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Fracture initiation in Viscoelastic Solids

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FRACTURE INITIATION IN VISCOELASTIC SOLIDS

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This thesis is approved as a creditable and independent investigation by a candidate for the degree, Master of Science, and is acceptable as meeting the thesis requirements for this degree, but without implying that the conclusions reached by the candidate are necessarily the conclusions of the major department.

Thesis Adviser   Date

Head, Mechanical Engineering Department   Date

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TABLE OF SYMBOLS

Symbol	Meaning	Units
English		
a	= Loading Constant	1/time
A	= Constant = $\frac{a\tau_o}{s}$	none
b	= Loading Constant = $\frac{\sigma_o\sqrt{2}\,(1-\nu-2\nu^2)}{\pi E}$	none
B	= Loading Constant = $\omega\tau_o$	radians
c	= Loading Constant = 2b	none
C	= Integration Constant	
D	= Loading Constant = $\frac{k}{s}$	none
e	= Exponent = 2.71828...	none
E	= Modulus of Elasticity	force/unit area
E_e	= Rubbery Modulus of Elasticity	force/unit area
E_g	= Glassy Modulus of Elasticity	force/unit area
E_o	= Maxwell Model Spring Modulus of Elasticity	force/unit area
E_1	= Modulus of Elasticity for Spring in Section 1 of Models 2 or 3	force/unit area
E_2	= Modulus of Elasticity for Spring in Section 2 of Models 2 or 3	force/unit area

Symbol	Meaning	Units
f	= Loading Constant = $\frac{\sigma_o \sqrt{2}\,(1-\nu)}{2E}$	none
$F(\tau)$	= Reciprocal Moduli Weighting Function of Relaxation Time	unit area/time.force
g	= Loading Constant = $2f$	none
G	= Modulus of Rigidity = $\frac{E}{2(1+\nu)}$	force/unit area
$G(\tau)$	= Moduli Weighting Function of Relaxation Time	force/time.unit area
$H(t)$	= Heaviside Unit Step Function of Time	none
J	= Creep Compliance Function of Time	unit area/force
k	= Loading Constant	none
K	= Loading Constant = $\frac{a\tau_2}{s}$	none
L	= Crack Length Function of Time	Length
L^o	= Initial Crack Length Constant	length
m	= Constitutive Constant = $\frac{E_1}{\tau_2}$	force/time.unit area
M	= 2-D Loading Ratio = $\frac{\sigma_o}{\sigma_G}$	none

Symbol	Meaning	Units
n	= Constitutive Constant = $E_1 + E_2$	force/unit area
N	= 3-D Loading Ratio $\frac{\sigma_o}{\sigma_G}$	none
P	= Strain Rate Constant = $\frac{\sqrt{\delta^o}\tau_2 \dot{\epsilon}_L + \epsilon_L}{cH(t)}$	none
q	= Strain Rate Constant = $\frac{\tau_o \sqrt{\delta^o} \dot{\epsilon}_L}{cH(t)}$	none
Q	= Constitutive Constant = $\frac{n}{E_2}$	none
r	= Coordinate Axis in Direction of Crack Length	length
s	= Loading Constant	none
t	= Time Variable	time
Δt	= Time Change Function of Time = $t - t_*$	time
t_*	= Time at Initial Fracture Propagation	time
$u^o(tip)$	= Displacement of "Cut Ends Cigar" Crack Tip in Direction of z-Coordinate	length
U	= Loading Constant = $\omega\tau_2$	radians
v	= Constitutive Constant = $\frac{E_1}{E_2}$	none

Symbol	Meaning	Units
W	= Relaxation Moduli Function of Time	force/unit area
X	= Crack Length Function of Time = $\frac{L}{L^o}$	none
Y	= Yield Point	force/unit area
Z	= Strain Rate Constant = $\frac{P}{Q}$	none

Greek

Symbol	Meaning	Units
α	= Dimension Constant	none
γ_{rz}, $\gamma_{r\theta}$ & $\gamma_{z\theta}$	= 3-D Shearing Strain in Direction of Coordinate Axis	none
γ_{xy}	= 2-D Shearing Strain in Direction of Coordinate Axis	none
δ	= Strain Averaging Distance Function of Time = $\frac{\Delta}{L}$	none
δ^o	= Strain Averaging Distance Constant = $\frac{\Delta}{L^o}$	none
Δ	= Strain Averaging Distance Constant	length
ϵ	= Total or Normalized Strain	none
ϵ_d	= Strain of Damper	none
ϵ_L	= Limiting Strain at Time of Initial Crack Propagation	none

Symbol	Meaning	Units
ϵ_s	= Strain of Spring	none
ϵ_r, ϵ_z & ϵ_θ	= 3-D Strains in Direction of Coordinate Axis	none
ϵ_x & ϵ_y	= 2-D Strains in Direction of Coordinate Axis	none
ϵ_o	= Constant Applied Strain	none
ϵ_1	= Principal Strain or Strain of Section 1 in Models 2 or 3	none
ϵ_2	= Strain of Section 2 in Models 2 or 3	none
ϵ_{d2}	= Strain of Section 2 Damper in Models 2 or 3	none
ϵ_{s2}	= Strain of Section 2 Spring in Models 2 or 3	none
$\langle {}^o\epsilon_1 \rangle$	= Averaged Elastic Principal Strain	none
$\langle \epsilon_1 \rangle$	= Averaged Viscoelastic Strain	none
η	= Viscosity Coefficient for Dampers	force.time/unit area
η_o	= Maxwell Model Damper Viscosity Coefficient	force.time/unit area
η_2	= Model 2 Damper Viscosity Coefficient	force.time/unit area

Symbol	Meaning	Units
θ_1	= Time = $\frac{t}{\tau_o}$	none
θ_2	= Time = $\frac{t}{\tau_2}$	none
θ_3	= Time = $\frac{t - t_*}{\tau_o}$	none
θ_4	= Time = $\frac{t - t_*}{\tau_2}$	none
ν	= Poisson's Ratio	none
π	= Pi = 3.14159...	none
ρ	= Crack Length Function of Time = $\frac{r}{L}$	none
σ	= Stress	force/unit area
σ_d	= Stress of Damper	force/unit area
σ_G	= Griffith Stress	force/unit area
σ_s	= Stress of Spring	force/unit area
σ_r, σ_θ & σ_z	= 3-D Stress in Direction of Coordinate Axis	force/unit area
σ_x & σ_y	= 2-D Stress in Direction of Coordinate Axis	force/unit area
σ_o	= Applied Constant Stress	force/unit area
σ_1	= Stress of Section 1 in Models 2 or 3	force/unit area
σ_2	= Stress oı Section 2 in Models 2 or 3	force/unit area
σ_{d2}	= Stress of Section 2 Damper in Models 2 or 3	force/unit area

Symbol	Meaning	Units
σ_{s2}	= Stress of Section 2 Spring in Models 2 or 3	force/unit area
τ	= Relaxation Time = $\frac{\eta}{E}$	time
τ_o	= Maxwell Model Relaxation Time = $\frac{\eta_o}{E_o}$	time
τ_2	= Relaxation Time of Section 2 in Models 2 or 3 = $\frac{\eta_2}{E_2}$	time
ϕ	= Arbitrary History of Loading	none
ψ	= Material Function of Time	none
ω	= Loading Constant	radians/time

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INTRODUCTION

The subject of this paper is to derive generalized analytical expressions to predict fracture in viscoelastic polymer materials. To aid in understanding behavioral characteristics of viscoelastic materials, a basic background of viscoelastic models and their responses are presented and briefly discussed.

Fracture from two types of cracks are considered; the first is a penny-shaped crack in a three-dimensional solid and the second is a crack the thickness of the material in a two-dimensional solid. Except for the constants, the same equations describe fracture in both 2-D and 3-D solids. The fracture equations were developed using expressions from Sneddon, Wnuk, Griffith and the theory of elasticity. By experimental observation /1/, fracture in viscoelastic materials occurs in three stages: latent, slow propagation, and rapid propagation. The normalized expression describing strain during the latent stage of fracture resulting from any type of loading for the 3-D polymer solid is

$$\epsilon = \frac{N}{\phi(0)}\left[\phi + \int_0^t \dot{\psi}(t-\tau)\phi(\tau)d\tau\right]$$

The hypothesized expression describing 3-D polymer solid crack length during the slow stage of fracture propagation resulting from any type of loading is

$$\dot{\epsilon}_L = \frac{c}{\sqrt{\delta^{\circ}}}\left[\phi x^{\frac{1}{2}} + \frac{\phi}{2} x^{-\frac{1}{2}}\dot{x} + \int_0^t \frac{\partial}{\partial t}\dot{\psi}(t-\tau)\phi(\tau)x^{\frac{1}{2}}(\tau)\,d\tau + \dot{\psi}(0)\phi x^{\frac{1}{2}}\right]$$

And for the rapid propagation, no expression was developed since at this point in time of fracture, the useful life of the material is already exhausted.

CHAPTER 1

VISCOELASTIC MODELS

This chapter presents four physical models (see Figure (1-1)) for the purpose of analytically representing the behavior of real viscoelastic materials. The model behavior patterns are first described in constitutive equations which are then used to derive creep and relaxation tests. Creep and relaxation tests for different models are mathematically and graphically compared. Viscoelastic operators for determining viscoelastic stresses and strains, creep compliance and relaxation moduli, are derived for use in subsequent chapters. Now that the viscoelastic models' responses are known, the relative ease of mathematical computation and response similarities are pointed out. Finally, ideal generalized models are discussed along with stating the simpler models' adequacy and simplicity.

1-1 Constitutive Equations

Constitutive equations relate stress, strain, stress rate and strain rate and include terms characterizing the material properties. One can synthesize the constitutive equations for viscoelastic models by first summing the stress or strains or their rates for each section of the model and then substituting the constitutive equations for the springs and dampers, which are respectively

$$\sigma_s = E\epsilon_s \tag{1-1}$$

$$\sigma_d = \eta\dot{\epsilon}_d \tag{1-2}$$

Consider the Maxwell Model where the total strain of the model is the sum of the strain for the spring and the damper.

$$\epsilon = \epsilon_s + \epsilon_d \tag{1-3}$$

The stress or stress rates for the model, spring and damper are identical, which gives

$$\begin{aligned} \sigma &= \sigma_s = \sigma_d \\ \dot{\sigma} &= \dot{\sigma}_s = \dot{\sigma}_d \end{aligned} \tag{1-4}$$

Before one can substitute the damper constitutive equation into the equation (1-3) of total strain, the total strain rates must be determined. Obtain these strain rates by taking the time derivative of equation (1-3).

$$\dot{\epsilon} = \dot{\epsilon}_s + \dot{\epsilon}_d \tag{1-5}$$

The time derivative of equation (1-1) is

$$\dot{\sigma}_s = E\dot{\epsilon}_s \tag{1-6}$$

Now put equations (1-6) and (1-2) into equation (1-5) which yields

$$\dot{\epsilon} = \frac{\dot{\sigma}_s}{E} + \frac{\sigma_d}{\eta} \tag{1-7}$$

Since the stresses and stress rates in the spring and damper sections are the same as expressed by equation (1-4); equation (1-7) becomes

$$\dot{\epsilon} = \frac{\dot{\sigma}}{E} + \frac{\sigma}{\eta} \tag{1-8}$$

the constitutive equation for the Maxwell Model.

The Voigt-Kelvin constitutive equation is derived by summing the stresses of the spring and damper.

$$\sigma = \sigma_s + \sigma_d \tag{1-9}$$

This model's equal strains and strain rates of the spring and damper are expressed by

$$\epsilon = \epsilon_s = \epsilon_d$$
$$\dot{\epsilon} = \dot{\epsilon}_s = \dot{\epsilon}_d \tag{1-10}$$

By substituting equations (1-1), (1-2) and (1-10) into equation (1-9), the Voigt-Kelvin constitutive equation is

$$\sigma = E\epsilon + \eta\dot{\epsilon} \tag{1-11}$$

The Model 2 constitutive equation is obtained by summing the strains of the spring and Voigt-Kelvin Model sections which are respectively denoted by subscripts 1 and 2.

$$\epsilon = \epsilon_1 + \epsilon_2 \tag{1-12}$$

Also, the stresses and stress rates in the two sections are identical.

$$\sigma = \sigma_1 = \sigma_2$$
$$\dot{\sigma} = \dot{\sigma}_1 = \dot{\sigma}_2 \tag{1-13}$$

First, determine the strain of section 2. From the Voigt-Kelvin constitutive equation (1-11) and equation (1-13), the strain of section 2 is

$$\epsilon_2 = \frac{\sigma - \eta_2\dot{\epsilon}_2}{E_2} \tag{1-14}$$

From equation (1-12), the strain rate of section 2 is

$$\dot{\epsilon}_2 = \dot{\epsilon} - \dot{\epsilon}_1 \tag{1-15}$$

and from equations (1-6) and (1-13), the strain rate of section 1 is

$$\dot{\epsilon}_1 = \frac{\dot{\sigma}}{E_1} \tag{1-16}$$

Now by substituting equations (1-15) and (1-16) into equation (1-14), the strain of section 2 is

$$\epsilon_2 = \frac{1}{E_2}\left[\sigma + \eta_2\left(\frac{\dot{\sigma}}{E_1} - \dot{\epsilon}\right)\right] \tag{1-17}$$

and from equations (1-1) and (1-13), the strain of section 1 is

$$\epsilon_1 = \frac{\sigma}{E_1} \tag{1-18}$$

Finally, the sum of the strains from sections 1 and 2, expressed by equations (1-17) and (1-18), gives the constitutive equation for Model 2.

$$\dot{\epsilon} + \frac{\epsilon}{\tau_2} = \frac{\dot{\sigma}}{E_1} + \frac{n\,\sigma}{E_1\,\eta_2} \tag{1-19}$$

Derive the Model 3 constitutive equation by summing the stresses of the spring and Maxwell Model sections which are respectively denoted by subscripts 1 and 2.

$$\sigma = \sigma_1 + \sigma_2 \tag{1-20}$$

Also the strains and strain rates for the two sections are identical.

$$\epsilon = \epsilon_1 = \epsilon_2$$
$$\dot{\epsilon} = \dot{\epsilon}_1 = \dot{\epsilon}_2 . \tag{1-21}$$

From equations (1-5) and (1-21), the strain rate of the Maxwell section is

$$\dot{\epsilon} = \dot{\epsilon}_{s2} + \dot{\epsilon}_{d2} \tag{1-22}$$

Solve for the strain rates of the spring and damper in section 2. The spring strain rate from equation (1-6) is

$$\dot{\epsilon}_{s2} = \frac{\dot{\sigma}_{s2}}{E_2} \tag{1-23}$$

where the stress rate of this spring is evaluated by taking the time derivative of equation (1-20) and by applying equation (1-4) to section 2.

$$\dot{\sigma}_{s2} = \dot{\sigma} - \dot{\sigma}_1 \qquad (1\text{-}24)$$

Evaluate section 1 stress rate by applying equations (1-6) and (1-21) to equation (1-24).

$$\dot{\sigma}_{s2} = \dot{\sigma} - E_1\dot{\epsilon} \qquad (1\text{-}25)$$

Substitute equation (1-25) into equation (1-23) and the strain rate for spring of section 2 takes the form

$$\dot{\epsilon}_{s2} = \frac{\dot{\sigma} - E_1\dot{\epsilon}}{E_2} \qquad (1\text{-}26)$$

From equation (1-2), the strain rate of the damper in section 2 is

$$\dot{\epsilon}_{d2} = \frac{\sigma_{d2}}{\eta_2} \qquad (1\text{-}27)$$

where the stress of the damper in the Maxwell section is determined from equation (1-20) using equation (1-4).

$$\sigma_{d2} = \sigma - \sigma_1 \qquad (1\text{-}28)$$

Evaluate section 1 stress by applying equations (1-1) and (1-21) to equation (1-28).

$$\sigma_{d2} = \sigma - E_1\epsilon \tag{1-29}$$

Substitute equation (1-29) into equation (1-27) and the strain rate for the damper of section 2 takes the form

$$\dot{\epsilon}_{d2} = \frac{\sigma - E_1\epsilon}{\eta_2} \tag{1-30}$$

Finally, by substituting equations (1-26) and (1-30) into (1-22) and simplifying, the Model 3 constitutive equation is

$$\dot{\epsilon} + \frac{m\epsilon}{n} = \frac{\dot{\sigma}}{n} + \frac{\sigma}{n\tau_2} \tag{1-31}$$

1-2 Creep

Viscoelastic models are subjected to standardized tests to determine and compare model behavioral characteristics. A standard creep test shows a model's elastic extension responses as a function of time due to a known applied static load.

Before exploring the creep of viscoelastic models, it is important that two types of elastic moduli be understood: glassy and rubbery. A single elastic spring will extend instantaneously upon sudden application of a load. Thus the stresses and strains are related by an instantaneous elastic modulus -- this is called a glassy modulus. When dealing with viscoelastic models, the glassy modulus is the acting elastic modulus for only a short time after the load has been applied. A single spring or a complex arrangement of springs and dampers may have a glassy modulus. The elastic extensions during the short lifetime of the glassy modulus does not include any extensions due to dampers.

The rubbery modulus relates stresses and strains during a long time period and includes those extensions allowed after some time by dampers.

Creep Equations

The equation expressing creep for a viscoelastic model is determined by substituting a constant stress into its constitutive equation and solving for the resulting strain. While solving the creep equation, a constant of integration arises. Initial conditions are applied to both the model and the general solution of creep and the two are then equated to solve for the integration constant.

