULTIPLIER THETA =       -0.098

ECONOMIC LOT SIZES QOP SUBJECT TO CONSTRAINTS
QOP( 1)=       205.8
QOP( 2)=        69.8
QOP( 3)=       209.8
QOP( 4)=       274.3
QOP( 5)=       186.5

AVERAGE ANNUAL COST SUBJECT TO CONSTRAINTS = $   9198.318
```

Figure 3-3b. Computer output for the five product sample problem subject to two constraints, Case 2.

```
MAX VALUE OF CAPITAL INVESTMENT FOR INVENTORY = $ 40000.000

MAX NUMBER OF ORDERS PER YEAR =      30.000

AVERAGE ANNUAL COST (UNRESTRICTED OPTIMUM ) = $    9137.108

UNRESTRICTED ECONOMIC LOTSIZES Q
Q( 1)=      158.1
Q( 2)=       61.2
Q( 3)=      200.0
Q( 4)=      300.0
Q( 5)=      210.0

MINIMUM POSSIBLE VALUE OF CAPMAX*ORDMAX =     1661077.00

CAPMAX*ORDMAX =    1200000.00
```

Figure 3-3c. Computer output for the five produce sample problem subject to two constraints, Case 3.

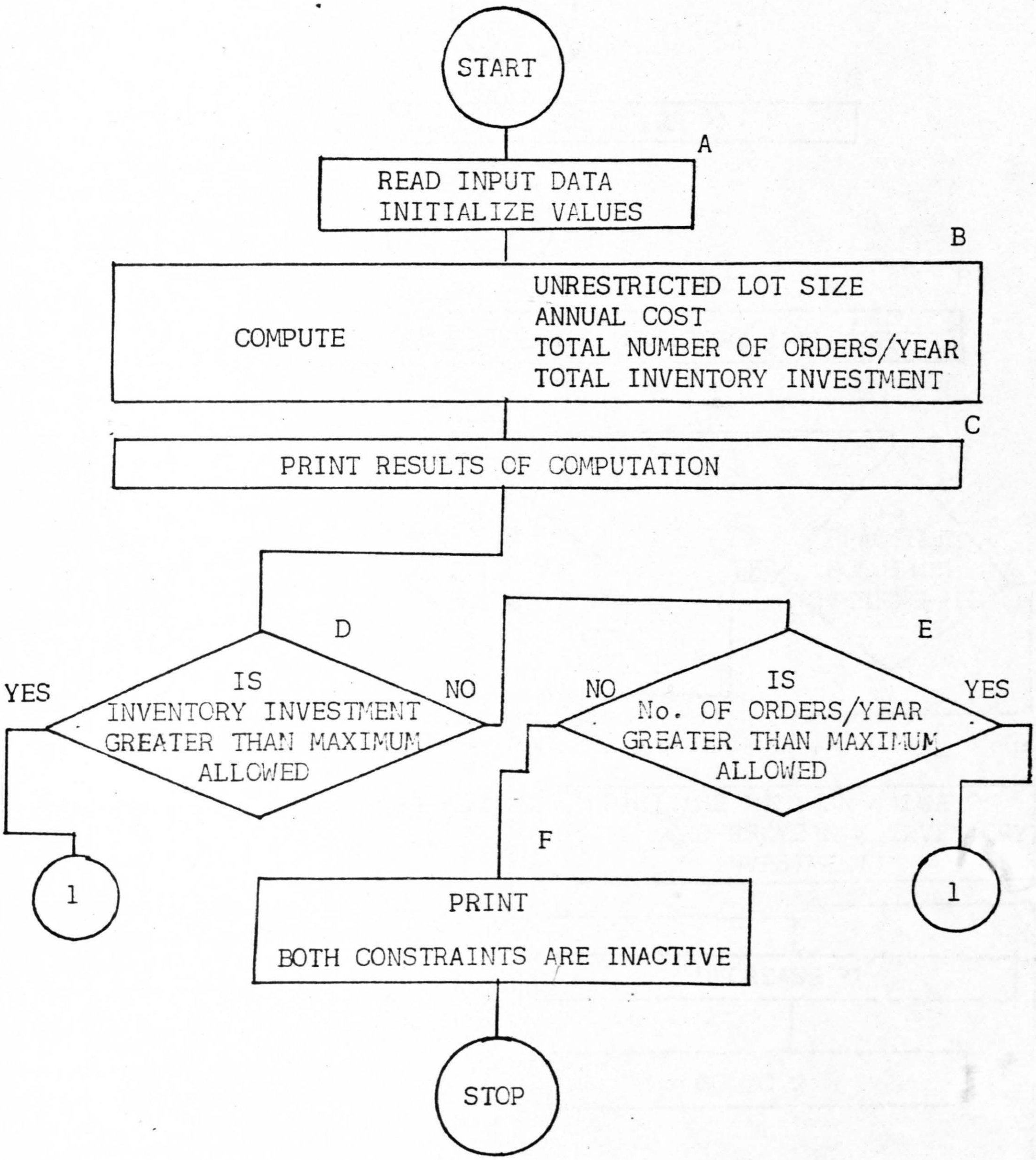
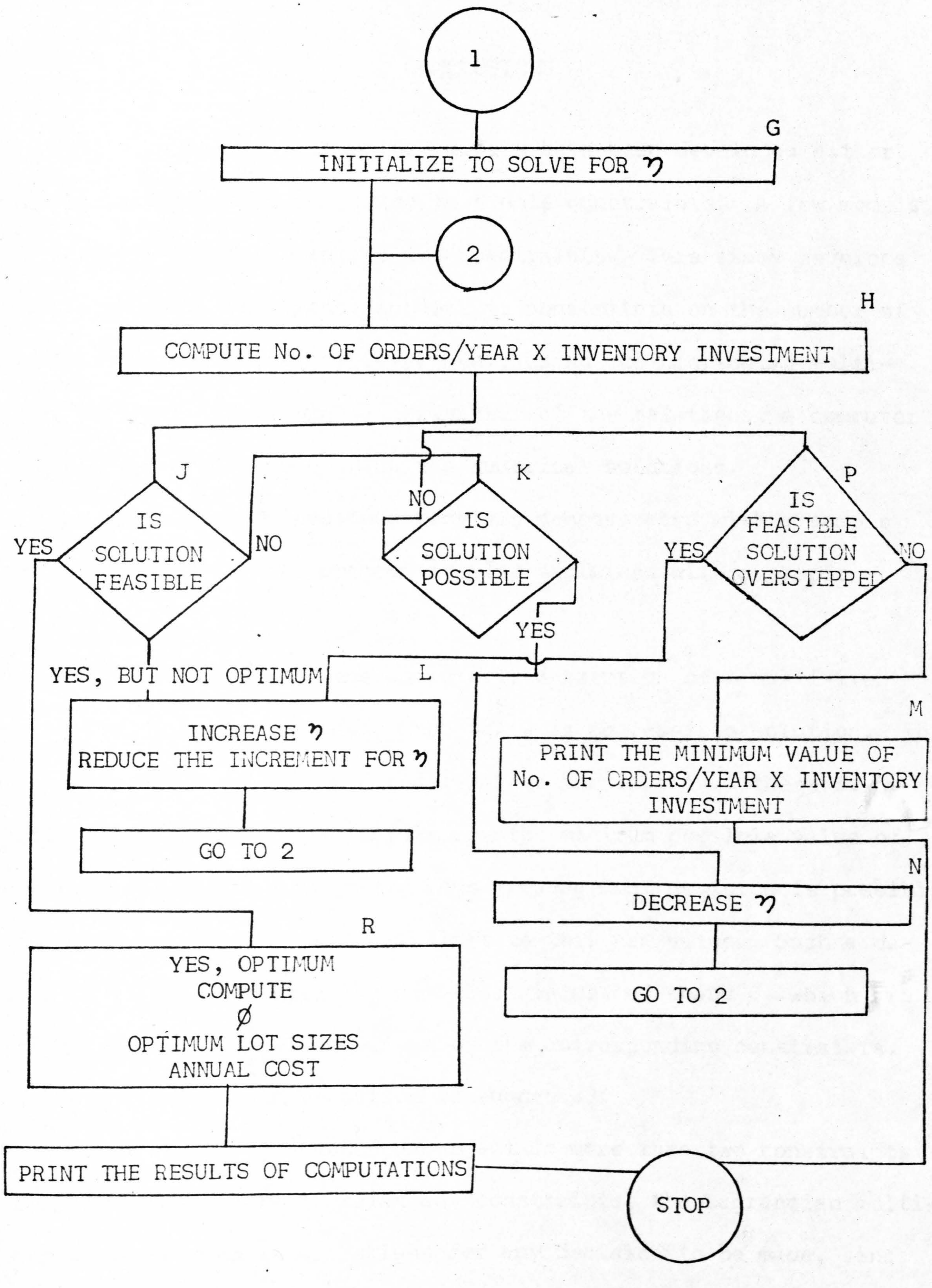


Figure 3-1. Flow chart for computing optimum lot sizes and annual cost for multi-item inventory problems subject to the constraints.



1
G
INITIALIZE TO SOLVE FOR η
2
H
COMPUTE No. OF ORDERS/YEAR X INVENTORY INVESTMENT
J
IS SOLUTION FEASIBLE
YES
NO
K
NO
IS SOLUTION POSSIBLE
YES
P
IS FEASIBLE SOLUTION OVERSTEPPED
YES
NO
YES, BUT NOT OPTIMUM
L
INCREASE η
REDUCE THE INCREMENT FOR η
GO TO 2
M
PRINT THE MINIMUM VALUE OF No. OF ORDERS/YEAR X INVENTORY INVESTMENT
N
DECREASE η
GO TO 2
R
YES, OPTIMUM
COMPUTE
Ø
OPTIMUM LOT SIZES
ANNUAL COST
PRINT THE RESULTS OF COMPUTATIONS
STOP

CHAPTER 4

CONCLUSIONS

Several multi-item inventory models have been developed either without limitations or subjected to single constraints. A few models have been developed subject to two constraints. This study develops a multi-item inventory model subject to constraints on the number of orders and on the inventory investment, using the Lagrangian multiplier technique. Following a derivation of the relations, a computer program is developed to provide the numerical solutions.

The usage of the developed model is demonstrated with a sample problem and the computer program is also explained with a sample problem.

