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On the exponential Diophantine equation $(4m^2+1)^x + (21m^2-1)^y = (5m)^z$

Nobuhiro Terai*

Division of Mathematical Sciences, Department of Integrated Science and Technology
Faculty of Science and Technology, Oita University, Oita, Japan
terai-nobuhiro@oita-u.ac.jp

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Abstract

Let m be a positive integer. Then we show that the exponential Diophantine equation $(4m^2+1)^x + (21m^2-1)^y = (5m)^z$ has only the positive integer solution $(x, y, z) = (1, 1, 2)$ under some conditions. The proof is based on elementary methods and Baker's method.

Keywords: Exponential Diophantine equation, integer solution, lower bound for linear forms in two logarithms.

MSC: 11D61

1. Introduction

Let a, b, c be fixed relatively prime positive integers greater than one. The exponential Diophantine equation

$$a^x + b^y = c^z \tag{1.1}$$

in positive integers x, y, z has been actively studied by a number of authors. It is known that the number of solutions (x, y, z) of equation (1.1) is finite, and all solutions can be effectively determined by means of Baker's method of linear forms in logarithms.

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Equation (1.1) has been investigated in detail for Pythagorean numbers a, b, c, too. Jeśmanowicz [8] conjectured that if a, b, c are Pythagorean numbers, i.e., positive integers satisfying $a^2+b^2=c^2$, then (1.1) has only the positive integer solution $(x, y, z) = (2, 2, 2)$ (cf. [14, 17, 22]). As an analogue of Jeśmanowicz' conjecture, the author proposed that if a, b, c, p, q, r are fixed positive integers satisfying $a^p+b^q=c^r$ with $a, b, c, p, q, r \geq 2$ and $\gcd(a, b) = 1$, then (1.1) has only the positive integer solution $(x, y, z) = (p, q, r)$ except for a handful of triples (a, b, c) (cf. [6, 12, 13, 15, 21, 24]). This conjecture has been proved to be true in many special cases. This conjecture, however, is still unsolved.

In Terai [23], the author showed that if m is a positive integer such that $1 \leq m \leq 20$ or $m \not\equiv 3 \pmod 6$, then the Diophantine equation

$$(4m^2+1)^x + (5m^2-1)^y = (3m)^z \tag{1.2}$$

has only the positive integer solution $(x, y, z) = (1, 1, 2)$. The proof is based on elementary methods and Baker's method. Suy-Li [20] proved that if $m \geq 90$ and $3 \mid m$, then equation (1.2) has only the positive integer solution $(x, y, z) = (1, 1, 2)$ by means of the result of Bilu-Hanrot-Voutier [3] concerning the existence of primitive prime divisors in Lucas-numbers. Finally, Bertók [1] has completely solved equation (1.2) including the remaining cases $20 < m < 90$. His proof can be done by the help of exponential congruences. This is a nice application of Bertók and Hajdu [2].

More generally, several authors have studied the Diophantine equation

$$(pm^2+1)^x + (qm^2-1)^y = (rm)^z \tag{1.3}$$

under some conditions, where p, q, r are positive integers satisfying $p + q = r^2$:

- (Miyazaki-Terai [16], 2014) $(m^2+1)^x + (qm^2-1)^y = (rm)^z$, $1 + q = r^2$,
- (Terai-Hibino [25], 2015) $(12m^2+1)^x + (13m^2-1)^y = (5m)^z$,
- (Terai-Hibino [26], 2017) $(3pm^2-1)^x + (p(p-3)m^2+1)^y = (pm)^z$,
- (Fu-Yang [7], 2017) $(pm^2+1)^x + (qm^2-1)^y = (rm)^z$, $r \mid m$,
- (Pan [19], 2017) $(pm^2+1)^x + (qm^2-1)^y = (rm)^z$, $m \equiv \pm 1 \pmod r$,
- (Murat [18], 2018) $(18m^2+1)^x + (7m^2-1)^y = (5m)^z$,
- (Kizildere et al. [10], 2018) $((q+1)m^2+1)^x + (qm^2-1)^y = (rm)^z$, $2q+1 = r^2$.