Substitution of a constant stress into the Maxwell Model constitutive equation (1-8) yields

$$\dot{\epsilon} = \frac{\sigma_o}{\eta} \tag{1-32}$$

Separate variables and integrate to obtain the general solution.

$$\epsilon = \frac{\sigma_o t}{\eta} + C \tag{1-33}$$

Establish the initial conditions. At time zero, a constant load was applied and the only instantaneous extension allowed by the glassy modulus is from the spring, thus

$$t = o$$
$$\sigma = \sigma_o \qquad (1\text{-}34)$$
$$E_g = E$$

Applying the initial conditions to the general solution equation (1-33) gives

$$\epsilon(o) = C \qquad (1\text{-}35)$$

Now consider the initial response of a model. The model's immediate stresses and strains must be related by the glassy modulus of elasticity as follows:

$$\sigma = E_g \epsilon \qquad (1\text{-}36)$$

From equation (1-36) and the initial conditions equation (1-34), the Maxwell Model instantaneous strain is

$$\epsilon(o) = \frac{\sigma_o}{E} \qquad (1\text{-}37)$$

Equate equations (1-35) and (1-37) to solve for the constant.

$$C = \frac{\sigma_o}{E} \qquad (1\text{-}38)$$

By substituting the constant equation (1-38) into equation (1-33), the final creep equation for the Maxwell Model becomes

$$\epsilon = \frac{\sigma_o t}{\eta} + \frac{\sigma_o}{E} \tag{1-39}$$

For the Voigt-Kelvin Model, substitute a constant stress into its constitutive equation (1-11) to obtain

$$\dot{\epsilon} + \frac{\epsilon}{\tau} = \frac{\sigma_o}{\eta} \tag{1-40}$$

Since equation (1-40) is a linear first-order differential equation, the general solution is

$$\epsilon = \frac{\sigma_o}{E} + Ce^{\frac{-t}{\tau}} \tag{1-41}$$

The initial conditions are a constant stress and no extension which are related by an infinite glassy modulus.

$$\begin{aligned} t &= o \\ \sigma &= \sigma_o \\ E_g &= \infty \\ \epsilon(o) &= 0 \end{aligned} \tag{1-42}$$

Applying the initial conditions to the general solution for creep, equation (1-41) gives

$$\epsilon(o) = C + \frac{\sigma_o}{E} \tag{1-43}$$

The initial elastic behavior of the model follows equation (1-36). By applying the initial conditions, equation (1-42), to equation (1-36), the initial model strain is

$$\epsilon(o) = 0 \qquad (1\text{-}44)$$

Equate equations (1-43) and (1-44) to solve for the integration constant.

$$C = -\frac{\sigma_o}{E} \qquad (1\text{-}45)$$

Substitute equation (1-45) into the general solution equation (1-41) to obtain the creep equation of the Voigt-Kelvin Model.

$$\epsilon = \frac{\sigma_o}{E}\left[1 - e^{\frac{-t}{\tau}}\right] \qquad (1\text{-}46)$$

For Models 2 and 3, follow a procedure similar to the previous two models. Substitute a constant stress into Models 2 and 3 constitutive equations (1-19) and (1-31) and solve the resulting first-order linear differential equations for their general solutions. The two models initial conditions are the constant stress and glassy moduli that permit instantaneous strains. For Models 2 and 3 the initial conditions are respectively

$$t = o$$
$$\sigma = \sigma_o \qquad (1\text{-}47)$$
$$E_g = E_1$$

and

$$t = o$$
$$\sigma = \sigma_o \qquad (1\text{-}48)$$
$$E_g = n$$

Apply the corresponding initial conditions to the general solutions and Models 2 and 3, then equate the two applications of initial conditions for each model to solve for the general solutions' constants C which yield

$$\epsilon = \frac{n\sigma_o}{E_1 E_2}\left[1 - \frac{E_1}{n} e^{\frac{-t}{\tau_2}}\right] \qquad (1\text{-}49)$$

and

$$\epsilon = \frac{\sigma_o}{E_1}\left[1 - \frac{E_2}{n} e^{\frac{-mt}{n}}\right] \qquad (1\text{-}50)$$

respectively as the equations of creep for Models 2 and 3.

Creep Compliance

The creep compliance of a viscoelastic model is the resulting strain per unit of applied stress. It is a simple matter to obtain the creep compliance for each of the models discussed in this chapter by dividing the creep equations (1-39), (1-46), (1-49) and (1-50) by a unit of constant stress.

The creep compliances are:

Maxwell Model

$$J = \frac{1}{E} + \frac{t}{\eta} \tag{1-51}$$

Voigt-Kelvin Model

$$J = \frac{1}{E}\left[1 - e^{\frac{-t}{\tau}}\right] \tag{1-52}$$

Model 2

$$J = \frac{n}{E_1 E_2}\left[1 - \frac{E_1}{n} e^{\frac{-t}{\tau_2}}\right] \tag{1-53}$$

Model 3

$$J = \frac{1}{E_1}\left[1 - \frac{E_2}{n} e^{\frac{-m}{n} t}\right] \tag{1-54}$$

Creep compliance to viscoelastic models serves the same purpose as the modulus of elasticity does with elastic solids. Creep compliance, if known, allows one to determine the resulting strain of a model upon application of any type of stress. Creep compliances of viscoelastic models permit quick mathematical comparisons of the different behavioral characteristics.

Creep Curves

Creep curves are used for quick visual comparisons of viscoelastic model responses. Graph (1-1) shows the creep curves for the models discussed in this chapter. The curves are plots of the creep equations; but the curves may also be explained by observing model behavior.

As one can see by observing the curves, the Maxwell Model represents a linear viscoelastic fluid. Immediately after loading, the model extends via the spring only and then after some time, the damper flows with a constant velocity, linearly increasing the strain.

The Voigt-Kelvin Model represents a viscoelastic solid with no immediate response. When the stress is applied, the strain that would occur instantly if no damper was present is approached exponentially.

Models 2 and 3 represent viscoelastic material known as the standard linear solid. Both models have identically shaped creep curves and permit the instantaneous strain of the Maxwell Model and then increase the strain exponentially like the Voigt-Kelvin Model.

1-3 Stress Relaxation

Stress relaxation is another test used to characterize a viscoelastic model. The relaxation test shows how a model's stress decays as a function of time after a known constant strain has been applied.

Relaxation Equations

Viscoelastic model stress relaxation is derived in a fashion similar to creep, except now a constant strain is imposed and the resulting stress is determined using the model constitutive equation. As with creep, constants of integration are solved for by applying the initial conditions to both the general solution of relaxation and the model and then equating the two.

The Maxwell Model is derived by substituting a constant strain into the constitutive equation (1-8).

$$\dot{\sigma} + \frac{\sigma}{\tau} = 0 \tag{1-55}$$

Separate variables, and integrate; this gives the general solution to equation (1-55).

$$\sigma = Ce^{\frac{-t}{\tau}} \tag{1-56}$$

Similar to equation (1-34), the initial conditions are

$$t = 0$$
$$\epsilon = \epsilon_o \tag{1-57}$$
$$E_g = E$$

which, when applied to the general solution equation (1-56), yields

$$\sigma(o) = C \tag{1-58}$$

and when applied to the initial model behavior equation

(1-36) yields

$$\sigma(o) = E\epsilon_o \tag{1-59}$$

Equate equations (1-58) and (1-59) to obtain

$$C = E\epsilon_o \tag{1-60}$$

Substitute equation (1-60) into the general solution equation (1-56) for the relaxation equation of the Maxwell Model.

$$\sigma = E\epsilon_o e^{\frac{-t}{\tau}} \tag{1-61}$$

Voigt-Kelvin models do not experience any stress relaxation because of the physical arrangement of the spring and damper connected in parallel. A strain relaxation does occur, however, if the model is given an initial strain and then allowed to relax under zero stress. To obtain the strain relaxation equation, substitute zero stress into the constitutive equation (1-11) and solve for the strain.

$$\dot{\epsilon} + \frac{\epsilon}{\tau} = 0 \tag{1-62}$$

The general solution for the strain is

$$\epsilon = Ce^{\frac{-t}{\tau}} \tag{1-63}$$

The initial conditions are

$$t = 0$$
$$\epsilon(o) = \epsilon_o \tag{1-64}$$

and when applied to the general solution equation (1-63), the value of the integration constant is

$$C = \epsilon_o \tag{1-65}$$

Substituting the constant equation (1-65) into equation (1-63) gives the strain relaxation for the Voigt-Kelvin Model.

$$\epsilon = \epsilon_o \, e^{\frac{-t}{\tau}} \tag{1-66}$$

Model 2 and 3 relaxation equations are derived by a procedure similar to the Maxwell Model. A constant strain is substituted into both models' constitutive equations (1-19) and (1-31) and the resulting first-order linear differential equations are solved for their general solutions. Except for the constant strain here, instead of stress in the case of creep equations, the initial conditions for Models 2 and 3 are the same as equations (1-47) and (1-48). They are respectively

$$\begin{aligned} t &= 0 \\ \epsilon &= \epsilon_o \\ E_g &= E_1 \end{aligned} \tag{1-67}$$

and

$$\begin{aligned} t &= 0 \\ \epsilon &= \epsilon_o \\ E_g &= n \end{aligned} \tag{1-68}$$

Apply the corresponding initial conditions to the general solutions and Models 2 and 3, then equate the two applications of initial conditions for each model to solve for the general solutions' constants C which yield

$$\sigma = \frac{E_1 \epsilon_o}{n} \left[E_2 + E_1 e^{\frac{-nt}{\eta_2}} \right] \qquad (1\text{-}69)$$

and

$$\sigma = \epsilon_o \left[E_1 + E_2 e^{-\frac{t}{\tau_2}} \right] \qquad (1\text{-}70)$$

respectively as the relaxation equations for Models 2 and 3.

Relaxation Moduli

Viscoelastic model stress relaxation modulus is the resulting stress per unit of applied strain. Relaxation moduli are obtained by dividing the relaxation equations (1-61), (1-66), (1-69) and (1-70) by a unit of strain. Note that the Voigt-Kelvin Model does not have any stress relaxation, just strain relaxation under zero stress. The relaxation moduli are as follows:

Maxwell Model

$$W = Ee^{\frac{-t}{\tau}} \qquad (1\text{-}71)$$

Voigt-Kelvin Model

$$W = e^{\frac{-t}{\tau}} \qquad (1\text{-}72)$$

Model 2

$$W = \frac{E_1}{n}\left[E_2 + E_1 e^{\frac{-nt}{\eta_2}}\right] \qquad (1\text{-}73)$$

Model 3

$$W = E_1 + E_2 e^{\frac{-t}{\tau_2}} \qquad (1\text{-}74)$$

Relaxation Curves

The relaxation curves shown in Graphs (1-2) and (1-3) are used for quick visual comparisons of different viscoelastic model responses. The curves are plots of the relaxation equations; but they may also be graphed by observing model behavior.

After being strained, the Maxwell Model has a high initial stress that is allowed to relax exponentially to zero as the damper flows.

The Voigt-Kelvin Model strain is relaxed as the stress of the spring returns the model to its neutral position at an exponential rate controlled by the damper.

After being strained, Models 2 and 3 have a high initial stress. The stress exponentially decreases as the dampers flow to a level maintained by the two springs in series for Model 2 and to that maintained by the section spring for Model 3.

1-4 Model Equivalence and Comparison

Any viscoelastic material described by Model 2 can also be described by Model 3; they are equivalent. A comparison of creep and relaxation curves and equations for the two models reveals identical responses and forms. The models differ only in the magnitude and arrangements of their parameters.

However, the creep computations associated with Model 2 are simpler. This is logical since Model 2 is composed of two series connected sections and the strains of each individual section need only to be added to obtain the model creep. In fact this may be shown by adding the strain responses of the spring and series connected Voigt-Kelvin Model to an applied constant stress. The result is the same creep equation (1-49) previously and independently obtained for Model 2. Creep of any generalized model composed of sections connected in series is the sum of their individual section responses. Such a simple addition of strains for each section is not possible with Model 3 where the sections are connected in parallel.

Similarly, relaxation is easier to compute when models have their sections connected in parallel by an argument similar to that used in creep. The Model 3 relaxation equation (1-70) may be derived independently by summing the stress responses of the spring and the Maxwell Model sections to an applied constant stress. The relaxation of any generalized model composed of sections connected in parallel is the sum of their individual section responses. The computational advantages pointed out here may be used when deriving the material function equation (2-15) for some example.

1-5 Relaxation Times

It is important to understand relaxation times when dealing with generalized models or applications of visco-elastic materials. The relaxation time for viscoelastic models is defined as the ratio of a viscosity coefficient to a modulus of elasticity. This time is the quickness of a viscoelastic response to reach static equilibrium after an imposed stress or strain. During a relaxation test, for example, the stress in a Maxwell Model relaxes to $\frac{1}{e}$ of its initial value after a time period equal to the model's relaxation time. This is shown by the ratio of the Maxwell relaxation equation (1-61) evaluated at the relaxation time to it evaluated at time zero.

$$\frac{\sigma(\tau)}{\sigma(0)} = \frac{1}{e} \qquad (1\text{-}75)$$

Similar stress ratios exist for the other models after a relaxation time passes.

Of the models discussed so far, the relaxation time span may be a few seconds or several weeks depending on the magnitudes of the model's modulus of elasticity and viscosity coefficient. But for each model there exists only one discrete relaxation time. Real materials and the models of the next section possess a continuous distribution of relaxation times that may range widely.

1-6 Generalized Models

Since polymer type materials respond in a manner represented by a continuous distribution of relaxation times, it does not appear that Model 2 would accurately represent real polymer materials. A model with a spring and n number of Voigt-Kelvin sections connected in series would represent polymer creep response, for example, more accurately than Model 2. This model is the generalized Voigt-Kelvin solid shown in Figure (1-2) and has the following creep equation obtained by summing the creep of each section:

$$\epsilon = \frac{\sigma_o}{E_1} + \sigma_o \sum_{i=2}^{n} \frac{1}{E_i}\left[1 - e^{\frac{-t}{\tau_i}}\right] \qquad (1\text{-}76)$$

If some mechanism possessed a continuous distribution of retardation times, it would simulate real material perfectly.

The creep equation for such a mechanism is given by the integral of Voigt-Kelvin elements from zero to infinity.

$$\epsilon = \frac{\sigma_o}{E_1} + \sigma_o \int_0^\infty F(\tau)\left[1 - e^{\frac{-t}{\tau}}\right] d\tau \qquad (1\text{-}77)$$

The Voigt-Kelvin moduli in equation (1-76) is replaced by a continuous weighting function in the integral of equation (1-77). This weighting function not only replaces the moduli, but may be used to emphasize any retardation times desired.

The perfect simulation equation (1-77), the creep of real materials, may be closely approximated by the Model 2 creep equation (1-49). Consider the amount of some real material's creep in one day. The spring and those Voigt-Kelvin elements in equation (1-77) with short relaxation times, relative to the one day experiment, will all respond in a manner that approaches the spring in section 1 of Model 2. The small creep of those elements in the perfect model with relatively long relaxation times may be neglected. And the average response of the elements whose relaxation times are near the same magnitude as the time of the experiment is approximated by section 2 of Model 2. The disadvantage of the Model 2 approximations of the perfect model may be offset by the easier mathematical treatment permitted by Model 2.

Similarly the generalized Maxwell solid shown in Figure (1-2) with n parallel connected Maxwell sections represents the stress relaxation of real materials better than Model 3. The Maxwell solid relaxation equations for n and a continuous distribution of relaxation times are respectively

$$\sigma = E_1 \epsilon_o + \epsilon_o \sum_{i=2}^{n} E_i \, e^{\frac{-t}{\tau_i}} \qquad (1\text{-}78)$$

and

$$\sigma = E_1 \epsilon_o + \epsilon_o \int_0^{\infty} G(\tau) e^{\frac{-t}{\tau}} \, d\tau \qquad (1\text{-}79)$$

CHAPTER 2

3-D POLYMER FRACTURE IN LATENT STAGE

The main objectives of this chapter are: one, to develop a general relationship of stress and strain at the edge of a penny shaped stationary crack in polymers; and two, to predict when the crack will grow. The penny shaped crack shown in Figure (2-1) has a cross-sectional shape similar to that of a cigar. Initially, Sneddon's tensile stress equations are the basis for deriving elastic strain expressions in the region of the crack tip for any arbitrary loading. However, at the crack tip, the strain is undefined; this problem is solved by averaging the strain over a small area in front of the crack using Wnuk's relationship of crack length to a small distance preceding the crack. Then the Elastic-Viscoelastic Correspondence Principle employs a viscoelastic operator and converts the strain from elastic to viscoelastic. The maximum strain any material permits is determined by applying a Griffith stress and is used to normalize the viscoelastic strain. The latent stage ends when the normalized strain becomes unity, the material separates and the crack propagates. Finally, examples of strains are computed for two types of viscoelastic materials subjected to three types of stress loading patterns.

2-1 Averaged Maximum Elastic Strain

The starting point for developing strain relations is Sneddon's /2/ elastic stress equations in the plane $z = 0$ of the penny-shaped crack.

$$\sigma_z = \frac{2\sigma}{\pi}\left[\frac{1}{\sqrt{\rho^2 - 1}} - \sin^{-1}\frac{1}{\rho}\right]$$

$$\sigma_r = \frac{2\sigma}{\pi}\left[\frac{1}{\sqrt{\rho^2 - 1}} - (\nu + \tfrac{1}{2})\sin^{-1}\frac{1}{\rho}\right] \qquad (2\text{-}1)$$

$$\sigma_\theta = \frac{2\sigma}{\pi}\left[\frac{2\nu}{\sqrt{\rho^2 - 1}} - (\nu + \tfrac{1}{2})\sin^{-1}\frac{1}{\rho}\right]$$

$$\tau_{rz} = \tau_{r\theta} = \tau_{\theta z} = 0$$

Since the Sneddon equations do not include the inertia effects, the analysis of this paper must be restricted to a slowly propagating crack. The applied load σ will be expressed only in units of time t by substituting

$$\sigma = \sigma_0 \phi \qquad (2\text{-}2)$$

Because this paper deals only with expressions at the crack tip, equation (2-1) may be simplified. At the crack tip, $\rho = 1$ which is an isolated singularity point. Expand $\frac{1}{\sqrt{\rho^2 - 1}}$ and $\sin^{-1}\frac{1}{\rho}$ of equation (2-1) in a Taylors series about the point $\rho = 1$.

$$\frac{1}{\sqrt{\rho^2 - 1}} = \frac{1}{\sqrt{2}}\left[(\rho - 1)^{-\frac{1}{2}} - \frac{(\rho - 1)^{\frac{1}{2}}}{1!\,2^2} + \frac{3(\rho - 1)^{\frac{3}{2}}}{2!\,2^4} - + \ldots\right] \qquad (2\text{-}3)$$

$$\sin^{-1}\frac{1}{\rho} = \frac{\pi}{2} - \frac{\sqrt{2}(\rho - 1)^{\frac{1}{2}}}{1!} + \frac{5\sqrt{2}(\rho - 1)^{\frac{3}{2}}}{3!\,2} + \ldots$$

