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# AMENABILITY OF CAYLEY GRAPHS THROUGH USE OF FØLNER'S CONDITIONS

by

Nikita Paulick

A Thesis

Submitted to the Faculty of Graduate Studies  
through the Department of Mathematics and Statistics  
in Partial Fulfillment of the Requirements for  
the Degree of Master of Science at the  
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Windsor, Ontario, Canada

2020

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AMENABILITY OF CAYLEY GRAPHS  
THROUGH USE OF FØLNER'S CONDITIONS

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## **Author's Declaration of Originality**

I hereby certify that I am the sole author of this thesis and that no part of this thesis has been published or submitted for publication.

I certify that to the best of my knowledge, my thesis does not infringe upon anyone's copyright nor violate any proprietary rights and that any ideas, techniques, quotations, or any other material from the work of other people included in my thesis, published or otherwise, are fully acknowledged in accordance with the standard referencing practices. Furthermore, to the extent that I have included copyrighted materials that surpasses the bounds of fair dealing within the meaning of the Canada Copyright Act, I certify that I have obtained a written permission of the copyright owner(s) to include such materials in my thesis and have included copies of such copyright clearances to my appendix.

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## Abstract

In this thesis we will study the definitions and properties relating to groups and Cayley graphs, as well as the concept of amenability. We will discuss McMullen's theorem that states that an infinite tree  $X$ , with every vertex having degree equal to 2 is amenable, otherwise if every vertex has degree greater than 2,  $X$  is non-amenable. We will also examine how if  $G$  is a finitely generated group acting on a set  $X$ , where  $A$  and  $B$  are two finite symmetric generating sets of  $G$ , then the Cayley graph  $\text{Cay}_A(G, X)$  is amenable if and only if  $\text{Cay}_B(G, X)$  is amenable. We will show that  $(G, X)$  satisfies Følner's condition if and only if for every finitely generated subgroup  $H$  of  $G$ ,  $\text{Cay}(H, X)$  is amenable. We will prove that for a finitely generated group  $G$ ,  $(G, X)$  is amenable if and only if  $\text{Cay}(G, X)$  is amenable; this is derived from the fact that  $(G, X)$  and  $\text{Cay}(G, X)$  have the same Følner's sequences.

## Acknowledgements

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**General Notation**

- $\mathbb{N} = \{1, 2, 3, \dots\}$  natural numbers
- $\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$  integers
- $\mathbb{R}$  real numbers
- $\mathbb{C} = \{x + iy : x, y \in \mathbb{R}\}$  complex numbers
- $\mathbb{F} = \mathbb{R}$  or  $\mathbb{C}$  field of real or complex numbers
- $\Re(x + iy) = x$  real part of  $x + iy$
- $\Im(x + iy) = y$  imaginary part of  $x + iy$
- $\in$  belongs to
- $a := b$   $a$  equals to  $b$  by definition
- $Y \subset X$   $Y$  is a subset of  $X$  (this does not exclude  $Y = X$ )
- $X - Y = X \setminus Y = \{x \in X : x \notin Y\}$
- $f \geq 0$   $f(x) \geq 0$  for all  $x$
- $A \Delta B = (A - B) \cup (B - A)$  symmetric difference between sets
- $a_n = O(n)$  : there exists an  $M > 0$  such that  $|a_n| \leq Mn$ , for all  $n \in \mathbb{N}$
- $\ell^\infty$  sequence space whose elements are bounded sequences
- $\|\cdot\|_\infty$  infinity norm
- $D_n$  dihedral group of order  $n$
- $|A|$  cardinality of a set  $A$
- $1_X$  characteristic function
- $B_r$  ball of radius  $r$
- $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt, \quad x > 0$

## CHAPTER 1

# Introduction and Preliminaries

### 1.1. Introduction

The theoretical background on the amenability of Cayley graphs using Følner's conditions will be studied in detail throughout this thesis. The topic of this thesis originated from the works of von Neumann [23] in 1929, when he defined amenable groups in terms of finitely additive invariant measures, which was then extended by Day [6–8] through generalizations and modifications of von Neumann's definition in 1949.

This thesis takes its roots from many important mathematical conjectures and theorems from the last century. It all began in the works of Banach and Tarski [2] wherein their paper they give a construction of a paradoxical decomposition. They provide a proof that given a ball in 3-dimensional space, there is a way of decomposing it into finitely many disjoint pieces that can be rearranged to form two balls of the same size as the original. Though this result seems counterintuitive, it is essentially a property of the isometry group of  $\mathbb{R}^3$ , as well as a statement about measure theory. It has been shown that this paradox does not exist on smaller dimensions  $n = 1$  or  $2$ . The reasoning behind this dichotomy is due to the fact that isometry groups of  $\mathbb{R}$  and  $\mathbb{R}^2$  are amenable, while that of  $\mathbb{R}^3$  is not. It is from this point in time that it is thought to be the birth of the concept of amenable groups.

John von Neumann gave the original definition of amenable groups in response to the Banach-Tarski paradox, in which he defines them to be those finite groups which have a finitely additive invariant probability measure (mean) on their subsets. This definition of amenability was extended by Mahlon Day to relate this concept onto semigroups. Day does this by using techniques from functional analysis by defining

invariant means by use of linear functionals. In particular, he introduces the concept of left amenable, which has been shown to coincide directly with the von Neumann definition. The extension of the Markov-Kakutani fixed point theorem to amenable groups is also a result due to Day [6]. The Markov-Kakutani fixed point theorem will be discussed within this thesis.

Over the years there have been many other alternative characterizations of the amenability of a group. The most widely used is that provided by Erling Følner in 1955 [10]. What is now referred to as *Følner's conditions* have been used to prove that if a group possess a Følner's sequence with respect to its action on itself, the group is amenable. Many authors use this as the base definition of amenability.

The subject of our study is comprised of two topics found in two different fields of mathematics, namely the concept of amenability, originally defined in the field of analysis, and Cayley graphs, which arose in the field of group theory. Before we move forward in this thesis it is best to give a brief overview of what these concepts mean and how they are tied together.

First it is pertinent to mention the concept of graphs, as the main results of this thesis will stem from the foundations of graph theory. We encounter graphs in our everyday lives, even though we may not be aware of it. In a very general setting, a graph is composed of two fundamental elements: vertices, and edges. The edges of a graph are what connect vertices together, which can be used, for example, to represent the bandwidth of a wired connection in a network, or to related webpages through various links.

Graphs have rich applications in various fields, not just mathematics. They have been used to encode all sorts of different structures and properties in physics, chemistry, biology, linguistics, computer science and countless other fields. Graph theory is a relatively new area of research taking its origins in 1736 wherein the famous mathematician Leonhard Euler published the first paper in the field. Euler's paper regarding one of the most historically notable problems in mathematics known as the Seven Bridges of Königsberg, proved to have no solution [3]. The problem proposed

asks if the seven bridges found in the city of Königsberg could all be traversed in a single trip while crossing each bridge once, and only once, with the additional requirement that the trip ends in the same place that it began. Euler's technique of analysis shown in his paper is considered to be the birth of graph theory. The field has since expanded to become a powerful tool in almost every branch of science, and an active field of research in mathematics today.

It is from the concepts and properties found in the field of graph theory that this thesis takes its roots. A few particular types of graphs that we concern ourselves with in this thesis are known as Cayley graphs, defined by Arthur Cayley in 1878. Cayley first created the notion of such graphs for finite groups as a way to encode the abstract structure of a group. His conventions allow for a given group, typically with a finite generating set, to form Cayley graphs with respect to the chosen generating set. Cayley's construction for such graphs comes from properties of groups and it is to this end that we see how group theory is closely tied with graph theory. These notions will be discussed in this thesis, and will be vital in characterizing the amenability of Cayley graphs.

We give a brief overview of the various sections in this thesis. In the remaining of Chapter 1 we introduce the basic definitions and properties needed to move forward in our work. This includes discussing the foundations of graph theory in Section 1.2, the notions needed to utilize the properties in free groups, discussed in Section 1.3, and the foundations of finitely generated groups found in Section 1.4. This introductory chapter serves as an aid in recalling definitions and preliminary materials needed for the later chapters.

In Section 2.1 we introduce the concept of expansion on graphs, and provide some examples to aid our understanding. We define what it means for a graph to be amenable and define how this is related to the measure of the expansion of a graph. Section 2.2 is used to examine alternative graph expansions, as well as provide interesting relations between them all, and discuss certain properties that make them more valuable in certain cases.

Chapter 3 takes an in depth look at the basic concepts of amenable groups and groups actions in order to prepare for Chapter 4. Section 3.1 discusses positive linear functionals, which are needed to construct the notion of means on Banach spaces over the complex field found in Section 3.2. One of the main results in this chapter is found in Theorem 3.2.4 where we illustrate the conditions needed for a linear functional to be a mean. This result is what leads us to the concept of invariant means in Section 3.3, ultimately rendering the definition of an amenable group. We provide a proof of the Markov-Kakutani fixed point theorem, and use it as a tool in the proof of Theorem 3.3.6 wherein we show how every abelian group is amenable.

Section 3.4 is of vital importance to our work because it is where we first introduce Rosenblatt's characterization of the existence of invariant means in Theorem 3.4.2. This theorem defines Følner's conditions, as well as provides equivalent conditions under which Følner's notion relates to invariant means. From here we are able to define amenable group actions by utilizing any of the equivalent conditions provided by Rosenblatt. We also show how in a finitely generated group, Følner's net can be replaced with a Følner's sequence.

Chapter 4 is the highlight of this thesis. It incorporates all of the results from the previous chapters, especially those found in Chapter 3 from Rosenblatt's theorem, in order to demonstrate the main results of our work. In Section 4.1 we introduce the notion of Cayley graphs and provide interesting examples and consequences of their properties. Section 4.2 contains two of the most important theorems in this thesis, namely Theorem 4.2.7 and Theorem 4.2.12. We illustrate that when a group  $G$  acts on a set  $X$ , with  $H$  being a finitely generated subgroup of  $G$ , the amenability of  $(G, X)$  will be characterized in terms of the amenability of  $\text{Cay}(H, X)$ . In addition, we see how this theorem can be strengthened by showing that  $(G, X)$  and  $\text{Cay}(G, X)$  have the same Følner's sequences when  $G$  is finitely generated. We provide Example 4.2.13 as an interesting demonstration of how we may use the Cayley graph of  $\mathbb{Z}^n$  in order to obtain a Følner's sequence for  $\mathbb{Z}^n$ .

Chapter 5 contains a brief conclusion of our work, as well as recent developments of the subject at hand, and provides a few ideas that can be used for future study.

Appendix A provides the remaining technical details of the proof of Rosenblatt's theorem found in Chapter 3.

**Author's contribution.** The materials found in this thesis are taken primarily from [16], [21], [22] and [24]. Throughout the remaining sections, other relevant sources are mentioned and referred to for further reading. The author's main contribution is found in the provision and expansion of details throughout this thesis. This includes supplementing the main results with extensive detail. Among the results whose proofs have been substantially expanded, we can mention Theorem 2.1.7, Examples 2.2.6 and 2.2.7, Lemma 3.2.3, Theorems 3.3.8, and 3.4.2, Lemma 4.2.2, Theorem 4.2.12 and Example 4.2.13.

## 1.2. Basic Notions of Graph Theory

In this section we discuss some of the underlying concepts of graph theory used in this thesis. Among the topics discussed we can mention finite and infinite graphs, connectivity, paths and trees.

Let  $V$  be a non-empty set. We denote by  $V^{(2)}$  the set of all unordered pairs (that is, two element subsets) of  $V$ , i.e.:

$$V^{(2)} = \{\{v_1, v_2\} : v_1, v_2 \in V, v_1 \neq v_2\}.$$

For example, if  $V = \{1, 2, 3, 4\}$ , then

$$V^{(2)} = \{\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}\}.$$

DEFINITION 1.2.1. A *graph*  $G$  is a pair of sets,  $(V, E)$ , such that  $V \neq \emptyset$  and  $E$  is a subset of  $V^{(2)}$ .

We call  $V(G)$  the set of *vertices*, and  $E(G)$  the set of *edges* of the graph  $G$ . For simplicity we can identify  $V(G)$  as  $V$ , and  $E(G)$  as  $E$ .

REMARK 1.2.2. An edge  $\{v_1, v_2\}$  is usually denoted by  $v_1v_2$ . To help us envision what this means, geometrically we can imagine the vertices of a graph as points in the plane and its edges as lines (they need not necessarily be straight) joining these vertices. It is also important to note that for our purposes we will be dealing with undirected edges. This means that our edge, say  $v_1v_2$ , can also be represented as  $v_2v_1$  to show that there is no direction specified between the two vertices.

DEFINITION 1.2.3. A graph is called *finite* if  $V$  is finite, and hence  $E$  is also finite. If  $V$  is infinite, the graph is called *infinite*.

DEFINITION 1.2.4. An *empty graph* is one with no edges (that is,  $E = \emptyset$ ). The empty graph with only one vertex is called the *trivial graph*.



REMARK 1.2.5. In this thesis we do not assume graphs to be finite unless stated so. However, we always assume graphs are locally finite in the sense that each vertex is connected to at most a finite number of other vertices.

DEFINITION 1.2.6. If  $(V, E)$  is a graph, then a *subgraph* of graph  $(V, E)$ , is another graph formed from a subset of the vertices and edges of  $(V, E)$ . The vertex subset must include all endpoints of the edge subset, but may also include additional vertices.

Given a set  $X$ ,  $|X|$  is used to denote the number of its elements, sometimes referred to as the cardinality of set  $X$ .

DEFINITION 1.2.7. The *order* of a graph  $(V, E)$  is  $|V|$ , and its *size* is  $|E|$ .

For the remainder of this section, we assume  $(V, E)$  is a given graph.

DEFINITION 1.2.8. Two vertices  $v$  and  $w$  are called *neighbours* or *adjacent*, if they are connected by an edge.

The following definition will be of particular importance for our study of expansion of graphs.

DEFINITION 1.2.9. If we take  $F \subset V$  to be a set of vertices, then a vertex is called a *neighbor* of  $F$  if  $v$  is not in  $F$  but  $v$  is *connected* to a vertex in  $F$  by some edge. The set of all neighbours of  $F$  is called the *border* of  $F$ , and is denoted by  $b(F)$ .

If  $F = \{v\}$  consists of a single vertex, then we talk about the neighbours of  $v$ , and denote the border of  $\{v\}$  by  $b(v)$ .

REMARK 1.2.10. There is, in general, no relation between  $|b(F)|$  and  $|b(F^c)|$ . For example, if  $F$  is a single vertex of a triangle, then  $|b(F)| = 2$  and  $|b(F^c)| = 1$ .

DEFINITION 1.2.11. Let  $F$  be a set of vertices. The *boundary* of  $F$ , denoted  $\partial F$ , is the set of all edges connecting  $F$  to  $V - F$ .

Note that  $\partial F = \partial(V - F)$ . Since a neighbouring point of  $F$  can be connected with more than one edge to  $F$ , it follows immediately from our definitions that

$$|b(F)| \leq |\partial F|,$$

with equality holding if and only if every neighbouring point of  $F$  is connected to  $F$  with exactly one edge.

For a vertex  $v$ ,  $|b(v)|$  is called the *degree* or *valency* of  $v$ , and is denoted by  $\deg(v)$ . Thus the degree of a vertex is the number of its neighbours which is the same as the number of edges incident on  $v$ , hence

$$\deg(v) = |b(v)| = |\partial(v)|.$$

DEFINITION 1.2.12. If every vertex of a graph has finite degree the graph is called *locally finite*.

DEFINITION 1.2.13. A *path* is a finite graph with vertices  $\{v_1, v_2, \dots, v_n\}$ ,  $n \geq 2$ , and with edges  $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n\}$ . The number  $n$  is called the *length* of the path.

DEFINITION 1.2.14. A *cycle* is a finite graph with vertices  $\{v_1, v_2, \dots, v_n\}$ ,  $n \geq 3$ , whose edges are  $\{v_1v_2, v_2v_3, \dots, v_{n-1}v_n, v_nv_1\}$ . The number  $n$  is called the *length* of the cycle.

With our understanding of the basic concepts of graph theory thus far, we are now able to state the following theorem.

THEOREM 1.2.15. *Let  $X = (V, E)$  be a graph (finite or infinite). Then the following conditions are equivalent:*

- (i) *For every partition of  $V$  into two sets,  $F$  and  $G$ , there is an edge with one end in  $F$  and one end in  $G$ .*
- (ii) *Every two vertices in  $V$  can be connected by a path.*

PROOF. First suppose (i) holds. Let  $v$  and  $w$  be two vertices, we need to show that they can be connected by a path. Let  $F$  be the set consisting of  $v$  and all those

vertices that can be connected to  $v$  be a path. If  $w \in F$  then we have nothing to show. Otherwise,  $V - F$  is nonempty, and hence  $F$  and  $V - F$  form a partition of  $V$ . By (i), there is an edge with one end, say  $u$ , in  $F$  and one end, say  $u'$  in  $V - F$ . Since  $v$  can be connected to  $u$  by a path, it follows that  $v$  can also be connected to  $u'$  by a path, contradicting the definition of  $F$ .

Next, suppose that (ii) holds. Let  $F$  and  $G$  form a partition of  $V$  and  $v \in F$  and  $w \in G$ . By (ii) there is a path  $v_1 = v, v_2, \dots, v_n = w$  joining  $v$  to  $w$ . Let  $1 \leq j < n$  be the largest index for which  $v_j \in F$ . Thus  $v_{j+1} \in G$  and  $v_j v_{j+1}$  is an edge with one end in  $F$  and one end in  $G$  as required.  $\square$

DEFINITION 1.2.16. A graph satisfying the equivalent conditions in the above theorem is called *connected*.

DEFINITION 1.2.17. A *component* is a connected subgraph in which no vertex is connected to a vertex outside the subgraph.

REMARK 1.2.18. Every graph can be partitioned into its components, or *connected components*. This follows from the fact that the relation  $v$  is connected to  $w$  by a path is an equivalence relation, and hence partitions the graph into its connected components.

DEFINITION 1.2.19. For  $v, w \in V$ , we define the *distance*  $d(v, w)$  to be the minimum length of the paths (if they exist) connecting  $v$  to  $w$ . If  $(V, E)$  is connected, then  $d$  is a distance (or metric) on  $V$ , provided we put  $d(v, v) = 0$  for all  $v \in V$ .

