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On two four term arithmetic progressions with equal product

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This paper is dedicated to Richard K. Guy in his 104th year in honour of his many and varied contributions to mathematics.

Abstract

We investigate when two four-term arithmetic progressions have an equal product of their terms. This is equivalent to studying the (arithmetic) geometry of a non-singular quartic surface. It turns out that there are many polynomial parametrizations of such progressions, and it is likely that there exist polynomial parametrizations of every positive degree. We find *all* such parametrizations for degrees 1 to 4, and give examples of parametrizations for degrees 5 to 10.

1. Introduction

The problem considered in this paper was first drawn to my attention by Richard Guy and Alex Fink, who asked which n-term arithmetic progressions can have equal product of their terms. For example, when $n = 5$, Fink observed that the two progressions

$$(4+t^5, 3+2t^5, 2+3t^5, 1+4t^5, 5t^5), \quad (t+4t^6, 2t+3t^6, 3t+2t^6, 4t+t^6, 5t)$$

have equal product. There is some literature on the subject. Gabovich [5] gives infinitely many examples of two such 4-term progressions. For general n, the only known example of two arithmetic progressions with equal product of terms is given by

$$(n+1)(n+2)\dots(2n) = 2\cdot 6\cdot 10\cdot\dots\cdot(4n-2);$$

in fact, Saradha, Shorey and Tijdeman [9, 10] show that other than this example, solutions in positive integers $x>y$, $n>2$, to

$$x(x+d_1)...(x+(n-1)d_1) = y(y+d_2)...(y+(n-1)d_2),$$

for fixed integers $0<d_1<d_2$, are finite in number, and can be effectively determined. Choudhry [2–4] gives several results, including the construction for a fixed positive integer n of two arithmetic progressions of length n with equal product of terms. Further, he describes infinitely many pairs of 5-term progressions with equal product, and also constructs five 4-term progressions, all having equal product of terms.

Here, we investigate the case $n=4$. The defining equation is that of a quartic surface, and we study the geometry of this surface. By computing the Néron-Severi group of the surface over $\mathbb{C}$, we can determine infinitely many parametrizations for the problem, and in particular, can determine all parametrizations of a given degree that correspond to curves lying on the surface of arithmetic genus 0. The number of such parametrized curves increases rapidly, with attendant computational difficulties. Here, we simply give all such parametrizations of degrees $1,2,3,4$, and examples of parametrizations for degrees $5,...,10$.

2. A quartic surface

Consider two four-term arithmetic progressions with equal products, which by homogeneity we may take in the form $\{a-3d,\ a-d,\ a+d,\ a+3d\}$ and $\{b-3c,\ b-c,\ b+c,\ b+3c\}$. Then

$$V: (a^2-9d^2)(a^2-d^2) = (b^2-9c^2)(b^2-c^2).$$

This equation defines a non-singular quartic surface V. Symmetries of V occur with sign changes of the coordinates, under the mapping $(a,b,c,d)\to(b,a,d,c)$, and under the mapping $(a,b,c,d)\to(3d,3c,b,a)$, generating a symmetry group of order 32. The surface contains the twenty $\mathbb{Q}$-rational straight lines shown in Table 1.

Accordingly, there is a rich geometry of V over the rationals. Denote by $\mathrm{NS}(V(K))$ the Néron-Severi group of the surface V over the field K; then we expect $\mathrm{NS}(V(\mathbb{Q}))$ to be a sizeable subgroup of $\mathrm{NS}(V(\mathbb{C}))$. For reference, the action of the symmetries on the $\mathbb{Q}$-rational straight lines is given in the Appendix.

There are four real lines defined over $\mathbb{Q}(\sqrt{3})$ (see Table 2) and eight imaginary lines (see Table 3).

It is straightforward by considering linear parametrizations to see that this is the full list of lines on the surface V. The intersection matrix $\{(l_i\cdot l_j)\}$ of the 32 lines has rank 19.

l_1: $a = 3d$, $b = 3c$	l_2: $a = 3d$, $b = c$	l_3: $a = 3d$, $b = -c$	l_4: $a = 3d$, $b = -3c$
l_5: $a = d$, $b = 3c$	l_6: $a = d$, $b = c$	l_7: $a = d$, $b = -c$	l_8: $a = d$, $b = -3c$
l_9: $a = -d$, $b = 3c$	l_{10}: $a = -d$, $b = c$	l_{11}: $a = -d$, $b = -c$	l_{12}: $a = -d$, $b = -3c$
l_{13}: $a = -3d$, $b = 3c$	l_{14}: $a = -3d$, $b = c$	l_{15}: $a = -3d$, $b = -c$	l_{16}: $a = -3d$, $b = -3c$
l_{17}: $a = b$, $c = d$	l_{18}: $a = b$, $c = -d$	l_{19}: $a = -b$, $c = d$	l_{20}: $a = -b$, $c = -d$

Table 1: Twenty $\mathbb{Q}$-rational straight lines on V

l_{21}: $a = \sqrt{3}c$, $b = \sqrt{3}d$	l_{22}: $a = \sqrt{3}c$, $b = -\sqrt{3}d$	l_{23}: $a = -\sqrt{3}c$, $b = \sqrt{3}d$	l_{24}: $a = -\sqrt{3}c$, $b = -\sqrt{3}d$

Table 2: Four real straight lines on V

l_{25}: $a = ib$, $c = id$	l_{26}: $a = ib$, $c = -id$	l_{27}: $a = -ib$, $c = id$	l_{28}: $a = -ib$, $c = -id$
l_{29}: $a = i\sqrt{3}c$, $b = i\sqrt{3}d$	l_{30}: $a = i\sqrt{3}c$, $b = -i\sqrt{3}d$	l_{31}: $a = -i\sqrt{3}c$, $b = i\sqrt{3}d$	l_{32}: $a = -i\sqrt{3}c$, $b = -i\sqrt{3}d$

Table 3: Eight imaginary straight lines on V

Various conics arise as the residual intersection of V with a plane passing through two of the straight lines. Denote by Π a hyperplane section of the surface V, so that Π has genus 3, and $\Pi^2 = 2 \cdot \text{genus}(\Pi) - 2 = 4$. Then the effective divisor $\Pi - l_i - l_j$ has self-intersection $(\Pi - l_i - l_j)^2 = -4 + 2(l_i \cdot l_j)$, so consequently has genus 0 if and only if $(l_i \cdot l_j) = 1$.

If $\Pi - l_i - l_j$ is irreducible, then its intersection pairing with l_k is non-negative, so $((l_i + l_j) \cdot l_k) \leq 1$. Conversely, if $\Pi - l_i - l_j$ is reducible, then necessarily it is linearly equivalent to $l_m + l_n$ for lines l_m, l_n, and now its intersection pairing with l_n equals $(l_m \cdot l_n) - 2 \leq -1$, that is, $((l_i + l_j) \cdot l_n) \geq 2$. Hence $\Pi - l_i - l_j$ is irreducible if and only if $((l_i + l_j).l_k) \leq 1$ for all lines l_k.