A major advantage of the computerized solution of a multi-item two constraint problem occurs, when there is no feasible solution. In such a case, the computer first determines the fact that there is no feasible solution. It then prints out the minimum possible value of the product of the constraints. Thus a management decision is possible as to whether to relax the constraints to this new value. Such a decision could be based upon the numerical values of η and ϕ, which are interpreted as the implicit values of the corresponding constraints.

The following recommendation is suggested:

A model should be developed subject to more than two constraints. If it becomes necessary to relax any constraints, the Lagrangian multipliers could be used as guidelines for any decision to be made, since they could be interpreted as shadow prices of the constraints.

APPENDIX A

Unrestricted Economic Lot Sizes

$$\overset{*}{Q}_j = \sqrt{\frac{2\,\lambda_j\,A_j}{C_j \times I_j}} \qquad j = 1,2,\ldots..n$$

$$\overset{*}{Q}_1 = \sqrt{\frac{2 \times 1000 \times 10}{100 \times 0.20}} = 100$$

$$\overset{*}{Q}_2 = \sqrt{\frac{2 \times 7{,}200 \times 2}{0.20 \times 40}} = 60$$

$\overset{*}{Q}_1$ = Unrestricted Economic Lot Size of Product X_1.

$\overset{*}{Q}_2$ = Unrestricted Economic Lot Size of Product X_2.

The optimal annual cost $K = \sum_{j=1}^{2} \frac{\lambda_j}{\overset{*}{Q}_j} \times A_j + \frac{1}{2} \times 0.20 \times \sum_{j=1}^{2} Q_j \times C_j$

$= 100 \times 10 + 120 \times 2 + \frac{1}{2} \times 0.20 \times 100$

$\times 100 + \frac{1}{2} \times 0.20 \times 60 \times 40$

$= \$2{,}480$

EOQ's Subject to the Limitation on the Number of Orders Alone

h = Maximum Allowable number of Orders per year = 200

$h_1 = \lambda_1/Q_1 = 10{,}000/Q_1$

$h_2 = \lambda_2/Q_2 = 7{,}200/Q_2$

$h_1 + h_2 \leq h$

$$\overset{*}{Q}_j = \sqrt{\frac{2 \times \lambda_j \times (A_j - \eta)}{I_j \times C_j}} \qquad j = 1,2.....n$$

$$\overset{*}{Q}_1 = 10 \times \sqrt{10 \times (10 - \eta)}$$

$$\overset{*}{Q}_2 = 10 \times \sqrt{18 \times (2 - \eta)}$$

when $\eta = 0$ $\quad Q_1 = 100 \quad h_1 = 100$

$Q_2 = 60 \quad h_2 = 120$

this corresponds to unrestricted optimum condition.

But $h_1 + h_2 = 220$

$> h$

hence, the constraint is violated.

$\overset{*}{\eta}$ was evaluated by using a trial and error method. The computer program for this is listed below.