DEFINITION 1.2.20. Let  $k \geq 2$  be any integer, then a graph is called *k-regular* if each vertex has degree  $k$ , (that is, has exactly  $k$  neighbours).

DEFINITION 1.2.21. A *tree* is a connected graph (finite or infinite) which has no cycles.

DEFINITION 1.2.22. A *complete graph* is one in which every two vertices are neighbours, in other words, every vertex is connected to any other vertex.

Thus, if  $|V| = m$ , then such a graph is  $(m - 1)$ -regular, and a graph of order  $|V| = m$  is complete if and only if it has the size  $|E| = \binom{m}{2}$ , which is the maximal possible size of order  $m$  graphs.

DEFINITION 1.2.23. Two graphs  $(V, E)$  and  $(V', E')$  are called *isomorphic* if there exists a bijection  $f : V \rightarrow V'$  such that two vertices  $v, w \in V$  are adjacent if and only if  $f(v), f(w) \in V'$  are adjacent.

We call  $f$  an *isomorphism* between  $\mathcal{G}$  and  $\mathcal{G}'$ , and we write  $\mathcal{G} \cong \mathcal{G}'$ , where the isomorphism  $f$  is sometimes called an *edge preserving isomorphism*.

REMARK 1.2.24. Graph isomorphism is an equivalence relation between graphs. Isomorphic graphs share certain structural properties: for example, the number of components in a graph is invariant under isomorphism.

### 1.3. Free Groups

In this section we briefly recall the notion of free groups and discuss some basic examples. For more details and proofs we refer to Gallian [11].

DEFINITION 1.3.1. Let  $G$  be a set together with a binary operation that assigns to each ordered pair of elements  $(a, b)$  of  $G$ , an element in  $G$  denoted by  $ab$ . We say  $G$  is a *group* under this operation if the following properties are satisfied:

1. *Associativity*:  $(ab)c = a(bc)$  for all  $a, b, c$  in  $G$ .
2. *Identity*: There is an element  $e$  (called the *identity*) in  $G$  such that  $ae = ea = a$  for all  $a$  in  $G$ .
3. *Inverses*: For each element  $a$  in  $G$ , there is an element  $a^{-1}$  in  $G$  (called the *inverse* of  $a$ ) such that  $aa^{-1} = a^{-1}a = e$ .

A group is said to be *abelian* if for any two elements  $a, b \in G$   $ab = ba$ .

A group is said to be *cyclic* if there is an element  $a \in G$  such that  $G = \{a^n | n \in \mathbb{Z}\}$ .

Now that we have defined what it means to be a group, we may introduce a few definitions that will lead to the idea of a free group, as defined by Hewitt and Ross [15].

DEFINITION 1.3.2. Let  $X$  be any nonempty set. For each  $x \in X$  we associate a symbol  $x^{-1}$  outside the set  $X$ , such that if  $x \neq y$ , then  $x^{-1} \neq y^{-1}$ . Let

$$X^{-1} = \{x^{-1} : x \in X\},$$

and let  $e$  be a symbol such that  $e \notin X \cup X^{-1}$ . By convention, we define  $x^0 = e$  for every  $x \in X$ . A *word* in  $X$  is either a finite formal product of elements of  $X$ :

$$x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n},$$

where  $r_k = \pm 1$ , or is the empty word  $e$ .

It is important to note that  $x_k$  need not be distinct in the definition of a word.

DEFINITION 1.3.3. A word is *reduced* if it is either the empty word or if  $r_k = r_{k+1}$  whenever  $x_k = x_{k+1}$ .

DEFINITION 1.3.4. The *length* of a reduced word  $x = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$  is  $n$ . The length of the empty word,  $e$ , is 0.

To better facilitate our understanding of reduced words and their lengths, lets take a look at an example.

EXAMPLE 1.3.5. Given the following words:

$$a = x_4 x_2 x_2^{-1} x_1 x_1 x_5, \quad b = x_2^{-1} x_2 x_1^{-1} x_1 x_3, \quad c = x_{10} x_4 x_4 x_4 x_3,$$

it is clear that  $c$  is the only reduced word. If we were to do the necessary cancellations when entries within a word are next to their inverses, then we would have  $a = x_4 x_1 x_1 x_5$  and  $b = x_3$ . Thus, the newly reduced words  $a, b$  and  $c$  have respective lengths of 4, 1 and 5.

Consider  $F$  to be the set of all reduced words in  $X$ . Two words

$$x = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} \quad \text{and} \quad y = y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m},$$

belonging to  $F$ , are defined as equal if

$$n = m \quad x_i = y_i \quad r_i = s_i, \quad \text{for all } 1 \leq i \leq n.$$

From here it is clear that no reduced word of length greater than or equal to 1 is equal to  $e$ .

To define the notion of a free group on  $X$ , we will define a group structure on  $F$ . If  $x = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$  and  $y = y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m}$  both belong to  $F$ , then their product  $xy$  is defined in the following way. Consider

$$z = x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n} y_1^{s_1} y_2^{s_2} \cdots y_m^{s_m}.$$

If  $z$  is reduced, we will now define it as  $xy$ . However, if it is not reduced, then clearly  $x_n = y_1$  and  $r_n = -s_1$ , leaving us with

$$z = x_1^{r_1} x_2^{r_2} \cdots x_{n-1}^{r_{n-1}} y_2^{s_2} \cdots y_m^{s_m}.$$

If this  $z$  is now reduced, it is defined as  $xy$ , if not, we will continue in this way until we finally reach a reduced word and thus define it as  $xy$ . Using this process of multiplication,  $F$  is now called the *free group generated by  $X$* .

REMARK 1.3.6. The adjective *free* for the name of the group  $F$  can be justified by the fact that we assume no relation holds between the elements of  $X$  and  $X^{-1}$  other than the obvious relation  $xx^{-1} = x^{-1}x = e$ .

Given a word like  $x_1^{r_1} x_2^{r_2} \cdots x_n^{r_n}$ , its inverse is defined as

$$x_n^{-r_n} x_{n-1}^{-r_{n-1}} \cdots x_1^{-r_1},$$

whereas the identity element of  $F$  is  $e$ . Using an inductive argument, it can be shown that for all  $x, y, z \in F$  the product law on  $F$  is associative:  $(xy)z = x(yz)$ .

Note that by its definition, a free group is nonabelian if  $X$  has more than one element.

EXAMPLE 1.3.7. Consider the following words  $x = x_1^{-1}x_2$  and  $y = x_2^{-1}x_1^{-1}$ . Then

$$xy = x_1^{-1}x_2x_2^{-1}x_1^{-1} = x_1^{-1}x_1^{-1},$$

where as

$$yx = x_2^{-1}x_1^{-1}x_1^{-1}x_2.$$

Thus  $xy \neq yx$ .

For convenience we can introduce the power notation where we can group repetitions of entries within a word together. For example:

$$xxy^{-1}y^{-1}y^{-1} = x^2y^{-3} \quad \text{and} \quad yyx^{-1}x^{-1} = y^2x^{-2}.$$

EXAMPLE 1.3.8. The infinite cyclic group  $\{x^n : n \in \mathbb{Z}\}$  is the free group on a set of a single element  $X = \{x\}$ . This group is isomorphic with the group of integers  $\mathbb{Z}$  under the usual addition operation, because all integers can be written by repeatedly adding or subtracting the number 1.

EXAMPLE 1.3.9. Let  $X = \{x, y\}$  be a set with two elements. We denote the free group on  $X$  by  $F_2$ . This is called the free group on two generators. This group is nonabelian and contains elements of the form

$$e, \quad x, \quad y, \quad x^2y, \quad x^2yx^{-1}, \quad y^3xy^{-1}x, \quad \dots$$

THEOREM 1.3.10. (*Universal Property of Free Groups*) Let  $F$  be the free group on a set  $X$  and let  $\iota : X \rightarrow F$ ,  $\iota(x) = x$  be the canonical injection of  $X$  into  $F$ . If  $G$  is a group and  $f : X \rightarrow G$  is any map, then there exists a unique homomorphism  $\phi : F \rightarrow G$  such that

$$\phi \circ \iota = f.$$

This theorem can be viewed as creating an extension of the map  $f : X \rightarrow G$  by way of a group homomorphism  $\phi : F \rightarrow G$ . If we denote  $f(x)$  by  $\tilde{x}$ , then  $\phi$  sends a word  $x_1^{r_1}x_2^{r_2} \cdots x_n^{r_n}$  in  $X$  to the corresponding product  $\tilde{x}_1^{r_1}\tilde{x}_2^{r_2} \cdots \tilde{x}_n^{r_n}$  in  $G$ .

DEFINITION 1.3.11. If  $F$  is a free group that is generated by a set  $X$ , then the *rank* of the free group  $F$  is the cardinality of  $X$ . For a finite set  $X$  with  $k$  elements, we say that  $F$  is a finitely generated free group and we will denote  $F$  as  $F_k$ .

Our next result is a consequence of the universality property from Theorem 1.3.10, showing that for every cardinal number  $k$  there is, up to isomorphism, exactly one free group of rank  $k$ .

**THEOREM 1.3.12.** *Let  $F$  be the free group generated by the set  $X$ , and  $F'$  be the free group generated by the set  $X'$ . Then  $F$  and  $F'$  are isomorphic if and only if  $X$  and  $X'$  have the same cardinality.*

**PROOF.** Suppose that  $|X| = |X'|$ . Then since  $X$  and  $X'$  have the same cardinalities, there exists a one-to-one correspondence

$$f_1 : X \rightarrow X' \quad \text{with} \quad f_1^{-1} : X' \rightarrow X.$$

Since  $F$  and  $F'$  are free groups, there exist unique homomorphic extensions  $\phi_1, \phi_2$  of  $f_1$  and  $f_1^{-1}$  respectively. Now, since  $\phi_2$  extends the inverse of the function which  $\phi_1$  extends, then the restriction of

$$\phi_2 \circ \phi_1 : F' \rightarrow F,$$

to  $i(X) \subset F$  is the identity function:

$$\begin{aligned} \phi_2 \circ \phi_1(i(x)) &= \phi_2(i'(f_1(x))) \\ &= i \circ f_1^{-1}(f_1(x)) \\ &= i(x). \end{aligned}$$

Thus,  $\phi_2 \circ \phi_1|_{i(X)} = I$ . Since extensions from the basis in free groups are unique, we can conclude that  $\phi_2 \circ \phi_1 = id_F$ . Similarly,  $\phi_1 \circ \phi_2 = id_{F'}$ . Thus  $\phi_1$  must be an isomorphism and  $F \cong F'$ .

Now suppose that  $F \cong F'$ . Let  $Hom(F, \mathbb{Z}_2)$  be the set of homomorphisms between  $F$  and  $\mathbb{Z}_2$ . Similarly, let  $Hom(F', \mathbb{Z}_2)$  be the set of homomorphisms between  $F'$  and  $\mathbb{Z}_2$ . Since  $F \cong F'$ , the similarity of structures ensures that

$$|Hom(F, \mathbb{Z}_2)| = |Hom(F', \mathbb{Z}_2)|.$$



There are exactly  $2^{|X|}$  functions between  $F$  and  $\mathbb{Z}_2$ , and thus we can conclude that

$$2^{|X|} = |\text{Hom}(F, \mathbb{Z}_2)| = |\text{Hom}(F', \mathbb{Z}_2)| = 2^{|X'|}.$$

Thus  $|X| = |X'|$ . □

Our final theorem in this section states that subgroups of free groups are themselves free groups.

**THEOREM 1.3.13.** (*Nielsen-Schreier*) *If  $G$  is a subgroup of a free group  $F$ , then  $G$  is isomorphic to a free group. That is, there exists a subset  $S$  of elements of  $G$  such that every element in  $G$  is a product of members of  $S$  and their inverses, and such that  $S$  satisfies no nontrivial relations.*

We refer to [1] for a proof of the above theorem.

### 1.4. Finitely Generated Groups

In this section we introduce the concept of finitely generated groups which will play a key role in Chapter 4. In particular we will examine the relation between groups with generators and free groups.

**DEFINITION 1.4.1.** Let  $G$  be a group and let  $X$  be a subset of  $G$ . Let  $\{H_i : i \in I\}$  be the family of all subgroups of  $G$  which contain  $X$ . Then

$$\bigcap_{i \in I} H_i,$$

is called the *subgroup of  $G$  generated by  $X$*  and is denoted as  $\langle X \rangle$ .

It is important to note that the same group can be generated by different sets of generators. For example,  $X$ ,  $X \cup X^{-1}$  and  $X \cup \{e\}$  will all generate the same subgroup of  $G$ .

**DEFINITION 1.4.2.** If  $X = \{a_1, \dots, a_n\}$  we will write  $\langle a_1, \dots, a_n \rangle$  in place of  $\langle X \rangle$ .  
If

$$G = \langle a_1, \dots, a_n \rangle,$$

where  $a_i \in G$ , then  $G$  is said to be *finitely generated*. If  $a \in G$  the subgroup  $\langle a \rangle$  is called the *cyclic subgroup* generated by  $a$ .

**THEOREM 1.4.3.** *Let  $G$  be a group and  $X \subset G$ . Then  $\langle X \rangle$  consists of finite products*

$$a_1^{n_1} a_2^{n_2} \cdots a_t^{n_t},$$

where  $a_i \in X$  and  $n_i \in \mathbb{Z}$ . In particular for every  $a \in G$

$$\langle a \rangle = \{a^n : n \in \mathbb{Z}\}.$$

The above theorem gives an explicit description of elements of  $\langle X \rangle$ , and the proof can be found in [17].

The following theorem summarizes some of the important facts about finitely generated groups.

**THEOREM 1.4.4.**

- (i) *Every homomorphic image of a finitely generated group is finitely generated.*
- (ii) *Every set of generators of a finitely generated group contains a finite subsystem which is an irreducible system of generators of the group.*
- (iii) *There is a finitely generated group with a subgroup which is not finitely generated.*
- (iv) *Let  $\{G_\alpha\}$  be a family of groups and let  $G = \prod_\alpha G_\alpha$  be the product of  $G_\alpha$ . Then  $G$  is finitely generated if and only if each  $G_\alpha$  is finitely generated and  $G_\alpha = \{e\}$  for all but a finite number  $\alpha$ .*

Next, we examine the relation between groups with generators and free groups. To begin, we will note that the subgroup  $\langle X \rangle$  of  $G$  can be identified with a quotient of the free group  $F$  on  $X$ . If  $f : X \rightarrow G$  is the identity map from  $X$  into  $G$ , then by Theorem 1.3.10 (Universal Property of Free Groups),  $f$  extends uniquely to a group homomorphism  $\phi : F \rightarrow G$  with

$$\phi(F) = \langle X \rangle.$$

If  $N = \ker \phi$ , we obtain

$$\langle X \rangle \cong F/N,$$

as a result of the first isomorphism theorem. We call the elements of  $N$  *relations* among the generators. To make things simpler when dealing with  $N$ , since it may be large, we will say that a set of words  $R = \{w_1, w_2, \dots\}$  is a set of *defining relations* for  $\langle X \rangle$  if  $R \subset N$  and  $N$  is the smallest normal subgroup of  $F$  containing  $R$ .

EXAMPLE 1.4.5. The dihedral group  $D_n$  of order  $2n$ , is generated by two elements, say  $x$  and  $y$ , with the relations:

$$x^n = e, \quad y^2 = e, \quad xyxy = e.$$

The elements  $x^n, y^2, xyxy$  form a set of defining relations for  $D_n$ .

The notation  $\langle x_1, \dots, x_m; r_2, \dots, r_k \rangle$  is used to represent the group that is generated by the elements  $x_1, \dots, x_m$  with the defining relations:  $r_2, \dots, r_k$ . Now, we may rewrite  $D_n$  as:

$$D_n = \langle x, y; x^n, y^2, xyxy \rangle.$$

EXAMPLE 1.4.6. Consider the finitely generated group

$$\langle x, y; xyx^{-1}y^{-1} \rangle.$$

This group is abelian and called the *free abelian group* of two generators  $\{x, y\}$ .

## CHAPTER 2

### Expansion Constants

In this chapter we discuss the expansion constant  $\gamma(X)$  of a graph  $X$ , and show how it can be used to define the amenability of a graph, which will be a major topic moving forward in this thesis. In Theorem 2.1.6 we show that the expansion of a graph is determined by the expansion of its connected components. Theorem 2.1.7, due to McMullen, shows that  $d$ -regular infinite trees have expansion  $\gamma(X) = d - 2$ . In Section 2.2 we discuss alternative expansion constants which are more suitable for finite graphs. An example of such expansion constants is the Cheeger constant  $h(X)$ . These expansion constants are computed for some specific examples.

#### 2.1. Graph Expansions and Amenable Graphs

Let  $X = (V, E)$  be a graph. Recall that for any set of vertices  $F$  of  $X$ , the *border of  $F$* ,  $b(F)$ , is the set of all vertices which are not in  $F$ , but are connected to some vertex in  $F$  by an edge (we say that the vertices in  $b(F)$  have distance 1 from  $F$ ).

DEFINITION 2.1.1. We define the *expansion*  $\gamma(X)$  of  $X$  by

$$\gamma(X) = \inf \left\{ \frac{|b(F)|}{|F|} : F \subset V, 0 < |F| < \infty \right\}.$$

REMARK 2.1.2.

- (a) Note that, according to the above definition, the expansion of a finite graph is zero, since for  $F = V$ , we have  $b(F) = \emptyset$ , and hence

$$\frac{|b(F)|}{|F|} = 0,$$

implying that  $\gamma(X) = 0$ .

- (b) Isomorphic graphs have the same expansion, this follows from the fact that an isomorphism preserves borders of sets.

The following definition is given by McMullen [21].

DEFINITION 2.1.3. A graph  $X = (V, E)$  is *amenable* if  $\gamma(X) = 0$ .

Intuitively, an amenable graph is one which contains “arbitrarily large sets with small borders.” We have seen that all finite graphs have zero expansion, and hence are amenable. On the other hand, if  $X$  is *nonamenable*, then  $\gamma(X) > 0$  and hence

$$0 < \gamma(X) \leq \frac{|b(F)|}{|F|},$$

for every finite subset  $F$  of  $V$ . Thus

$$|b(F)| \geq \gamma(X)|F|,$$

meaning that in a nonamenable graph, the size of the border of any finite set,  $F$ , of vertices is greater than or equal to a definite fraction,  $\gamma(X)$ , of the size of the set itself.