In the vicinity of the crack tip, all terms in equation (2-3) may be neglected except $\frac{(\rho - 1)^{-\frac{1}{2}}}{\sqrt{2}}$; therefore

$$\frac{1}{\sqrt{\rho^2 - 1}} = \frac{(\rho - 1)^{-\frac{1}{2}}}{\sqrt{2}} \qquad (2\text{-}4)$$

$$\sin^{-1}\frac{1}{\rho} = 0$$

at the crack tip. Hence equations (2-1) applied near the crack tip simplifies, upon substitution of equation (2-4) to

$$\sigma_z = \frac{\sqrt{2}\,\sigma}{\pi\sqrt{\rho - 1}}$$

$$\sigma_r = \frac{\sqrt{2}\,\sigma}{\pi\sqrt{\rho - 1}} \qquad (2\text{-}5)$$

$$\sigma_\theta = \frac{2\sqrt{2}\,\nu\,\sigma}{\pi\sqrt{\rho - 1}}$$

Now elastic strain may be evaluated from the three dimensional stress-strain relations -- Hooke's Law yields

$$\epsilon_z = \frac{1}{E}\left[\sigma_z - \nu(\sigma_r + \sigma_\theta)\right]$$

$$\epsilon_r = \frac{1}{E}\left[\sigma_r - \nu(\sigma_z + \sigma_\theta)\right] \qquad (2\text{-}6)$$

$$\epsilon_\theta = \frac{1}{E}\left[\sigma_\theta - \nu(\sigma_z + \sigma_r)\right]$$

$$\gamma_{rz} = \frac{\tau_{rz}}{G} \qquad \gamma_{r\theta} = \frac{\tau_{r\theta}}{G} \qquad \gamma_{\theta z} = \frac{\tau_{\theta z}}{G}$$

The strain at the crack tip is given by combining equations (2-2), (2-5) and (2-6)

$$\epsilon_z = \frac{b\,\phi}{\sqrt{\rho - 1}}$$

$$\epsilon_r = \frac{b\,\phi}{\sqrt{\rho - 1}} \qquad (2\text{-}7)$$

$$\epsilon_\theta = 0$$

$$\gamma_{rz} = \gamma_{r\theta} = \gamma_{\theta z} = 0$$

Determine the principal strains by substituting equations (2-7) into the following determinant from the theory of elasticity:

$$\begin{vmatrix} \epsilon_r - \epsilon & \gamma_{r\theta} & \gamma_{rz} \\ \gamma_{r\theta} & \epsilon_\theta - \epsilon & \gamma_{\theta z} \\ \gamma_{rz} & \gamma_{\theta z} & \epsilon_z - \epsilon \end{vmatrix} = 0$$

expanding the determinant into a cubic equation and solving for the roots. The maximum root or principal strain is

$$\epsilon_1 = \frac{b\phi}{\sqrt{\rho - 1}} = \frac{b\phi\sqrt{L}}{\sqrt{r - L}} \qquad (2\text{-}8)$$

which still has the singularity point at the crack tip. Eliminate the singularity point by averaging ϵ_1 over a small area on the plane $z = 0$ in front of the crack tip; see Figure (2-1). The averaging area is a narrow strip of width Δ wrapped around the crack edge. The small magnitude of Δ is related to the crack tip displacement by Wnuk /3/ (see Figure (2-1) and equation (2-21)) and because of Δ's small size, the averaging area may be approximated by

$$\text{Averaging Area} \approx 2\pi L\Delta \qquad (2\text{-}9)$$

Because the relationship of averaging distance to crack tip displacement is used, this paper assumes the penny-shaped crack looks like Wnuk's cigar-shaped crack in Figure (2-1).

Using the law of the mean for double integrals from calculus and equation (2-9), the averaged strain, denoted by $\langle {}^{o}\epsilon_1 \rangle$, becomes

$$\langle {}^{o}\epsilon_1 \rangle = \frac{1}{2\pi L \Delta} \int_{L}^{L+\Delta} \int_{0}^{2\pi} \epsilon_1 r\,dr\,d\theta \tag{2-10}$$

After substituting equation (2-8) into (2-10) and integrating, the results are

$$\langle {}^{o}\epsilon_1 \rangle = \frac{c\phi}{3} \left[\left(\frac{\Delta}{L}\right)^{\frac{1}{2}} + 3\left(\frac{\Delta}{L}\right)^{-\frac{1}{2}} \right] \tag{2-11}$$

Since Δ is small,

$$\frac{\Delta}{L} \ll 1$$

Δ/L is negligible compared to L/Δ and equation (2-11) simplifies to

$$\langle {}^{o}\epsilon_1 \rangle = \frac{c\phi\sqrt{L}}{\sqrt{\Delta}} \tag{2-12}$$

which is the averaged maximum elastic strain. Equation (2-12) may be expressed in dimensionless terms by multiplying by

$$\sqrt{\frac{L^{o}}{L^{o}}}$$

which yields

$$\langle {}^{o}\epsilon_1 \rangle = \frac{c\phi\sqrt{X}}{\sqrt{\delta^{o}}} \tag{2-13}$$

the average maximum elastic strain. Let the pre-superscript o on the strain denote elastic as opposed to viscoelastic relations in the next section.

2-2 Averaged Maximum Viscoelastic Strain

Introducing the elastic-viscoelastic correspondence principle taken from Wnuk /4/ converts elastic strain relations into viscoelastic relations.

$$\langle \epsilon_1 \rangle = \langle {}^{o}\epsilon_1 \rangle + \int_0^t \dot{\psi}\,(t - \tau)\langle {}^{o}\epsilon_1(\tau) \rangle \, d\tau \qquad (2\text{-}14)$$

where $\dot{\psi}$ is a material function characterizing a viscoelastic sample. For viscoelastic polymers, $\dot{\psi}$ simplifies to

$$\dot{\psi} = \frac{\dot{J}(t)}{J(o)} \qquad (2\text{-}15)$$

for polymer type materials. The examples of strain from viscoelastic models shown later in this chapter use the creep compliances derived in Chapter 1 with arbitrary values for the elastic and viscosity coefficients. For the creep compliances of real materials, one may first decide what model represents the material and then determine the proper magnitudes of the coefficients through experimentation. By substituting equation (2-13) into equation (2-14) one obtains the general expression for viscoelastic strain at the crack tip

$$\langle \epsilon_1 \rangle = \frac{c}{\sqrt{\delta^o}} \left[\phi \sqrt{X} + \int_o^t \dot{\psi}(t - \tau)\phi(\tau)\sqrt{X(\tau)}\, d\tau \right] \quad (2\text{-}16)$$

This chapter deals with strains during the latent stage, i.e. at times before material separation. For time $t \leqslant t_*$, the crack length is constant and equal to the initial length L^o and therefore $X = 1$. Equation (2-16) simplifies to

$$\langle \epsilon_1 \rangle = \frac{c}{\sqrt{\delta^o}} \left[\phi + \int_o^t \dot{\psi}(t - \tau)\phi(\tau) d\tau \right] \quad (2\text{-}17)$$

2-3 Normalized Viscoelastic Strain

The strain equation (2-17) for the latent stage may be normalized by dividing by the maximum or limiting strain at the crack tip which is experienced as the material begins to separate. The limiting strain will be determined by applying a Griffith stress /5/.

$$\sigma_G = \sqrt{\frac{\pi E Y u^o(tip)}{2(1 - \nu^2)L^o}} \quad (2\text{-}18)$$

Griffith stress is known for different materials and is that stress just required to produce material separation. By applying an instantaneous Griffith stress at time $t = 0$ to equation (2-17), the limiting strain is obtained

$$\epsilon_L = \frac{c\phi(o)}{\sqrt{\delta^o}} \quad (2\text{-}19)$$

The Griffith stress must be included within $\frac{c\phi(o)}{\sqrt{\delta^o}}$ of equation (2-18) and may be identified. To do this expand δ^o.

$$\delta^o = \frac{\Delta}{L^o} \tag{2-20}$$

Wnuk /3/ relates delta to the displacement in the z- direction at the crack tip and the crack tip displacement with the Griffith stress

$$\Delta = \alpha u^o(\text{tip})$$

$$u^o(\text{tip}) = \frac{2(1 - \nu^2)\sigma_o^2 L^o}{\pi EY} \tag{2-21}$$

Using these expressions and equation (2-20), the limiting strain equation (2-19) takes the form

$$\epsilon_L = \frac{c\phi(o)}{\sigma_o}\sqrt{\frac{\pi EY}{2(1 - \nu^2)\alpha}} \tag{2-22}$$

Finally, the limiting strain with the Griffith stress identified is obtained after multiplying and dividing equation (2-22) by $\sqrt{\frac{u^o(\text{tip})}{L^o}}$ and simplifying, using equations (2-18) and (2-21), to

$$\epsilon_L = \frac{c\phi(o)}{N\sqrt{\delta^o}} \tag{2-23}$$

Now if equation (2-17) is divided by equation (2-23), the normalized viscoelastic strain for the latent stage is

$$\epsilon = \frac{N}{\phi(o)}\left[\phi + \int_o^t \dot{\psi}\ (t - \tau)\,\phi(\tau)d\tau\right] \qquad (2\text{-}24)$$

This equation is the ratio of existing strain as a function of time to the maximum strain at material separation and is always less than or equal to unity during the latent stage. When the normalized strain becomes larger than unity the latent stage is ended and the material at the crack tip separates, increasing the crack length. Remember that the strain in equation (2-24) is valid only at the tip of a constant length crack.

2-4 Examples

To demonstrate the theoretical derivation's effectiveness, the strains for two types of viscoelastic materials under various loading patterns will be calculated and graphed. Then the validity of the theoretical results may be determined by comparison with experimental results.

The Maxwell Model and Model 2 of Chapter 1 with the following arbitrary history of loading patterns will serve as examples for graphing equation (2-24):

$$\phi = H(t) \qquad (2\text{-}25)$$

$$\phi = s + at \qquad (2\text{-}26)$$

$$\phi = s + k\ \sin\omega t \qquad (2\text{-}27)$$

The magnitude of the first loading pattern, the Heaviside unit function, is taken as unity at time zero and may be thought of as time $t = 0+$. This defines ϕ, and therefore allows equation (2-24) to take on a finite magnitude, at time $t = 0$ for $H(t)$ as

$$\phi(o) = 1 \tag{2-28}$$

Now examples of normalized viscoelastic strain will be calculated and graphed.

Maxwell Model with $\phi = H(t)$

First evaluate the material function using equation (2-15). The time derivative of the creep compliance equation (1-51) is

$$\dot{J} = \frac{1}{\eta_o} \tag{2-29}$$

The creep compliance evaluated at time $t = 0$ is

$$J(o) = \frac{1}{E_o} \tag{2-30}$$

Now the material function is evaluated by substituting equations (2-29) and (2-30) into equation (2-15)

$$\dot{\psi} = \frac{1}{\tau_o} \tag{2-31}$$

By substituting equations (2-28), (2-25) and (2-31) into equation (2-24), the normalized strain takes the form

$$\epsilon = N\left[H(t) + \frac{1}{\tau_o}\int_0^t H(\tau)d\tau\right] \qquad (2\text{-}32)$$

The solution to equation (2-32) is

$$\epsilon = N\left[1 + \theta_1\right] \qquad (2\text{-}33)$$

Maxwell Model with $\phi = s + at$

For the linear loading pattern, substitute equations (2-26) and (2-31) into the normalized strain equation (2-24) and solve; the solution is

$$\epsilon = N\left[1 + \theta_1\left\{1 + A\left(1 + \frac{\theta_1}{2}\right)\right\}\right] \qquad (2\text{-}34)$$

and is graphed for different magnitudes of loadings in Graphs (2-2A & -2B).

Maxwell Model with $\phi = s + k\sin\omega t$

Similarly for the sinusoidal loading, substitute equations (2-27) and (2-31) into equation (2-24) and solve. The solution is

$$\epsilon = N\left[1 + \theta_1 + D\left(\sin B\theta_1 + \frac{1}{B} - \frac{1}{B}\cos B\theta_1\right)\right] \qquad (2\text{-}35)$$

and is graphed for different magnitudes of loading in Graphs (2-3A, -3B & -3C).

Model 2 with $\phi = H(t)$

The solution to the normalized strain is calculated by following a procedure similar to that for the Maxwell Model. Evaluate the material function equation (2-15) by taking the derivative of the creep compliance equation (1-53) and dividing by the creep compliance evaluated at time t = 0.

$$\dot{\psi} = \frac{E_1}{\eta_2} e^{\frac{-t}{\tau_2}} \qquad (2\text{-}36)$$

Substitute equations (2-36), (2-25) and (2-28) into equation (2-24) and the normalized strain takes the form

$$\epsilon = N\left[H(t) + \frac{E_1}{\eta_2}\int_0^t e^{\frac{-t+\tau}{\tau_2}} H(\tau)d\tau\right] \qquad (2\text{-}37)$$

The solution to equation (2-37) is

$$\epsilon = N\left[1 + v(1 - e^{-\theta_2})\right] \qquad (2\text{-}38)$$

Equation (2-38) is graphed for different loadings in Graphs (2-4A & -4B).

Model 2 with $\phi = s + at$

Substitute equations (2-26) and (2-36) into equation (2-24) to evaluate the normalized strain. The result is

$$\epsilon = \left[N\ 1 + K(1 + v)\,\theta_2 + v(1 - K)(1 - e^{-\theta_2}) \right] \quad (2\text{-}39)$$

which is graphed for different magnitudes of loading in Graphs (2-5A & -5B).

Model 2 with $\phi = s + k \sin \omega t$

And for the last example of viscoelastic fracture during the latent stage, substitute equations (2-27) and (2-36) into equation (2-24) and solve. The solution is

$$\epsilon = N \left[1 + D \sin U\theta_2 + v \left\{ 1 - e^{-\theta_2} + \frac{DU}{1 + U^2} \left(e^{-\theta_2} - \cos U\theta_2 + \frac{1}{U} \sin U\theta_2 \right) \right\} \right] \quad (2\text{-}40)$$

which is graphed for different loading magnitudes in Graphs (2-6A, -6B, -6C & -6D).

Graphs

Graphs of each previous example show the model responses to different loadings and how each response alters with changes in magnitudes of relaxation times and other constants. In general, the normalized strain curve asymptotes and mean values in the case of sinusoidal loading, are monotone increasing with time for the Maxwell Model, due to the series connected damper, and are constant for Model 2. Ignoring fatigue effects, the useful life of those materials whose normalized strain never reaches unity is infinite. The penny-shaped cracks will grow for those examples where normalized strain curves become unity and larger. The latent stage graphs are not a valid analysis for magnitudes larger than unity, beyond which the fracture propagation analysis must be used to determine the useful life of the sample.

CHAPTER 3

3-D POLYMER FRACTURE IN SLOW PROPAGATION

During the slow growth stage of fracture, the crack propagation is assumed to be a function of the strain rate. The general equation of motion expressing crack length is derived by taking the time derivative of the viscoelastic strain expression of Chapter 2. Examples of crack growth are computed for two types of viscoelastic materials subjected to a constant stress.

3-1 Strain Rate Hypothesis

To solve for the equation of motion during the slow fracture propagation stage, hypothesize that the crack tip movement is a function of the strain rate. The time derivative of equation (2-16), using Leibniz' rule for differentiating the integral, is

$$\dot{\epsilon}_L = \frac{c}{\sqrt{\delta^o}} \left[\dot{\phi}\, X^{\frac{1}{2}} + \frac{\phi}{2} X^{-\frac{1}{2}} \dot{X} + \int_0^t \frac{\partial}{\partial t} \dot{\psi}\,(t - \tau) \phi(\tau) X^{\frac{1}{2}}(\tau) d\tau + \dot{\psi}\,(o) \phi X^{\frac{1}{2}} \right] \qquad (3\text{-}1)$$

This is the general equation of motion for viscoelastic polymer materials. Notice the principal strain was replaced by the limiting strain since during crack growth the strain at the crack tip is at its maximum and the material is separating.

3-2 Examples

The same models used in Section 2-4 will be used here with the Heaviside unit loading to demonstrate fracture propagation through a viscoelastic sample during the slow stage.

Maxwell Model with $\phi = H(t)$

Upon substitution of equations (2-31) and (2-25) into equation (3-1) the equation of motion becomes

$$\dot{\epsilon}_L = \frac{cH(t)}{\sqrt{\delta^o}}\left[\frac{\dot{X}X^{-\frac{1}{2}}}{2} + \frac{X^{\frac{1}{2}}}{\tau_o}\right] \tag{3-2}$$

The solution to the equation of motion is obtained by separating variables and integrating from $t = t_*$ and $X = 1$ to $t = t$ and $X = X$

$$X = q + (1 - q)e^{-\theta_3} \tag{3-3}$$

which is graphed in Graph (3-1). With the application of a constant load, the strain rate is assumed to be constant for a given material.

Model 2 with $\phi = H(t)$

By substituting equations (2-36) and (2-25) into equation (3-1) and taking the partial derivative of the integrand, the equation of motion becomes

$$\dot{\epsilon}_L = \frac{c}{\sqrt{\delta^o}}\left[\frac{H(t)}{2} X^{-\frac{1}{2}}\dot{X} - \frac{E_1 e^{\frac{-t}{\tau_2}}}{\eta_2 \tau_2}\int_0^t e^{\frac{\tau}{\tau_2}} H(\tau)X^{\frac{1}{2}}(\tau)d\tau + \frac{E_1}{\eta_2} H(t)X^{\frac{1}{2}}\right] \qquad (3\text{-}4)$$

The integral equation (3-4) solution is derived by substituting equations (2-36) and (2-25) into (2-16), remembering the crack tip strain during fracture propagation is the limiting strain, and solving for the integral.

$$\int_0^t e^{\frac{\tau}{\tau_2}} H(\tau)X^{\frac{1}{2}}(\tau)d\tau = \frac{\eta_2}{E_1} e^{\frac{t}{\tau_2}}\left[\frac{\epsilon_L\sqrt{\delta^o}}{c} - H(t)X^{\frac{1}{2}}\right] \qquad (3\text{-}5)$$

Now if equations (3-5) and (3-4) are combined and simplified, the resulting differential equation is

$$\dot{X} + \frac{2Q}{\tau_2} X - \frac{2P}{\tau_2} X^{\frac{1}{2}} = 0 \qquad (3\text{-}6)$$

Equation (3-6) is solved by separating variables and integrating from $t = t_*$ and $X = 1$ to the present $t = t$ and $X = X$. The solution is

$$X = Z + (1 - Z)e^{-Q\theta_4} \qquad (3\text{-}7)$$

which is plotted in Graphs (3-2A & -2B) for different values of P and Q.

Graphs

Graphs of both examples above show how the crack length increases with time for constant loadings and how the crack growth is altered with changes in magnitudes of the relaxation times and other constants.

As long as the constant q in the Maxwell Model (or Z in Model 2) is unity or greater, the crack will grow at a predictable rate. When q (or Z) is less than unity, the crack length decreases with increasing time. It may be verified that for stresses equal to or greater than the Griffith stress the crack is unstable and thus the range of negative slopes $\frac{dX}{d\theta_3}$ $\left(\text{or } \frac{dX}{d\theta_4}\right)$ is interpreted here as the instability or rapid propagation range, cf. Cherepanov /6/.