We note that equation (1.2), which was completely resolved by Terai, Suy-Li and Bertók, is the first equation shown that equation (1.3) has only the trivial solution $(x, y, z) = (1, 1, 2)$ without any assumption on m. All known results for the above-mentioned equations need congruence relations or inequalities on m.

In this paper, we consider the exponential Diophantine equation

$$(4m^2+1)^x + (21m^2-1)^y = (5m)^z \tag{1.4}$$

with m positive integer. Denote $v_p(n)$ by the exponent of p in the factorization of a positive integer n. Our main result is the following:

Theorem 1.1. *Let m be a positive integer. Suppose that $v_5(4m^2+1) = v_5(21m^2-1) = 1$ only if $m \equiv \pm 1 \pmod{10}$. Then equation* (1.4) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

This paper is organized as follows. When m is even or m is odd in (1.4) with $y \geq 2$, we show Theorem 1.1 by using elementary methods such as congruence methods and the quadratic reciprocity law. When m is odd in (1.4) with $m \equiv \pm 2$ (mod 5) and $y = 1$, we show Theorem 1.1 by applying a lower bound for linear forms in two logarithms due to Laurent [11]. The proof of the case $m \equiv \pm 1 \pmod 5$ uses the Primitive Divisor Theorem due to Zsigmondy [27]. That of the case $m \equiv 0$ (mod 5) is based on a result on linear forms in p-adic logarithms due to Bugeaud [5].

2. Preliminaries

In order to obtain an upper bound for a solution of Pillai's equation, we need a result on lower bounds for linear forms in the logarithms of two algebraic numbers. We will introduce here some notations. Let α_1 and α_2 be real algebraic numbers with $|\alpha_1| \geq 1$ and $|\alpha_2| \geq 1$. We consider the linear form

$$\Lambda = b_2 \log \alpha_2 - b_1 \log \alpha_1,$$

where b_1 and b_2 are positive integers. As usual, the *logarithmic height* of an algebraic number α of degree n is defined as

$$h(\alpha) = \frac{1}{n}\left(\log |a_0| + \sum_{j=1}^{n} \log \max\left\{1, \left|\alpha^{(j)}\right|\right\}\right),$$

where a_0 is the leading coefficient of the minimal polynomial of α (over $\mathbb{Z}$) and $(\alpha^{(j)})_{1 \leq j \leq n}$ are the conjugates of α. Let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\left\{h(\alpha_i), \frac{|\log \alpha_i|}{D}, \frac{1}{D}\right\},$$

for $i \in \{1, 2\}$, where D is the degree of the number field $\mathbb{Q}(\alpha_1, \alpha_2)$ over $\mathbb{Q}$. Define

$$b' = \frac{b_1}{D \log A_2} + \frac{b_2}{D \log A_1}.$$

We choose to use a result due to Laurent [11, Corollary 2], with $m = 10$ and $C_2 = 25.2$.

Proposition 2.1 (Laurent [11]). *Let Λ be given as above, with $\alpha_1 > 1$ and $\alpha_2 > 1$. Suppose that α_1 and α_2 are multiplicatively independent. Then*

$$\log|\Lambda| \geq -25.2D^4 \left(\max\left\{\log b' + 0.38, \frac{10}{D}\right\}\right)^2 \log A_1 \log A_2.$$

Next, we shall quote a result on linear forms in p-adic logarithms due to Bugeaud [5]. Here we consider the case where $y_1 = y_2 = 1$ in the notation from [5, p. 375].

Let p be an odd prime. Let a_1 and a_2 be non-zero integers prime to p. Let g be the least positive integer such that

$$\mathrm{ord}_p(a_1^g - 1) \geq 1, \quad \mathrm{ord}_p(a_2^g - 1) \geq 1,$$

where we denote the p-adic valuation by $\mathrm{ord}_p(\,\cdot\,)$. Assume that there exists a real number E such that

$$1/(p-1) < E \leq \mathrm{ord}_p(a_1^g - 1).$$

We consider the integer

$$\Lambda = a_1^{b_1} - a_2^{b_2},$$

where b_1 and b_2 are positive integers. We let A_1 and A_2 be real numbers greater than 1 with

$$\log A_i \geq \max\{\log|a_i|, E\log p\} \quad (i = 1, 2),$$

and we put $b' = b_1/\log A_2 + b_2/\log A_1$.