If one of the component lines is $\mathbb{Q}$-rational, then by symmetry we can assume l_i is one of l_1, l_2, l_{17}. Only $\Pi - l_1 - l_j$, for $j = 17, 20, 26, 27$, are acceptable under the above criteria. Only $\Pi - l_2 - l_j$, for $j = 21, 24, 30, 31$, are acceptable. Only $\Pi - l_{17} - l_j$, for $j = 1, 6, 11, 16, 18, 19, 21, 24, 29, 32$, are acceptable.

If no component line is $\mathbb{Q}$-rational, then we have only $\Pi - l_i - l_j$ for $(i, j) =$

(21, 22), (21, 23), (21, 25), (21, 28), (25, 26), (25, 27), (25, 29), (25, 32), (29, 30), (29, 31).

It follows that there are precisely two equivalence classes of such $\mathbb{Q}$-rational conics, typified by $\Pi - l_1 - l_{17}$ ($\sim \Pi - l_6 - l_{20}$), and $\Pi - l_{17} - l_{19}$.
The plane $a + b = c + d$ cuts the surface in the two lines l_6, l_{20}, and the residual conic

$$4a^2 + 7ab + 2b^2 - 11ac - 7bc + 9c^2 = 0,$$

with parametrization

$$a : b : c : d \;=\; 3s^2 + s + 2 \;:\; -s^2 - 3s - 8 \;:\; s^2 - 3s - 2 \;:\; s^2 + s - 4. \tag{2.1}$$

This conic lies in an equivalence class under symmetry of order 16.

The plane $c = d$ cuts V in l_{17}, l_{19}, and the conic

$$a^2 + b^2 = 10c^2,$$

with parametrization

$$a : b : c : d \;=\; 3s^2 - 2s - 3 \;:\; s^2 + 6s - 1 \;:\; s^2 + 1 \;:\; s^2 + 1, \tag{2.2}$$

lying in an equivalence class of order 4. In this manner we recognise twenty $\mathbb{Q}$-rational conics on V, the residual intersections of the following planes:

Q_1:	$a + b = c + d$	Q_2:	$a + b = c - d$
Q_3:	$a + b = -c + d$	Q_4:	$a + b = -c - d$
Q_5:	$a - b = c + d$	Q_6:	$a - b = c - d$
Q_7:	$a - b = -c + d$	Q_8:	$a - b = -c - d$
Q_9:	$a - b = 3(c - d)$	Q_{10}:	$a - b = 3(c + d)$
Q_{11}:	$a - b = -3(c + d)$	Q_{12}:	$a - b = 3(-c + d)$
Q_{13}:	$a + b = 3(c - d)$	Q_{14}:	$a + b = 3(c + d)$
Q_{15}:	$a + b = -3(c + d)$	Q_{16}:	$a + b = 3(-c + d)$
Q_{17}:	$a = b$	Q_{18}:	$a = -b$
Q_{19}:	$c = d$	Q_{20}:	$c = -d$

Table 4: Twenty $\mathbb{Q}$-rational conics on V

A plane intersection does not of course necessarily contain a straight line, but may give rise to two conics. A straightforward (machine) computation shows that plane intersections delivering two conics arise precisely for the planes (writing $i = \sqrt{-1}$, $r = \sqrt{3}$):

$$a - (1 - i)c + rd = 0, \qquad \text{and} \qquad a + 2(1 - i)c - ird = 0,$$

together with symmetries and conjugates. The first plane intersection here comprises the two conics

$$Q_0: \qquad a - (1 - i)c + rd = 0, \quad b^2 + (2r - 5)c^2 + (2i + 2)cd - 2rid^2 = 0;$$

$$Q_0': \qquad a-(1-i)c+rd=0, \quad b^2+(-2r-5)c^2+(-2i-2)cd+2rid^2=0;$$

and Q_0 has parametrization

$$(a,b,c,d)=((-1+r)(3u^2-(3+r)uv-v^2),(1+i)(ru^2+(-4+2r)uv+v^2),\\ (1+i)(ru^2-v^2),(-1+r)(u^2+(1+r)uv-v^2)).$$

Further, the surface V is fibred by curves of genus 1. Consider the intersection of V with the family of planes

$$a-d=t(b-c). \tag{2.3}$$

The intersection contains the line $l_6 : \{a=d,\ b=c\}$, together with residual cubic curve

$$b^3(-1+9t^4)+b^2c(-1-27t^4)+9bc^2(1+3t^4)+\\ 9c^3(1-t^4)-36a(b-c)^2t^3+44a^2(b-c)t^2-16a^3t=0.$$

This cubic contains points such as $\mathcal{O}_t(a,b,c,d)=(t,1,-1,-t)$, the point where (2.3) meets the skew line $\{a+d=0=b+c\}$, and so is an elliptic curve over $\mathbb{Q}(t)$. The locus of $\mathcal{O}_t$ as t varies is the line l_{11}. A cubic model of the above curve is

$$E_t : V^2=U^3+67t^2U^2+1440t^4U+36t^2(1+277t^4+t^8), \tag{2.4}$$

with mappings

$$(U,V)=\big(-4t(-2a+7bt-7at^4+2bt^5)/(b+c-2at^3+bt^4-ct^4), \tag{2.5}\\ 2t(t^4-1)(-b^2-10bc-9c^2-40a^2t^2+82abt^3-82act^3-42b^2t^4+82bct^4\\ +20a^2t^6-28abt^7+28act^7+9b^2t^8-18bct^8+9c^2t^8)/(b+c-2at^3+bt^4-ct^4)^2\big),$$

and

$$a:b:c:d = -36t^2(1+t^4)(7+2t^4)-2(4+59t^4)U-5t^2U^2+2t(7+2t^4)V :\\ -36t(1+t^4)(2+7t^4)-2t^3(59+4t^4)U-5tU^2+2(2+7t^4)V :\\ 4t(2+509t^4-43t^8)+2t^3(101-4t^4)U+5tU^2+2(-2+3t^4)V :\\ 4t^2(-43+509t^4+2t^8)+2(-4+101t^4)U+5t^2U^2+2t(3-2t^4)V. \tag{2.6}$$