```
      ETA = - 0.650
   13 RES = 1000.0/SQRT(100.00 - 10.0 * ETA) + 720.0/SQRT (36.0 - 18.0
                                  * ETA)
      IF (RES - 200.5) 10, 11, 12
   12 ETA = ETA - 0.01
      GO TO 13
   10 IF (RES - 199.5) 14, 11, 11
   14 ETA = ETA + 0.01
      GO TO 13
   11 WRITE (12, 6) ETA
    6 FORMAT (F 10.5)
      END
```

From the computer output $\eta = -0.69$

$\eta = -0.69$ satisfied the limitation on the number of orders.

The annual cost $K = 97 \times 10 + 103 \times 2$

$$+ \frac{1}{2} \times 0.20\ (103 \times 100)$$

$$+ \frac{1}{2} \times 0.20 \times (70 \times 40)$$

$$= 970 + 206 + 1030 + 280$$

$$= \$2,486.00$$

EOQ's Subject to the Constraint on the Capital Inventment Alone

D = Maximum Allowable dollar investment for inventory

= \$12,000.00

$$\overset{*}{Q}_j = \sqrt{\frac{2 \times \lambda_j \times A_j}{C_j\ (I_j - 2\phi)}} \qquad j = 1,2.....n$$

when $\phi = 0$ $\quad Q_1 = 100 \quad D_1 = C_1 \times Q_1 \quad D_1 = 100 \times 100 = \$10,000$

$Q_2 = 60 \quad D_2 = C_2 \times Q_2 \quad = 40 \times 60 = 2,400$

This corresponds to unrestricted optimum condition. But we find that $D_1 + D_2 = \$12,400$

hence, the constraint is violated.

$D_1 = C_1 \times Q_1$

$$= C_1 \times \sqrt{\frac{2 \times A_1 \times C_1}{(I_1 - 2\phi)}} = \frac{4470}{\sqrt{(0.20 - 2\phi)}}$$

$$D_2 = C_2 \times Q_2$$

$$= C_2 \times \sqrt{\frac{2 \times A_2 \times C_2}{I_2 - 2\emptyset}} = \frac{1072}{\sqrt{(0.20 - 2 \times \emptyset)}}$$

$D_1 + D_2 = \$12,000$ Limiting Condition

$$\frac{4470}{\sqrt{(0.20 - 2\emptyset)}} + \frac{1072}{\sqrt{(020 - 2\emptyset)}} = 12,000$$

Solving for $\emptyset$ we get

$\emptyset = - 0.014 / 2$

$\emptyset = - 0.007$

$\overset{*}{Q}_1 = 96$ $D_1 = 96 \times 100 = 9,600$

$\overset{*}{Q}_2 = 58$ $D_2 = 58 \times 40 = 2,320$

$D_1 + D_2 =$ $\$11,920$

We find that this case satisfies the constraint.

Minimum annual cost

$$K = \frac{10,000}{96} \times 10 + \frac{7,200}{58} \times 2$$

$$+ \frac{1}{2} \times 0.20 \times 96 \times 100 + \frac{1}{2} \times 0.20 \times 58 \times 40$$

$$= 1042 + 248 + 960 + 232$$

$$= \$2,482$$

EOQ's Subject to Both Constraints

$D = \$12{,}000$

$h = 200$

$$\overset{*}{Q}_j = \sqrt{\frac{2\lambda_j\,(A_j - \overset{*}{\eta})}{C_j \times (I_j - 2\overset{*}{\phi})}} \qquad j = 1,2\ldots.n$$

The case when $\phi = 0$ and $\eta = 0$ refers to the unrestricted optimum condition. In such a case, both the constraints are violated.

when $\eta = 0$, the constraint on the number of orders is violated.

when $\phi = 0$, the constraint on capital is violated.

To satisfy both the constraints,

both η and ϕ have to be negative.

therefore, $D - \sum_{j=1}^{2} C_j \times Q_j = 0$

$$h - \sum_{j=1}^{2} \lambda_j / Q_j = 0$$

Solving these two equations results in $\overset{*}{Q}_1 = 85$

$\overset{*}{Q}_2 = 87$

Lot sizes $\overset{*}{Q}_1$ and $\overset{*}{Q}_2$ satisfy both the constraints.

The minimum annual cost $K = \$2{,}540$

$\overset{*}{\eta} = -9.297$

$\overset{*}{\phi} = -0.167$

APPENDIX B