Proposition 2.2 (Bugeaud [5]). *With the above notation, if a_1 and a_2 are multiplicatively independent, then we have the upper estimate*

$$\mathrm{ord}_p(\Lambda) \leq \frac{36.1g}{E^3(\log p)^4}\big(\max\{\log b' + \log(E\log p) + 0.4,\, 6E\log p,\, 5\}\big)^2 \log A_1 \log A_2.$$

The following is a direct consequence of an old version of the Primitive Divisor Theorem due to Zsigmondy [27]:

Proposition 2.3 (Zsigmondy [27]). *Let A and B be relatively prime integers with $A > B \geqslant 1$. Let $\{a_k\}_{k\geqslant 1}$ be the sequence defined as*

$$a_k = A^k + B^k.$$

If $k > 1$, then a_k has a prime factor not dividing $a_1 a_2 \cdots a_{k-1}$, whenever $(A, B, k) \neq (2, 1, 3)$.

3. Proof of Theorem 1.1

In this section, we give a proof of Theorem 1.1.

3.1. The case where m is odd and $m \equiv \pm 1 \pmod 5$

Lemma 3.1. *Let m be a positive integer such that m is odd and $m \equiv \pm 1 \pmod 5$. Suppose that $v_5(4m^2+1) = v_5(21m^2-1) = 1$. Then equation* (1.4) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof. If $v_5(4m^2+1) = v_5(21m^2-1) = 1$, then $\gcd(4m^2+1, 21m^2-1) = 5$. Put $A = (4m^2+1)/5$ and $B = (21m^2-1)/5$. Then $\gcd(A, B) = 1$ and $AB \not\equiv 0 \pmod 5$. In view of $(5m)^x < (4m^2+1)^x < (5m)^z$ from (1.4), it follows that the inequality $z > x$ holds. Equation (1.4) can be written as

$$5^y B^y = 5^x(5^{z-x}m^z - A^x)$$

with $AB \not\equiv 0 \pmod 5$. This implies that $x = y$. Then equation (1.4) becomes

$$a_x = A^x + B^x = 5^{z-x}m^z.$$

Apply Proposition 2.3 with $A = (4m^2+1)/5$ and $B = (21m^2-1)/5$. Note that $\gcd(A, B) = 1$. Since $a_1 = 5m^2$, it follows that $x = 1$, which yields $(y, z) = (1, 2)$. $\square$

Lemma 3.2. *In* (1.4), *y is odd.*

Proof. When $m = 1$, we see that $v_5(4m^2+1) = v_5(21m^2-1) = 1$. By Lemma 3.1, we may suppose that $m \geq 2$. It follows that $z \geq 2$ from (1.4). Taking (1.4) modulo m^2 implies that $1 + (-1)^y \equiv 0 \pmod{m^2}$ and hence y is odd. $\square$

3.2. The case where m is even

Lemma 3.3. *If m is even, then equation* (1.4) *has only the positive integer solution $(x, y, z) = (1, 1, 2)$.*

Proof. If $z \leq 2$, then $(x, y, z) = (1, 1, 2)$ from (1.4). Hence we may suppose that $z \geq 3$. Taking (1.4) modulo m^3 implies that

$$1 + 4m^2x - 1 + 21m^2y \equiv 0 \pmod{m^3},$$

so

$$4x + 21y \equiv 0 \pmod m,$$

which is impossible, since y is odd and m is even. We therefore conclude that if m is even, then equation (1.4) has only the positive integer solution $(x, y, z) = (1, 1, 2)$. $\square$

3.3. The case where m is odd and $m \equiv \pm 2 \pmod 5$

By Lemma 3.3, we may suppose that m is odd with $m \geq 3$. Let (x, y, z) be a solution of (1.4).