We note that the torsion subgroup of $E(\mathbb{C}(t))$ is trivial. The curve E_t at (2.4) is singular at $t=0,\infty,\pm1,\pm i$, and at the eight roots of $243t^8+1711t^4+243=0$. The discriminant of (2.4) is

$$-144(t-1)^2t^4(t+1)^2(t^2+1)^2(243t^8+1711t^4+243),$$

and we have the following Kodaira classification types, with the corresponding decomposition of the intersection (see Table 5) together with type I_1 nodal cubics at each root of $243t^8+1711t^4+243=0$. Shioda's fundamental formula [11] results in

$$20 \geq \text{rank NS}(V(\mathbb{C})) = \text{rank } E_t(\mathbb{C}(t))+2+2(3-1)+4(2-1)+8(1-1),$$

whence rank $E_t(\mathbb{C}(t)) \leq 10$.

t	$=$	0	IV	l_5	$+$	l_7	$+$	l_8
t	$=$	∞	IV	l_2	$+$	l_{10}	$+$	l_{14}
t	$=$	1	I_2			l_{17}	$+$	Q_7
t	$=$	-1	I_2			l_{20}	$+$	Q_1
t	$=$	i	I_2			l_{26}	$+$	conic
t	$=$	$-i$	I_2			l_{27}	$+$	conic

Table 5: Singular decompositions of E_t

Theorem 2.1. *$NS(V(\mathbb{C}))$ is a $\mathbb{Z}$-module of rank 19, with basis the divisor classes of the 18 lines l_1, l_2, l_3, l_4, l_5, l_7, l_8, l_{10}, l_{11}, l_{16}, l_{17}, l_{18}, l_{20}, l_{21}, l_{22}, l_{25}, l_{26}, l_{29}, and the conic Q_0.*

We prove Theorem 2.1 in several steps. It is known that $\mathrm{NS}(V(\mathbb{C}))$ is generated over $\mathbb{Z}$ by (i) a fibre of E_t, the zero section, the fibre components that do not meet the zero section; and (ii) sections that form a basis of $E_t(\mathbb{C}(t))$. For (i), we have the ten generators l_2, l_5, l_7, l_8, l_{10}, l_{11}, l_{17}, l_{20}, l_{26}, l_{27}. For (ii), we shall show $E_t(\mathbb{C}(t))$ has rank 9, so that indeed rank $\mathrm{NS}(V(\mathbb{C})) = 19$. It will then remain to determine an explicit basis.

The straight lines and conic Q_0 provide us with the following 9 independent points in $E_t(\mathbb{C}(t))$:

pullback	point on $E_t(\mathbb{C}(t))$
l_1	$J_1 = (-15t^2,\ 6t^5 + 6t)$;
l_4	$J_2 = (-18t^2,\ 6t^5 - 6t)$;
l_{16}	$J_3 = (-30t^2,\ -6t^5 - 6t)$;
l_{18}	$J_4 = (4t^4 - 10t^3 - 10t^2 - 10t + 4,$ $-8t^6 + 30t^5 - 58t^4 + 60t^3 - 58t^2 + 30t - 8)$;
l_{21}	$J_5 = (2rt^3 - 18t^2 + 2rt,\ 6t^5 + 2rt^4 + 12t^3 + 2rt^2 + 6t)$;
l_{22}	$J_6 = (4rt^3 - 18t^2 - 4rt,\ -6t^5 - 16rt^4 + 12t^3 + 16rt^2 - 6t)$;
l_{25}	$J_7 = (-4t^4 + 10it^3 - 10t^2 - 10it - 4,$ $8it^6 + 30t^5 - 58it^4 - 60t^3 + 58it^2 + 30t - 8i)$;
l_{29}	$J_8 = (-4rit^3 - 18t^2 - 4rit, -6t^5 - 16rit^4 - 12t^3 - 16rit^2 - 6t)$;
Q_0	$J_9 = ((r+3)(i+1)t^3 - 2(r+10)t^2 + (3r+5)(i-1)t + 4(r+2)i,$ $6t^5 + (5r+9)(i-1)t^4 + 2(5r+11)it^3 - 7(r+1)(i+1)t^2$ $-6(4r+7)t + 4(3r+5)(i-1))$

Table 6: Points on $E_t(\mathbb{C}(t))$

That the points J_i, $i = 1, \ldots, 9$, are linearly independent on E_t follows from the height-pairing matrix

$$M = \begin{pmatrix} \frac{8}{3} & 0 & \frac{4}{3} & 2 & \frac{2}{3} & \frac{4}{3} & 2 & \frac{4}{3} & \frac{4}{3} \\ 0 & \frac{2}{3} & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ \frac{4}{3} & 0 & \frac{8}{3} & 2 & \frac{4}{3} & \frac{2}{3} & 2 & \frac{2}{3} & \frac{2}{3} \\ 2 & 0 & 2 & 3 & 1 & 1 & 2 & 1 & \frac{3}{2} \\ \frac{2}{3} & 0 & \frac{4}{3} & 1 & \frac{5}{3} & \frac{1}{3} & 1 & \frac{1}{3} & -\frac{1}{6} \\ \frac{4}{3} & 0 & \frac{2}{3} & 1 & \frac{1}{3} & \frac{5}{3} & 1 & \frac{2}{3} & \frac{1}{6} \\ 2 & 0 & 2 & 2 & 1 & 1 & 3 & 1 & \frac{1}{2} \\ \frac{4}{3} & 0 & \frac{2}{3} & 1 & \frac{1}{3} & \frac{2}{3} & 1 & \frac{5}{3} & \frac{7}{6} \\ \frac{4}{3} & \frac{1}{3} & \frac{2}{3} & \frac{3}{2} & -\frac{1}{6} & \frac{1}{6} & \frac{1}{2} & \frac{7}{6} & \frac{7}{3} \end{pmatrix}$$

of determinant $\frac{8}{9}$. It follows that rank $E_t(\mathbb{C}(t)) \geq 9$.

We now have that the divisor classes of the following 19 curves are independent in the Néron-Severi group $\mathrm{NS}(V, \mathbb{C})$:

$$l_1, l_2, l_3, l_4, l_5, l_7, l_8, l_{10}, l_{11}, l_{16}, l_{17}, l_{18}, l_{20}, l_{21}, l_{22}, l_{25}, l_{26}, l_{29}, Q_0. \tag{2.7}$$

(Note: the conic $ac = bd$ cuts V in the divisor

$$l_1 + l_6 + l_{11} + l_{16} + l_{17} + l_{20} + l_{26} + l_{27} \sim 2\,\Pi \sim l_1 + l_2 + l_3 + l_4 + l_5 + l_6 + l_7 + l_8,$$

which allows us up to linear equivalence to replace l_{27} by l_3.)

Lemma 2.2. *NS*$(V(\mathbb{C}))$ *has rank* 19.

Proof. We follow closely the exposition of Kloosterman [6] to which the reader is referred for full details.