CHAPTER 4

2-D POLYMER FRACTURE

The 2-D case of viscoelastic fracture is, as one might expect, similar to the 3-D case except for some constants. The procedure followed here is like that in Chapters 2 and 3; only now, start with Sneddon's stress equations for the 2-D case. The elastic strain is averaged over a small distance, instead of area, using the same Wnuk expression for the small distance preceding the crack. The same Elastic-Viscoelastic Correspondence Principle is used, but to normalize the viscoelastic strain, the 2-D Griffith stress expression replaces the 3-D counterpart. No examples were computed since the results would be identical to the 3-D case except for the assumed values of the constants and should be interpreted in the same fashion as before.

4-1 Averaged Maximum Elastic Strain

Sneddon's /2/ elastic stress equations along the crack line are

$$\sigma_x = \frac{\sigma}{\sqrt{2\rho}}$$

$$\sigma_y = \frac{\sigma}{\sqrt{2\rho}} \qquad (4\text{-}1)$$

$$\tau_{xy} = 0$$

which has an isolated singularity at the crack tip point $\rho = 0$ i.e. when $r = 0$. To obtain the strains at the crack tip, use the two-dimensional Hooke's Law.

$$\epsilon_x = \frac{1}{E}\left[\sigma_x - \nu\sigma_y\right]$$

$$\epsilon_y = \frac{1}{E}\left[\sigma_y - \nu\sigma_x\right] \qquad (4\text{-}2)$$

$$\gamma_{xy} = \frac{\tau_{xy}}{G}$$

By substituting the stress equations (4-1) at the crack tip into equations (4-2) the strains at the crack tip are

$$\epsilon_x = \frac{f\phi}{\sqrt{\rho}}$$

$$\epsilon_y = \frac{f\phi}{\sqrt{\rho}} \qquad (4\text{-}3)$$

$$\gamma_{xy} = 0$$

From the theory of elasticity, the principal strains are determined by substituting equations (4-3) into the following determinant:

$$\begin{vmatrix} \epsilon_x - \epsilon & \gamma_{xy} \\ \gamma_{xy} & \epsilon_y - \epsilon \end{vmatrix} = 0$$

expanding the determinant into a quadratic equation and solving for the roots. The maximum root or principal strain is

$$\epsilon_1 = \frac{f\phi}{\sqrt{\rho}} = \frac{f\phi\sqrt{L}}{\sqrt{r}} \qquad (4\text{-}4)$$

Now eliminate the singularity point $r = 0$ by averaging the maximum principal strain over a small interval distance Δ, Δ is taken from Wnuk /3/ ahead of the crack tip.

$$\langle\epsilon_1\rangle = \frac{1}{\Delta}\int_0^{\Delta} \epsilon_1 \, dr \qquad (4\text{-}5)$$

Substitute equation (4-4) into the averaging equation (4-5) and integrate to obtain the mean value of strain as

$$\langle\epsilon_1\rangle = \frac{g\phi\sqrt{L}}{\sqrt{\Delta}} \qquad (4\text{-}6)$$

And as expected, equation (4-6) is identical to equation (2-12) except the loading constant g replaces c. And because of the similarity between the averaged principal elastic strain equations (4-6) and (2-12), the viscoelastic strain relations for the two-dimensional case may be obtained from the three-dimensional case by replacing the loading constant c by g. Thus the viscoelastic strain equation (2-16) becomes

$$\langle \epsilon_1 \rangle = \frac{g}{\sqrt{\delta^o}}\left[\phi X^{\frac{1}{2}} + \int_0^t \dot{\psi}\,(t - \tau)\,\phi(\tau) X^{\frac{1}{2}}(\tau)\,d\tau\right] \tag{4-7}$$

4-2 Latent Stage Normalized Strain and Slow Propagating Stage Equation

The procedure used to derive the normalized strain for the 2-D case is identical to that of Section 2-3. Mathematical terms applicable to the 2-D materials are displacement at the crack tip from Wnuk /7/

$$u^o(\text{tip}) = \frac{(1 - \nu^2)\,\sigma_o^2 \pi L^o}{2EY} \tag{4-8}$$

and the Griffith stress from Griffith /5/

$$\sigma_G = \sqrt{\frac{2EY\,u^o(\text{tip})}{\pi(1 - \nu^2)L^o}} \tag{4-9}$$

which correspond respectively to 3-D equations (2-21) and (2-18). The resulting normalized strain for the 2-D latent stage is

$$\epsilon = \frac{M}{\phi(o)}\left[\phi + \int_0^t \dot{\psi}\,(t - \tau)\,\phi(\tau)\,d\tau\right] \tag{4-10}$$

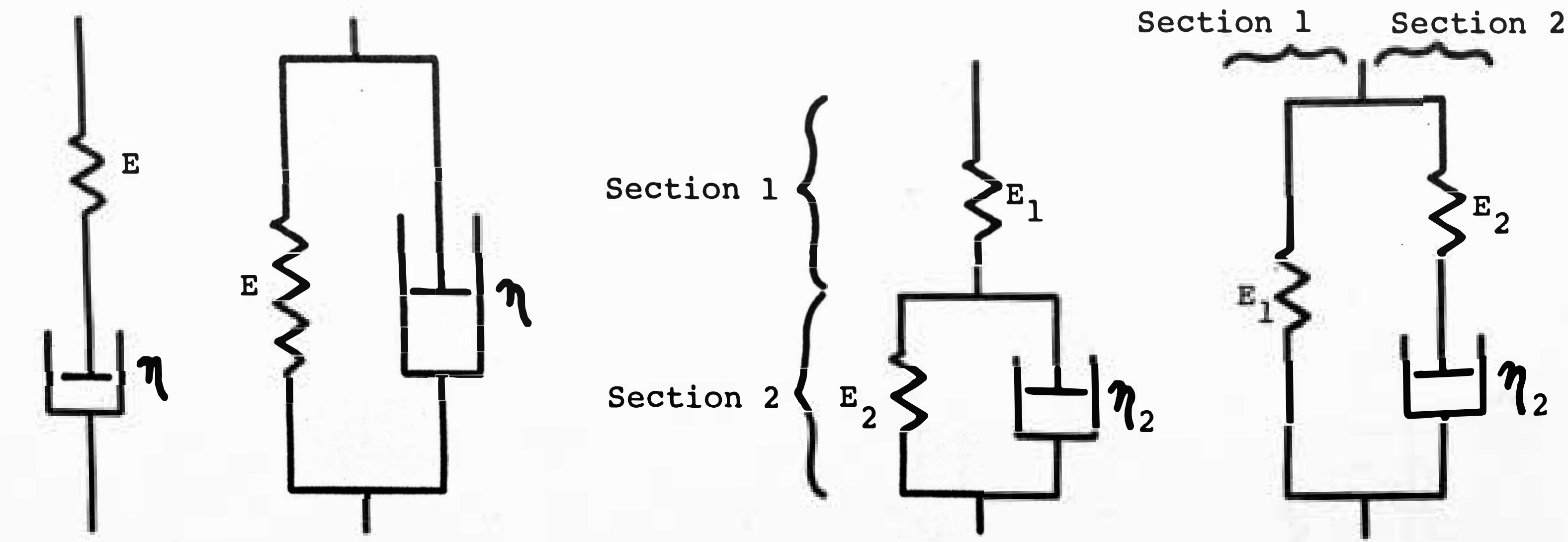
The 2-D equation of fracture propagation is found by taking the time derivative of equation (4-7) as in Section 3-1.

$$\dot{\epsilon}_L = \frac{g}{\sqrt{\delta^o}}\left[\dot{\phi}\, X^{\frac{1}{2}} + \frac{\phi\, X^{-\frac{1}{2}}\dot{X}}{2} + \int_0^t \frac{\partial}{\partial t}\dot{\psi}\,(t-\tau)\, \phi(\tau) X^{\frac{1}{2}}(\tau)\, d\tau + \dot{\psi}\,(o)\,\phi\, X^{\frac{1}{2}}\right] \quad (4\text{-}11)$$

All examples and graphs for 2-D materials are the same as those of Chapters 2 and 3 except the constants c and N are replaced by g and M respectively.

FIGURE 1-1

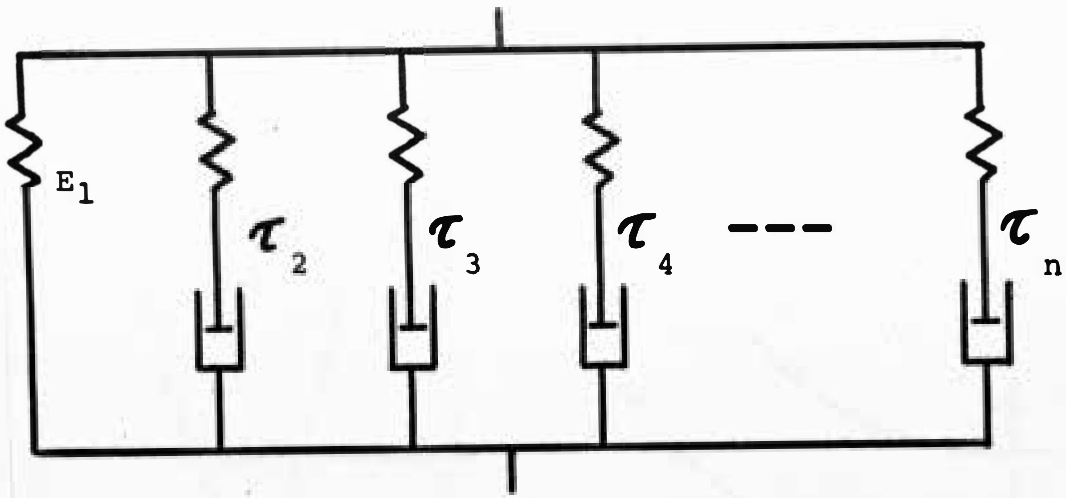
VISCOELASTIC MODELS



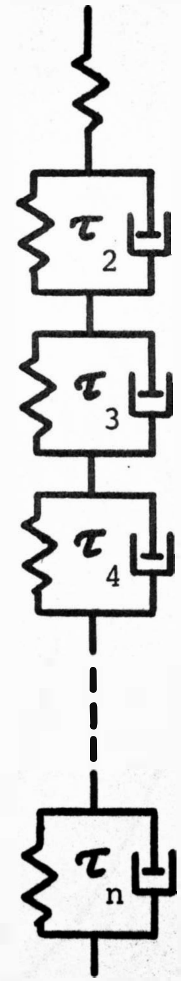
Maxwell   Voigt-Kelvin   Model 2   Model 3

FIGURE 1-2

GENERALIZED VISCOELASTIC MODELS



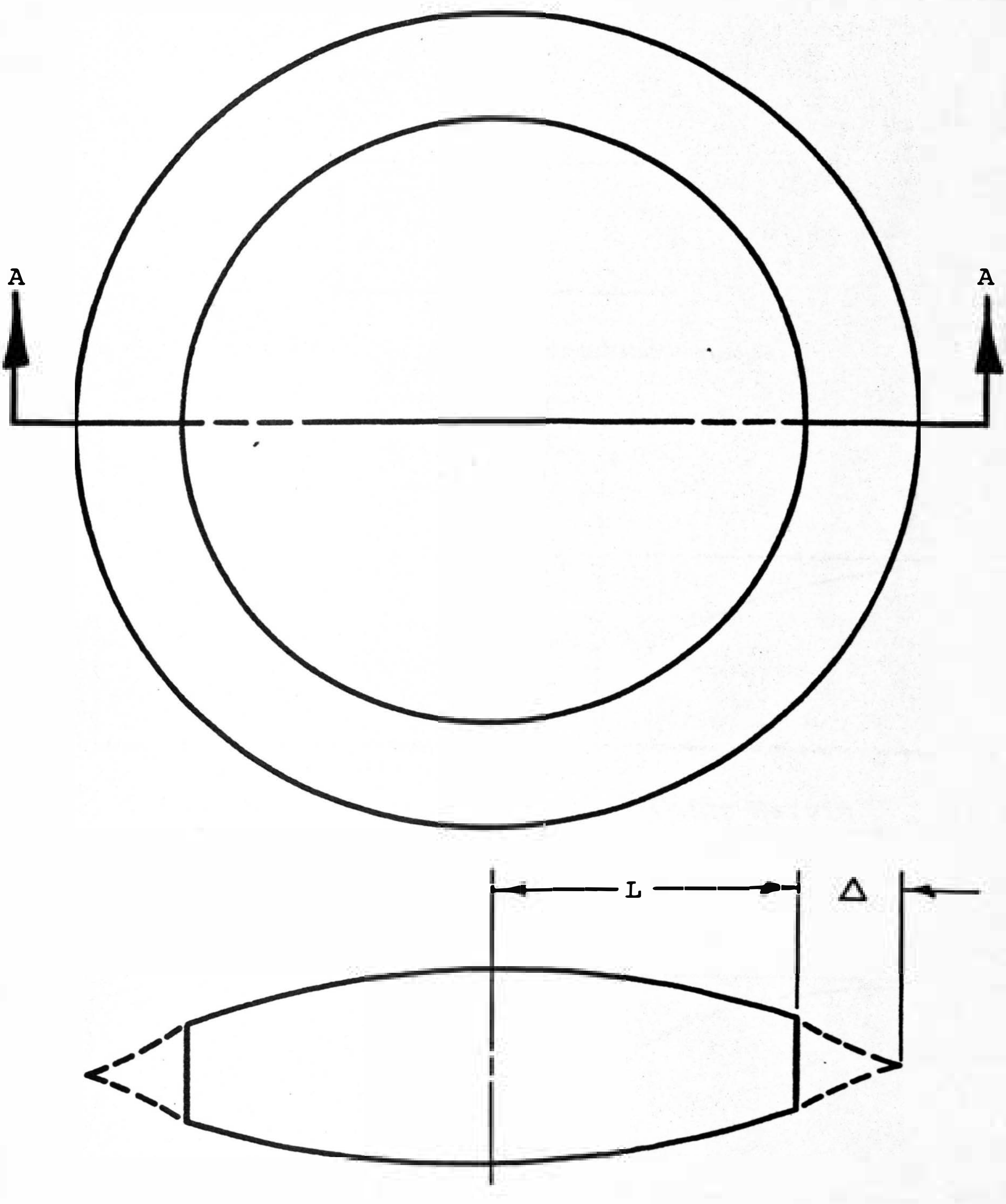
Generalized Maxwell Solid



Generalized Voigt-
Kelvin Solid

FIGURE 2-1

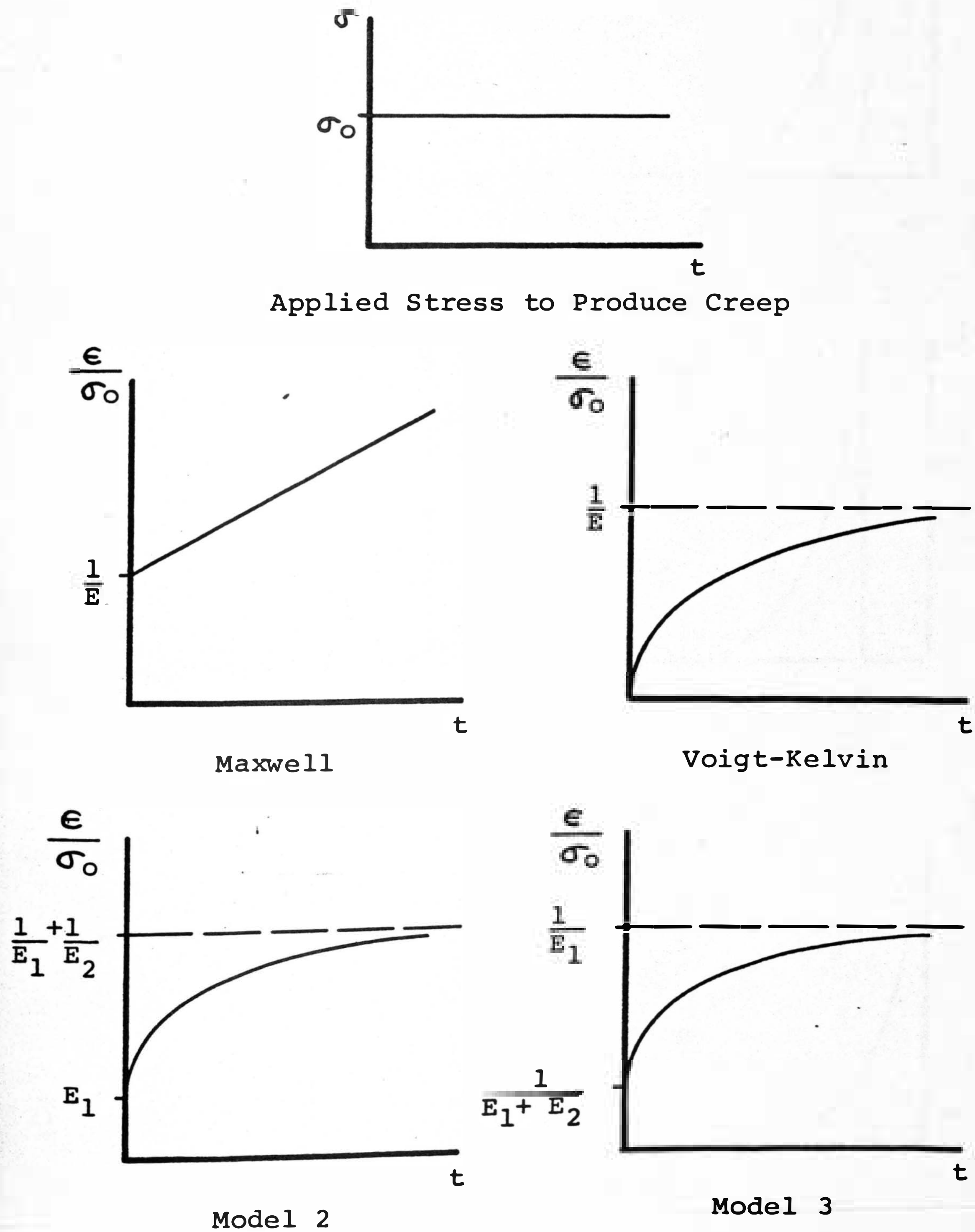
WNUK'S "CUT ENDS CIGAR" MODEL OF A CRACK