Lemma 3.4. *If m is odd and $m \equiv \pm 2 \pmod 5$, then $y = 1$ and x is odd.*

Proof. Suppose that $m \equiv \pm 2 \pmod 5$, i.e., $m^2 \equiv -1 \pmod 5$. Then $\left(\frac{21m^2-1}{4m^2+1}\right) = 1$ and $\left(\frac{5m}{4m^2+1}\right) = -1$, where $\left(\frac{*}{*}\right)$ denotes the Jacobi symbol. Indeed,

$$\left(\frac{21m^2-1}{4m^2+1}\right) = \left(\frac{m^2-6}{4m^2+1}\right) = \left(\frac{4m^2+1}{m^2-6}\right) = \left(\frac{25}{m^2-6}\right) = 1$$

and

$$\left(\frac{5m}{4m^2+1}\right) = \left(\frac{5}{4m^2+1}\right)\left(\frac{m}{4m^2+1}\right) = \left(\frac{4m^2+1}{5}\right)\left(\frac{4m^2+1}{m}\right) =$$

$\left(\frac{-3}{5}\right)\left(\frac{1}{m}\right) = (-1) \cdot 1 = -1$, since $m^2 \equiv -1 \pmod 5$. In view of these, z is even from (1.4).

Suppose that $y \geq 2$. Taking (1.4) modulo 8 implies that

$$5^x \equiv (5m)^z \equiv 1 \pmod 8,$$

so x is even.

On the other hand, since $m^2 \equiv -1 \pmod 5$, taking (1.4) modulo 5 implies that

$$2^x + 3^y \equiv 0 \pmod 5,$$

which contradicts the fact that x is even and y is odd. Hence we obtain $y = 1$. Then, taking (1.4) modulo 8 implies that $5^x + 4 \equiv (5m)^z \equiv 1 \pmod 8$, so x is odd. □

From Lemma 3.4, it follows that $y = 1$ and x is odd. If $x = 1$, then we obtain $z = 2$ from (1.4). From now on, we may suppose that $x \geq 3$. Hence our theorem is reduced to solving Pillai's equation

$$c^z - a^x = b \tag{3.1}$$

with $x \geq 3$, where $a = 4m^2 + 1$, $b = 21m^2 - 1$ and $c = 5m$.

We now want to obtain a lower bound for x.

Lemma 3.5. $x \geq \frac{1}{4}(m^2 - 21)$.

Proof. Since $x \geq 3$, equation (3.1) yields the following inequality:

$$(5m)^z = (4m^2+1)^x + 21m^2 - 1 \geq (4m^2+1)^3 + 21m^2 - 1 > (5m)^3.$$

Hence $z \geq 4$. Taking (3.1) modulo m^4 implies that

$$1 + 4m^2 x + 21m^2 - 1 \equiv 0 \pmod{m^4},$$

so $4x + 21 \equiv 0 \pmod{m^2}$. Hence we obtain our assertion. □

We next want to obtain an upper bound for x.

Lemma 3.6. $x < 2521 \log c$.

Proof. From (3.1), we now consider the following linear form in two logarithms:

$$\Lambda = z \log c - x \log a \quad (> 0).$$

Using the inequality $\log(1+t) < t$ for $t > 0$, we have

$$0 < \Lambda = \log(\frac{c^z}{a^x}) = \log(1 + \frac{b}{a^x}) < \frac{b}{a^x}. \tag{3.2}$$

Hence we obtain

$$\log \Lambda < \log b - x \log a. \tag{3.3}$$

On the other hand, we use Proposition 2.1 to obtain a lower bound for Λ. It follows from Proposition 2.1 that

$$\log \Lambda \geq -25.2 \left(\max\{\log b' + 0.38, 10\}\right)^2 (\log a)(\log c), \tag{3.4}$$

where $b' = \frac{x}{\log c} + \frac{z}{\log a}$.

We note that $a^{x+1} > c^z$. Indeed,

$$a^{x+1} - c^z = a(c^z - b) - c^z = (a-1)c^z - ab \geq 4m^2 \cdot 25m^2 - (4m^2+1)(21m^2-1) > 0.$$

Hence $b' < \frac{2x+1}{\log c}$.