```
                        DISK OPERATING SYSTEM/360 FORTRAN   360N-FO-451  CL 3-7
      DIMENSION Q(10),H(10),C(10),A(10),RS(10),QOP(10)
      READ(11,101)M
  101 FORMAT(I10)
      READ(11,102)BINT,CAPMAX,ORDMAX
  102 FORMAT(3F10.3)
      READ(11,103)(C(I),I=1,M)
  103 FORMAT(7F10.3)
      READ(11,104)(A(I),I=1,M)
  104 FORMAT(7F10.3)
      READ(11,105)(RS(I),I=1,M)
  105 FORMAT(7F10.3)
      DELTA=100.0
      WRITE(12,1115)CAPMAX
 1115 FORMAT(1H0,'MAX VALUE OF CAPITAL INVESTMENT FOR INVENTORY = $',F10
     C.3)
      WRITE(12,1116)ORDMAX
 1116 FORMAT(1H0,'MAX NUMBER OF ORDERS PER YEAR = ',F10.3)
      DO 11 I=1,M
   11 Q(I)=SQRT(2.0*     RS(I)*A(I)/C(I)/BINT)
      D=0.0
      DO 22 I=1,M
   22 D=D&C(I)*Q(I)
      DO 33 I=1,M
   33 H(I)=     RS(I)/Q(I)
      SH=0.0
      DO  44 I=1,M
   44 SH=SH&H(I)
      TAC=0.0
      DO 55 I=1,M
   55 TAC=TAC&     RS(I)*A(I)/Q(I)&0.5*BINT*C(I)*Q(I)
      WRITE(12,106)TAC
  106 FORMAT('0AVERAGE ANNUAL COST (UNRESTRICTED OPTIMUM ) = $ ',F10.3)
      WRITE(12,1112)
 1112 FORMAT(1H0,'UNRESTRICTED ECONOMIC LOTSIZES Q ')
      DO 66 I=1,M
   66 WRITE(12,107)I,Q(I)
  107 FORMAT(3H Q(,I2,2H)=,F10.1)
      IF(D-CAPMAX)108,108,109
  108 KT=1
      GO TO 110
  109 KT=2
      GO TO 110
  110 IF(SH-ORDMAX)111,111,112
  111 MT=1
      GO TO 113
  112 MT=2
      GO TO 113
  113 IF(KT-1)114,114,115
  114 IF(MT-1)116,116,115
  116 WRITE(12,117)
  117 FORMAT(32H    BOTH CONSTRAINTS ARE INACTIVE)
      GO TO 999
  115 Y=CAPMAX*ORDMAX
      Z=D*SH&1.0
```