Let Y be a smooth projective surface defined over $\mathbb{Q}$, with Néron-Severi group $\mathrm{NS}(Y)$. Suppose that p is a prime of good reduction, and denote by $\overline{Y}$ the reduction of Y modulo p. It is known that $\mathrm{NS}(Y)$ modulo torsion together with the intersection pairing on $\mathrm{NS}(Y)$ forms a lattice. Denote by $\Delta(\mathrm{NS}(Y_K))$ the discriminant of a Gram matrix of the Néron-Severi lattice $\mathrm{NS}(Y_K)$ of Y over K with respect to the pairing. Proposition 4.2 of Kloosterman tells us that $\Delta(\mathrm{NS}(Y_{\overline{\mathbb{Q}}}))$ and $\Delta(\mathrm{NS}(\overline{Y}_{\overline{\mathbb{F}_p}}))$ differ by a square.

The idea therefore (originally suggested by van Luijk) is to find two distinct primes p_1, p_2 of good reduction for which the rank of the Néron-Severi lattices is the same, but for which the discriminants of the lattices differ by a non-square. It will follow that the rank of $\mathrm{NS}(Y_{\overline{\mathbb{Q}}})$ is at least one less than the rank of $\mathrm{NS}(\overline{Y}_{\overline{\mathbb{F}_{p_1}}})$.

We quote two further results from Kloosterman. Here, q is a prime power, and l a prime with $(l, q) = 1$.

Conjecture 4.3 (Tate Conjecture).
Let $Y/\mathbb{F}_q$ be a smooth surface with Néron-Severi rank $\rho(Y)$. Let F_q be the automorphism of $H^2_{\text{ét}}(Y, \mathbb{Q}_l))$ induced by the Frobenius automorphism of $\mathbb{F}_q$. Let $Q(t)$ be $\det(I - tF_q | H^2_{\text{ét}}(Y, \mathbb{Q}_l))$. Then $\rho(Y)$ equals the number of reciprocal zeroes of $Q(t)$ of the form $q\zeta$, with ζ a root of unity.

Conjecture 4.6 (Artin-Tate Conjecture).
Let $Y/\mathbb{F}_q$ be a smooth surface with Néron-Severi rank $\rho(Y)$. Let F_q be the automorphism of $H^2_{\text{ét}}(Y, \mathbb{Q}_l))$ induced by the Frobenius automorphism of $\mathbb{F}_q$. Let $Q_q(t)$ be $\det(I - tF_q | H^2_{\text{ét}}(Y, \mathbb{Q}_l))$. Then

$$\lim_{s\to 1} \frac{Q_q(q^{-s})}{(1-q^{1-s})^{\rho'(Y)}} = \frac{(-1)^{\rho'(Y)-1}\#\mathrm{Br}(Y)\Delta(\mathrm{NS}(Y_{\mathbb{F}_q}))}{q^{\alpha(Y)}(\#NS(Y_{\mathbb{F}_q})_{\text{tor}})^2},$$

where $\alpha(Y) = \chi(Y, O_Y) - 1 + \dim \mathrm{Pic}^0(Y)$, $\mathrm{Br}(Y)$ is the Brauer group of Y, $\mathrm{NS}(Y_{\mathbb{F}_q})$ is the subgroup of $\mathrm{NS}(Y_{\overline{\mathbb{F}_q}})$ generated by $\mathbb{F}_q$-rational divisors, and $\rho'(Y) = \operatorname{rank} \mathrm{NS}(Y_{\overline{\mathbb{F}_q}})$.

These Conjectures are known to be true when $(q, 6) = 1$ and $Y/\mathbb{F}_q$ is an elliptic $K3$ surface, as in the case we are considering.

Again from Kloosterman, Proposition 4.7, the order of $\mathrm{Br}(Y)$ is a square, and with the hypothesis that $\rho(Y) = \rho'(Y)$, then the Artin-Tate Conjecture gives the following:

$$\Delta(\mathrm{NS}(Y_{\overline{\mathbb{F}_q}})) \equiv (-1)^{\rho'(Y)-1} q^{\alpha(Y)} \lim_{s\to 1} \frac{Q_q(q^{-s})}{(1-q^{1-s})^{\rho'(Y)}} \bmod \mathbb{Q}^{*\,2}.$$

In our case, at the primes of good reduction $p = 37, 61$, the known 19 independent divisor classes are defined over $\mathbb{F}_p$. By counting the points on V over $\mathbb{F}_p$ and $\mathbb{F}_{p^2}$ we compute

$$Q_{37}(x) = (1-37x)^{20}(1+38x+1369x^2), \quad Q_{61}(x) = (1-61x)^{20}(1+118x+3721x^2).$$

We have $\rho(Y) = \rho'(Y) = 20$. We thus get

$$\Delta(\mathrm{NS}(Y_{\overline{\mathbb{F}_p}})) \equiv -p^{\alpha(Y)} \lim_{s\to 1} \frac{Q_p(p^{-s})}{(1-p^{1-s})^{20}} \bmod \mathbb{Q}^{*\,2}.$$

Hence

$$\Delta(\mathrm{NS}(Y_{\overline{\mathbb{F}_{37}}})) \equiv -37^{\alpha(Y)}(1 + \frac{38}{37} + 1) \equiv -7 \cdot 37^{\alpha(Y)-1} \bmod \mathbb{Q}^{*\,2};$$
$$\Delta(\mathrm{NS}(Y_{\overline{\mathbb{F}_{61}}})) \equiv -61^{\alpha(Y)}(1 + \frac{118}{61} + 1) \equiv -3 \cdot 5 \cdot 61^{\alpha(Y)-1} \bmod \mathbb{Q}^{*\,2}.$$

Consequently, the two discriminants do not differ by a perfect square, and it follows that the rank of $\mathrm{NS}(Y_{\overline{\mathbb{Q}}})$ is at least one less than the rank of $\mathrm{NS}(Y_{\overline{\mathbb{F}_{37}}})$, so must equal 19. □

Corollary 2.3. *The group $E_t(\mathbb{C}(t))$ has rank nine, and the points $J_1, \ldots, J_9$ listed in Table 6 form a basis.*

Proof. The previous computation implies the rank is 9. That the $\{J_i\}$ form a basis follows from Lemma 2.5 of Kuwata [7]. The first criterion in the Lemma implies that the index of the subgroup in $E_t(\mathbb{C}(t))$ generated by the J_i can be divisible only by 2 or 3. It is a straightforward computation to determine that for $\epsilon_i = 0, 1$, not all zero, none of the points $\sum_{i=1}^{9} \epsilon_i J_i$ can lie in $2E_t(\mathbb{C}(t))$; and for $\epsilon_i = 0, \pm 1$, not all zero, none of the points $\sum_{i=1}^{9} \epsilon_i J_i$ can lie in $3E_t(\mathbb{C}(t))$. □

It remains to determine a $\mathbb{Z}$-basis for $\mathrm{NS}(V, \mathbb{C})$.