REMARK 2.1.4. For infinite graphs, there is in general no relation between amenability and connectivity of the graph: connected graphs can be amenable or nonamenable (Theorem 2.1.7), and disconnected graphs can also be amenable or nonamenable (Theorem 2.1.6).

To compute  $\gamma(X)$  for an infinite graph, we need only consider its components as the next theorem will show, but first we need an important lemma.

LEMMA 2.1.5. *If  $x_1, \dots, x_n$  and  $x'_1, \dots, x'_n$  are positive numbers such that none of  $x_i$  are zero, then*

$$\frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\}.$$

PROOF. This follows by a simple induction. The case of  $n = 2$  is easy to check. For the general case we write

$$\frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_{n-1}}, \frac{x'_n}{x_n} \right\} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\}.$$

□

THEOREM 2.1.6. *If  $X$  is an infinite graph and  $\{X_i\}_{i \in I}$  are its components, then*

$$\gamma(X) = \inf_{i \in I} \gamma(X_i).$$

PROOF. Let  $i \in I$  and  $V_i$  be the set of vertices of  $X_i$ . Since  $X_i$  is a component of  $X$ , if  $F$  is a finite proper subset of  $V_i$ , the border points of  $F$  in  $X$  (if any) are all inside  $V_i$ , and from this it follows that  $\gamma(X) \leq \gamma(X_i)$ . Thus we have

$$\gamma(X) \leq \inf_{i \in I} \gamma(X_i).$$

To prove the reverse inequality, let  $F$  be a finite, nonempty subset of vertices of  $X$ . Then there are a finite set of indices  $i_1, \dots, i_n$  such that

$$F \subset V_{i_1} \cup \dots \cup V_{i_n}.$$

Let  $x_j$  be the number of elements of  $F \cap V_{i_j}$ , and  $x'_j$  be the number of its bordering points in  $V_{i_j}$ . If  $b(F)$  is the border of  $F$  in  $X$ , then

$$|b(F)| = x'_1 + \dots + x'_n,$$

and we have

$$\frac{|b(F)|}{|F|} = \frac{x'_1 + \dots + x'_n}{x_1 + \dots + x_n} \geq \min \left\{ \frac{x'_1}{x_1}, \dots, \frac{x'_n}{x_n} \right\} \geq \inf_{i \in I} \gamma(X_i).$$

The inequality  $\gamma(X) \geq \gamma(X_i)$  follows, and hence  $\gamma(X) \geq \inf \gamma(X_i)$ . □

THEOREM 2.1.7 (McMullen). *Let  $X$  be an infinite tree in which every vertex has a fixed degree  $d \geq 2$ . If  $d = 2$  then  $X$  is amenable, otherwise it is nonamenable with expansion  $\gamma(X) = d - 2$ .*

PROOF. First suppose  $d = 2$ . Let  $v_1$  be an arbitrary vertex. We know that  $v_1$  is connected to two other vertices  $v_0$  and  $v_2$ . Since  $v_2$  is already connected to  $v_1$ , it follows that  $v_2$  must be connected to only one other vertex  $v_3$ . Since  $X$  has no cycle,  $v_3 \neq v_0$ . We can now repeat the argument and conclude that  $v_3$  is connected to only one other vertex  $v_4 \notin \{v_0, v_1, v_2, v_3\}$ . Continuing in this way, we can find a vertex  $v_n$  which is connected to  $v_{n-1}$  and  $v_{n+1}$  where  $v_{n+1} \notin \{v_0, v_1, v_2, \dots, v_n\}$ . Let

$$F_n = \{v_1, \dots, v_n\}.$$

Then  $b(F_n) = \{v_0, v_{n+1}\}$ , and hence:

$$\frac{|b(F_n)|}{|F_n|} = \frac{2}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus  $\gamma(X) = 0$ , and  $X$  is amenable.

Next suppose that  $d \geq 3$ . We label the vertices of  $X$  by the following process. Let  $v_1$  be an arbitrary vertex. We know  $v_1$  is neighbor to  $d$  other vertices  $v_{11}, v_{12}, \dots, v_{1d}$ . Pick one vertex from this group, say  $v_{1i}$  where  $1 \leq i \leq d$ . Then  $v_{1i}$  is neighbor to  $d - 1$  other vertices (since  $|b(v_{1i})| = d$  and  $v_{1i}$  is already connected to  $v_1$  by an edge). Let us denote these vertices

$$v_{1i1}, v_{1i2}, \dots, v_{1id-1}.$$

Note that since the graph is acyclic, we have

$$\{v_{1i1}, v_{1i2}, \dots, v_{1id-1}\} \cap \{v_{11}, v_{12}, \dots, v_{1d}\} = \emptyset.$$

This argument can be repeated for any  $v_{1ij}, 1 \leq i \leq d, 1 \leq j \leq d - 1$  and one can then argue that  $v_{1ij}$  is neighbor to  $d - 1$  vertices

$$v_{1ij1}, v_{1ij2}, \dots, v_{1ijd-1},$$

such that

$$\left( \{v_1, v_{11}, v_{12}, \dots, v_{1d}\} \bigcup_{i=1}^d \{v_{1i1}, \dots, v_{1id-1}\} \right) \cap \{v_{1ij1}, v_{1ij2}, \dots, v_{1ijd-1}\} = \emptyset.$$

Note that since a tree is connected and acyclic, all vertices of the tree will be labeled uniquely in this way. Now if  $v \in V$  there exists a path, say  $v_1 v_2 \cdots v_m$  such that  $v_m = v$ , which connects  $v_1$  to  $v$ . Since  $v_2$  is neighbor to  $v_1$  we must have  $v_2 \in \{v_{11}, \dots, v_{1d}\}$ , say  $v_2 = v_{11}$ . Since  $v_3$  is neighbor to  $v_2 = v_{11}$ , we must have  $v_3 \in \{v_{111}, \dots, v_{11d-1}\}$ , say  $v_3 = v_{111}$ . If we continue this argument we see that  $v_n$  can be labelled without loss of generality, as  $v_{11\dots 1}$ , where the number of indices is  $n$ .

In a tree whose vertices are labelled as above, if a vertex is labelled by  $m$  indices, let us say that ‘it is on the level  $m$  of the tree.’ No two elements on the same level of a tree can be neighbours since a tree is acyclic. Thus if  $v$  is located on level  $m \geq 2$ , from  $d$  neighbouring points of  $v$ , one is located at the lower level  $m - 1$ , and the other  $d - 1$  are located at the higher level  $m + 1$ .

Now we can show that  $\gamma(X) \geq d - 2$ , using an induction on  $|F|$  where  $F$  is a finite subset of the set of vertices  $V$ . Suppose  $F = \{v\}$  is a singleton. Then

$$\frac{|b(F)|}{|F|} = \frac{d}{1} \geq d - 2.$$

Suppose that by an induction hypothesis, for any finite set  $F$  of vertices with  $|F| \leq n$  we have shown that

$$\frac{|b(F)|}{|F|} \geq d - 2.$$

Let

$$F' = \{v_1, \dots, v_{n+1}\},$$

be a set consisting of  $n + 1$  vertices. Each  $v_i$  must appear on a certain level of the tree. Suppose  $v_{n+1}$  appears in the highest level  $m$ . This means that from the  $d$  bordering points of  $v_{n+1}$ , at most one point (that is, the one at level  $m - 1$ ), can be among

$$\{v_1, \dots, v_n\} \cup b(\{v_1, \dots, v_n\}).$$

Set  $F = \{v_1, \dots, v_n\}$ . Since

$$|b(F)| \cap |b(v_{n+1})| \leq 1,$$



and  $|b(v_{n+1})| = d$ , we have

$$|b(F')| \leq (|b(F)| - 1) + d - 1 = |b(F)| + d - 2.$$

Thus, using the induction hypothesis we can write

$$\frac{|b(F')|}{|F'|} \geq \frac{|b(F)| + d - 2}{|F| + 1} \geq \frac{|F|(d - 2) + d - 2}{|F| + 1} \geq d - 2.$$

At this point we have shown that

$$\gamma(X) \geq d - 2.$$

To show that  $\gamma(X) = d - 2$ , it suffices to find a sequence of finite sets  $F_n$  for which

$$\frac{|b(F_n)|}{|F_n|} \rightarrow d - 2 \quad \text{as } n \rightarrow \infty.$$

Consider the set with  $n$  elements given by

$$F_n = \{v_1, v_{11}, v_{111}, \dots, v_{111\dots 1}\}.$$

Then

$$|b(F_n)| = 2(d - 1) + (n - 2)(d - 2) = n(d - 2) + 2.$$

Hence

$$\frac{|b(F_n)|}{|F_n|} = \frac{n(d - 2) + 2}{n} \rightarrow d - 2 \quad \text{as } n \rightarrow \infty.$$

as we wanted to show.  $\square$

Using Theorem 2.1.6 we can state McMullen's result for the more general case of infinite acyclic graphs (i.e, removing the condition of connectivity).

**COROLLARY 2.1.8.** *Suppose  $X$  is an infinite acyclic graph in which every vertex has degree  $d \geq 2$ . If  $d = 2$  then  $X$  is amenable, otherwise it is nonamenable with  $\gamma(X) = d - 2$ .*

**PROOF.** Each component of  $X$  is a tree in which every vertex has degree  $d$ . The result follows immediately from Theorems 2.1.6 and 2.1.7.  $\square$

## 2.2. Alternative Graph Expansions

In this section we discuss other expansion constants such as the Cheeger constant,  $h(X)$ . In addition, we examine the isoperimetric constants  $h'(X)$  and  $h_0(X)$ , with interesting inequalities that show the relation between them all. We also show that  $h(X) = \gamma(X)$  if  $X$  is a regular tree, illustrating how the Cheeger constant can also be used to define the expansion of an infinite tree.

As previous literature shows [4] there are various ways to define expansion constants for graphs, and there is no uniform terminology or notation in this regard. Most of these definitions are “essentially” equivalent at least for regular graphs, in the sense that if  $\gamma_1$  and  $\gamma_2$  are two such constants, then for some suitable numbers  $M_1 > 0$  and  $M_2 > 0$ , we have

$$\gamma_1(X) \leq M_2 \gamma_2(X) \quad \text{and} \quad \gamma_2(X) \leq M_1 \gamma_1(X).$$

The basic idea underlying the definitions of most of these constants is that if a graph has an expansion  $\gamma$  then every finite subset of the graph is guaranteed to ‘expand’ (in some sense or the other) at least by the amount  $\gamma$ . Although most definitions involve finite graphs only, we formulate two such definitions in a setting that can be applied to infinite locally finite graphs as well. We have already encountered one such expansion constant in Section 2.1 which we have denoted by  $\gamma(X)$  (see Definition 2.1.1).

**DEFINITION 2.2.1.** Let  $X = (V, E)$  be a locally finite graph. The *Cheeger constant* of  $X$  is defined by

$$h(X) = \inf \left\{ \frac{|\partial F|}{\min(|F|, |V - F|)} : \emptyset \neq F \subsetneq V, |F| < \infty \right\}.$$

The Cheeger constant has also been called the *edge expansion constant*. We remark that Cheeger defined his constant originally for Riemannian manifolds, see [5].

If we view  $X$  as a transmission network, then  $h(X)$  is a measure of the quality of transmission: the larger the value of  $h(X)$ , the better the ratio between outgoing edges to the number of vertices in a finite subset of  $V$ .

Note that when  $V$  is infinite, then

$$\min(|F|, |V - F|) = |F|,$$

for every finite subset  $F$  of  $V$ , and hence

$$h(X) = \inf \left\{ \frac{|\partial F|}{|F|} : \emptyset \neq F \subsetneq V, |F| < \infty \right\}.$$

REMARK 2.2.2.

- (a) Note that  $\partial F = \partial(V - F)$ . This is true since if  $e \in \partial F$ , then  $e$  is connecting a vertex from  $F$  to one in  $V - F$ . Then by definition,  $e \in \partial(V - F)$ , where the reverse argument also works. Thus the term  $\min(|F|, |V - F|)$  instead of  $|F|$  in the definition of  $h(X)$  tells us that in Cheeger's definition of expansion, if communication between  $F$  and  $V - F$  is effective in one direction, say from  $F$  to  $V - F$  due to the large ratio produced from

$$\frac{|\partial F|}{|F|},$$

then we have a "good" communication network, even if the flow of information from  $V - F$  to  $F$  is not as good, due to the smaller ratio

$$\frac{|\partial F|}{|V - F|}.$$

In other words, the network is treated as an undirected network.

- (b) As we alluded to earlier, there are numerous variations of the definition of an expansion constant found in the literature. For example, one may insist that a good network is one in which two-sided communications are performed effectively

over the entire network, and define

$$h_0(X) = \inf \left\{ \frac{|\partial F|}{|F|} : \emptyset \neq F \subsetneq V, |F| < \infty \right\}.$$

Then it follows from the definition that, for finite networks, in general,

$$h_0(X) \leq h(X),$$

while for infinite networks

$$h_0(X) = h(X).$$

The following definition of our next expansion constant can be found in [4].

**DEFINITION 2.2.3.** Let  $X = (V, E)$ . Then we define

$$h'(X) = \inf \left\{ \frac{|b(F)|}{\min(|F|, |V - F|)} : \emptyset \neq F \subsetneq V, |F| < \infty \right\}.$$

A comparison between  $h(X)$ ,  $h'(X)$  and  $\gamma(X)$  shows that in general

$$(2.2.1) \quad \gamma(X) \leq h'(X) \leq h(X),$$

due to the fact that

$$|\partial F| \geq |b(F)|.$$

As we will show later with examples, the inequalities however can be strict.

Note that for infinite graphs,  $h'(X)$  coincides with  $\gamma(X)$ :

$$h'(X) = \gamma(X).$$

For finite graphs however,  $h(X)$ ,  $h'(X)$  and  $\gamma(X)$  need not be equal.

**REMARK 2.2.4.** If there is an upper bound  $k$  on the number of edges going out from any vertex in  $V$ , then  $|\partial F| \leq k|b(F)|$ , and therefore,

$$h'(X) \leq h(X) \leq kh'(X).$$

It's not surprising then that the theory of expansions work well with both expansion constants  $h(X)$  and  $h'(X)$ . Thus, even though  $h'(X)$  is quantitatively different from

the Cheeger constant  $h(X)$ , qualitatively it leads to the same results in most cases of interest.

Next we compute the expansions of some finite graphs. As for finite graphs,  $\gamma(X) = 0$ , we concentrate on other definitions of expansions.

EXAMPLE 2.2.5. Let  $X = (V, E)$  with

$$V = \{1, 2, 3, 4, 5\}, \quad \text{and} \quad E = \{13, 14, 15, 23, 24, 25\}.$$

Then a direct computation shows that

$$h'(X) = 1, \quad \text{and} \quad h(X) = 1.$$

This shows that from our inequality in (2.2.1), equality holds even though we have a finite graph.  $\square$

The next example shows that even for a highly connected graph such as a complete graph,  $h'(X)$  may poorly reflect the expansion.

EXAMPLE 2.2.6. Let  $K_n$  be the complete graph of  $n$  vertices. Then

$$h_0(K_n) = h'(K_n) = 1, \quad h(K_n) = n - \left\lfloor \frac{n}{2} \right\rfloor.$$

For the first assertion, we note that if  $|F| = \ell$ , then

$$|\partial F| = \ell(n - \ell),$$

so

$$\begin{aligned} h_0(K_n) &= \min_{\emptyset \neq F \subsetneq V} \frac{|\partial F|}{|F|} \\ &= \min_{1 \leq \ell \leq n-1} \frac{\ell(n - \ell)}{\ell} \\ &= 1. \end{aligned}$$

Also, if  $|F| = \ell$  then  $|b(F)| = n - \ell$ , and thus

$$\begin{aligned} h'(K_n) &= \inf_{\emptyset \neq F \subsetneq V} \left\{ \frac{|b(F)|}{\min\{|F|, |V - F|\}} \right\} \\ &= \inf_{1 \leq \ell \leq n-1} \left\{ \frac{n - \ell}{\min\{\ell, n - \ell\}} \right\} \\ &= \inf \begin{cases} \frac{n - \ell}{\ell} & \text{if } \ell \leq \lfloor \frac{n}{2} \rfloor, \\ 1 & \text{if } \ell > \lfloor \frac{n}{2} \rfloor. \end{cases} \end{aligned}$$

Similarly, when  $|F| = \ell$ , then

$$\begin{aligned} h(K_n) &= \inf_{\emptyset \neq F \subsetneq V} \left\{ \frac{|\partial F|}{\min\{|F|, |V - F|\}} \right\} \\ &= \inf_{1 \leq \ell \leq n-1} \left\{ \frac{\ell(n - \ell)}{\min\{\ell, n - \ell\}} \right\}. \end{aligned}$$

However,

$$\min\{\ell, n - \ell\} = \begin{cases} \ell & \text{if } \ell \leq \lfloor \frac{n}{2} \rfloor, \\ n - \ell & \text{if } \ell > \lfloor \frac{n}{2} \rfloor. \end{cases}$$

Thus

$$\begin{aligned} h(K_n) &= \inf_{1 \leq \ell \leq n-1} \begin{cases} n - \ell & \text{if } \ell \leq \lfloor \frac{n}{2} \rfloor, \\ \ell & \text{if } \ell > \lfloor \frac{n}{2} \rfloor. \end{cases} \\ &= n - \lfloor \frac{n}{2} \rfloor \sim \lfloor \frac{n}{2} \rfloor. \quad \square \end{aligned}$$

The following shows an example of a minimally connected graph for which the constants  $h(X)$  and  $h'(X)$  indicate a poor expansion, as expected.