SECTION A-A

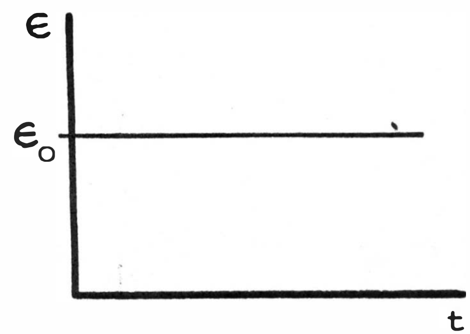
GRAPH 1-1

CREEP CURVES OF VISCOELASTIC MODELS

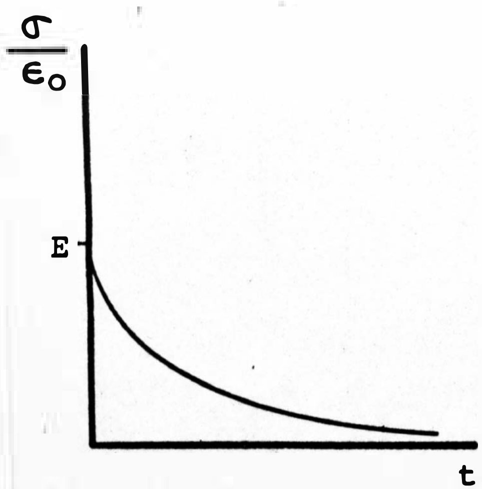


GRAPH 1-2

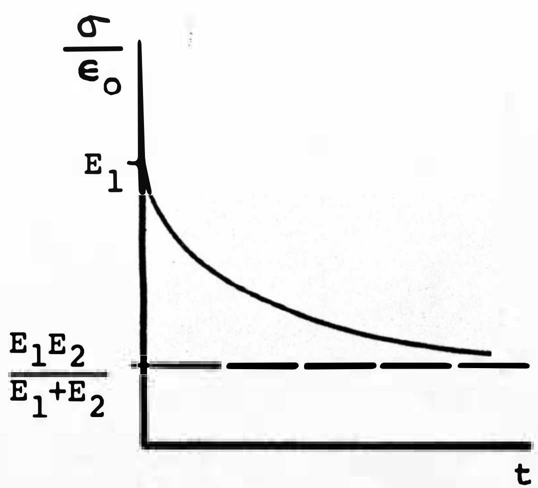
STRESS RELAXATION CURVES OF VISCOELASTIC MODELS



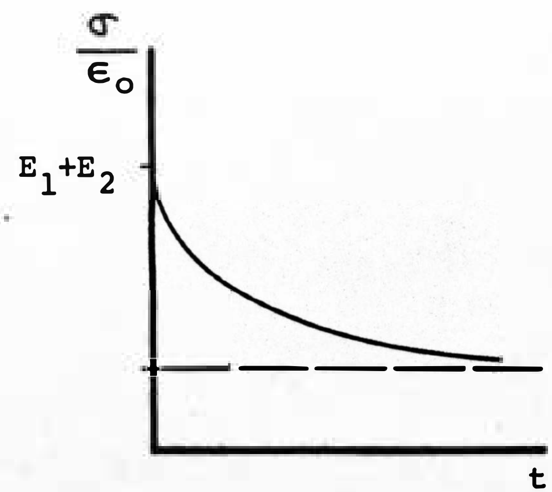
Applied Strain to Produce Relaxation



Maxwell



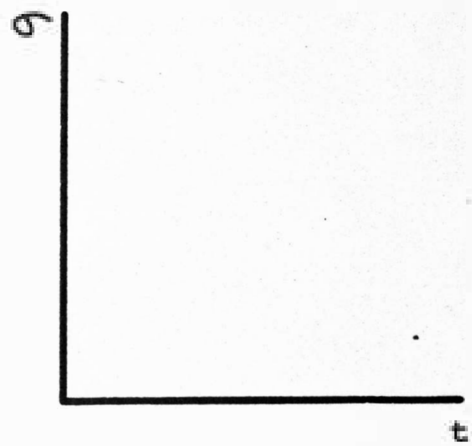
Model 2



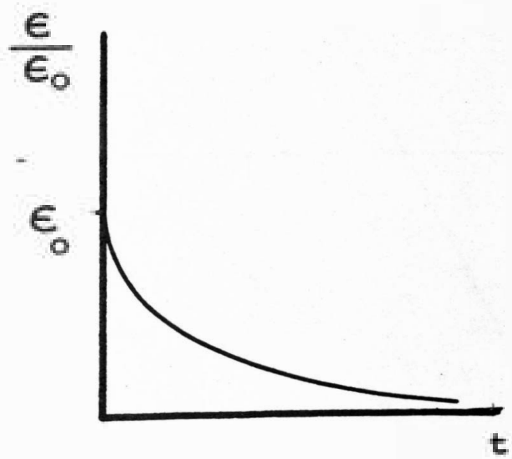
Model 3

GRAPH 1-3

STRAIN RELAXATION CURVE OF VISCOELASTIC MODEL



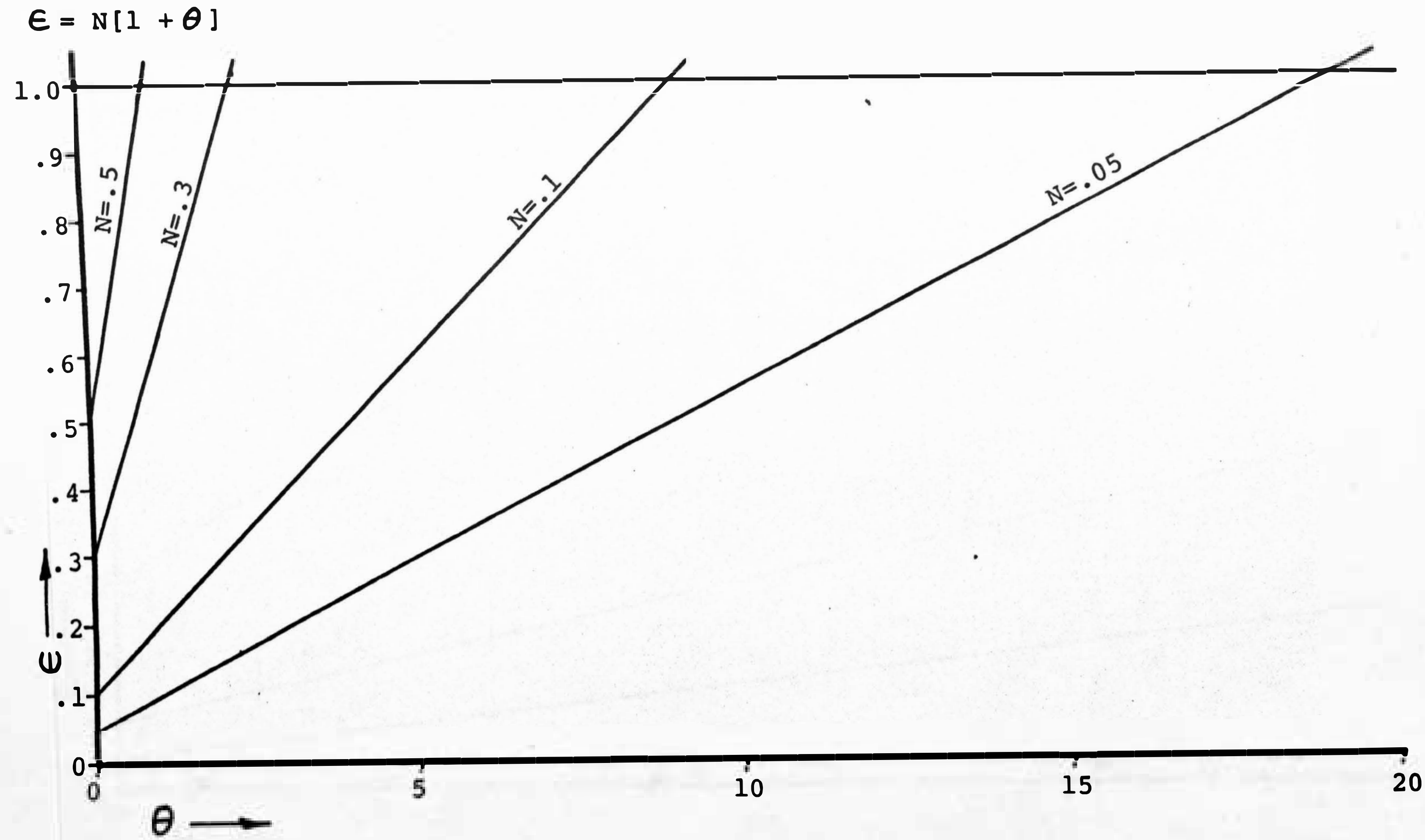
No Applied Stress During Relaxation



Voigt - Kelvin

GRAPH 2-1 LATENT STAGE

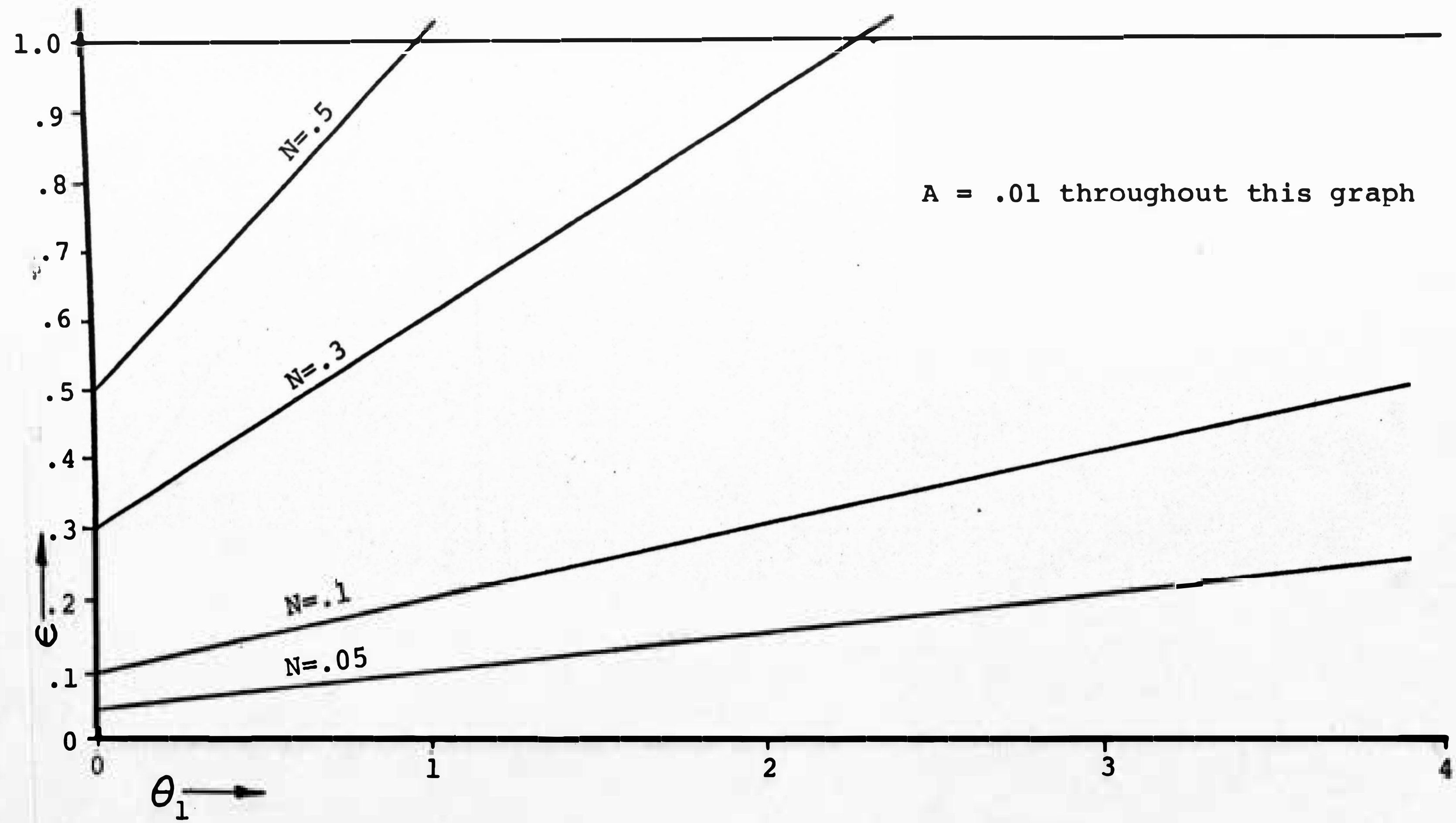
MAXWELL MODEL NORMALIZED STRAIN WITH $\phi = H(t)$ AND VARIOUS N



GRAPH 2-2A LATENT STAGE

MAXWELL MODEL NORMALIZED STRAIN WITH $\phi = s + at$ AND VARIOUS N

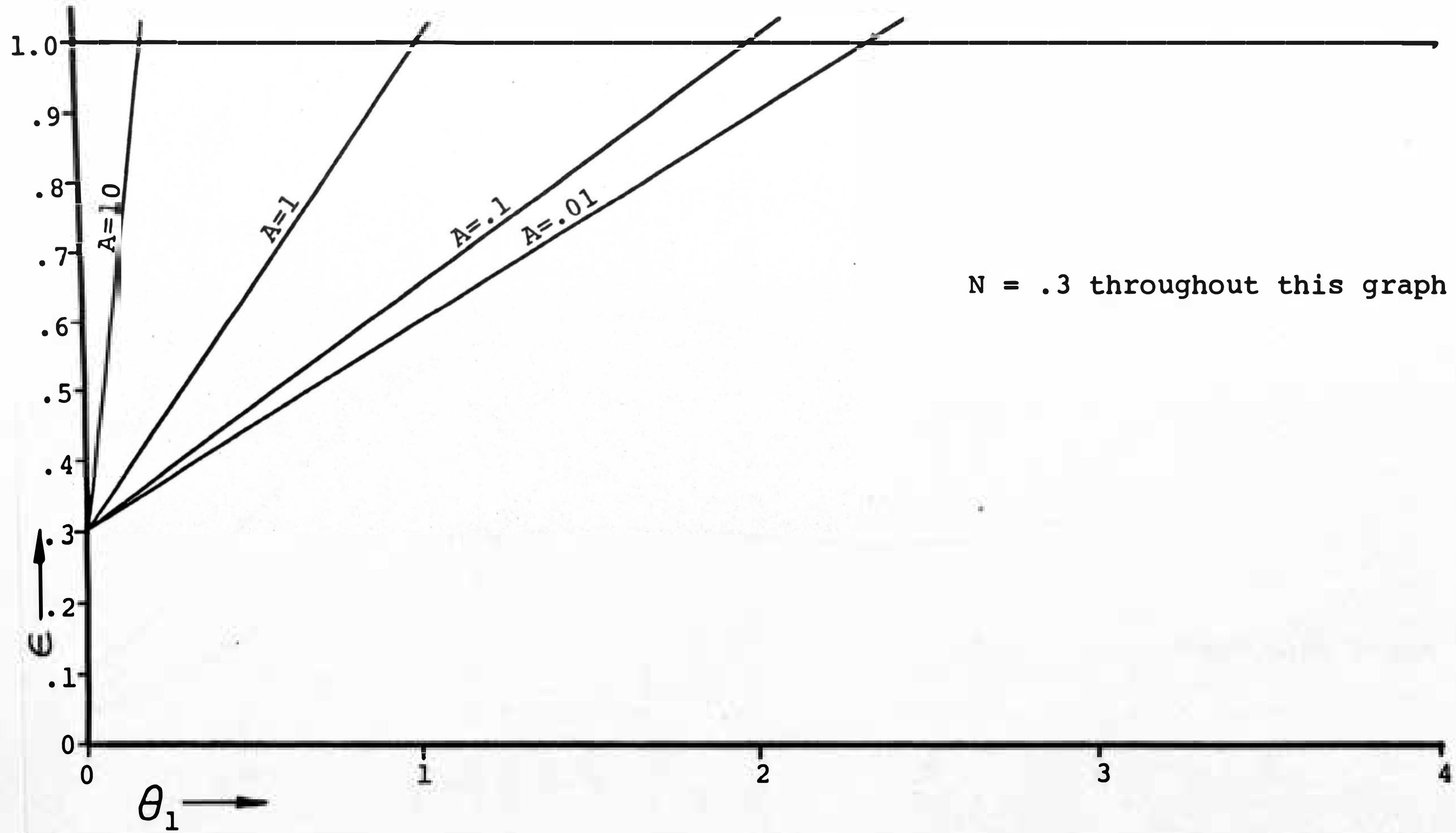
$$\epsilon = N\left[1 + \theta_1\left\{1 + A\left(1 + \frac{\theta_1}{2}\right)\right\}\right]$$



GRAPH 2-2B LATENT STAGE

MAXWELL MODEL NORMALIZED STRAIN WITH ϕ = s + at AND VARIOUS A

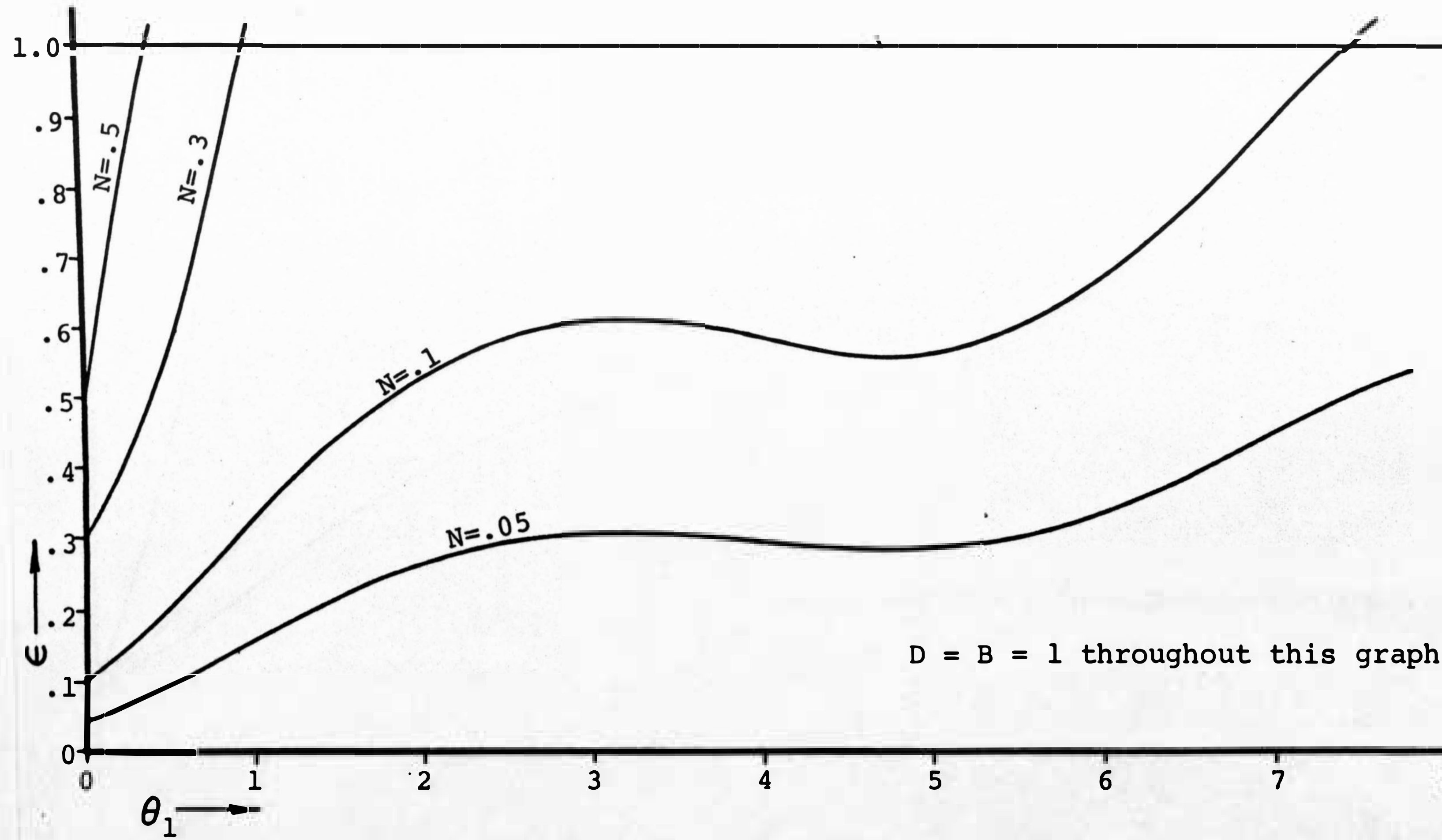
$$\epsilon = N\left[1 + \theta_1\left\{1 + A\left(1 + \frac{\theta_1}{2}\right)\right\}\right]$$