The divisors at (2.7) form a basis over $\mathbb{Q}$. Let $D \sim c_1 l_1 + c_2 l_2 + \cdots + c_{26} l_{26} + c_{29} l_{29} + c_0 Q_0$, which notationally we abbreviate to $(c_1, c_2, \ldots, c_{26}, c_{29}, c_0)$, lie in $\mathrm{NS}(V, \mathbb{C})$ for $c_i \in \mathbb{Q}$. Demanding integer intersection with each of the 32 straight lines and Q_0 gives a system of equations for the coefficients c_i that implies D is a $\mathbb{Z}$-linear combination of the following divisors:

$$l_1, l_2, l_3, l_4, l_5, l_7, l_{10}, l_{17}, l_{18}, l_{20}, l_{21}, l_{22}, l_{25}, l_{26}, l_{29}, Q_0, \tag{2.8}$$

and

$$\begin{aligned} D_1 &\sim \frac{1}{4}(0, 0, 1, -1, 0, -1, 1, 0, 0, 0, 0, -2, 0, 0, 2, 2, 0, -2, 0), \\ D_2 &\sim \frac{1}{4}(1, -3, 2, 0, -1, 1, 0, -1, 1, 0, 2, 0, 0, 0, -2, 0, -2, 2, 0), \\ D_3 &\sim \frac{1}{8}(0, 1, 1, 3, 3, -5, -1, 2, -1, 1, -2, 0, -2, -4, 4, -4, 4, 0, 0). \end{aligned}$$

The divisor $\Delta \sim aD_1 + bD_2 + cD_3$ for $a, b, c \in \mathbb{Z}$ satisfies

$$\Delta^2 = -4a^2 + \frac{5}{2}ab - \frac{7}{2}b^2 + \frac{3}{2}ac + \frac{7}{2}bc - \frac{33}{8}c^2,$$

which, being equal to $2 \cdot \mathrm{genus}(\Delta) - 2$, lies in $2\mathbb{Z}$. Thus c is even, and D is a $\mathbb{Z}$-linear combination of the divisors at (2.8) and of $(d_1, d_2, d_3) = (D_1, D_2, 2D_3 + l_2 - l_{26})$. Now

$$\begin{aligned} 4d_1 &\sim -2l_9 + 2l_{13} + 2l_{15} + 2l_{16} + 2l_{19} + 2l_{22} + l_{25} - l_{28} - 5l_{29} - 3l_{32}, \\ 4d_2 &\sim -2l_3 + 4l_4 - 6l_9 + 4l_{12} + 4l_{15} + 4l_{16} - 2l_{19} - 8l_{22} - 4l_{23} + 2l_{24} \\ &\quad + l_{25} + 3l_{28} + 2l_{29} + 5l_{30} + 3l_{31} - 2l_{32} - 4Q_0, \\ 4d_3 &\sim -2l_3 + 10l_4 - 8l_9 + 8l_{13} + 6l_{15} + 14l_{16} + 3l_{22} - l_{23} + 4l_{24} + 4l_{28} \\ &\quad - 9l_{29} - 10l_{30} - 10l_{31} - 9l_{32}, \end{aligned}$$

linear equivalences which express the divisors $4d_i$ of degree 0 in terms of divisors which meet E_t. Each induces a divisor of points $(4d_i.E_t)$ on E_t of degree 0, and we can compute the image of these divisors under the Jacobian mapping jac from the group of divisors on E_t of degree 0, to E_t.

We first identify the following intersections on E_t.

l	$(l.E_t)$	l	$(l.E_t)$
l_1	J_1	l_{21}	J_5
l_3	$-J_2+J_3$	l_{22}	J_6
l_4	J_2	l_{23}	J_1-J_6
l_9	J_2+J_3	l_{24}	J_3-J_5
l_{11}	$\mathcal{O}$	l_{25}	J_7
l_{12}	J_1-J_2	l_{28}	$J_1+J_3-J_7$
l_{13}	$-J_2$	l_{29}	J_8
l_{15}	J_1+J_2	l_{30}	$-J_1-J_2-J_4+J_5+J_6+J_7-J_8+2J_9$
l_{16}	J_3	l_{31}	$J_1+J_2+J_3+J_4-J_5-J_6-J_7+J_8-2J_9$
l_{18}	J_4	l_{32}	J_1-J_8
l_{19}	$J_1+J_3-J_4$	Q_0	J_9

Table 7: Intersections on E_t

Using the above table,

$$\begin{aligned} \mathrm{jac}(4d_1.E_t) &= -2J_2+J_3-2J_4+2J_6+2J_7-2J_8, \\ \mathrm{jac}(4d_2.E_t) &= J_1-2J_2+2J_3-2J_6+2J_8, \\ \mathrm{jac}(4d_3.E_t) &= 2(J_2+J_3-2J_5+2J_6-2J_7). \end{aligned} \tag{2.9}$$

The assumption that $ad_1+bd_2+cd_3$, $a,b,c\in\mathbb{Z}$, exists as divisor implies that $\mathrm{jac}((a\,4d_1+b\,4d_2+c\,4d_3).E_t) = 4\,\mathrm{jac}((ad_1+bd_2+cd_3).E_t)\in 4E_t(\mathbb{C}(t))$, that is

$$\begin{aligned} bJ_1-2(a+b-c)J_2+(a+2b+2c)J_3-2aJ_4-4cJ_5+2(a-b+2c)J_6 \\ +2(a-2c)J_7-2(a-b)J_8\in 4E_t(\mathbb{C}(t)). \end{aligned}$$

The deduction is that $a,b\equiv 0 \bmod 4$, $c\equiv 0 \bmod 2$. A set of $\mathbb{Z}$-generators is now the divisors at (2.8) and $4d_1$, $4d_2$, $2d_3$; equivalently, the divisors

$$l_1,l_2,l_3,l_4,l_5,l_7,l_8,l_{10},l_{11},l_{17},l_{18},l_{20},l_{21},l_{22},l_{25},l_{26},l_{29},Q_0,$$

and

$$d_4=2d_3\sim\frac{1}{2}(0,5,1,3,3,-5,-1,2,-1,1,-2,0,-2,-4,4,-4,0,0,0).$$

Assume that d_4 exists as a divisor in $\mathrm{NS}(V,\mathbb{C})$. From (2.9), we have $\mathrm{jac}(2d_4.E_t) = \mathrm{jac}(4d_3.E_t) = 2(J_2+J_3-2J_5+2J_6-2J_7)$, so that the divisor $d_5=d_4-l_9+l_{21}-l_{22}+l_{25}$ of degree 0 satisfies $\mathrm{jac}(2d_5.E_t)=0$. Since E has trivial torsion, it follows that $\mathrm{jac}(d_5.E_t)=0$. Hence from properties of the Jacobian mapping, $d_5.E_t\sim 0$ on E_t. Thus there exists a function f_t on E_t having divisor $d_5.E_t$, and induced by a function f on V. Then $(f)-d_5$ is a divisor not meeting E_t, which therefore is a sum of the singular components of E_t; equivalently, a sum of the singular straight line components of E_t. We deduce

$$d_5\sim c_2l_2+c_5l_5+c_7l_7+c_8l_8+c_{10}l_{10}+c_{14}l_{14}+c_{17}l_{17}+c_{20}l_{20}+c_{26}l_{26}+c_{27}l_{27}.$$

However $1 = d_5.l_{17} = -2c_{17}$, impossible. Thus d_5 cannot exist as divisor, and $\mathrm{NS}(V, \mathbb{C})$ has $\mathbb{Z}$-basis as required. This completes the proof of Theorem 2.1.