**EXAMPLE 2.2.7.** Let  $C_n$  be the cycle of  $n$  vertices, that is:

$$V = \{1, 2, \dots, n\} \quad \text{and} \quad E = \{12, 23, \dots, (n-1)n, n1\}.$$

Then we claim that

$$h_0(C_n) = \frac{2}{n-1} \quad \text{and} \quad h(C_n) = \frac{2}{\lfloor \frac{n}{2} \rfloor} \approx \frac{4}{n}.$$

If  $F$  is any proper subset of  $V$ , such that  $|F| = \ell$  then  $|\partial F| \geq 2$ , and hence

$$h_0(X) = \min_{\emptyset \neq F \subsetneq V} \frac{|\partial F|}{|F|} \geq \min_{1 \leq \ell \leq n-1} \frac{2}{\ell} = \frac{2}{n-1}.$$

If however, we choose

$$F = \{1, 2, \dots, n-1\},$$

then

$$\frac{|\partial F|}{|F|} = \frac{2}{n-1},$$

hence the assertion that

$$h_0(C_n) = \frac{2}{n-1},$$

follows.

Concerning  $h(C_n)$ , we observe that if  $F$  is any proper subset of  $V$ , then  $|\partial F| \geq 2$ , and

$$\min\{|F|, |V - F|\} \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

Therefore, we are left with

$$h(C_n) \geq \frac{2}{\left\lfloor \frac{n}{2} \right\rfloor}.$$

Now if  $F$  is a half cycle, then  $|\partial F| = 2$  and

$$\min\{|F|, |V - F|\} = \left\lfloor \frac{n}{2} \right\rfloor,$$

and thus

$$h(C_n) \leq \frac{2}{\left\lfloor \frac{n}{2} \right\rfloor}.$$

It follows that

$$h(C_n) = \frac{2}{\left\lfloor \frac{n}{2} \right\rfloor} \sim \frac{4}{n} \rightarrow 0. \quad \square$$

**THEOREM 2.2.8.** *If  $X$  is an infinite tree in which every vertex has degree  $d \geq 2$ , then  $h(X) = \gamma(X)$ . In other words,  $h(X) = 0$  if  $d = 2$  and  $h(X) = d - 2$  if  $d \geq 3$ .*

PROOF. Note that since  $V$  is infinite,  $|V - F| = \infty$  for each finite subset of  $V$ , and hence

$$h(X) = \inf_{\emptyset \neq F, |F| \leq \infty} \frac{|\partial F|}{|F|}.$$

If  $d = 2$  then the proof is the same as in Theorem 2.1.7. In fact, if  $F_n$  is as in that proof, then  $|\partial F_n| = 2$ , and hence

$$\frac{|\partial F_n|}{|F_n|} = \frac{2}{n} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus  $\gamma(X) = 0$ .

On the other hand, if  $d \geq 3$ , then by choosing  $F_n = \{v_1, v_{11}, v_{111}, \dots, v_{111\dots 1}\}$ , (as in the proof), then

$$|F_n| = n \quad \text{and} \quad |b(F_n)| = |\partial F_n| = n(d-2) + 2,$$

hence

$$\frac{|\partial F_n|}{|F_n|} = \frac{n(d-2) + 2}{n} \rightarrow d-2 \quad \text{as } n \rightarrow \infty,$$

which proves that  $h(X) \leq d-2$ . But since

$$h(X) \geq \gamma(X) \quad \text{and} \quad \gamma(X) = d-2,$$

by Theorem 2.1.7, it follows that  $h(X) = d-2$ .

□



## CHAPTER 3

### Amenable Groups and Amenable Group Actions

In this chapter we define the basic concepts of group actions, positive linear functionals and amenability. We will use them to lay the foundation towards our next chapter, which is the main goal in this thesis. We will examine how linear functionals are used to define a state over a Banach space, which leads to the notion of invariant means and amenability. We will discuss the characterizations of amenability in terms of Følner's condition which is needed in Chapter 4.

#### 3.1. Group Actions

This section is used to introduce what it means for a group  $G$  to act on a set  $X$ , and that the existence of such an action is equivalent to the existence of a group homomorphism. Some terminology related to group actions will also be defined, such as transitive, faithful and free actions, as well as orbits and fixed points.

**DEFINITION 3.1.1.** Let  $G$  be a group (not necessarily topological) and let  $X$  be a nonempty set. We say  $G$  *acts* in  $X$  (or  $X$  is a  $G$ -*space*) if there exists a map

$$G \times X \rightarrow X, \quad (s, x) \mapsto s \cdot x,$$

such that for all  $s, t \in G$  and all  $x \in X$ , we have

- (i)  $s \cdot (t \cdot x) = (st) \cdot x$ ,
- (ii)  $e \cdot x = x$ .

The properties of the group action imply that

$$s^{-1} \cdot (s \cdot x) = (s^{-1}s) \cdot x = x,$$

and hence for every  $s \in G$ , the map

$$X \rightarrow X, \quad x \mapsto s \cdot x,$$

is a bijection, and its inverse is the map  $x \mapsto s^{-1} \cdot x$ .

DEFINITION 3.1.2.  $G$  acts *transitively* in  $X$  if for every  $x, y \in X$ , there exists some  $s \in G$  such that  $s \cdot x = y$ .

REMARK 3.1.3. Let  $S(X)$  denote the *group of bijections* of  $X$ , where the group elements are bijective maps from  $X$  to  $X$ , and the group operation is the composition of maps. This group is also known as the group of *permutations* of  $X$ , or the group of *symmetries* of  $X$ .

The existence of an action of  $G$  on  $X$  is equivalent to the existence of a group homomorphism

$$\pi : G \rightarrow S(X).$$

In the special case that  $X$  is a linear space, and each  $\pi(s)$  is a *linear* bijection on  $X$ , then the group homomorphism

$$\pi : G \rightarrow GL(X),$$

is called a linear *representation* of  $G$  on  $X$ . (Here  $GL(X)$  denotes the group of linear isomorphisms on  $X$  under the composition operation).

EXAMPLE 3.1.4. Let  $G$  be a group and  $H$  be subgroup of  $G$  (not necessarily normal). We can define an equivalence relation on  $G$  by defining  $x \sim y$  if  $x^{-1}y \in H$ . We denote the set of all equivalence classes thus obtained by  $G/H$ , and the class of an element  $y \in G$  by  $yH$ . We call  $G/H$  the left coset space of  $G$  modulo  $H$ . Then  $G$  acts by left multiplication on  $G/H$ , as follows:

$$G \times G/H \rightarrow G/H, \quad (x, yH) \mapsto xyH.$$

This action is well-defined and transitive: given  $xH, yH \in G/H$ , we have

$$z(xH) = yH \quad \text{for } z = yx^{-1} \in G.$$

As we shall see later,  $G$ -spaces of the form  $G/H$  are typical among transitive  $G$ -spaces.

Next we discuss some additional terminology related to the group actions.

DEFINITION 3.1.5. For every  $x \in X$ , the set  $Gx = \{s \cdot x : s \in G\}$  is called the *orbit* of  $x$ .

We can interpret this definition of an orbit of element  $x$  as “everything that can be reached from  $x$  by some action of an element in  $G$ .”

DEFINITION 3.1.6. For every element  $x \in X$ , the *stabilizer subgroup* of  $x$  is the set of all elements in  $G$  that leave  $x$  fixed:

$$\text{Stab}_G(x) = \{s \in G : s \cdot x = x\}.$$

Some authors call the stabilizer subgroup as the *isotropy group* of  $x$ . We may also view this definition as “the set of all elements of  $G$  which do not move  $x$  when they act on  $x$ .”

DEFINITION 3.1.7. We say  $x \in X$  is a *fixed point* of the action of  $G$  on  $X$  if  $s \cdot x = x$  for all  $s \in G$ .

DEFINITION 3.1.8. We say that  $G$  acts *faithfully* in  $X$  if  $s \in G$  and  $s \cdot x = x$  for all  $x \in X$ , then  $s = e$ .

DEFINITION 3.1.9. We say that  $G$  acts *freely* in  $X$  if the relation  $sx = tx$  implies  $s = t$ . (This is equivalent to the property that the stabilizer of any  $x \in X$  is just  $e$ .)

EXAMPLE 3.1.10. Let  $G$  be a group,  $N$  a normal subgroup of  $G$ , and consider the canonical action of  $G$  on  $G/N$  as that from Example 3.1.4.

- (i) This action has no fixed points since the action of every  $x \notin N$  leaves no points in  $G/N$  fixed.
- (ii) This action is not faithful since the action of every  $x \in N$  preserves every point of  $G/N$ .
- (iii) This action is not free since if  $x \neq x'$  and  $x^{-1}x' \in N$ , then  $xyN = x'yN$  for all  $yN \in G/N$ .

REMARK 3.1.11. The relation “ $y$  belongs to the same orbit of  $x$ ” is an equivalence relation on  $X$ , where the equivalence classes are the orbits of elements of  $X$ . We denote  $X/G$  the set of all such orbits. The action of  $G$  on  $X$  is transitive if and only if  $X/G$  is reduced to a single element.

Next we will discuss the notion of isomorphisms between  $G$ -spaces.

DEFINITION 3.1.12. Let  $X$  and  $Y$  be two  $G$ -spaces. We say that  $X$  and  $Y$  are *isomorphic* if there exists a bijection  $\Phi : X \rightarrow Y$  such that

$$\Phi(s \cdot x) = s \cdot \Phi(x) \quad \text{for all } s \in G, x \in X.$$

The Example 3.1.4 is typical among *transitive*  $G$ -spaces. In fact we have the following result.

THEOREM 3.1.13. *Let  $G$  be a group acting transitively on a set  $X$ . If  $x_0 \in X$  is arbitrary and  $H$  be the stabilizer subgroup of  $x_0$ , i.e.,*

$$H = \{s \in G : s \cdot x_0 = x_0\}.$$

*Then  $X$  is isomorphic to  $G/H$  as  $G$ -spaces.*

PROOF. Define a map  $\phi : G \rightarrow X$  by  $\phi(s) = s \cdot x_0$ . Then  $\phi$  is a surjection of  $G$  onto  $X$  that is constant on each left cosets  $sH$  of  $H$ . Hence  $\phi$  induces a bijection  $\Phi : G/H \rightarrow X$  defined by

$$\Phi(sH) = \phi(s) = s \cdot x_0.$$

It is easy to verify that for all  $s \in G$ ,  $tH \in G/H$ :

$$\Phi(s \cdot tH) = \Phi(stH) = st \cdot x_0 = s \cdot \Phi(tH).$$

In other words, the  $G$ -space  $X$  is isomorphic to  $G/H$ . □

### 3.2. Positive Linear Functionals and States

In this section we define a mean on a Banach space over the complex field and examine some of its properties. The focal point in this section is Theorem 3.2.4 which illustrates the conditions needed for a linear functional to be a mean.

Let  $X$  be a set, and  $\ell^\infty(X)$  denote the set of all complex-valued bounded functions on  $X$ . We denote the supremum norm on  $\ell^\infty(X)$  by  $\|\cdot\|_\infty$ . We recall that  $\ell^\infty(X)$  is a Banach space over the field of complex numbers  $\mathbb{C}$ . Moving forward in this thesis we will denote the function which takes the constant value 1 on  $X$  by  $1_X$ .

**DEFINITION 3.2.1.** Let  $V$  be a vector space over the complex field. A *linear functional* on  $V$  is a linear mapping

$$f : X \longrightarrow \mathbb{C}.$$

If  $V$  is a normed space, then the collection of all continuous linear functionals on  $V$  is called the *dual space of  $V$*  and is denoted as  $V^*$ .

**DEFINITION 3.2.2.** A *mean* or a *state* on  $\ell^\infty(X)$ , is a continuous linear functional  $m \in \ell^\infty(X)^*$  such that

- (i)  $m(f) \geq 0$  whenever  $f \geq 0$ ,
- (ii)  $\|m\| = 1$ .

Note here that condition (i) is called the *positivity* property of a mean and is usually written as  $m \geq 0$ . Thus a mean is a positive linear functional on  $\ell^\infty(X)$  of norm 1.

The positivity of a linear functional has several consequences, some of which are listed in the following lemma.

LEMMA 3.2.3. *Let  $m$  be a positive linear functional in  $\ell^\infty(X)^*$ .*

- (i) *If  $f \in \ell^\infty(X)$  is real-valued, then  $m(f) \in \mathbb{R}$ .*
- (ii) *If  $f, g$  are real-valued and  $f \leq g$ , then  $m(f) \leq m(g)$ .*
- (iii)  *$m(1_X) = \|m\|$ .*

PROOF. (i) If  $f$  is real-valued, then

$$f = f^+ - f^-,$$

where  $f^\pm$  are positive functions, representing the positive and negative parts of  $f$ , that is:

$$f^+ = \frac{|f| + f}{2}, \quad f^- = \frac{|f| - f}{2}.$$

Since  $m(f^\pm)$  are both positive, it follows that  $m(f) = m(f^+) - m(f^-) \in \mathbb{R}$ .

(ii) Since  $g - f \geq 0$ , we have  $m(g - f) \geq 0$ , and thus  $m(g) - m(f) \geq 0$ , or  $m(f) \leq m(g)$ .

(iii) Since  $\|1_X\|_\infty = 1$ , we have

$$\begin{aligned} \|m\| &= \sup_{\|f\|_\infty \leq 1} |m(f)| \\ &\geq |m(1_X)| \\ &= m(1_X), \end{aligned}$$

where in the last equation we used the positivity of  $m$ . Thus

$$\|m\| \geq m(1_X).$$

Next, let  $\|f\|_\infty \leq 1$ , and write  $f = f_1 + if_2$ . Since

$$-\|f\|_\infty 1_X \leq f_1 \leq \|f\|_\infty 1_X,$$

it follows from (ii) that

$$-\|f\|_\infty m(1_X) \leq m(f_1) \leq \|f\|_\infty m(1_X),$$

or equivalently,

$$|m(f_1)| \leq \|f\|_\infty m(1_X).$$

Similarly,

$$|m(f_2)| \leq \|f\|_\infty m(1_X).$$

Thus, using the arithmetic-geometric mean inequality, we can write

$$\begin{aligned} |m(f)| &= |m(f_1) + im(f_2)| \\ &= \sqrt{m(f_1)^2 + m(f_2)^2} \\ &\leq \frac{|m(f_1)| + |m(f_2)|}{2} \\ &\leq \frac{2\|f\|_\infty m(1_X)}{2} \\ &= \|f\|_\infty m(1_X). \end{aligned}$$

Thus

$$|m(f)| \leq \|f\|_\infty m(1_X).$$

Since  $f \in \ell^\infty(X)$  is arbitrary, it follows that

$$\|m\| \leq m(1_X).$$

The equality  $\|m\| = m(1_X)$  now follows.  $\square$

Our next result provides us with an alternative characterization of a mean which can give us an understanding of why the term is so aptly named.

**THEOREM 3.2.4.** *A linear functional  $m \in \ell^\infty(X)^*$  is a mean if and only if for all real-valued functions  $f \in \ell^\infty(X)$ , we have*

$$(3.2.2) \quad \inf f \leq m(f) \leq \sup f.$$

PROOF. Suppose  $m$  is a mean. If  $f \in \ell^\infty(X)$  is real-valued, then by Lemma 3.2.3(ii), the inequalities

$$(\inf f)1_X \leq f \leq (\sup f)1_X,$$

imply that

$$(\inf f)m(1_X) \leq m(f) \leq (\sup f)m(1_X).$$

But since  $m$  is a mean,  $\|m\| = 1$ , and by Lemma 3.2.3(iii), we have  $m(1_X) = 1$ . Thus we obtain

$$\inf f \leq m(f) \leq \sup f.$$

Conversely, if (3.2.2) holds, then  $m(f) \geq 0$  whenever  $f \geq 0$ , and thus  $m$  is a positive linear functional. By Lemma 3.2.3(iii),  $\|m\| = m(1_X)$ . However, (3.2.2) implies that  $m(1_X) = 1$ . Thus  $\|m\| = 1$ , and hence  $m$  is a mean.  $\square$

Let  $\ell_{\mathbb{R}}^\infty(X)$  denote the set of all real-valued bounded functions on  $X$ . Then  $\ell_{\mathbb{R}}^\infty(X)$  under the supremum norm is a Banach space over the field of real numbers  $\mathbb{R}$ . Since every complex-valued function can be written as the sum of its real and imaginary parts, we have the identity

$$(3.2.3) \quad \ell^\infty(X) = \ell_{\mathbb{R}}^\infty(X) + i\ell_{\mathbb{R}}^\infty(X).$$

In particular,  $\ell_{\mathbb{R}}^\infty(X) \subset \ell^\infty(X)$ .

It will be useful to have the notion of a mean on  $\ell_{\mathbb{R}}^\infty(X)$  as well. Since this space is a *real* Banach algebra, we may state the definition as follows.

DEFINITION 3.2.5. A continuous linear functional  $m : \ell_{\mathbb{R}}^\infty(X) \rightarrow \mathbb{R}$ , such that  $\|m\| = 1$ , and  $m(f) \geq 0$  whenever  $f \geq 0$ , is called a *mean* on  $\ell_{\mathbb{R}}^\infty(X)$ .

It follows from the identity (3.2.3) that the existence of a mean on either of  $\ell_{\mathbb{R}}^\infty(X)$  or  $\ell^\infty(X)$  implies the existence of the mean on the other space. More precisely we are led to the following theorem.



THEOREM 3.2.6. *There is a mean on  $\ell^\infty(X)$  if and only if there is a mean on  $\ell_{\mathbb{R}}^\infty(X)$ .*

PROOF. If  $m$  is a mean on  $\ell^\infty(X)$ , then clearly, by restriction we get a mean on  $\ell_{\mathbb{R}}^\infty(X)$ .