Put $M = \frac{x}{\log c}$. Combining (3.3) and (3.4) leads to

$$x \log a < \log b + 25.2 \left(\max\left\{\log\left(2M + \frac{1}{\log c}\right) + 0.38, 10\right\}\right)^2 (\log a)(\log c),$$

so

$$M < 1 + 25.2 \left(\max\left\{\log\left(2M + \frac{1}{2}\right) + 0.38, 10\right\}\right)^2,$$

since $\log c = \log(5m) \geq \log 15 > 2$. We therefore obtain $M < 2521$. This completes the proof of Lemma 3.6. $\square$

We are now in a position to prove Theorem 1.1. It follows from Lemmas 3.5, 3.6 that

$$\frac{1}{4}(m^2 - 21) < 2521 \log 5m.$$

Hence we obtain $m \leq 269$. From (3.2), we have the inequality

$$\left|\frac{\log a}{\log c} - \frac{z}{x}\right| < \frac{b}{x a^x \log c},$$

which implies that $\left|\frac{\log a}{\log c} - \frac{z}{x}\right| < \frac{1}{2x^2}$, since $x \geq 3$. Thus $\frac{z}{x}$ is a convergent in the simple continued fraction expansion to $\frac{\log a}{\log c}$.

On the other hand, if $\frac{p_r}{q_r}$ is the r-th such convergent, then

$$\left|\frac{\log a}{\log c} - \frac{p_r}{q_r}\right| > \frac{1}{(a_{r+1}+2)q_r^2},$$

where a_{r+1} is the $(r+1)$-st partial quotient to $\frac{\log a}{\log c}$ (see e.g. Khinchin [9]). Put $\frac{z}{x} = \frac{p_r}{q_r}$. Note that $q_r \le x$. It follows, then, that

$$a_{r+1} > \frac{a^x \log c}{bx} - 2 \ge \frac{a^{q_r} \log c}{bq_r} - 2. \tag{3.5}$$

Finally, we checked by Magma [4] that inequality (3.5) does not hold for any r with $q_r < 2521\log(5m)$ in the range $3 \le m \le 269$.

3.4. The case $m \equiv 0 \pmod 5$

Let m be a positive integer with $m \equiv 0 \pmod 5$. Let (x, y, z) be a solution of (1.4). Taking (1.4) modulo $m(\ge 5)$ implies that y is odd. Here, we apply Proposition 2.2. For this we set $p := 5, a_1 := 4m^2+1, a_2 := 1-21m^2, b_1 := x, b_2 := y$, and

$$\Lambda := (4m^2+1)^x - (1-21m^2)^y.$$

Then we may take $g = 1, E = 2, A_1 = 4m^2+1, A_2 := 21m^2-1$. Hence we have

$$2z \le \frac{36.1}{8(\log 5)^4}\big(\max\{\log b' + \log(2\log 5) + 0.4,\ 12\log 5\}\big)^2 \log(4m^2+1)\log(21m^2-1),$$

where $b' := \frac{x}{\log(21m^2-1)} + \frac{y}{\log(4m^2+1)}$. Suppose that $z \ge 4$. We will observe that this leads to a contradiction. Taking (1.4) modulo m^4 implies that

$$4x + 21y \equiv 0 \pmod{m^2}.$$

In particular, we see that $M := \max\{x, y\} \ge m^2/25$. Therefore, since $z \ge M$ and $b' \le \frac{M}{\log m}$, we obtain

$$\begin{aligned} 2M \le \frac{36.1}{8(\log 5)^4}&\left(\max\left\{\log\left(\frac{M}{\log m}\right) + \log(2\log 5) + 0.4,\ 12\log 5\right\}\right)^2 \\ &\times \log(4m^2+1)\log(21m^2-1). \end{aligned} \tag{3.6}$$

If $m \ge 122009$, then

$$2M \le \frac{36.1}{8(\log 5)^4}\left(\log\left(\frac{M}{\log m}\right) + \log(2\log 5) + 0.4\right)^2 \log(4m^2+1)\log(21m^2-1).$$

Since $m^2 \le 25M$, the above inequality gives

$$2M \le 0.7\left(\log M - \log(\log 122009) + 1.6\right)^2 \log(100M+1)\log(525M-1).$$

We therefore obtain $M \le 3386$, which contradicts the fact that $M \ge m^2/25 \ge 595447844$.