In the Appendix, we give a matrix expressing the divisor classes of the 32 lines as linear combinations of this generating set.

3. Rational parametrizations

That part of the Néron-Severi Group defined over $\mathbb{Q}$ is seen to be generated by the divisor classes of

$$l_1, l_2, l_3, l_4, l_5, l_7, l_8, l_{10}, l_{11}, l_{16}, l_{17}, l_{18}, l_{20},$$

which set we denote by $\{C_i\}$, $i = 1, \dots, 13$, with

$$\begin{aligned}
l_{21} + l_{21}^{\mathrm{conj}} &\sim l_3 + l_4 + l_7 + l_8 - l_{17} - l_{20}, \\
l_{22} + l_{22}^{\mathrm{conj}} &\sim l_1 + l_2 - l_5 - l_7 - 2l_8 + l_{10} + l_{11} + l_{17} + l_{20}, \\
l_{25} + l_{25}^{\mathrm{conj}} &\sim l_1 - l_7 - l_{10} + l_{16} + l_{17} + l_{20}, \\
l_{26} + l_{26}^{\mathrm{conj}} &\sim l_2 + l_3 + l_4 + l_5 + l_7 + l_8 - l_{11} - l_{16} - l_{17} - l_{20}, \\
l_{29} + l_{29}^{\mathrm{conj}} &\sim l_1 + 2l_2 + l_3 + l_4 - l_8 + l_{10} - l_{16} - l_{17} - l_{20}, \\
l_{30} + l_{30}^{\mathrm{conj}} &\sim -l_2 - l_5 + l_{11} + l_{16} + l_{17} + l_{20}.
\end{aligned}$$

The associated intersection matrix is

	l_1	l_2	l_3	l_4	l_5	l_7	l_8	l_{10}	l_{11}	l_{16}	l_{17}	l_{18}	l_{20}
l_1	-2	1	1	1	1	0	0	0	0	0	1	0	1
l_2	1	-2	1	1	0	0	0	1	0	0	0	0	0
l_3	1	1	-2	1	0	1	0	0	1	0	0	0	0
l_4	1	1	1	-2	0	0	1	0	0	1	0	1	0
l_5	1	0	0	0	-2	1	1	0	0	0	0	0	0
l_7	0	0	1	0	1	-2	1	0	1	0	0	1	0
l_8	0	0	0	1	1	1	-2	0	0	1	0	0	0
l_{10}	0	1	0	0	0	0	0	-2	1	0	0	1	0
l_{11}	0	0	1	0	0	1	0	1	-2	0	1	0	1
l_{16}	0	0	0	1	0	0	1	0	0	-2	1	0	1
l_{17}	1	0	0	0	0	0	0	0	1	1	-2	1	0
l_{18}	0	0	0	1	0	1	0	1	0	0	1	-2	1
l_{20}	1	0	0	0	0	0	0	0	1	1	0	1	-2

Putting $\Gamma \sim x_1 C_1 + x_2 C_2 + \dots + x_{13} C_{13}$, we have

$$\begin{aligned}
&\deg(\Gamma)^2 - 4(\Gamma.\Gamma) = \deg(\Gamma)^2 - 8(\mathrm{genus}(\Gamma) - 1) = \\
&(x_1 - x_2 - x_3 + x_4 - x_5 + x_6 - x_7 + x_8 + x_9 + x_{10} - x_{11} - x_{12} - x_{13})^2 \\
&+ 2(x_1 - x_4 - x_6 - x_8 + x_9 + x_{10} - x_{11} + x_{12} - x_{13})^2
\end{aligned}$$

$$\begin{aligned}
&+2(x_1 - x_4 + x_6 + x_8 - x_9 + x_{10})^2 \\
&+2(x_1 - x_2 - x_5 - x_9 - x_{10})^2 + 2(x_1 - x_3 + x_7 + x_9 - x_{10})^2 \\
&+2(x_2 - x_4 - x_5 + x_6 - x_8)^2 + 2(x_3 - x_4 - x_6 + x_7 + x_8)^2 \\
&+2(x_{11} - x_{12} + x_{13})^2 + 4(x_{11} - x_{13})^2 + 4(x_5 - x_7)^2 + 4(x_2 - x_3)^2 + 4x_{12}^2
\end{aligned}$$

which is in a machine computable form if we wish to determine (via the coefficients x_i) the curves Γ of genus 0 and given degree $\deg(\Gamma)$. Putting

$$\begin{aligned}
m_1 &= x_1 - x_2 - x_3 + x_4 - x_5 + x_6 - x_7 + x_8 + x_9 + x_{10} - x_{11} - x_{12} - x_{13}, \\
m_2 &= x_1 - x_2 - x_5 - x_9 - x_{10}, \\
m_3 &= x_2 - x_4 - x_5 + x_6 - x_8, \\
m_4 &= x_1 - x_3 + x_7 + x_9 - x_{10}, \\
m_5 &= x_3 - x_4 - x_6 + x_7 + x_8, \\
m_6 &= x_1 - x_4 + x_6 + x_8 - x_9 + x_{10}, \\
m_7 &= x_1 - x_4 - x_6 - x_8 + x_9 + x_{10} - x_{11} + x_{12} - x_{13}, \\
m_8 &= x_{11} - x_{12} + x_{13}, \\
m_9 &= x_2 - x_3, \\
m_{10} &= x_5 - x_7, \\
m_{11} &= x_{11} - x_{13}, \\
m_{12} &= x_{12}, \\
m_{13} &= \deg(\Gamma),
\end{aligned}$$

we have to tabulate the finitely many solutions to the equation

$$m_1^2 + 2\sum_{i=2}^{8} m_i^2 + 4\sum_{i=9}^{12} m_i^2 = \deg(\Gamma)^2 - 4(\Gamma.\Gamma) \tag{3.1}$$