Conversely, suppose  $m'$  is a mean on  $\ell_{\mathbb{R}}^\infty(X)$ . For each  $f = f_1 + if_2$ , define

$$m(f) = m'(f_1) + im'(f_2).$$

Then  $m$  is linear since

$$\begin{aligned} m(f + \alpha g) &= m(f_1 + if_2 + (\alpha_1 + i\alpha_2)(g_1 + ig_2)) \\ &= m(f_1 + \alpha_1 g_1 - \alpha_2 g_2 + i(f_2 + \alpha_1 g_2 + \alpha_2 g_1)) \\ &= m'(f_1 + \alpha_1 g_1 - \alpha_2 g_2) + im'(f_2 + \alpha_1 g_2 + \alpha_2 g_1) \\ &= m'(f_1) + \alpha_1 m'(g_1) - \alpha_2 m'(g_2) + im'(f_2) + i\alpha_1 m'(g_2) + i\alpha_2 m'(g_1) \\ &= m'(f_1) + im'(f_2) + (\alpha_1 + i\alpha_2)(m'(g_1) + im'(g_2)) \\ &= m(f) + \alpha m(g). \end{aligned}$$

The linear functional is continuous on  $\ell^\infty(X)$ , since for every  $f \in \ell^\infty(X)$ :

$$\begin{aligned} |m(f)| &= |m'(f_1) + im'(f_2)| \\ &\leq |m'(f_1)| + |m'(f_2)| \\ &\leq \|f_1\|_\infty + \|f_2\|_\infty \\ &\leq 2\|f\|_\infty. \end{aligned}$$

That is  $\|m\| \leq 2$ . It follows from the definition of  $m$  that (3.2.2) holds for  $m$  if and only if it holds for  $m'$ . Thus  $m$  is a mean.  $\square$

### 3.3. Amenable Groups

In this section we define the notion of a left invariant mean. The purpose of this section is to tie the idea of the amenability of a group  $G$  to the existence of a left

invariant mean over  $\ell^\infty(G)$ . This will lead to our definition of Følner's conditions that will be shown later to be equivalent to a group's amenability (Theorem 3.3.11). We also examine a special case in Theorem 3.3.6 to show how every abelian group is amenable through use of the Markov-Kakutani Theorem [18, 20].

Let  $G$  be a group. If  $s \in G$  and  $f \in \ell^\infty(G)$ , we define the left and right translates of  $f$  by  $s$ , by the following relations:

$$L_s f(t) = f(st), \quad R_s f(t) = f(ts) \quad \text{where } s, t \in G.$$

The notion of a mean on  $\ell^\infty(G)$  and  $\ell_{\mathbb{R}}^\infty(G)$  have been defined in Section 3.2. With the presence of the group action, we may now define the notion of a left invariant mean.

DEFINITION 3.3.1. A mean  $m \in \ell^\infty(G)^*$  is called *left invariant* (LIM) if

$$m(L_s f) = m(f),$$

for all  $f \in \ell^\infty(G)$  and all  $s \in G$ . A similar definition holds for left invariant means on  $\ell_{\mathbb{R}}^\infty(G)$ .

Similar to Theorem 3.2.6 we have the following result.

THEOREM 3.3.2. *There is a left invariant mean on  $\ell^\infty(G)$  if and only if there is a left invariant mean on  $\ell_{\mathbb{R}}^\infty(G)$ .*

For the proof it suffices to note that if  $m$  and  $m'$  are the two means on  $\ell^\infty(G)$  and  $\ell_{\mathbb{R}}^\infty(G)$  as in the proof of Theorem 3.2.6, then  $m$  is left invariant if and only if  $m'$  is so.

In 1924 Banach and Tarski [2] proved the existence of paradoxical decompositions in  $\mathbb{R}^n$  ( $n \geq 3$ ), better known as Banach-Tarski paradox. In his study of the Banach-Tarski theorem in 1929, von Neumann [23] introduced and studied a class of groups known as *amenable groups*. There are several ways to define amenable groups, and one such way is to use the existence of invariant means as our next definition will show.

DEFINITION 3.3.3. A group  $G$  is amenable if there exists a left invariant mean on  $\ell^\infty(G)$ .

The term ‘amenable’ (German: mittlebar, French: moyenable) was introduced by Day [6, 7] as a pun between a mean and the naive meaning as “being agreeable”.

EXAMPLE 3.3.4. Every finite group  $G$  is amenable. In fact, it is easy to check that the continuous linear functional

$$m : \ell^\infty(G) \rightarrow \mathbb{C}, \quad m(f) = \frac{1}{|G|} \sum_{x \in G} f(x),$$

satisfies the inequalities in (3.2.2), and therefore  $m$  is a mean. The left invariance of  $m$  follows directly from its definition.

Our objective is to show that every abelian group is amenable. To this end we shall need the following well-known fixed point theorem from functional analysis.

THEOREM 3.3.5 (Markov-Kakutani). *Let  $V$  be a topological vector space and  $K$  be a nonempty compact, convex subset of  $X$ . Suppose  $\mathcal{F}$  is a family of continuous mappings  $T : K \rightarrow K$ , such that*

- (i)  $T((1 - \lambda)x + \lambda y) = (1 - \lambda)Tx + \lambda Ty$ , for all  $x, y \in K, \lambda \in [0, 1]$ .
- (ii)  $TS = ST$  for all  $S, T \in \mathcal{F}$ .

*In that case, there is a point  $p \in K$  such that*

$$Tp = p \quad \text{for all } T \in \mathcal{F}.$$

A mapping  $T : K \rightarrow K$  satisfying (i) is called an *affine mapping* on  $K$ . Thus, the theorem can be rephrased as: *a commuting family of continuous affine mappings on a compact convex set has a fixed point.* Note that the theorem does not claim that the fixed point is unique.

PROOF. For each integer  $n \geq 1$ , and each  $T \in \mathcal{F}$ , let

$$T_n = \frac{1}{n}(I + T + T \dots + T^{n-1}),$$

where  $I$  is the identity map on  $K$ . Each  $T_n$  is a continuous affine mapping on  $K$ . As each  $T_n$  is continuous and affine, it follows that  $T_n(K)$  is a convex compact set. Moreover, since  $K$  is convex, it follows that

$$T_n(K) \subset K \quad \text{for all } T \in \mathcal{F}, n \geq 1.$$

Let us define

$$\mathcal{K} = \{T_n(K) : n \geq 1, T \in \mathcal{F}\}.$$

For  $S, T \in \mathcal{F}$ , since  $ST = TS$ , it follows that  $S_n T_n = T_n S_n$ , and hence

$$S_n T_n(K) \subset S_n(K) \cap T_n(K).$$

It follows that any finite subfamily of  $\mathcal{K}$  has a nonempty intersection. Since members of  $\mathcal{K}$  are compact sets, it follows that the intersection of all sets in  $\mathcal{K}$  must be nonempty, that is, there is a point

$$p \in \bigcap \mathcal{K} = \bigcap \{T_n(K) : n \geq 1, T \in \mathcal{F}\}.$$

We claim that  $p$  is the required fixed point of  $\mathcal{F}$ . We argue by contradiction. Suppose that  $T \in \mathcal{F}$  such that  $Tp \neq p$ . Thus,  $Tp - p \neq 0$ , and hence there is an open neighbourhood  $U$  of 0 (the origin of  $V$ ), such that

$$Tp - p \notin U.$$

Let  $n \geq 1$  be arbitrary. Since  $p \in T_n(K)$ , we can find a  $q \in K$  such that

$$p = T_n q = \frac{1}{n}(I + T + T \dots + T^{n-1})q.$$

Hence

$$Tp - p = \frac{1}{n}(T^n - I)q \notin U.$$

Since  $q \in K$ , it follows  $T^n q \in K$ , and hence  $n^{-1}(K - K)$  is not a subset of  $U$ , for every  $n \geq 1$ . On the other hand, since  $K - K$  is compact, it is a bounded subset of  $V$ , i.e., for larger enough  $n$  we must have  $n^{-1}(K - K) \subset U$ . We have reached

a contradiction. The contradiction arose from the assumption that  $p$  is not a fixed point of  $F$ . The proof is now complete.  $\square$

**THEOREM 3.3.6.** *Every abelian group is amenable.*

**PROOF.** To prove the existence of a left invariant mean, let  $M$  be the set of all means on  $\ell^\infty(G)$ . Thus  $M$  is a subset of the closed unit ball of  $\ell^\infty(G)^*$ .

To see that  $M$  is not empty, let  $g \in \ell^1(G)$  be any positive function of norm 1, i.e.,

$$\|g\|_1 = \sum_{x \in G} g(x) = 1.$$

Then the natural image of  $g$  in  $\ell^1(G)^{**} = \ell^\infty(G)^*$  defines a mean on  $\ell(G)^*$ . In fact, if  $\kappa(g)$  is the natural image, then the map

$$\kappa(g) : \ell^\infty(G) \rightarrow \mathbb{C}, \quad \langle \kappa(g), f \rangle = \sum_{x \in G} g(x)f(x),$$

satisfies the inequalities in (3.2.2), and therefore  $\kappa(g)$  is a mean.

So far we have shown that the set of all means,  $M$ , is not empty. It is easy to check that  $M$  is a  $w^*$ -closed convex subset of the closed unit ball of  $\ell^\infty(G)^*$ . Since by Alaoglu's theorem, the closed unit ball of  $\ell^\infty(G)^*$  is compact in the  $w^*$ -topology, it follows that  $M$  is compact in the  $w^*$ -topology.

Thus we have shown that the set  $M$  of all means on  $\ell^\infty(G)^*$  is a nonempty  $w^*$ -compact convex set. What remains to prove is that at least one member of  $M$  must be left invariant. To this end, we shall use the Markov-Kakutani fixed point theorem.

For each  $x \in G$  and  $m \in M$ , let  $T_x m \in M$  be defined by

$$\langle T_x m, f \rangle = \langle m, L_x f \rangle,$$

where  $f \in \ell^\infty(G)$ . Each mapping  $T_x : M \rightarrow M$ , defined above, is an affine mapping in the sense that  $T_x$  preserves convex combinations in  $M$ :

$$T_x(\lambda m_1 + (1 - \lambda)m_2) = \lambda T_x(m_1) + (1 - \lambda)T_x(m_2), \quad (0 \leq \lambda \leq 1).$$

Thus  $\{T_x\}_{x \in G}$  is a family of continuous affine mappings on the compact convex set  $M$ . Since by assumption, the group  $G$  is commutative, it follows that the family  $\{T_x\}_{x \in G}$  is a commuting family of mappings under composition, in the sense that

$$T_x T_y = T_{xy} = T_{yx} = T_y T_x.$$

Now, the Markov-Kakutani fixed point theorem implies that this family of mappings must have a fixed point, i.e., an element  $m \in M$  such that

$$T_x m = m \quad \text{for all } x \in G.$$

The element  $m$  thus obtained is the required left invariant mean, and hence  $G$  is abelian.  $\square$

Our next theorem extends the range of examples of amenable groups.

- THEOREM 3.3.7.** (i) *Any subgroup of an amenable group is itself amenable.*  
(ii) *The product of two amenable groups is an amenable group.*

We will omit the proof of this theorem but reference [25] for the technical details. This theorem is used to signify the class of amenable groups being larger than that of just finite or abelian groups. For example, if  $G_1$  is an abelian group and  $G_2$  is a finite group, then the product  $G_1 \times G_2$  is amenable, but is neither finite nor abelian.

The next theorem will show a useful relation between left invariant means and the existence of finitely additive invariant probability measures.

At this point we recall that a measure  $\mu$  on  $\mathcal{P}(X)$  is called invariant if:

$$\mu(s \cdot A) = \mu(A) \quad \text{for all } A \in \mathcal{P}(X) \text{ and all } s \in G.$$

A probability measure is one such that  $\mu(X) = 1$ . A finitely additive measure on  $\mathcal{P}(X)$  is a mapping  $\mu : \mathcal{P}(X) \rightarrow [0, \infty)$  such that:

- (i)  $\mu(\emptyset) = 0$ ,

(ii) if  $A_1, \dots, A_n$  is any finite number of pairwise disjoint elements of  $\mathcal{P}(X)$  then

$$\mu(\cup_{i=1}^n A_i) = \sum_{i=1}^n \mu(A_i).$$

**THEOREM 3.3.8.** *Let  $G$  be a group. Then there exists a mean  $m \in \ell_{\mathbb{R}}^{\infty}(G)^*$  if and only if there exists a finitely additive, probability measure  $\mu$  on  $\mathcal{P}(G)$ . Further,  $m$  is left invariant if and only if  $\mu$  is left invariant.*

**PROOF.** Suppose  $m \in \ell_{\mathbb{R}}^{\infty}(G)^*$  is a mean. We define a positive function  $\mu$  on  $\mathcal{P}(G)$  by  $\mu(B) = m(1_B)$ , for all  $B \in \mathcal{P}(G)$ ; in particular,

$$\mu(G) = m(1_G) = 1.$$

Finite additivity of  $\mu$  follows from the linearity of  $m$ . If  $m$  is left invariant, then  $\mu$  is left invariant since

$$\mu(sB) = m(1_{sB}) = m(L_{s^{-1}}1_B) = m(1_B) = \mu(B).$$

Next we prove the converse statement. Suppose  $\mu$  is a finitely additive, probability measure on  $\mathcal{P}(G)$ . Define

$$m : \ell_{\mathbb{R}}^{\infty}(G) \longrightarrow \mathbb{R}, \quad m(f) = \int_G f d\mu.$$

(For integration with respect to finitely additive measures, see Dunford and Schwartz [9, Chapter 3, section 6]). So in particular,  $m(1_G) = \mu(G) = 1$ . For each  $f \in \ell_{\mathbb{R}}^{\infty}(G)$ , the relation

$$(\inf f)1_G \leq f \leq (\sup f)1_G,$$

and the monotonicity of the integral implies that

$$(\inf f)\mu(G) \leq \int_G f d\mu \leq (\sup f)\mu(G).$$

Thus

$$m(f) = \int_G f d\mu \geq 0,$$

whenever  $f \geq 0$ . Moreover, if  $\mu$  is left invariant, then for each  $s \in G$  we have

$$m(L_s f) = \int_G L_s f d\mu = \int_G f d\mu.$$

We have shown that  $m$  is a left invariant mean on  $\ell_{\mathbb{R}}^{\infty}(G)^*$ .  $\square$

**COROLLARY 3.3.9.** *A group  $G$  is amenable if and only if there exists a finitely additive, left invariant probability measure on  $\mathcal{P}(G)$ .*

**EXAMPLE 3.3.10.** As an application of the above theorem we will show that the nonabelian free group on two generators  $F_2 = \langle a, b \rangle$  is nonamenable. We identify the elements of  $F_2$  with the set of reduced words in  $S = \{a, b, a^{-1}, b^{-1}\}$ . For  $x \in S$ , let  $E_x$  be the set of elements in  $F_2$  beginning with  $x$ . If  $\mu$  is a left invariant finitely additive probability measure on  $\mathcal{P}(F_2)$ , then

$$(3.3.4) \quad \mu(F_2) = \mu(\{e\}) + \mu(E_a) + \mu(E_b) + \mu(E_{a^{-1}}) + \mu(E_{b^{-1}}).$$

However,

$$a^{-1}E_a = \{e\} \cup E_a \cup E_b \cup E_{b^{-1}},$$

thus

$$(3.3.5) \quad \mu(a^{-1}E_a) = \mu(\{e\}) + \mu(E_a) + \mu(E_b) + \mu(E_{b^{-1}}).$$

Hence, using the left invariant of  $\mu$  we can write

$$(3.3.6) \quad \mu(F_2) = \mu(E_{a^{-1}}) + \mu(a^{-1}E_a) = \mu(E_{a^{-1}}) + \mu(E_a).$$

If we repeat this argument with  $E_b$  in place of  $E_a$ , we get

$$(3.3.7) \quad \mu(F_2) = \mu(E_{b^{-1}}) + \mu(E_b).$$

Substituting (3.3.6) and (3.3.7) into (3.3.4), we get

$$\mu(F_2) = \mu(\{e\}) + 2\mu(F_2),$$



which is impossible since, by assumption

$$\mu(F_2) = 1 \text{ and } \mu(\{e\}) \geq 0.$$

□

Amenability can be characterized more directly in terms of properties of the group  $G$  rather than  $\ell^\infty(G)$ . Følner [10] provides us with one such property as follows.

**THEOREM 3.3.11.** *For a group  $G$ , the following conditions are equivalent.*

- (i)  $G$  is amenable.
- (ii) For every  $\epsilon > 0$  and every finite set  $K \subset G$ , there is a finite set  $U \subset G$  such that for all  $x \in K$ ,

$$\frac{|xU \Delta U|}{|U|} < \epsilon \quad (\text{Følner's condition}).$$

Here of course,  $|\cdot|$  denotes the cardinality and  $\Delta$  the symmetric difference. The number  $|xU \Delta U|$  is a measure of how far apart  $xU$  and  $U$  are from each other, and the condition

$$\frac{|xU \Delta U|}{|U|} < \epsilon,$$

shows that  $xU$  and  $U$  are not very far apart compared to the size of  $U$ . We shall state and prove a more general form of Følner's theorem in Theorem 3.4.2.

### 3.4. Amenable Group Actions

In this section we will define an invariant mean for a  $G$ -space  $(G, X)$ . Using this definition will be key to understanding the equivalency of conditions that Rosenblatt has provided in Theorem 3.4.2 for such invariant means. The notion of amenable groups discussed in the previous section can now be extended to group actions within this section.

DEFINITION 3.4.1. Let  $G$  be a group acting on a set  $X$ . A mean for  $(G, X)$  is a positive linear functional  $m \in \ell^\infty(X)^*$  with norm 1. If  $m$  satisfies the condition

$$m(L_s f) = m(f) \quad \text{for all } s \in G, f \in \ell^\infty(X),$$

where  $(L_s f)(x) = f(s \cdot x)$ , with  $x \in X$ , then  $m$  is called an *invariant mean* for  $(G, X)$ .

We note that if  $X = G$ , this definition is consistent with the notion of an invariant mean on  $G$  given in Definition 3.3.1.

For the purpose of this thesis we shall need the following characterizations of the existence of invariant means due to Rosenblatt [24, Theorem 4.10]. Note that we shall state only a special case of Rosenblatt's result, which is nonetheless sufficient for our purposes in this thesis.

THEOREM 3.4.2. (*Rosenblatt*) *Let  $G$  be a group acting on a set  $X$ . The following statements are equivalent.*

- (i)  $(G, X)$  has an invariant mean.
- (ii) There exists a finitely additive, invariant probability measure defined on all subsets of  $X$ .
- (iii) For every  $\epsilon > 0$  and every finite subset  $A$  of  $G$ , there exists a finite set  $F \subset X$  such that for all  $a \in A$

$$(3.4.8) \quad \frac{|a \cdot F \Delta F|}{|F|} \leq \epsilon \quad (\text{Følner's condition}).$$

- (iv) There exists a net  $(F_\alpha)_\alpha$  of finite subsets of  $X$  such that for all  $s \in G$ ,

$$(3.4.9) \quad \lim_\alpha \frac{|s \cdot F_\alpha \Delta F_\alpha|}{|F_\alpha|} = 0 \quad (\text{Følner's net}).$$

PROOF. (i)  $\Leftrightarrow$  (ii): The equivalence of (i) and (ii) follows by a routine adaptation of the proof of Theorem 3.3.8. Since the adaptation presents no technical difficulties, we omit the details for brevity.