If $m < 122009$, then inequality (3.6) gives

$$\frac{2}{25}m^2 \le 251 \log(4m^2+1)\log(21m^2-1).$$

This implies that $m \le 882$. Hence all x, y and z are also bounded. It is not hard to verify by Magma [4] that there is no (m, x, y, z) under consideration satisfying (1.4). We conclude that $z \le 3$. In this case, we can easily show that $(x, y, z) = (1, 1, 2)$. This completes the proof of Theorem 1.1.

Remark 3.7. The values of m, a, b, c satisfying the condition of Theorem 1.1 with $1 \le m < 100$ are given in the table below.

m	a	b	c
1	5	$2^2 \cdot 5$	5
11	$5 \cdot 97$	$2^2 \cdot 5 \cdot 127$	$5 \cdot 11$
19	$5 \cdot 17^2$	$2^2 \cdot 5 \cdot 379$	$5 \cdot 19$
21	$5 \cdot 353$	$2^2 \cdot 5 \cdot 463$	$3 \cdot 5 \cdot 7$
29	$5 \cdot 673$	$2^2 \cdot 5 \cdot 883$	$5 \cdot 29$
31	$5 \cdot 769$	$2^2 \cdot 5 \cdot 1009$	$5 \cdot 31$
39	$5 \cdot 1217$	$2^2 \cdot 5 \cdot 1597$	$3 \cdot 5 \cdot 13$
49	$5 \cdot 17 \cdot 113$	$2^2 \cdot 5 \cdot 2521$	$5 \cdot 7^2$
51	$5 \cdot 2081$	$2^2 \cdot 5 \cdot 2731$	$3 \cdot 5 \cdot 17$
61	$5 \cdot 13 \cdot 229$	$2^2 \cdot 5 \cdot 3907$	$5 \cdot 61$
69	$5 \cdot 13 \cdot 293$	$2^2 \cdot 5 \cdot 4999$	$3 \cdot 5 \cdot 23$
71	$5 \cdot 37 \cdot 109$	$2^2 \cdot 5 \cdot 67 \cdot 79$	$5 \cdot 71$
79	$5 \cdot 4993$	$2^2 \cdot 5 \cdot 6553$	$5 \cdot 79$
81	$5 \cdot 29 \cdot 181$	$2^2 \cdot 5 \cdot 83^2$	$3^4 \cdot 5$
89	$5 \cdot 6337$	$2^2 \cdot 5 \cdot 8317$	$5 \cdot 89$
99	$5 \cdot 7841$	$2^2 \cdot 5 \cdot 41 \cdot 251$	$3^2 \cdot 5 \cdot 11$

Let m be a positive integer with $m \equiv \pm 1 \pmod{10}$. Suppose that $v_5(4m^2+1) = v_5(21m^2-1)$. Since $(4m^2+1)+(21m^2-1) = 25m^2$, we see that $\gcd(4m^2+1, 21m^2-1) = 5$ or 25 according as $v_5(4m^2+1) = v_5(21m^2-1) = 1$ or 2. Put $A = (4m^2+1)/5^e$ and $B = (21m^2-1)/5^e$ with $e = 1, 2$ according as $v_5(4m^2+1) = v_5(21m^2-1) = 1$ or 2. Then $\gcd(A, B) = 1$ and $AB \not\equiv 0 \pmod 5$. Though we apply Proposition 2.3 to the case $v_5(4m^2+1) = v_5(21m^2-1) = 2$, e.g., $m = 9, 41, 59, 191, 209$, etc., we can not obtain $x = 1$ unlike Theorem 1.1. Indeed, $a_x = A^x + B^x = 5^{z-2x}m^z$ and $a_1 = m^2$.

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