and then determine $(x_1, \ldots, x_{13}) = \mathbf{x}$ from $(m_1, \ldots, m_{13}) = \mathbf{m}$ by means of

$$\mathbf{x} = \frac{1}{4}\begin{pmatrix}
0 & 2 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & -2 & 1 \\
0 & 1 & 1 & -1 & 1 & -1 & 0 & -1 & 3 & 1 & 0 & -2 & 1 \\
0 & 1 & 1 & -1 & 1 & -1 & 0 & -1 & -1 & 1 & 0 & -2 & 1 \\
0 & 1 & -1 & -1 & -1 & -1 & 0 & -1 & 1 & -1 & 0 & -2 & 1 \\
-1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & 1 & 0 & -2 & 0 \\
0 & -1 & 1 & 1 & -1 & 1 & -1 & -1 & -2 & 0 & 0 & 0 & 0 \\
-1 & -1 & -1 & 1 & -1 & 1 & 0 & -1 & -1 & -3 & 0 & -2 & 0 \\
1 & 0 & 0 & 0 & 2 & 0 & -1 & 0 & 1 & 1 & 0 & 2 & 0 \\
1 & -1 & 1 & 1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 2 & 0 \\
0 & -1 & -1 & -1 & -1 & 1 & 1 & 1 & 0 & -2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 2 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & -2 & 2 & 0
\end{pmatrix}\mathbf{m}^t$$

This imposes congruence conditions on the m_i at (3.1), namely:

$$\begin{aligned} m_1 + m_{13} &\equiv 0 \bmod 2, \\ m_2 + m_3 + m_6 &\equiv 0 \bmod 2, \\ m_4 + m_5 + m_6 &\equiv 0 \bmod 2, \\ m_6 + m_7 + m_8 &\equiv 0 \bmod 2, \\ m_8 + m_{11} + m_{12} &\equiv 0 \bmod 2, \\ m_1 + m_3 + m_4 + m_8 &\equiv 0 \bmod 2, \\ m_1 + m_7 + m_9 + m_{10} &\equiv 0 \bmod 2, \end{aligned}$$

and

$$\begin{aligned} m_1 + 2m_6 + m_{13} &\equiv 0 \bmod 4, \\ m_1 - m_2 + m_3 + m_4 + m_5 - m_6 + m_8 - m_9 + m_{10} + 2m_{12} &\equiv 0 \bmod 4, \\ m_2 - m_3 - m_4 + m_5 - m_6 + m_7 + m_8 + 2m_9 &\equiv 0 \bmod 4. \end{aligned}$$

For $\mathbb{Q}$-rational curves of degree 1, we find (as expected) exactly the 20 known $\mathbb{Q}$-rational lines, falling into three equivalence classes under symmetry, with representatives l_1 (8 symmetries), l_2 (8 symmetries), and l_{17} (4 symmetries).

For $\mathbb{Q}$-rational curves of degree 2 we find the known conics, falling into the two equivalence classes $\Pi - l_1 - l_{17}$ (16 symmetries) and $\Pi - l_{17} - l_{18}$ (4 symmetries). Their parametrizations are given at (2.1) and (2.2).

There are 24 $\mathbb{Q}$-rational irreducible cubics, in three equivalence classes up to symmetry, with representatives $2\Pi - l_5 - l_{12} - l_{19} - l_{30} - l_{31}$, $2\Pi - l_{11} - l_{16} - l_{17} - l_{18} - l_{20}$, and $2\Pi - l_1 - l_{11} - l_{17} - l_{18} - l_{20}$ (8 symmetries each).

Equivalence class	Parametrization $(a : b : c : d)$
$2\Pi - l_5 - l_{12} - l_{19} - l_{30} - l_{31}$	$-5 + 21s^2$
	$5 + 3s^2$
	$-7s + 15s^3$
	$s + 15s^3$
$2\Pi - l_{11} - l_{16} - l_{17} - l_{18} - l_{20}$	$4 + s + 7s^2 + 6s^3$
	$6 + 7s + s^2 + 4s^3$
	$-2 + 3s + 7s^2 + 4s^3$
	$4 + 7s + 3s^2 - 2s^3$
$2\Pi - l_1 - l_{11} - l_{17} - l_{18} - l_{20}$	$3 + 7s + 7s^2 + s^3$
	$1 + 7s + 7s^2 + 3s^3$
	$1 + s + 3s^2 + s^3$
	$1 + 3s + s^2 + s^3$

Table 8: Rational cubics on V

There are 176 $\mathbb{Q}$-rational quartics in eight equivalence classes:

Equivalence class	Parametrization $(a:b:c:d)$
$\{0,0,0,-1,0,1,-1,1,2,-1,1,1,1\}$	$6-5s-11s^2-7s^3-s^4$
	$-12-21s-15s^2-5s^3-s^4$
	$4+s-3s^2-3s^3-s^4$
	$6+11s+11s^2+5s^3+s^4$
$\{0,0,0,1,1,1,2,-1,0,1,0,-1,0\}$	$3-7s-2s^2-20s^3+8s^4$
	$-3+3s-24s^2+16s^3-8s^4$
	$-1+7s-8s^4$
	$3-5s+2s^2+4s^3-8s^4$
$\{1,0,1,1,-1,0,0,0,0,0,0,1,1\}$	$12+27s+42s^2+23s^3+2s^4$
	$18+37s+18s^2+9s^3+4s^4$
	$6+7s-7s^3-4s^4$
	$4-9s-12s^2-7s^3-2s^4$
$\{0,-1,0,1,0,0,1,0,0,1,0,1,1\}$	$-3-18s-6s^2-4s^3-s^4$
	$9-4s-6s^2-6s^3-s^4$
	$-3+2s+12s^2+4s^3+s^4$
	$1-12s+2s^3+s^4$
$\{0,0,-1,1,0,0,1,0,0,1,0,1,1\}$	$12+27s-21s^2-149s^3-65s^4$
	$6+41s+27s^2+33s^3+65s^4$
	$6+25s+81s^2+41s^3-13s^4$
	$-4-15s-9s^2-59s^3-13s^4$
$\{1,0,1,0,-1,0,-2,1,1,0,1,1,1\}$	$-1+11s+3s^2+49s^3+10s^4$
	$3-s+9s^2+21s^3+40s^4$
	$1-s+13s^2-27s^3-10s^4$
	$-1+5s+s^2-s^3+20s^4$

Table 9: Rational quartics on V

The divisor $\{0,-1,0,0,0,1,1,0,0,1,0,1,1\}$ represents a $\mathbb{Q}$-rational quartic curve defined over $\mathbb{Q}$, but possessing no rational (indeed real) points; its parametrization may be given as

$$\begin{aligned} a:b:c:d = {} & i\sqrt{3}(1+s^2)(1-s-s^2) : \\ & i\sqrt{3}(1+s^2)(1+s-s^2) : \\ & 1-s+4s^2+s^3+s^4 : \\ & 1+s+4s^2-s^3+s^4. \end{aligned}$$