GRAPH 2-3A LATENT STAGE

MAXWELL MODEL NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS N

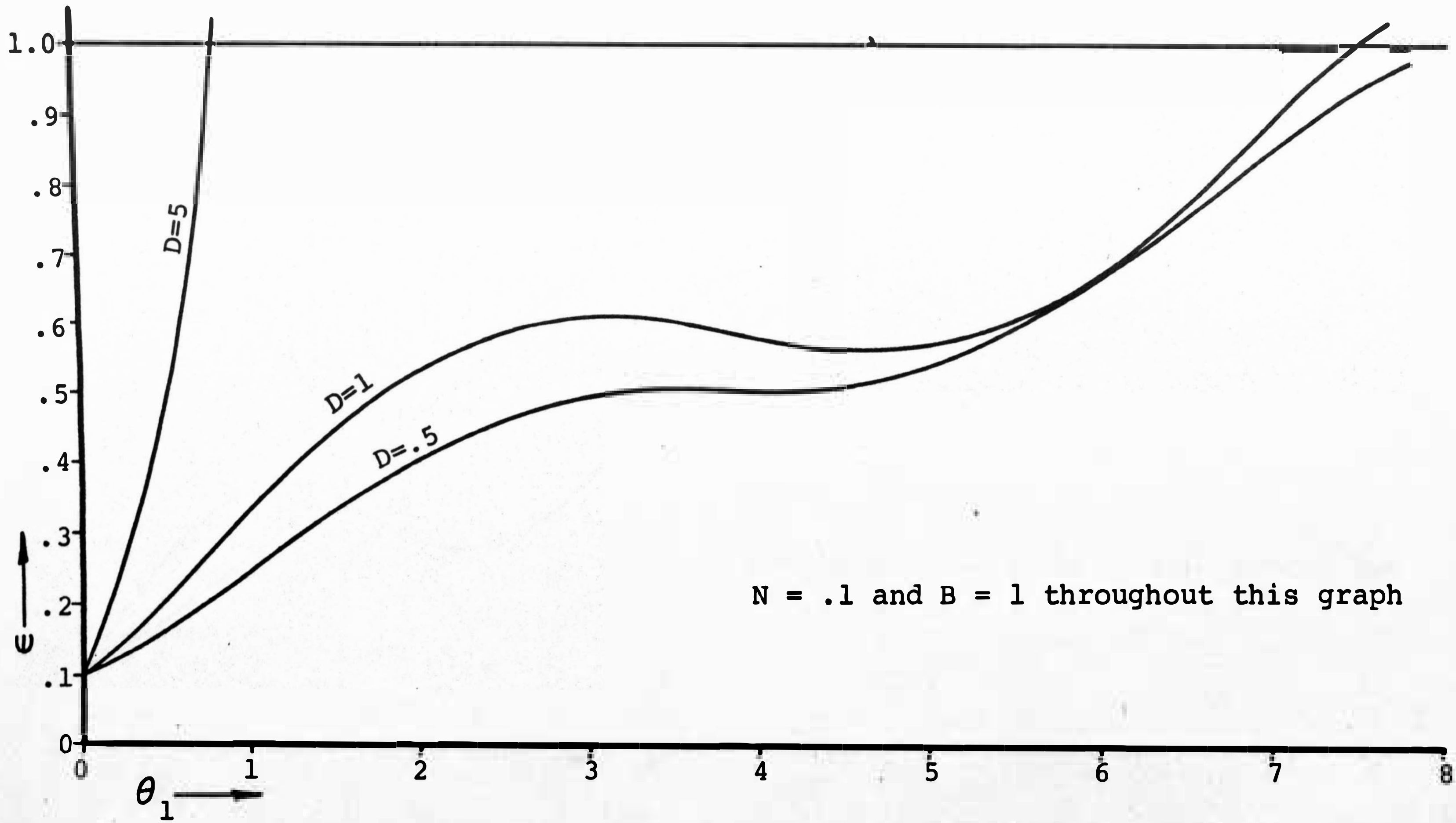
$$\epsilon = N\left[1 + \theta_1 + D\left(\sin B\theta_1 + \frac{1}{B} - \frac{1}{B}\cos B\theta_1\right)\right]$$



GRAPH 2-3B   LATENT STAGE

MAXWELL MODEL NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS D

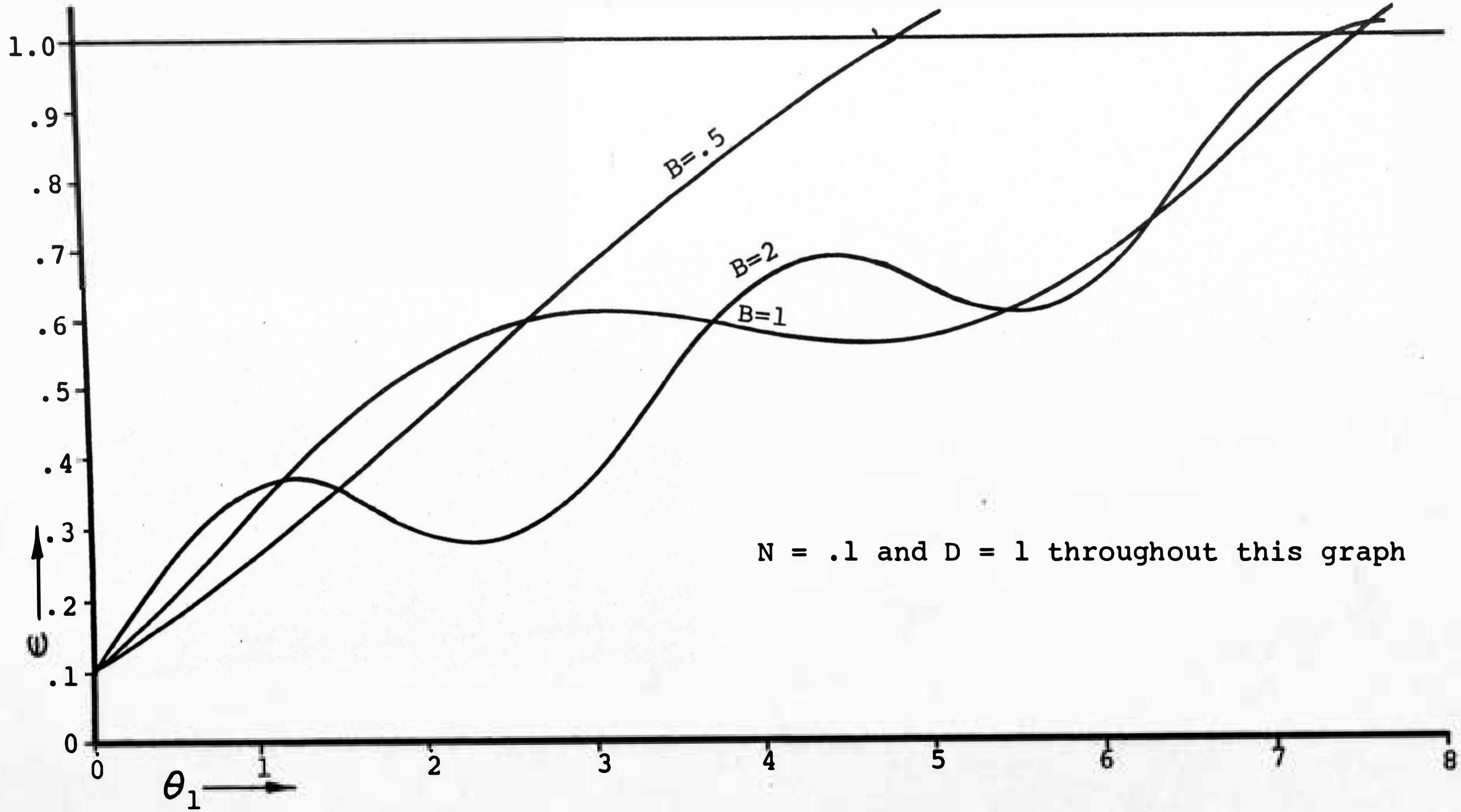
$$\epsilon = N\left[1 + \theta_1 + D\left(\sin B\,\theta_1 + \frac{1}{B} - \frac{1}{B}\cos B\,\theta_1\right)\right]$$



GRAPH 2-3C LATENT STAGE

MAXWELL MODEL NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS B

$$\epsilon = N\left[1 + \theta_1 + D\left(\sin B\theta_1 + \frac{1}{B} - \frac{1}{B}\cos B\theta_1\right)\right]$$

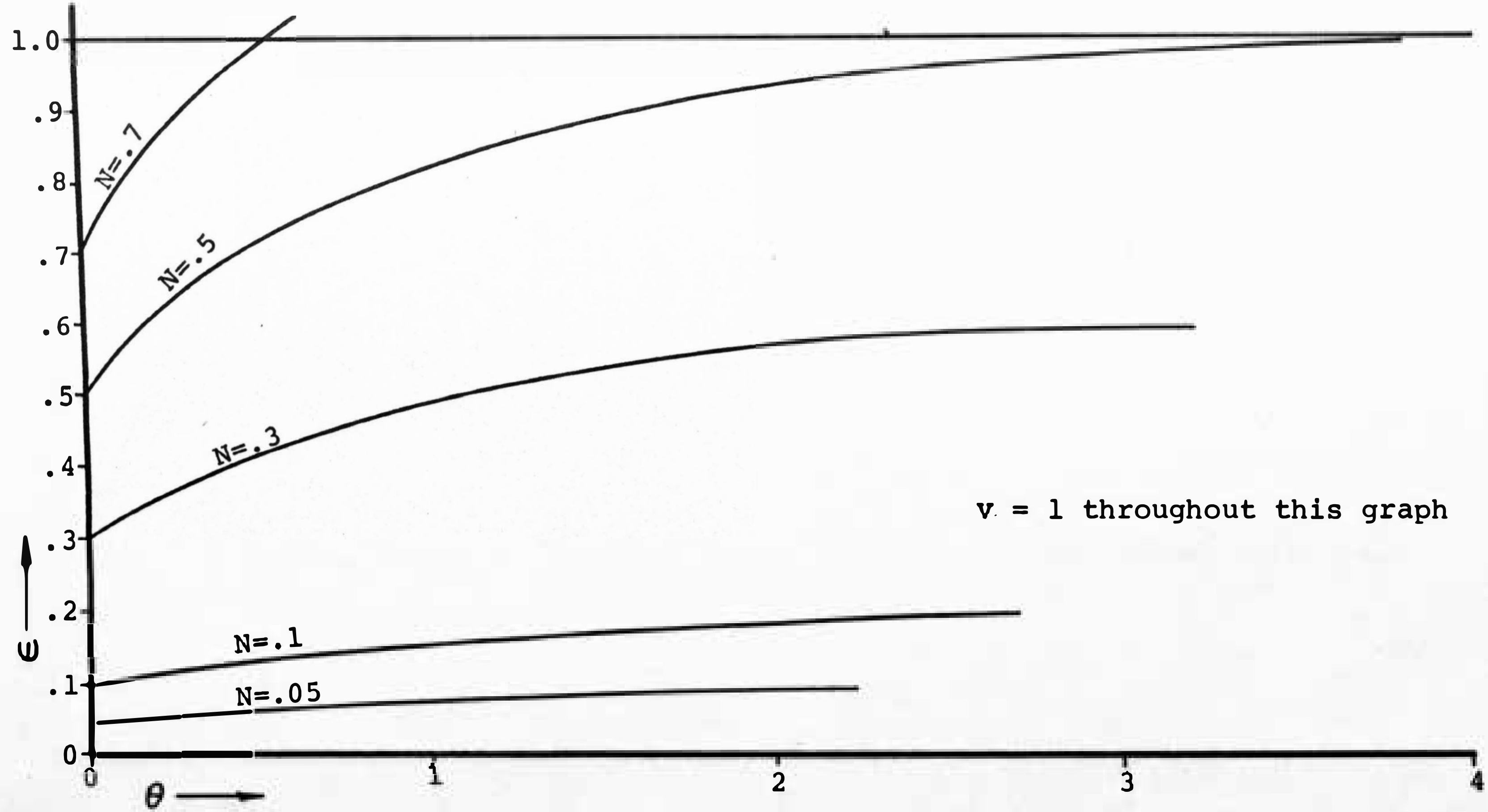


N = .1 and D = 1 throughout this graph

GRAPH 2-4A LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = H(t)$ AND VARIOUS N

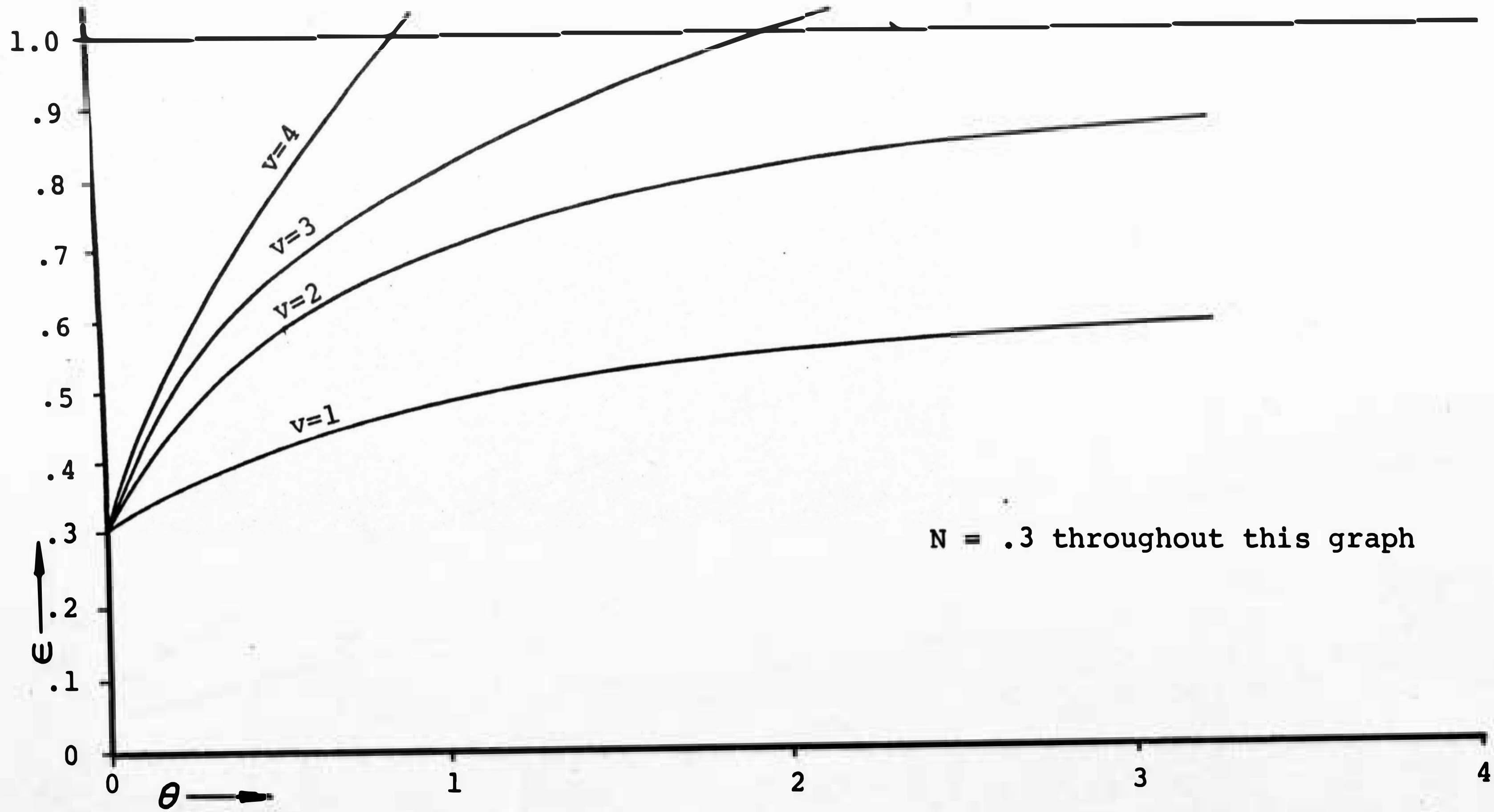
$$\epsilon = N\left[1 + v\left(1 - e^{-\theta}\right)\right]$$



GRAPH 2-4B LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH ϕ = H(t) AND VARIOUS v