```
12/24/70                FORTMAIN
      ETA=0.0
  120 V=0.0
      DO 77 I=1,M
   77 V=V&SQRT(2.0*     RS(I)*C(I)*(A(I)-ETA))
      W=0.0
      DO 88 I=1,M
   88 W=W&SQRT(C(I)*     RS(I)/2.0/(A(I)-ETA))
      RES=V*W
      IF(RES-Y)118,119,121
  121 IF(RES-Z)122,122,444
  444 IF(DELTA-0.01)123,123,118
  123 WRITE(12,124)Z
  124 FORMAT('0MINIMUM POSSIBLE VALUE OF CAPMAX*ORDMAX = ',F12.2)
      BUM=CAPMAX*ORDMAX
      WRITE(12,1114)BUM
 1114 FORMAT('0CAPMAX*ORDMAX = ',F12.2)
      GO TO 999
  122 ETA=ETA-DELTA
      Z=RES
      GO TO 120
  118 ETA=ETA&DELTA
      DELTA=DELTA/10.0
      ETA=ETA-DELTA
      GO TO 120
  119 WRITE(12,125)ETA
  125 FORMAT('0LAGRANGE MULTIPLIER ETA = ',F10.3)
      P=0.0
      DO 99 I=1,M
   99 P=P&SQRT(2.0*     RS(I)*C(I)*(A(I)-ETA))
      THETA=0.5*(BINT-(P**2.0/CAPMAX**2.0))
      WRITE(12,1111)THETA
 1111 FORMAT('0LAGRANGE MULTIPLIER THETA = ',F10.3)
      WRITE(12,1113)
 1113 FORMAT('0ECONOMIC LOT SIZES QOP SUBJECT TO CONSTRAINTS ')
      DO 222 I=1,M
      QOP(I)=SQRT(2.0*     RS(I)/C(I)*(A(I)-ETA)/(BINT-2.0*THETA))
  222 WRITE(12,126)I,QOP(I)
  126 FORMAT(1H ,4HQOP(,I2,2H)=,F10.1)
      TACP=0.0
      DO 333 I=1,M
  333 TACP=TACP&     RS(I)*A(I)/QOP(I)&0.5*BINT*C(I)*QOP(I)
      WRITE(12,127)TACP
  127 FORMAT('0AVERAGE ANNUAL COST SUBJECT TO CONSTRAINTS = $ ',F10.3)
  999 END
```

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