(iii)  $\Leftrightarrow$  (iv): First, assume (iii) holds. Let  $n \in \mathbb{N}$  and  $A$  be a finite subset of  $G$ , and put  $\alpha = (n, A)$ . We can order all such  $\alpha$  by the relation  $\alpha_1 \leq \alpha_2$  if and only if  $n_1 \leq n_2$

and  $A_1 \subset A_2$ . For each  $\alpha = (n, A)$ , let  $F_\alpha \subset X$  be a finite set such that for all  $a \in A$ :

$$\frac{|a \cdot F_\alpha \Delta F_\alpha|}{|F_\alpha|} \leq \frac{1}{n}.$$

The net  $\{F_\alpha\}$  thus obtained satisfies the condition in (iv). In fact, if  $s \in G$  and  $\epsilon > 0$ , we can choose  $n \in \mathbb{N}$  such that  $\frac{1}{n} < \epsilon$ , and choose  $A = \{s\}$ . Now if  $\beta = (n, \{s\})$ , then for all  $\alpha \geq \beta$ ,

$$\frac{|s \cdot F_\alpha \Delta F_\alpha|}{|F_\alpha|} \leq \frac{1}{n} \leq \epsilon,$$

which proves (iv).

Conversely, suppose that (iv) holds. Let  $A = \{a_1, \dots, a_n\}$  be a finite subset of  $G$ , and let  $\epsilon > 0$ . For each  $1 \leq i \leq n$ , let  $\alpha_i$  be chosen so that

$$\alpha \geq \alpha_i \implies \frac{|a_i \cdot F_\alpha \Delta F_\alpha|}{|F_\alpha|} \leq \epsilon.$$

If we choose  $\beta$  so that  $\beta \geq \alpha_i$  for all  $1 \leq i \leq n$ , then

$$\alpha \geq \beta \implies \frac{|a_i \cdot F_\alpha \Delta F_\alpha|}{|F_\alpha|} \leq \epsilon, \quad \text{for all } 1 \leq i \leq n.$$

This implies that (iii) holds. The most challenging part of this proof is showing the equivalence (i)  $\Leftrightarrow$  (iii). We show the details of the remainder of this proof in Appendix A.  $\square$

**REMARK 3.4.3.** We can interpret (iv) from above in the following way: for any group element  $s$ , the proportion of elements of  $F_\alpha$  that are moved by  $s$  and then divided by the size of  $F_\alpha$  tends to 0 as  $\alpha$  gets large.

**REMARK 3.4.4.** It should be noted that if  $G$  is finitely generated, then the Følner's net in (3.4.9) can be replaced with a Følner's sequence  $(F_n)_n$ . To see this, let  $S = S^{-1}$  be a set of generators of  $G$ , and for each  $n \in \mathbb{N}$ , let  $S_n$  be the set of all elements of  $G$  representable as words of length less than or equal to  $n$ . Then

$$|S_n| \leq |S|^n < \infty.$$

By Theorem 3.4.2(iii), there exists a finite set  $F_n \subset X$  such that

$$\frac{|s \cdot F_n \Delta F_n|}{|F_n|} < \frac{1}{n} \quad \text{for all } s \in S_n.$$

Then  $(F_n)_n$  is a Følner's sequence: in fact, for a given  $s_0 \in G$  and  $\epsilon > 0$ , let  $n_0 \in \mathbb{N}$  be such that

$$s_0 \in S_{n_0} \quad \text{and} \quad \frac{1}{n_0} < \epsilon.$$

Then for all  $n \geq n_0$ , we have  $s_0 \in S_n$ , and therefore

$$\frac{|s_0 \cdot F_n \Delta F_n|}{|F_n|} < \frac{1}{n} < \frac{1}{n_0} < \epsilon.$$

Let us observe that in the special case that  $X = G$  and the action of  $G$  is the group multiplication, the existence of an invariant mean on  $(G, G)$  is the same as the amenability of the group  $G$ . In view of this fact, it is natural to make the following definition.

**DEFINITION 3.4.5.** Let  $G$  act on a set  $X$ . We call  $(G, X)$  *amenable* if any one of the equivalent conditions in Theorem 3.4.2 holds.

**EXAMPLE 3.4.6.** Let  $G$  be an arbitrary group acting on a set  $X$ . We show that if  $|X| < \infty$ , then  $(G, X)$  is amenable. To this end we show that

$$m = \frac{1}{|X|} 1_X,$$

is an invariant mean for  $(G, X)$ . Clearly  $m$  is positive on  $\ell_{\mathbb{R}}^{\infty}(X)$ . Further,

$$\begin{aligned} \langle m, 1_X \rangle &= \frac{1}{|X|} \langle 1_X, 1_X \rangle \\ &= \frac{|X|}{|X|} \\ &= 1. \end{aligned}$$

If  $\phi \in \ell_{\mathbb{R}}^{\infty}(X)$  and  $s \in G$ , then using the fact that  $s \cdot X = X$ , since  $x = s \cdot (s^{-1} \cdot x)$ , we are left with

$$\begin{aligned}
\langle m, L_s \phi \rangle &= \frac{1}{|X|} \sum_{x \in X} 1_X(x) L_s \phi(x) \\
&= \frac{1}{|X|} \sum_{x \in X} \phi(s \cdot x) \\
&= \frac{1}{|X|} \sum_{x \in X} \phi(x) \\
&= \langle m, \phi \rangle.
\end{aligned}$$

Therefore  $m$  is invariant.

The relation between the amenability of  $G$  and the amenability of  $(G, X)$  is stated in the next theorem.

**THEOREM 3.4.7.** *Let  $G$  act on a set  $X$ . Then the following hold.*

- (i) *If  $G$  is amenable, then  $(G, X)$  is amenable.*
- (ii) *If  $(G, X)$  is amenable, then  $G$  need not be amenable in general.*

**PROOF.** (i) If  $m$  is a left invariant mean on  $G$ , then we can define an invariant mean for  $(G, X)$  as follows. Let  $x_0 \in X$  be arbitrary but fixed. For each  $f \in \ell_{\mathbb{R}}^{\infty}(X)$ , define a function  $\tilde{f} \in \ell_{\mathbb{R}}^{\infty}(G)$  by

$$\tilde{f}(s) = f(s \cdot x_0) \quad \text{for all } s \in G.$$

The following properties can be easily verified.

- (a)  $\widetilde{f + \alpha g} = \tilde{f} + \alpha \tilde{g}$ .
- (b)  $\widetilde{1_X} = 1_G$ .
- (c)  $\widetilde{L_s f} = L_s \tilde{f}$ .

Define

$$(3.4.10) \quad m' : \ell_{\mathbb{R}}^{\infty}(X) \longrightarrow \mathbb{R}, \quad m'(f) = m(\tilde{f}).$$

Then  $m'$  is linear since

$$\begin{aligned} m'(f + \alpha g) &= m(\widetilde{f + \alpha g}) \\ &= m(\widetilde{f} + \alpha \widetilde{g}) \\ &= m'(\widetilde{f}) + \alpha m'(\widetilde{g}). \end{aligned}$$

Clearly  $m' \geq 0$  and  $m'(1_X) = m(1_G) = 1$ . Furthermore  $\|m'\| \leq 1$  because if  $f \in \ell_{\mathbb{R}}^{\infty}(X)$ , then

$$\begin{aligned} |m'(f)| &= |m(\widetilde{f})| \\ &\leq \|\widetilde{f}\|_{\infty} \\ &\leq \|f\|_{\infty}. \end{aligned}$$

Thus  $\|m'\| = 1$ . Finally  $m$  is invariant because

$$\begin{aligned} m'(L_s f) &= m(\widetilde{L_s f}) \\ &= m(L_s \widetilde{f}) \\ &= m(\widetilde{f}) \\ &= m'(f). \end{aligned}$$

(ii) Let  $F_2$  be the free group on two generators,  $N$  be a subgroup of  $F_2$  of finite index, and let  $F_2$  act on the left coset space  $F_2/N$  by the natural action:

$$x \cdot (yN) = xyN.$$

We know that  $F_2$  is nonamenable from Example 3.3.10, however since  $F_2/N$  is finite, Example 3.4.6, tells us that  $(F_2, F_2/N)$  must be amenable.

It remains for us to show that  $F_2$  has a subgroup of finite index. In fact, we show  $F_2$  has a subgroup of index 2. Let  $F_2 = \langle a, b \rangle$ , so that  $a, b$  are generators of  $F_2$ , and let  $i : \{a, b\} \rightarrow F_2$  be the canonical injection. Now, let

$$f : \{a, b\} \longrightarrow \mathbb{Z}_2, \quad a \mapsto 0, \quad b \mapsto 1.$$

By the universal property of free groups (Theorem 1.3.10), there exists a unique homomorphism

$$\phi : \mathbb{F}_2 \longrightarrow \mathbb{Z}_2,$$

such that  $\phi \circ i = f$ , i.e.,

$$\phi(a) = 0, \quad \phi(b) = 1.$$

In particular, the homomorphism  $\phi$  is surjective, and hence  $F_2 / \ker \phi \cong \mathbb{Z}_2$ . That is,  $N := \ker \phi$  is a normal subgroup of  $F_2$  of index 2. □

## CHAPTER 4

### Amenability of Cayley Graphs

The purpose of this chapter is to highlight the main topic of this thesis. It is used to introduce the concept of Cayley graphs, and provide many special and interesting consequences of their properties. The main result of this thesis is Theorem 4.2.7: the amenability of  $(G, X)$  implies the amenability of  $\text{Cay}(H, X)$ , for every finitely generated subgroup  $H$  of  $G$ . We will see how in the case of finitely generated groups, this theorem can be extended in Theorem 4.2.12 through use of Følner's sequences.

#### 4.1. Basic Notions of Cayley Graphs

In this section we will introduce the notion of Cayley graphs, and see how they are used to encode the abstract structure of a group. We will give the definition of a Cayley graph and examine how they can be constructed to provide a geometric representation of certain groups. A few special remarks will be discussed based on our definition, and we will provide a few helpful examples to better facilitate our understanding.

**DEFINITION 4.1.1.** Suppose  $G$  acts on a set  $X$  and  $G$  is finitely generated by a finite symmetric set,  $A$ , of generators of  $G$  (not containing the identity). The *Cayley graph*  $\text{Cay}_A(G, X)$  is defined as follows: the vertices of the graph are the points in  $X$ , and two vertices  $x, y \in X$  are connected by an edge if  $a \cdot x = y$  for some  $a \in A$ .

Examining the definition given above, we can make a few interesting remarks.

**REMARK 4.1.2.**

- (a) What we have defined as  $\text{Cay}_A(G, X)$  is sometimes called the Schreier graph and denoted as  $\text{Sch}_A(G, X)$ , [27].



- (b) Of particular interest for us is the special case that  $X = G$  and the action of  $G$  on itself is the group multiplication. In this case,  $\text{Cay}_A(G, X)$  will be called the *Cayley graph* of  $G$  and denoted by  $\text{Cay}_A(G)$ .

The Cayley graph of a group  $\text{Cay}_A(G)$  is always connected since every element of  $G$  is connected to the identity  $e$ . The Cayley graph of a group with respect to a finite generating set  $A$  is always locally finite, even if the group itself is infinite. When  $A$  is finite, i.e.  $|A| = n$ , then the degree of every vertex in the Cayley graph does not exceed  $n$ .

- (c) It should be noted that the Cayley graph depends on the generating set  $A$ . For example, if  $G = \mathbb{Z}_6$  is the cyclic group of order 6, both  $A = \{1, 5\}$  and  $A' = \{1, 2, 3, 4, 5\}$  generate  $\mathbb{Z}_6$ , however, the Cayley graphs corresponding to these generating sets are not isomorphic, as we can see below in Figure 1.

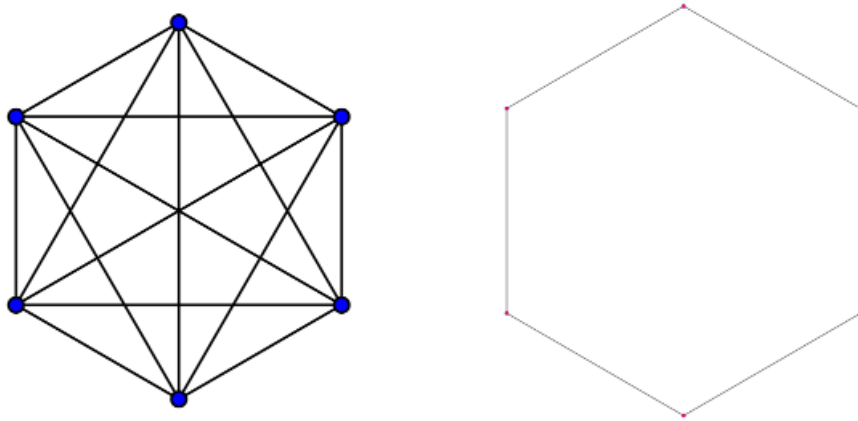
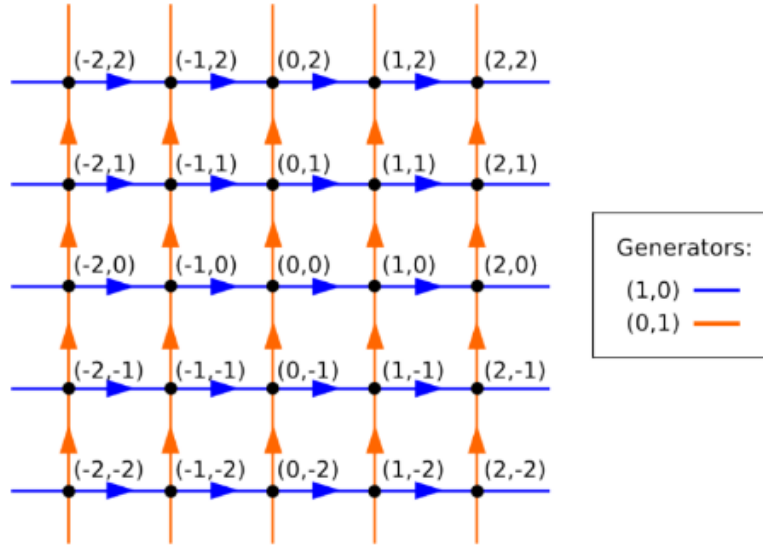


FIGURE 1. Cayley graphs of  $\mathbb{Z}_6$ .

EXAMPLE 4.1.3. Consider the abelian group  $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$ . This group is generated by  $\{(1,0), (0,1)\}$ . To represent the Cayley graph let us choose the symmetric set of generators

$$A = \{(1, 0), (0, 1), (-1, 0), (0, -1)\}.$$

Figure 2 represents  $\text{Cay}(\mathbb{Z}^2)$ , and later we will show that this graph has expansion zero and hence is amenable.

FIGURE 2. Cayley graph of  $\mathbb{Z}^2$ .

EXAMPLE 4.1.4. Let  $F_2$  be the free group on two generators  $\{a, b\}$ . The Cayley graph  $\text{Cay}(F_2)$  of  $F_2$  is a tree in which each vertex has degree  $d = 4$ , as seen in Figure 3. To justify this, note that two vertices corresponding to words  $x$  and  $y$  are connected by an edge if  $xy^{-1} \in A = \{a, b, a^{-1}, b^{-1}\}$ . In other words, a vertex  $y$  is connected to four other vertices  $ay, a^{-1}y, by,$  and  $b^{-1}y$ . Starting from  $e$  we can reach any other vertex by successive multiplication by elements from  $\{a, b, a^{-1}, b^{-1}\}$ . This implies that the Cayley graph of  $F_2$  is a 4-regular connected graph. A simple argument shows that the graph cannot contain any cycle (note that cycles must contain at least three edges), hence the graph must be a 4-regular tree. By Theorem 2.1.7,  $\text{Cay}(F_2)$  is nonamenable with the expansion constant

$$\gamma(\text{Cay}(F_2)) = 2.$$

REMARK 4.1.5. (a) Non-isomorphic groups may have the same Cayley graphs. For example, let  $K_n$  be the undirected complete graph of order  $n$ . Then  $K_n \cong \text{Cay}_A(G)$ , where  $G$  is an arbitrary group of order  $n$ , and  $A$  is the set of all non-identity elements of  $G$ . Thus, the number of Cayley representations for  $K_n$  is greater than or equal to the number of nonisomorphic groups of order  $n$ .

- (b) Some authors use a different convention to define  $\text{Cay}(G)$ : two vertices  $x, y \in G$  are connected by an edge if  $xa = y$  for some  $a \in A$ . The convention used in this thesis is consistent with the one used for group actions.
- (c) Although Cayley graphs depend on the generating sets  $A$ , the results in this thesis are in effect independent of a particular choice of  $A$  (see Theorem 4.2.3). For this reason we shall usually drop the subscript  $A$  from the notation and denote a Cayley graph by  $\text{Cay}(G, X)$ , when there is no fear of confusion.

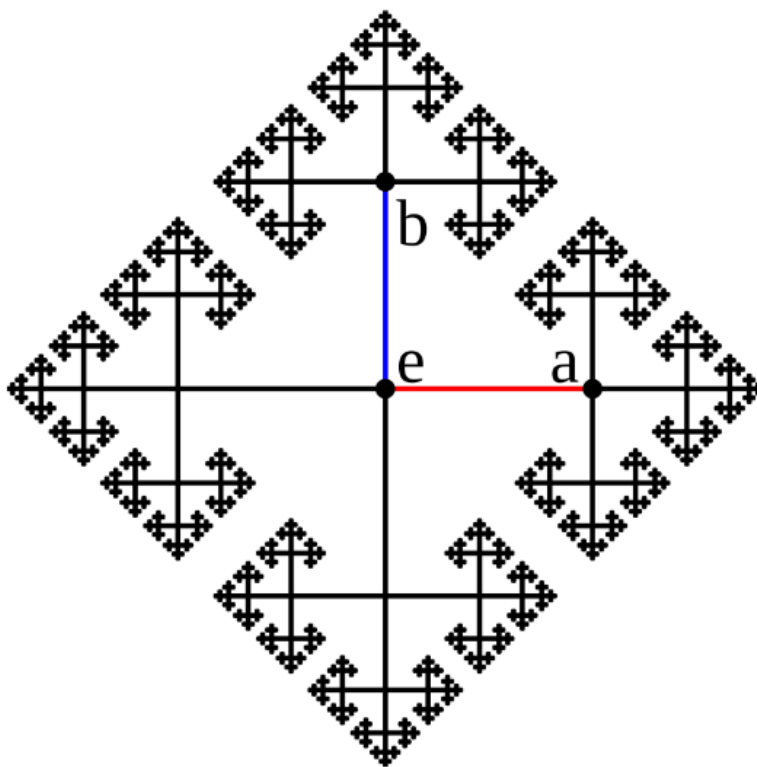


FIGURE 3. The Cayley graph of  $F_2$ , where each new edge is drawn at half the size to give fractal images.