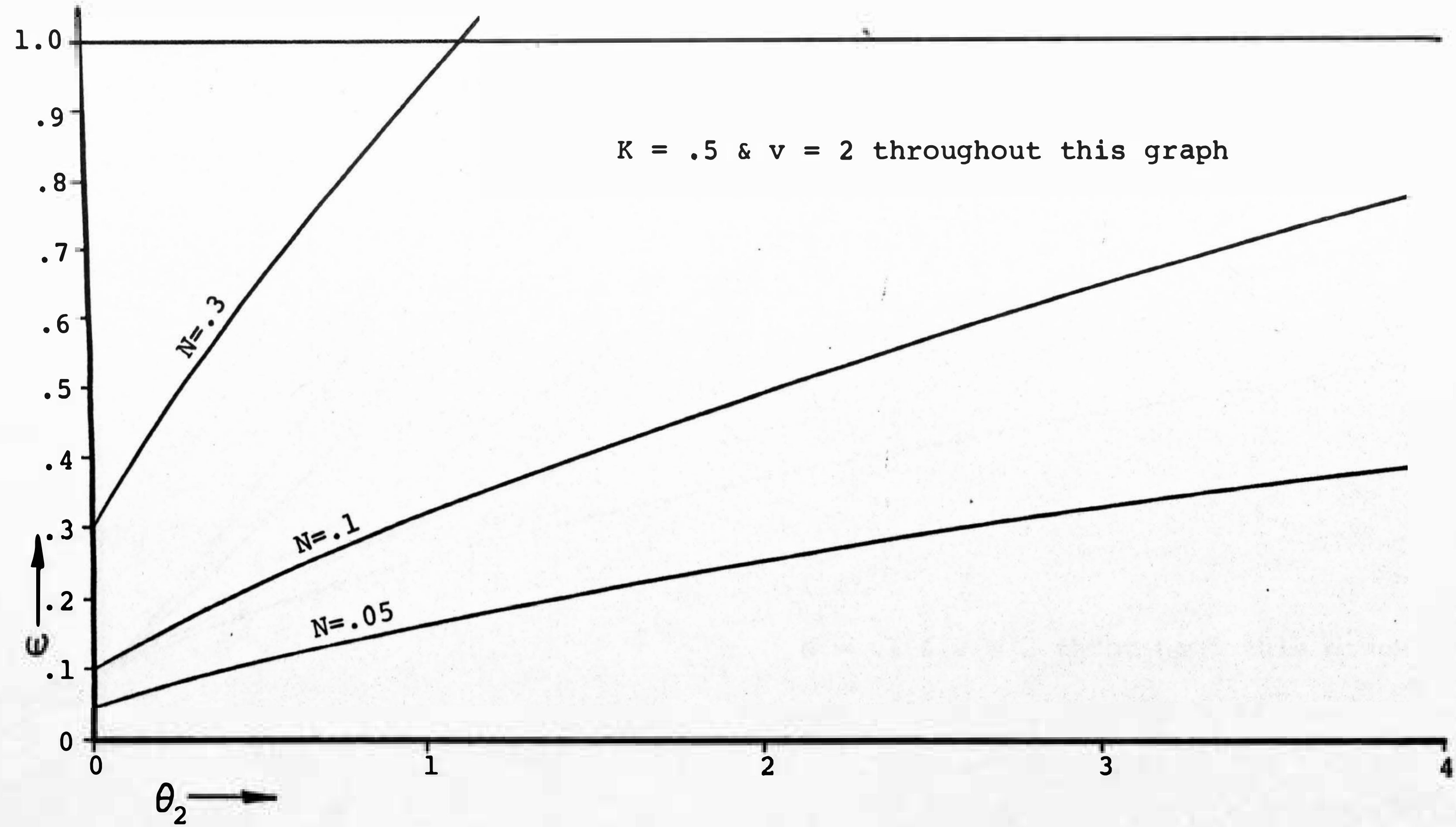
$$\epsilon = N\left[1 + v\left(1 - e^{-\theta}\right)\right]$$



GRAPH 2-5A   LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH ϕ = s + at AND VARIOUS N

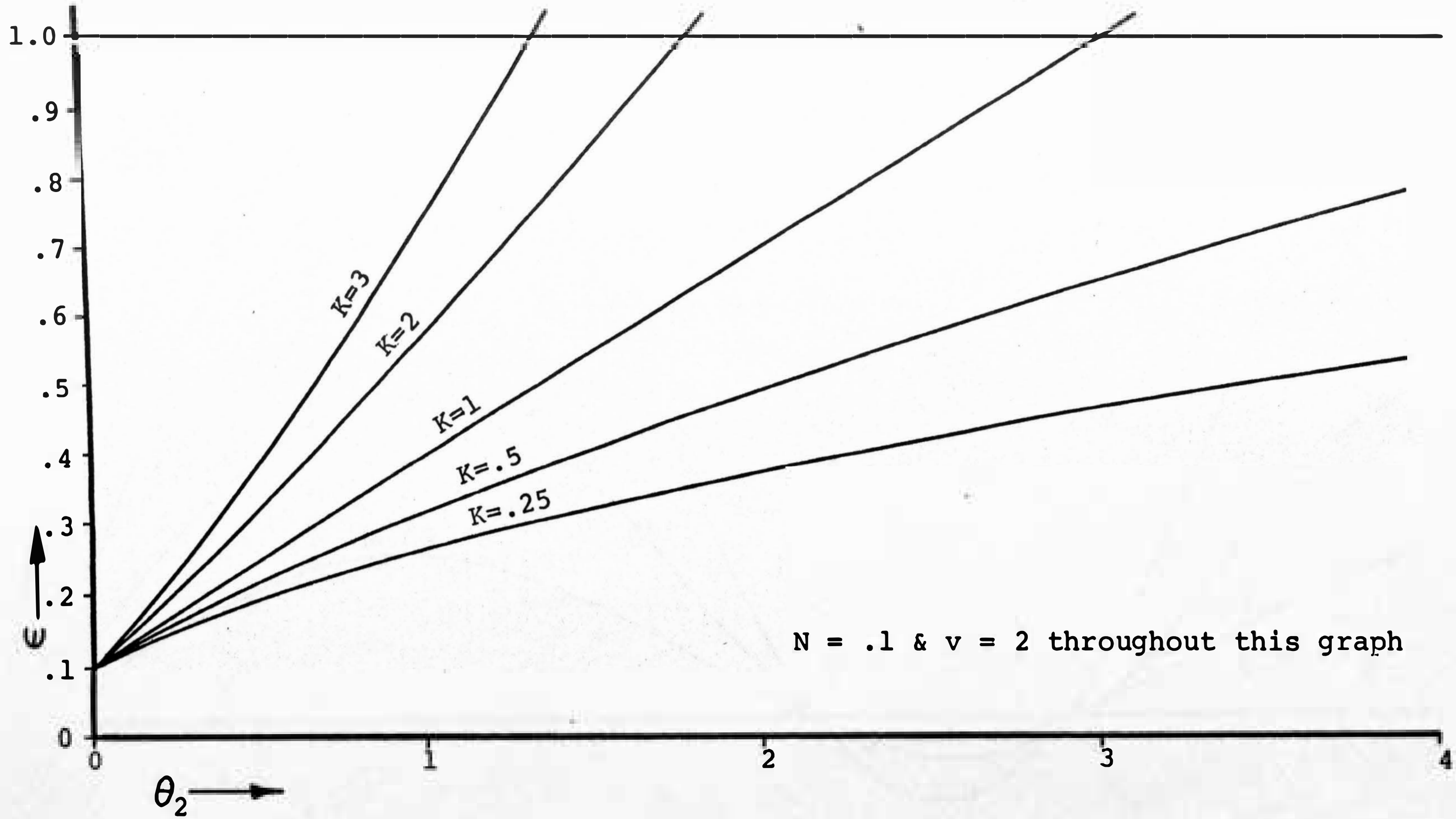
$$\epsilon = N\left[1 + K(1 + v)\,\theta_2 + v(1 - K)(1 - e^{-\theta_2})\right]$$



GRAPH 2-5B LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = s + at$ AND VARIOUS K

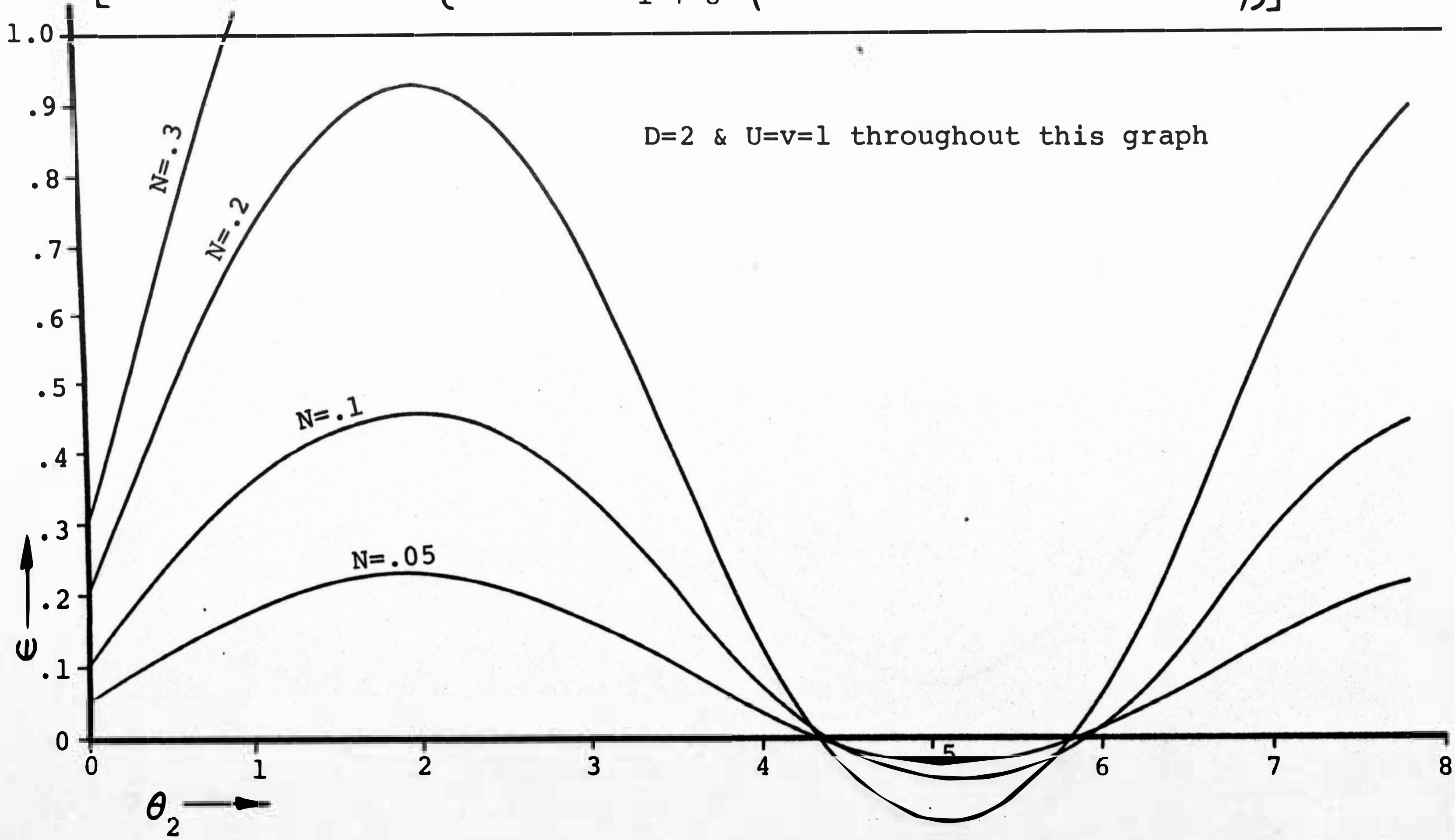
$$\epsilon = N\left[1 + K(1 + v)\theta_2 + v(1 - K)(1 - e^{-\theta_2})\right]$$



GRAPH 2-6A LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS N

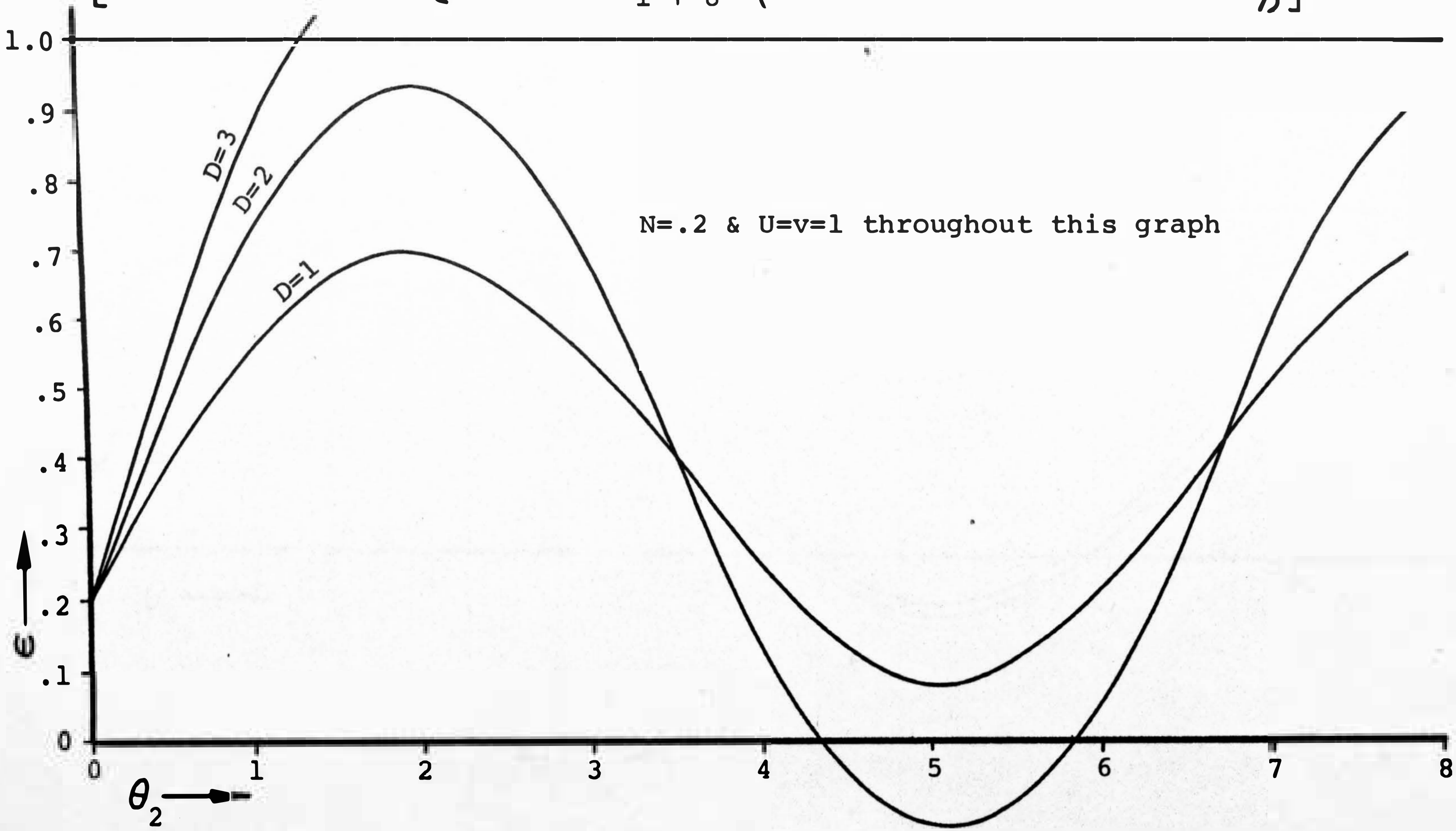
$$\epsilon = N\left[1 + D \sin U\theta_2 + v\left\{1 - e^{-\theta_2} + \frac{DU}{1 + U^2}\left(e^{-\theta_2} - \cos U\theta_2 + \frac{1}{D} \sin U\theta_2\right)\right\}\right]$$



GRAPH 2-6B LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS D

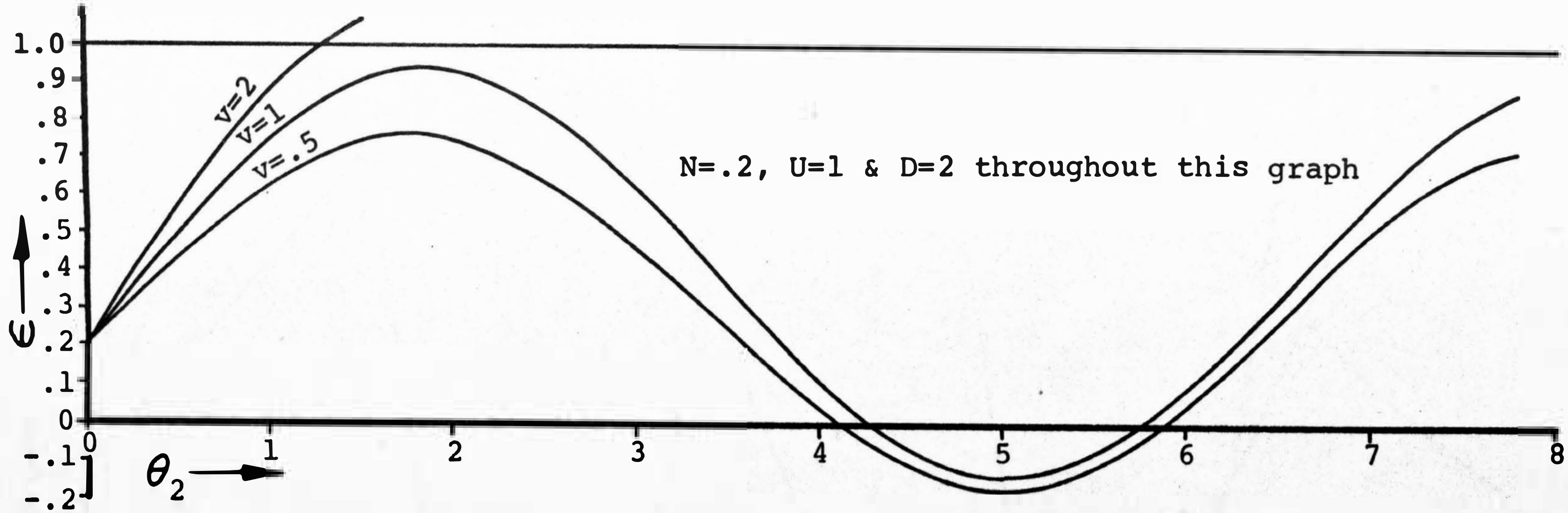
$$\epsilon = N\left[1 + D \sin U\theta_2 + v\left\{1 - e^{-\theta_2} + \frac{DU}{1+U^2}\left(e^{-\theta_2} - \cos U\theta_2 + \frac{1}{D}\sin U\theta_2\right)\right\}\right]$$