Similarly, the divisor $\{2,3,2,2,0,-1,-1,0,0,-1,-1,-1,0\}$ is represented by

$$\begin{aligned} a:b:c:d = {} & 3+7s-8s^2-7s^3+3s^4 : \\ & 3-7s-8s^2+7s^3+3s^4 : \\ & \sqrt{7/3}(1+s-s^2)(1+s^2) : \\ & \sqrt{7/3}(1-s-s^2)(1+s^2). \end{aligned}$$

The number of rationally parametrizable curves increases rapidly, and it seems likely that there are such curves of every positive degree. We content ourselves with listing just one rational parametrization for degrees 5 to 10.

$$(a,b,c,d) = (3s^5+5s,\ 5s^4+3,\ s^4-1,\ s^5-s);$$

$$\begin{aligned}(a,b,c,d) = (&27s^6+27s^5+19s^2+17s+6,\\ &27s^6+45s^5+36s^4-18s^3-39s^2-23s-4,\\ &9s^6-3s^5+12s^4+30s^3+35s^2+17s+4,\\ &9s^6-9s^5-36s^4-48s^3-31s^2-11s-2);\end{aligned}$$

$$\begin{aligned}(a,b,c,d) = (&s^7+16s^6+56s^5+85s^4+44s^3+s^2-11s-3,\\ &3s^7+11s^6-s^5-44s^4-85s^3-56s^2-16s-1,\\ &s^7+5s^6+9s^5+20s^4+25s^3+16s^2+4s+1,\\ &s^7+4s^6+16s^5+25s^4+20s^3+9s^2+5s+1);\end{aligned}$$

$$\begin{aligned}(a,b,c,d) = (&s^8-5s^7+26s^6-76s^5+137s^4-115s^3+16s^2+64s-24,\\ &s^8-3s^7-2s^6+46s^5-153s^4+277s^3-282s^2+156s-24,\\ &s^8-5s^7+10s^6-6s^5-17s^4+35s^3-30s^2+4s-8,\\ &s^8-7s^7+26s^6-60s^5+105s^4-137s^3+136s^2-80s+24);\end{aligned}$$

$$(a,b,c,d) = (s^9-33s^5-184s,\ s^8+47s^4+96,\ 3s^8+21s^4-32,\ s^9+7s^5+56s);$$

$$\begin{aligned}&(a,b,c,d) =\\ &(4s^{10}-25s^9+123s^8-355s^7+653s^6-610s^5+56s^4+720s^3-976s^2+640s-192,\\ &6s^{10}-31s^9+61s^8-15s^7-233s^6+538s^5-728s^4+760s^3-864s^2+544s-64,\\ &2s^{10}-5s^9-19s^8+155s^7-481s^6+930s^5-1208s^4+1080s^3-608s^2+160s-64,\\ &4s^{10}-31s^9+119s^8-285s^7+533s^6-762s^5+808s^4-560s^3+304s^2-256s+64).\end{aligned}$$

4. Appendix

For reference, we give here (in terms of subscript) the action of the sign-change symmetries on the $\mathbb{Q}$-rational lines, together with the action of the further two symmetries:

(a,b,c,d)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
(a,b,c,-d)	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4	18	17	20	19
(a,b,-c,d)	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13	18	17	20	19
(a,b,-c,-d)	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	17	18	19	20
(a,-b,c,d)	4	3	2	1	8	7	6	5	12	11	10	9	16	15	14	13	19	20	17	18
(a,-b,c,-d)	16	15	14	13	12	11	10	9	8	7	6	5	4	3	2	1	20	19	18	17
(a,-b,-c,d)	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	20	19	18	17
(-a,b,c,d)	13	14	15	16	9	10	11	12	5	6	7	8	1	2	3	4	19	20	17	18
(b,a,d,c)	1	5	9	13	2	6	10	14	3	7	11	15	4	8	12	16	17	18	19	20
(3d,3c,b,a)	6	5	8	7	2	1	4	3	14	13	16	15	10	9	12	11	17	19	18	20

Table 10: Action of the symmetries on the $\mathbb{Q}$-rational straight lines

The following matrix expresses the linear equivalence classes of the 32 straight lines on V in terms of the set of $\mathbb{Z}$-generators of Theorem 2.1.

	l_1	l_2	l_3	l_4	l_5	l_7	l_8	l_{10}	l_{11}	l_{16}	l_{17}	l_{18}	l_{20}	l_{21}	l_{22}	l_{25}	l_{26}	l_{29}	Q_0
l_1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_2	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_3	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_4	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_5	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
l_6	1	1	1	1	-1	-1	-1	0	0	0	0	0	0	0	0	0	0	0	0
l_7	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
l_8	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
l_9	0	0	0	1	0	0	1	-1	-1	1	0	0	0	0	0	0	0	0	0
l_{10}	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
l_{11}	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0
l_{12}	1	1	1	0	0	0	-1	0	0	-1	0	0	0	0	0	0	0	0	0
l_{13}	0	1	1	0	-1	0	-1	1	1	-1	0	0	0	0	0	0	0	0	0
l_{14}	0	-1	0	0	1	1	1	-1	0	0	0	0	0	0	0	0	0	0	0
l_{15}	1	1	0	1	0	-1	0	0	-1	0	0	0	0	0	0	0	0	0	0
l_{16}	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
l_{17}	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
l_{18}	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0
l_{19}	1	1	1	1	1	0	1	-1	-1	0	-1	-1	-1	0	0	0	0	0	0
l_{20}	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
l_{21}	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
l_{22}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
l_{23}	1	1	0	0	-1	-1	-2	1	1	0	1	0	1	0	-1	0	0	0	0
l_{24}	0	0	1	1	0	1	1	0	0	0	-1	0	-1	-1	0	0	0	0	0
l_{25}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
l_{26}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
l_{27}	0	1	1	1	1	1	1	0	-1	-1	-1	0	-1	0	0	0	-1	0	0
l_{28}	1	0	0	0	0	-1	0	-1	0	1	1	0	1	0	0	-1	0	0	0
l_{29}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
l_{30}	-1	1	1	0	0	1	0	0	-1	-1	-1	-1	-2	1	1	1	-1	-1	2
l_{31}	1	-2	-1	0	-1	-1	0	0	2	2	2	1	3	-1	-1	-1	1	1	-2
l_{32}	1	2	1	1	0	0	-1	1	0	-1	-1	0	-1	0	0	0	0	-1	0

Table 11: Linear equivalence classes of the lines in terms of the $\mathbb{Z}$-generators of Theorem 2.1

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