#### 4.2. Cayley Graphs and their Amenability

In this section we will demonstrate in Theorem 4.2.3 how the amenability of a Cayley graph is independent of its generating set. We show in Theorem 4.2.7 that when a group  $G$  acts on a set  $X$ , with  $H$  being a finitely generated subgroup of  $G$ , the

amenability of  $(G, X)$  will be characterized in terms of the amenability of  $\text{Cay}(H, X)$ . This theorem can be strengthened by showing that  $(G, X)$  and  $\text{Cay}(G, X)$  have the same Følner's sequences, when  $G$  is finitely generated.

**THEOREM 4.2.1.** *If  $H$  is a finitely generated subgroup of a group  $G$ , then the graphs  $\text{Cay}(H, G)$  and  $\text{Cay}(H)$  have the same expansion.*

**PROOF.** The components of the graph  $\text{Cay}(H, G)$  are of the form  $\text{Cay}(H, Hz)$  where  $Hz$  are the right cosets of  $H$  in  $G$ , with  $z \in G$ . Let  $F$  be a subset of  $H$ ,  $b(Fz)$  be the border of  $Fz$  in  $\text{Cay}(H, Hz)$ , and  $b(F)$  be the border of  $F$  in  $\text{Cay}(H)$ . Then it is easy to check that  $b(Fz) = b(F)z$ . It follows that the graphs  $\text{Cay}(H, Hz)$  and  $\text{Cay}(H)$  have the same expansion for every  $z \in G$ . Now the corollary follows from Theorem 2.1.6.  $\square$

**LEMMA 4.2.2.** *Let  $G$  be a finitely generated group and  $A$  be a finite generating subset of  $G$ . Let  $G$  act on a set  $X$  and  $F$  be a nonempty subset of  $X$ . Suppose that  $x \in F$  with*

$$a_1, \dots, a_n \in A \quad \text{and} \quad y = a_1 a_2 \cdots a_n \cdot x.$$

*Then*

$$(4.2.11) \quad y \in \bigcup_{j=0}^n b^{(j)}(F),$$

*where  $b^{(0)}(F) = F$  and  $b^{(j)}(F) = b(b(\cdots(b(F))\cdots))$ ,  $j$ -times.*

**PROOF.** We will use induction on  $n$ . The case  $n = 1$  follows immediately from the definition of  $b(F)$ .

Suppose, as an induction hypothesis that

$$(4.2.12) \quad a_2 a_3 \cdots a_n \cdot x \in \bigcup_{j=0}^{n-1} b^{(j)}(F).$$

To prove (4.2.11) assume that

$$(4.2.13) \quad y \notin \bigcup_{j=0}^{n-1} b^{(j)}(F).$$

We must then show that  $y \in b^{(n)}(F)$ . For every  $0 \leq j \leq n-2$ , (4.2.13) implies that

$$y \notin b^{(j)}(F) \cup b^{(j+1)}(F),$$

and hence

$$a_2 a_3 \cdots a_n \cdot x \notin b^{(j)}(F).$$

Now it follows from (4.2.12) that

$$a_2 a_3 \cdots a_n \cdot x \in b^{(n-1)}(F).$$

We know from (4.2.13) that

$$y \notin b^{(n-1)}(F),$$

thus we must have

$$y \in b(b^{(n-1)}(F)) = b^{(n)}(F),$$

as we wanted to show. □

The following theorem states that the amenability of a Cayley graph, when associated to a group action, is independent of its generating.

**THEOREM 4.2.3.** *Let  $G$  be a finitely generated group and  $A$  and  $B$  be two sets of generators of  $G$ . Suppose  $G$  acts on a set  $X$ . Then  $\text{Cay}_A(G, X)$  is amenable if and only if  $\text{Cay}_B(G, X)$  is amenable.*

**PROOF.** Let each  $a \in A$  be represented as a reduced word in  $B$ , and let  $M$  be the length of the longest such words. Similarly, we will express each  $b \in B$  as a reduced word in  $A$ , with  $N$  representing the length of the longest such words. Let  $F$  be a finite subset of  $X$  and  $b_A(F)$  and  $b_B(F)$  denote the border of  $F$  in  $\text{Cay}_A(G, X)$  and  $\text{Cay}_B(G, X)$ , respectively. Let  $y \in b_B(F)$ , so that  $y \notin F$  but  $y = b \cdot x$  for some  $b \in B, x \in F$ . If  $b = a_1 a_2 \cdots a_n$  is a representation of  $b$  as a reduced word in  $A$ , then

$n \leq N$ , and  $y = a_1 a_2 \cdots a_n \cdot x$ . Now from Lemma 4.2.2, we have

$$y \in \bigcup_{j=1}^n b_A^{(j)}(F),$$

noting that  $y \notin F = b_A^{(0)}(F)$ . Since the above holds for every  $y \in b_B(F)$ , we obtain

$$b_B(F) \subset \bigcup_{j=1}^N b_A^{(j)}(F),$$

and consequently

$$|b_B(F)| \leq \sum_{j=1}^N |b_A^{(j)}(F)|.$$

It follows from the definition of the border that

$$|b_A(b_A(F))| \leq |A| |b_A(F)|,$$

and by repeated application of this inequality we get

$$|b_A^{(j)}(F)| \leq |A|^{j-1} |b_A(F)| \quad \text{where } 1 \leq j \leq N.$$

Hence

$$|b_B(F)| \leq \left( \sum_{j=0}^{N-1} |A|^j \right) |b_A(F)|.$$

By a similar argument, changing the roles of  $A$  and  $B$ , we can obtain

$$|b_A(F)| \leq \left( \sum_{j=0}^{M-1} |B|^j \right) |b_B(F)|.$$

If we write

$$C_1 = \left( \sum_{j=0}^{M-1} |B|^j \right)^{-1} \quad \text{and} \quad C_2 = \left( \sum_{j=0}^{N-1} |A|^j \right),$$

then clearly both  $C_1$  and  $C_2$  are nonzero, and the last two inequalities can be written as

$$(4.2.14) \quad C_1 |b_A(F)| \leq |b_B(F)| \leq C_2 |b_A(F)|.$$

Upon dividing by  $|F|$  we get

$$(4.2.15) \quad C_1 \frac{|b_A(F)|}{|F|} \leq |b_B(F)| \leq C_2 \frac{|b_A(F)|}{|F|}.$$

If we denote the expansion of  $\text{Cay}_A(G, X)$  by  $\gamma_A$  and the expansion of  $\text{Cay}_B(G, X)$  by  $\gamma_B$ , then it follows from (4.2.15) that

$$C_1 \gamma_A \leq \gamma_B \leq C_2 \gamma_A.$$

The claim of the theorem follows immediately.  $\square$

REMARK 4.2.4. For the special case of  $\text{Cay}(G)$ , we can see that a Cayley graph's amenability is independent of its generating set through the works of Soardi [26] in Theorem 7.34, as well as Bekka et al. [4] in Example 3.6.2(ii), and Grigorchuk [12].

Our next lemma provides us with a way of estimating the cardinality of the border of a set  $F$ .

LEMMA 4.2.5. *Let  $G$  be a finitely generated group and  $A$  be a finite symmetric set of generators of  $G$ . Suppose  $G$  acts on a set  $X$  and  $\text{Cay}(G, X)$  is the corresponding Cayley graph. Then for every nonempty finite subset  $F$  of  $X$ ,*

$$(4.2.16) \quad \frac{1}{2|A|} \sum_{a \in A} |a \cdot F \Delta F| \leq |b(F)| \leq \frac{1}{2} \sum_{a \in A} |a \cdot F \Delta F|.$$

PROOF. For each  $a \in A$ ,

$$(4.2.17) \quad \begin{aligned} |a \cdot F \Delta F| &= |a \cdot F - F| + |F - a \cdot F| \\ &= |a \cdot F - F| + |a \cdot (a^{-1} \cdot F - F)| \\ &= |a \cdot F - F| + |a^{-1} \cdot F - F|. \end{aligned}$$

Since

$$\bigcup_{a \in A} a \cdot F,$$

contains all vertices that are adjacent to vertices in  $F$ , we have

$$\bigcup_{a \in A} (a \cdot F - F) = b(F).$$

Using (4.2.17) and the fact that  $A = A^{-1}$ , we obtain

$$|b(F)| \leq \sum_{a \in A} |a \cdot F - F| = \frac{1}{2} \sum_{a \in A} |a \cdot F \Delta F|.$$

To prove the first inequality in (4.2.16), we write

$$\sum_{a \in A} |a \cdot F \Delta F| = 2 \sum_{a \in A} |a \cdot F - F| \leq 2|A| \max_{a \in A} |a \cdot F - F| \leq 2|A| |b(F)|.$$

□

**REMARK 4.2.6.** The estimation of  $|b(F)|$  seen in the above lemma, has been discussed in the works of Følner [10] and Bekka et al. [4] in the case that  $G$  acts on itself.

**THEOREM 4.2.7.** *Let  $G$  be a group acting on a set  $X$ . Then  $(G, X)$  is amenable if and only if for every finitely generated subgroup  $H$  of  $G$ ,  $\text{Cay}(H, X)$  is amenable.*

**PROOF.** If  $(G, X)$  is amenable, then  $(H, X)$  is amenable since Følner's condition (3.4.8) for  $(G, X)$  clearly implies the Følner's condition for  $(H, X)$ . Now let  $A$  be a finite symmetric generating set for  $H$ . Følner's condition for  $(H, X)$  implies that for a given  $0 < \epsilon$  there exists a finite set  $F \subset X$  such that

$$\frac{|a \cdot F \Delta F|}{|F|} \leq \frac{\epsilon}{|A|} \quad (a \in A).$$

It follows from (4.2.16) that

$$\frac{|b(F)|}{|F|} \leq \frac{1}{2} \sum_{a \in A} \frac{|a \cdot F \Delta F|}{|F|} \leq \frac{1}{2} \sum_{a \in A} \frac{\epsilon}{|A|} = \frac{\epsilon}{2}.$$

Thus  $\text{Cay}(H, X)$  is amenable.

To prove the converse, suppose  $\text{Cay}(H, X)$  is amenable for every finitely generated subgroup  $H$  of  $G$ . We shall verify that  $(G, X)$  satisfies the Følner's condition. Let



$A \subset G$  be a finite set. By enlarging  $A$  if necessary, we may assume that  $A$  is symmetric. Let  $H$  be the subgroup of  $G$  generated by  $A$ . By assumption  $\text{Cay}(H, X)$  is amenable and hence there must exist a finite set  $F \subset X$  with the property that

$$\frac{|b(F)|}{|F|} \leq \frac{\epsilon}{2|A|}.$$

Then using (4.2.16), for each  $a \in A$ ,

$$\frac{|a \cdot F \Delta F|}{|F|} \leq 2|A| \frac{|b(F)|}{|F|} \leq 2|A| \frac{\epsilon}{2|A|} = \epsilon.$$

Thus Følner's condition holds for  $(G, X)$ , and  $(G, X)$  is amenable.  $\square$

If we apply Theorem 4.2.7 to the special case that  $X = G$  and we use Theorem 4.2.1, we can obtain the following result.

**COROLLARY 4.2.8.** *A group  $G$  is amenable if and only if  $\text{Cay}(H)$  is amenable for every finitely generated subgroup  $H$  of  $G$ .*

**EXAMPLE 4.2.9.** Let  $F_2$  be the free nonabelian group on 2 generators. This group is nonamenable (Example 3.3.10) and hence by Corollary 4.2.8  $\text{Cay}(F_2)$  has nonzero expansion. It is not difficult to verify that  $\text{Cay}(F_2)$  is a 4-regular infinite tree. As a result,  $\text{Cay}(F_2)$  has expansion  $\gamma = 4 - 2$  (Theorem 2.1.7).  $\square$

We may now state the following, which follows from Theorem 3.4.7(i) and Theorem 4.2.7.

**COROLLARY 4.2.10.** *If  $G$  is an amenable group acting on  $X$ , then  $\text{Cay}(H, X)$  is amenable for every finitely generated subgroup  $H$  of  $G$ .*

In preparation for our next main result, we state the following.

**LEMMA 4.2.11.** *Let  $G$  be a finitely generated group and  $A$  be a finite symmetric set of generators of  $G$ . Suppose  $G$  acts on a set  $X$  and  $F$  is a finite subset of  $X$ .*

(i) *For every  $a \in A$  and  $s \in G$ ,*

$$(4.2.18) \quad as \cdot F \Delta F \subset (as \cdot F \Delta s \cdot F) \cup (s \cdot F \Delta F).$$

(ii) For every  $a \in A$ ,

$$(4.2.19) \quad |b(a \cdot F)| \leq (3 + |A|)|b(F)|.$$

PROOF. (i) The inclusions

$$as \cdot F - F \subset (as \cdot F - s \cdot F) \cup (s \cdot F - F),$$

and

$$F - as \cdot F \subset (F - s \cdot F) \cup (s \cdot F - as \cdot F),$$

imply directly that

$$(as \cdot F - F) \cup (F - as \cdot F) \subset (as \cdot F \Delta s \cdot F) \cup (s \cdot F \Delta F),$$

which proves (4.2.18).

(ii) It follows from the equality

$$a \cdot F = (a \cdot F \cap F) \cup (a \cdot F \cap F^c),$$

that

$$(4.2.20) \quad b(a \cdot F) \subset b(a \cdot F \cap F) \cup b(a \cdot F \cap F^c).$$

Next we estimate the sizes of the two sets on the right side of (4.2.20). It is easy to check that

$$b(a \cdot F \cap F) \subset b(F) \cup (F - a \cdot F) = b(F) \cup a \cdot (a^{-1} \cdot F - F).$$

From this it follows that

$$(4.2.21) \quad |b(a \cdot F \cap F)| \leq |b(F)| + |a^{-1} \cdot F - F| \leq 2|b(F)|.$$

Since  $a \cdot F \cap F^c \subset b(F)$ , it follows that

$$b(a \cdot F \cap F^c) \subset b(b(F)) \cup b(F),$$

and hence

$$(4.2.22) \quad |b(a \cdot F \cap F^c)| \leq |b(b(F))| + |b(F)| \leq |A||b(F)| + |b(F)| = (|A| + 1)|b(F)|.$$

Combining (4.2.20), (4.2.21) and (4.2.22) we get

$$|b(a \cdot F)| \leq (3 + |A|)|b(F)|,$$

as we wanted to show.  $\square$

In the case of finitely generated groups, our next theorem improves the result of Theorem 4.2.7 by showing that  $(G, X)$  and  $\text{Cay}(G, X)$  have the same Følner's sequences. Recall that for finitely generated groups, we may work with Følner's sequences instead of Følner's net (Remark 3.4.4).

**THEOREM 4.2.12.** *Let  $G$  be a finitely generated group acting on a set  $X$  and  $A$  be a finite symmetric generating set of  $G$ . A sequence  $(F_n)_n$  of finite subsets of  $X$  is a Følner's sequence of  $(G, X)$  if and only if it is a Følner's sequence of  $\text{Cay}(G, X)$ . In particular,  $(G, X)$  is amenable if and only if  $\text{Cay}(G, X)$  is amenable.*

**PROOF.** First we will prove the 'only if' part. Let  $(F_n)_n$  be a Følner's sequence for  $(G, X)$ . Using (4.2.16) and (3.4.9), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} &\leq \frac{1}{2} \lim_{n \rightarrow \infty} \sum_{a \in A} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} \\ &= \frac{1}{2} \sum_{a \in A} \lim_{n \rightarrow \infty} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} \\ &= 0. \end{aligned}$$

Thus  $(F_n)_n$  is a Følner's sequence for  $\text{Cay}(G, X)$ .

To prove the 'if' part, suppose that  $(F_n)_n$  is a Følner's sequence for  $\text{Cay}(G, X)$ . Note that for each  $a \in A$ , (4.2.16), implies that

$$|a \cdot F_n \Delta F_n| \leq 2|A||b(F_n)|,$$

from which it follows that

$$\lim_{n \rightarrow \infty} \frac{|a \cdot F_n \Delta F_n|}{|F_n|} \leq 2|A| \lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} = 0.$$

It remains to show that the above limit holds if  $a \in A$  is replaced by an arbitrary  $s \in G$ . Let  $s = a_1 a_2 \cdots a_k$  ( $a_i \in A$ ) be an arbitrary but fixed element of  $G$ . Let also  $s_i = a_i a_{i+1} \cdots a_k$ , for  $i = 1, \dots, k$  so that  $s_1 = s$  and  $s_k = a_k$ . Put  $s_{k+1} = e$ . Then for each  $n \in \mathbb{N}$ , using (4.2.18) repeatedly, we can write

$$(4.2.23) \quad s \cdot F_n \Delta F_n \subset \sum_{i=1}^k (s_i \cdot F_n \Delta s_{i+1} \cdot F_n).$$

Furthermore, by letting  $C = 3 + |A|$ , and using (4.2.16) and (4.2.19), we get

$$(4.2.24) \quad \begin{aligned} |s_i \cdot F_n \Delta s_{i+1} \cdot F_n| &\leq \sum_{a \in A} |a s_{i+1} \cdot F_n \Delta s_{i+1} \cdot F_n| \\ &\leq 2|A| |b(s_{i+1} \cdot F_n)| \\ &\leq 2|A| C^{k-i} |b(F_n)|. \end{aligned}$$

It follows from (4.2.23) and (4.2.24) that

$$\begin{aligned} |s \cdot F_n \Delta F_n| &\leq \sum_{i=1}^k |s_i \cdot F_n \Delta s_{i+1} \cdot F_n| \\ &\leq 2|A| \sum_{i=1}^k C^{k-i} |b(F_n)|. \end{aligned}$$

If we let

$$R = 2|A| \sum_{i=1}^k C^{k-i} = \frac{2|A|(C^k - 1)}{C - 1},$$

then for each  $n \in \mathbb{N}$ , we obtain

$$|s \cdot F_n \Delta F_n| \leq R |b(F_n)|,$$

where  $R$  is independent of  $n$ . Thus

$$\lim_{n \rightarrow \infty} \frac{|s \cdot F_n \Delta F_n|}{|F_n|} \leq R \cdot \lim_{n \rightarrow \infty} \frac{|b(F_n)|}{|F_n|} = 0,$$

which completes the proof that  $(F_n)_n$  is Følner's sequence for  $(G, X)$ .  $\square$

An interesting application of the above theorem is that the 'geometry' of the graph  $\text{Cay}(G, X)$  can be used in finding a Følner's sequence for  $(G, X)$ . This is illustrated in the following example.