GRAPH 2-6C LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS v

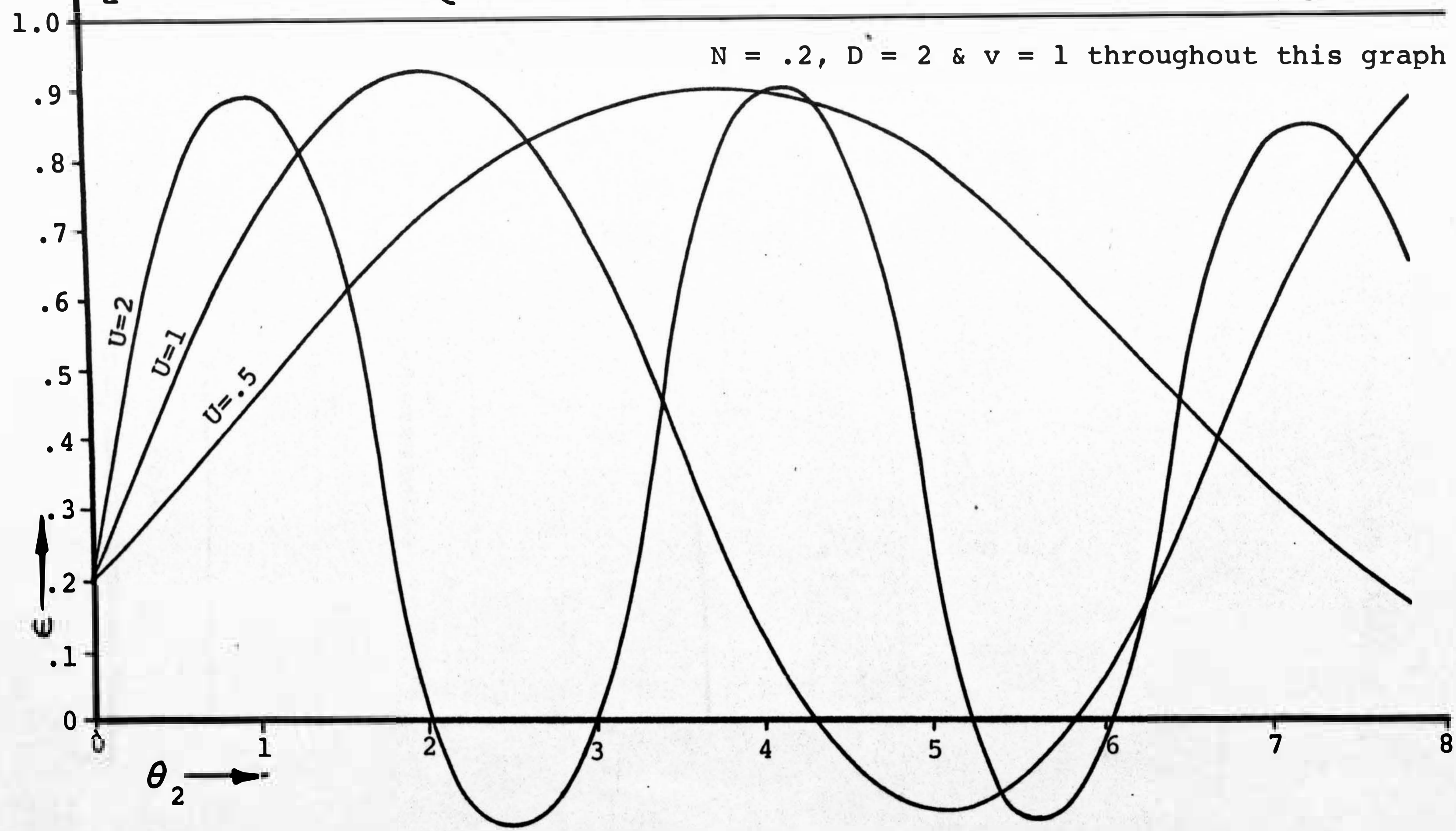
$$\epsilon = N\left[1 + D \sin U\theta_2 + v\left\{1 - e^{-\theta_2} + \frac{DU}{1 + U^2}\left(e^{-\theta_2} - \cos U\theta_2 + \frac{1}{D} \sin U\theta_2\right)\right\}\right]$$



GRAPH 2-6D LATENT STAGE

MODEL 2 NORMALIZED STRAIN WITH $\phi = s + k \sin \omega t$ AND VARIOUS U

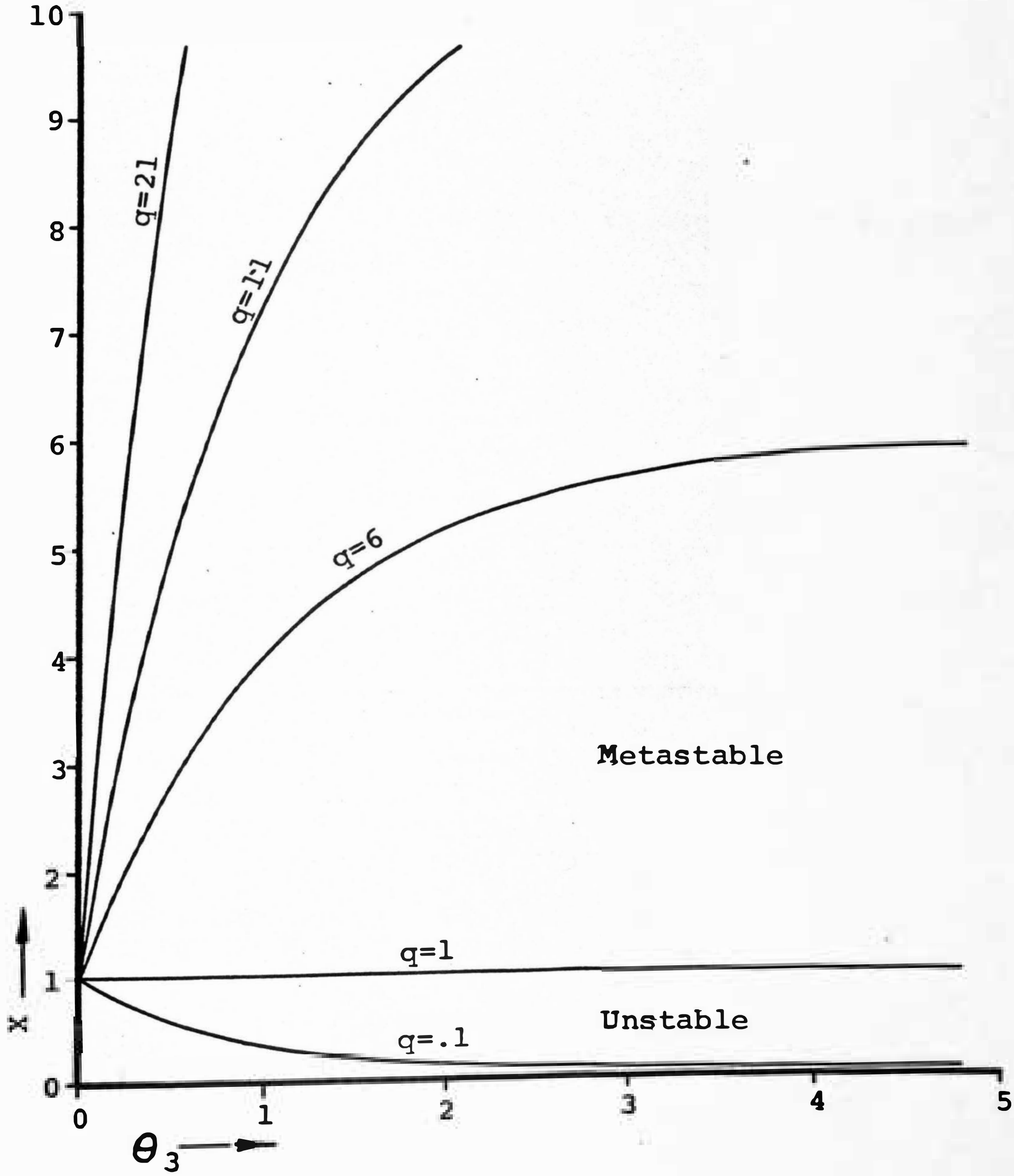
$$\epsilon = N\left[1 + D \sin U\theta_2 + v\left\{1 - e^{-\theta_2} + \frac{DU}{1 + U^2}\left(e^{-\theta_2} - \cos U\theta_2 + \frac{1}{D} \sin U\theta_2\right)\right\}\right]$$



GRAPH 3-1 FRACTURE PROPAGATION

MAXWELL MODEL CRACK LENGTH WITH $\phi = H(t)$ AND VARIOUS q

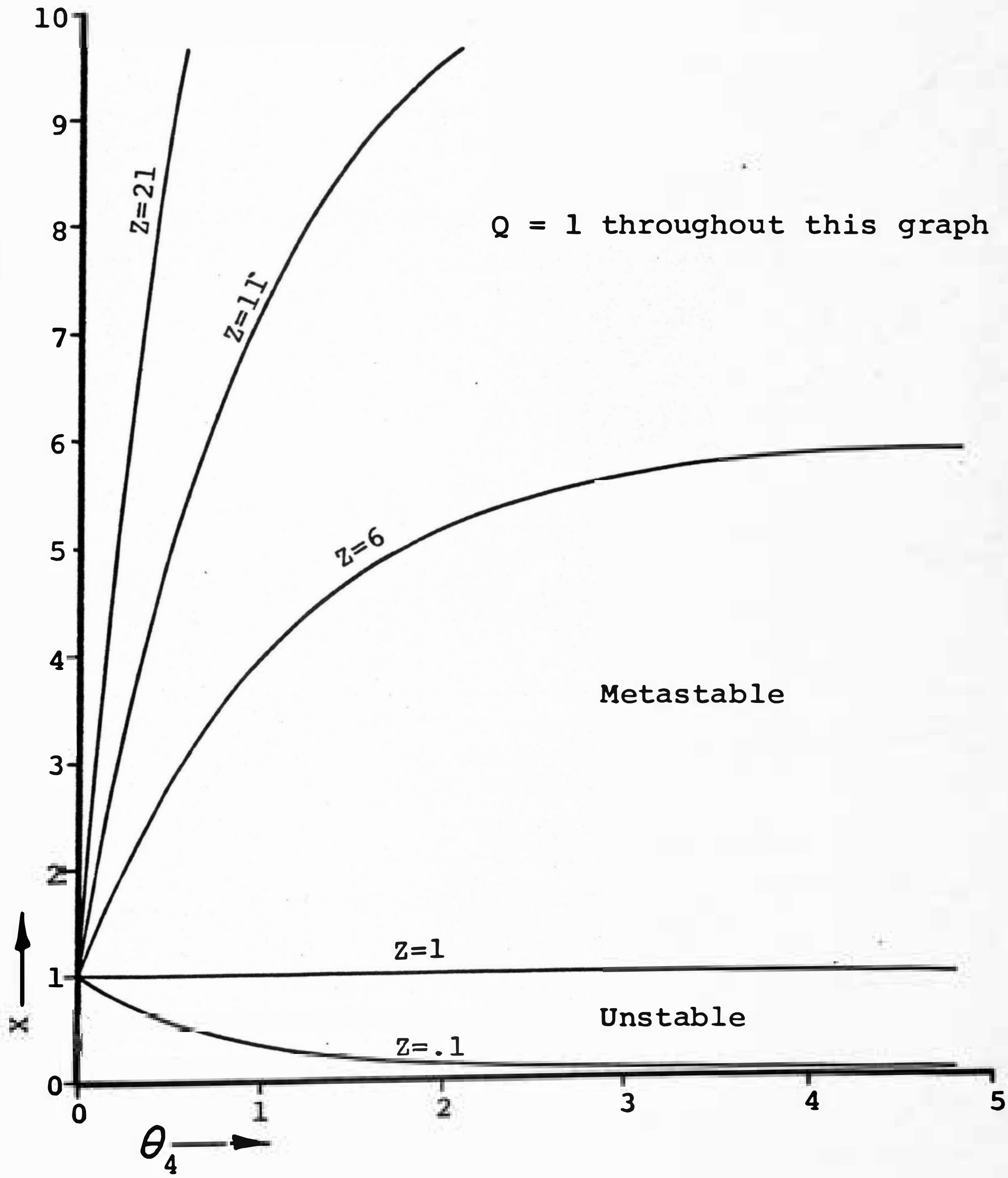
$$X = q + (1 - q)e^{-\theta_3}$$



GRAPH 3-2A FRACTURE PROPAGATION

MODEL 2 CRACK LENGTH WITH $\phi = H(t)$ AND VARIOUS Z

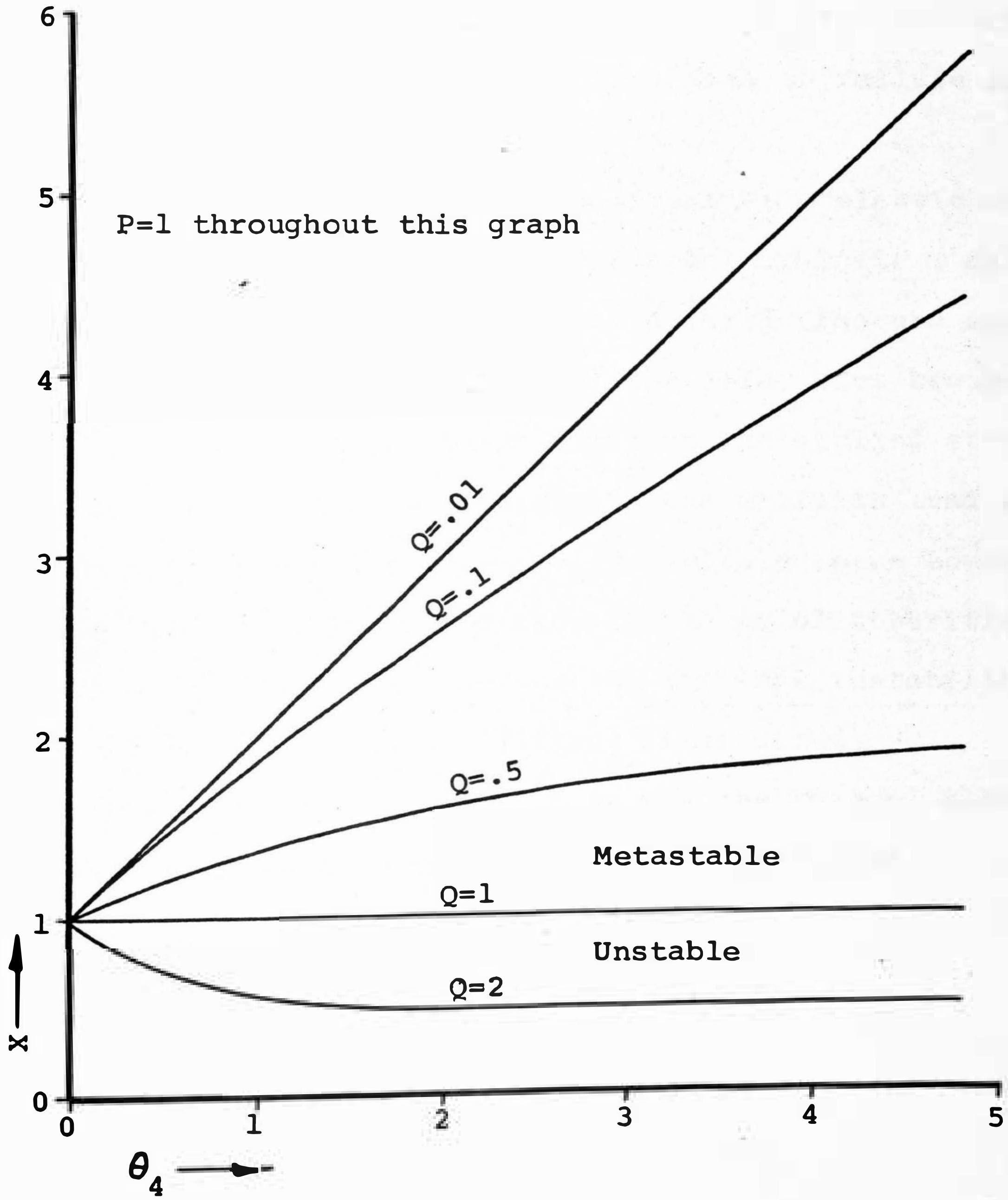
$$X = Z + (1 - Z)e^{-Q\theta_4}$$



GRAPH 3-2B FRACTURE PROPAGATION

MODEL 2 CRACK LENGTH WITH $\phi = H(t)$ AND VARIOUS Q

$$X = \frac{P}{Q} + \left(1 + \frac{P}{Q}\right) e^{-Q\theta_4}$$



CONCLUSIONS

The synthesized strain equations have been developed to predict fracture initiation in viscoelastic solids.

Delayed fracture is inevitable in loaded materials whose behavior is represented by the Maxwell Model. According to the theory developed, a Maxwell material will fracture under an arbitrarily small load if sufficient time to failure is allowed.

Contrary to this behavior, the linear viscoelastic solid represented by a standard 3-parameter model exhibits a distinct range of loading magnitudes for which delayed fracture may occur. Below a certain stress level the delay time becomes infinite and this sets the lower limit on the applied stress (endurance limit). The upper limit is the Griffith load at which fracture occurs instantaneously. Within these bounds the delay time is finite and a certain amount of subcritical propagation will take place before the terminal instability (equivalent to catastrophic fracture) is attained.

The time for fracture initiation and the rate of slow growth under various ranges of loading are predicted theoretically.

LITERATURE CITED

1. Mould, R.E., "Fundamental Phenomena in the Materials," vol. 4, Plenum Press, 1967.

2. Sneddon, N., "The Distribution of Stress in the Neighborhood of a Crack in an Elastic Solid," Proceedings of the Royal Society, London, England, vol. 187 Series A, 1946, p. 229.

3. Wnuk, M.P., "Effects of Time and Plasticity on Fracture," Brit. J. Appl. Phys., 1969, ser. 2, vol. 2, p. 1258, Printed in Great Britain.

4. Wnuk, M.P., and Knauss, W.G., "Delayed Fracture in Viscoelastic-Plastic Solids," Galcit SM 68-8, March, 1968, pp. 8-9.

5. Griffith, A., "The Phenomenon of Rupture and Flow in Solids," Philosophical Transactions of the Royal Society, London, England, vol. 221, Series A, 1921, p. 163.

6. Cherepanov, G.P., "Crack Propagation in Continuous Media," PMM, vol. 31, no. 3, 1967, p. 476.

7. Wnuk, M.P., "Similarity Between Creep Rupture in Viscoelastic Solid and Fatigue in Metals," Department of Mechanical Engineering, South Dakota State University, ONR Report 1, April 1970.