EXAMPLE 4.2.13. Consider the abelian group  $\mathbb{Z}^n$ , generated by

$$A = \{\pm e_i : 1 \leq i \leq n\},$$

where  $e_i$  is the  $n$ -tuple with 1 in the  $i$ th coordinate and 0 elsewhere. The Cayley graph  $\text{Cay}(\mathbb{Z}^n)$  is the infinite lattice in  $\mathbb{R}^n$  whose vertices are the points in  $\mathbb{Z}^n$  and whose edges are the line segments of unit length parallel to the axes, joining the vertices. By Corollary 4.2.8,  $\text{Cay}(\mathbb{Z}^n)$  is amenable. To construct a Følner's sequence, let  $F_r$  be the set of vertices of this graph that are on or inside the closed ball  $B_r$  in  $\mathbb{R}^n$  of radius  $r > 0$  and center 0. Let  $N(r) = |F_r|$  be the number of lattice points that belong to  $B_r$ . We recall that if  $|B_r|$  is the volume (i.e., the  $n$ -dimensional Lebesgue measure) of  $B_r$ , then

$$|B_r| = |B_1|r^n = \frac{\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2} + 1)}r^n.$$

We can find estimates of  $N(r)$  with the help of a classical argument due to Gauss (cf. Hardy and Wright [13], de la Harp [14], p. 5-6). Each

$$v = (m_1, \dots, m_n) \in \mathbb{Z}^n,$$

uniquely identifies a unit cell

$$S_v = [m_1 - 1, m_1] \times \cdots \times [m_n - 1, m_n],$$

in  $\mathbb{R}^n$  which has  $v$  as its upper-right corner. If  $v \in B_r$ , then  $S_v \subset B_{r+\sqrt{n}}$ , and hence

$$N(r) \leq |B_{r+\sqrt{n}}| = |B_1|(r + \sqrt{n})^n.$$

Similarly, if

$$S_v \cap B_{r-\sqrt{n}} \neq \emptyset, \quad (r > \sqrt{n}),$$

then  $S_v \subset B_r$ , and hence

$$N(r) \geq |B_{r-\sqrt{n}}| = |B_1|(r + \sqrt{n})^n.$$

Thus

$$(4.2.25) \quad |B_1|(r - \sqrt{n})^n \leq N(r) \leq |B_1|(r + \sqrt{n})^n,$$

from which it follows that

$$N(r) = |B_1|r^n + O(r^{n-1}).$$

Next we estimate  $|b(F_r)|$ . If  $w \in b(F_r)$ , then  $w \notin B_r$ , but  $w$  is connected by an edge to some vertex  $v \in B_r$ . Thus  $w \in B_{r+\sqrt{n}}$ . Then using (4.2.25),

$$|b(F_r)| \leq N(r + \sqrt{n}) - N(r) \leq |B_1|(r + 2\sqrt{n})^n - |B_1|(r - \sqrt{n})^n,$$

from which it follows that

$$|b(F_r)| = O(r^{n-1}).$$

Therefore we have

$$\lim_{r \rightarrow \infty} \frac{|b(F_r)|}{|F_r|} = \lim_{r \rightarrow \infty} \frac{O(r^{n-1})}{|B_1|r^n + O(r^{n-1})} = 0.$$

It follows that  $(F_r)_{r \in \mathbb{N}}$  is a Følner's sequence for both  $\mathbb{Z}^n$  and  $\text{Cay}(\mathbb{Z}^n)$ .  $\square$

## CHAPTER 5

### Conclusion and Future Work

In this thesis we have studied the notion of amenability in contexts of groups, group actions and graphs. We have shown how Følner's conditions can be used to establish connections between these three notions of amenability. Amenable groups are defined through the existence of left invariant means in Section 3.3. The most important examples of amenable groups are finite and abelian groups. Amenable group actions are discussed in Section 3.4. Various characterizations of this amenability is given in Rosenblatt's Theorem 3.4.2. The relation between amenable groups and amenable group actions is discussed in Theorem 3.4.7 where it is shown that amenable group actions are more general than amenable groups. Amenability of graphs is discussed in Section 2.1, whereby a graph is called amenable if it has zero expansion. The relations between amenable group actions and amenable graphs constitute two main parts of this thesis and are discussed in Chapter 4. The main results state that if  $G$  is a finitely generated group acting on a set  $X$ , then  $(G, X)$  is amenable if and only if  $\text{Cay}(G, X)$  is amenable.

An interesting question for future work is whether or not Følner's condition is the most efficient way of detecting the amenability of a graph, and how a new characterization of amenability can benefit other fields of study. It would also be interesting to see if fixed point theory can be extended towards the notion of amenable Cayley graphs, and if so, can it be used to characterize the amenability of Cayley graphs.

## Appendix A

### Completion of Rosenblatt's Theorem

This appendix will be used to give a proof of the equivalence between (i) and (iii) in the statement of Theorem 3.4.2. To this end, we shall need some preliminary definitions and results.

If  $f \in \ell_{\mathbb{R}}^1(X)$  and  $g \in \ell_{\mathbb{R}}^{\infty}(X)$ , then

$$\langle f, g \rangle = \sum_{x \in X} f(x)g(x),$$

denotes the natural duality between  $\ell_{\mathbb{R}}^1(X)$  and  $\ell_{\mathbb{R}}^{\infty}(X)$ . We denote by  $\ell_{\mathbb{R}^+}^1(X)$  the space of all positive functions in  $\ell_{\mathbb{R}}^1(X)$ , and

$$P_X = \left\{ f \in \ell_{\mathbb{R}^+}^1(X) : \langle f, 1_X \rangle = \sum_{x \in X} f(x) = 1 \right\}.$$

The following definition is due to M. M. Day [6].

DEFINITION A.0.1. We say that a net  $(f_{\alpha})$  in  $P_X$  converges to a  $w^*$ -invariance if for all  $s \in G$  and  $h \in \ell^{\infty}(X)$ , we have

$$(A.0.26) \quad \lim_{\alpha} \langle f_{\alpha}, h - \lambda(s)h \rangle = 0,$$

where  $\lambda(s)h \in \ell^{\infty}(X)$  is defined by  $\lambda(s)h(x) = h(s^{-1}x)$ .

The following result follows from the Hahn-Banach separation theorem. For a proof we refer to Rosenblatt [24, Lemma 2.1].

LEMMA A.0.2. *Let  $E$  be a vector space of real-valued functions on a set  $X$ . If  $\theta$  is a positive linear functional on  $E$ , then there exists a net  $(F_{\gamma})$  of finite sequences in  $X$*



and a net  $(D_\gamma)$  in  $\mathbb{N}$  such that for all  $f \in E$ ,

$$D_\gamma^{-1} \sum_{x \in F_\gamma} f(x) \rightarrow \langle \theta, f \rangle.$$

DEFINITION A.0.3. A net  $(x_\gamma)$  in a set  $X$  is called a *universal net* if for all  $B \subset X$ , either  $(x_\gamma)$  is eventually in  $B$  (that is, there exists  $\gamma_0$  such that  $x_\alpha \in B$  for all  $\gamma \geq \gamma_0$ ), or  $(x_\gamma)$  is eventually in  $X - B$ .

It can be proved that every net has a universal subset, and moreover, in a Hausdorff topological space, any universal net that has an accumulation point is convergent to that accumulation point [19, p. 81].

An easy consequence of the above observation is found in the following.

LEMMA A.0.4. *Given a net of real linear functionals  $(l_\alpha)$  on a linear space  $E$ , such that for each  $x \in E$ ,  $(l_\alpha(x))$ , is eventually bounded, there exists a real linear functional  $l$  on  $E$  and a subnet  $(l_\gamma)$  of  $(l_\alpha)$  such that  $l_\gamma \rightarrow l$  pointwise on  $E$ .*

PROOF. Let  $(l_\gamma)$  be a universal subnet of  $(l_\alpha)$ . Then  $(l_\gamma(x))$  is a universal net in  $\mathbb{R}$  for all  $x \in E$ . Since  $(l_\alpha(x))$  is eventually bounded,  $(l_\gamma(x))$  is also eventually bounded. But then  $(l_\gamma(x))$  has at least one accumulation point. Since  $(l_\gamma(x))$  is universal, it converges to its accumulation point. Define

$$l(x) = \lim_{\gamma} l_\gamma(x).$$

Because  $(l_\gamma(x))$  converges for each  $x \in E$ ,  $l$  is a well-defined linear functional on  $E$ . □

The following technical result is a variation of the fact that in a locally convex space, the closure of a convex set coincides with its weak closure.

LEMMA A.0.5. *Let  $(E, \tau)$  be a locally convex space,  $S$  be a set of linear mappings of  $E$ , and  $C \subset E$  a convex set. Then there exists a net  $(x_\alpha)$  in  $C$ , such that  $x_\alpha - Tx_\alpha \rightarrow 0$  weakly for all  $T \in S$  if and only if there exists a net  $(y_\gamma)$  in  $C$ , such that  $y_\gamma - Ty_\gamma \rightarrow 0$  in the  $\tau$ -topology for all  $T \in S$ .*

Note that no continuity assumption is made on  $S$ . For a proof of this lemma, see [24, Lemma 4.8].

Now we have all of the means to prove the equivalence of statements (i) and (iii) in Theorem 3.4.2. This fact follows from the next result.

LEMMA A.0.6. *The following statements are equivalent:*

- (i) *There is an invariant mean for  $(G, X)$ .*
- (ii) *There exists a net  $f_\alpha \in \ell_{\mathbb{R}^+}^1(X)$  such that the net*

$$\phi_\alpha := \frac{1}{\langle f_\alpha, 1_X \rangle} f_\alpha,$$

*converges to a  $w^*$ -invariance.*

- (iii) *Følner's condition holds for  $(G, X)$ .*

PROOF. For a finite sequence  $F$  in  $X$ , let  $\chi_F = \sum_{x \in F} 1_{\{x\}}$ , where  $1_{\{x\}}$  is the characteristic function at  $\{x\}$  (so  $\chi_F = 1_F$  if there are no repetitions in  $F$ ). Note that  $\chi_F \in \ell_{\mathbb{R}^+}^1(X)$ .

(i)  $\implies$  (ii) Let  $\phi$  be an invariant mean for  $(G, X)$ . By Lemma A.0.2, there exists a net of finite sequences  $(F_\alpha)$  in  $X$ , and a net  $(D_\alpha)$  in  $\mathbb{N}$  such that  $D_\alpha^{-1} \chi_{F_\alpha} \rightarrow \phi$  pointwise on  $\ell_{\mathbb{R}}^\infty(X)$ . Since  $\langle \phi, 1_X \rangle = 1$ , it follows that

$$D_\alpha^{-1} \langle \chi_{F_\alpha}, 1_X \rangle \rightarrow 1,$$

and hence

$$\langle \chi_{F_\alpha}, 1_X \rangle^{-1} \chi_{F_\alpha} \rightarrow \phi,$$

pointwise on  $\ell_{\mathbb{R}}^\infty(X)$ . For every  $\alpha$ , let

$$\phi_\alpha = \langle \chi_{F_\alpha}, 1_X \rangle^{-1} \chi_{F_\alpha},$$

so that  $\phi_\alpha \in P_X$  for all  $\alpha$ . Since for every  $h \in \ell_{\mathbb{R}}^\infty(X)$ , and every  $s \in G$ ,

$$\langle \phi, h - \lambda(s)h \rangle = 0,$$

we get

$$\lim_{\alpha} \langle \phi_{\alpha}, h - \lambda(s)h \rangle = 0.$$

In other words,  $\phi_{\alpha}$  converges to  $w^*$ -invariance.

(ii)  $\implies$  (i) Let  $(f_{\alpha})$  be as in (ii). Since each  $f_{\alpha}$  is a positive function, we have

$$\|f\|_1 = \langle f, 1_X \rangle,$$

and hence for each  $h \in \ell^{\infty}(X)$ , we can write

$$|\langle f_{\alpha}, 1_X \rangle^{-1} \langle f_{\alpha}, h \rangle| \leq \langle f_{\alpha}, 1_X \rangle^{-1} \|h\|_{\alpha} \|f\|_1 \leq \|h\|_{\infty}.$$

For every  $\alpha$ , let

$$\phi_{\alpha} = f_{\alpha} / \langle f_{\alpha}, 1_X \rangle.$$

Then  $\phi_{\alpha}$  is a positive linear functional on  $\ell_{\mathbb{R}}^{\infty}(X)$ , and by above, for all  $h \in \ell_{\mathbb{R}}^{\infty}(X)$ , the net  $(\langle h, \phi_{\alpha} \rangle)$  is bounded. By Lemma A.0.4, there exists a subnet  $(\phi_{\gamma})$  of  $(\phi_{\alpha})$  and a linear functional  $\phi$  on  $\ell_{\mathbb{R}}^{\infty}(X)$  such that  $\phi_{\gamma} \rightarrow \phi$  pointwise on  $\ell_{\mathbb{R}}^{\infty}(X)$ . But since  $\phi_{\alpha} \geq 0$  and  $\langle \phi_{\alpha}, 1_X \rangle = 1$  for all  $\alpha$ , we have  $\phi \geq 0$  and

$$\|\phi_{\alpha}\|_1 = \langle \phi, 1_X \rangle = 1.$$

Also since  $\langle \phi_{\alpha}, \lambda(s)h - h \rangle \rightarrow 0$ , we have  $\langle \phi, \lambda(s)h - h \rangle = 0$ . So  $\phi$  is  $G$ -invariant. We have shown that  $\phi$  is an invariant mean for  $(G, X)$ .

(iii)  $\implies$  (ii) Let  $(F_{\alpha})$  be a Følner net in  $G$ , and let  $\phi_{\alpha} \in P_X$  be defined by

$$\phi_{\alpha} = |F_{\alpha}|^{-1} 1_{F_{\alpha}} = \frac{1}{\langle 1_{F_{\alpha}}, 1_X \rangle} 1_{F_{\alpha}}.$$

We shall prove that  $(\phi_{\alpha})$  converges to a  $w^*$ -invariance. For any  $h \in \ell_{\mathbb{R}}^{\infty}(X)$  and  $s \in G$ ,

$$\begin{aligned} |\langle \phi_{\alpha}, h - \lambda(s)h \rangle| &= |F_{\alpha}|^{-1} \left| \sum_{x \in F_{\alpha}} (h(x) - h(s^{-1} \cdot x)) \right| \\ &\leq |F_{\alpha}|^{-1} \sum_{z \in F_{\alpha} \Delta s^{-1} \cdot F_{\alpha}} |h(z)| \\ &\leq |F_{\alpha}|^{-1} \sum_{z \in F_{\alpha} \Delta s^{-1} \cdot F_{\alpha}} \|h\|_{\infty} \end{aligned}$$

$$\leq \|h\|_\infty \frac{|(F_\alpha \Delta s^{-1} \cdot F_\alpha)|}{|F_\alpha|} \rightarrow 0$$

since  $(F_\alpha)$  is a Følner's net. We have proved that  $(\phi_\alpha)$  converges to  $w^*$ -invariance.

(i)  $\implies$  (iii) Suppose  $(G, X)$  has an invariant mean  $\phi$  and we shall prove that  $(G, X)$  satisfies the Følner's condition. As we saw before in the proof of (i)  $\implies$  (ii), Lemma A.0.2 implies that there exist a net of finite sequences  $(F_\alpha)$  in  $X$  and a net  $(D_\alpha)$  in  $\mathbb{N}$  such that

$$D_\alpha^{-1} \chi_{F_\alpha} \rightarrow \phi,$$

pointwise on  $\ell_\mathbb{R}^\infty(X)$ . Since  $\langle \phi, 1_X \rangle = 1$ , it follows that  $D_\alpha^{-1} \langle \chi_{F_\alpha}, 1_X \rangle \rightarrow 1$ , and hence

$$\langle \chi_{F_\alpha}, 1_A \rangle^{-1} \chi_{F_\alpha} \rightarrow \phi,$$

pointwise on  $\ell_\mathbb{R}^\infty(X)$ . For every  $\alpha$ , let

$$f_\alpha = \langle \chi_{F_\alpha}, 1_X \rangle^{-1} \chi_{F_\alpha},$$

so that  $f_\alpha \in P_X$  for all  $\alpha$ . Since for every  $h \in \ell_\mathbb{R}^\infty(X)$ , and every  $s \in G$ ,

$$\langle \phi, h - \lambda(s)h \rangle = 0,$$

we get

$$\lim_\alpha \langle f_\alpha, h - \lambda(s)h \rangle = 0.$$

Since

$$\langle f_\alpha, h - \lambda(s)h \rangle = \langle f_\alpha, -\lambda(s^{-1})f_\alpha, h \rangle,$$

it follows that for every  $h \in \ell_\mathbb{R}^\infty(X)$ , and every  $s \in G$

$$\lim_\alpha \langle f_\alpha - \lambda(s)f_\alpha, h \rangle = 0.$$

Now let

$$\mathcal{F}_X = P_X \cap \{f \in \ell_\mathbb{R}^1(X) : \text{supp } f \text{ is finite}\}.$$

Thus, we have a net  $(f_\alpha)$  in the convex set  $\mathcal{F}_X \subset \ell_{\mathbb{R}}^1(X)$  such that for all  $s \in G$ ,

$$f_\alpha - \lambda(s)f_\alpha \rightarrow 0 \text{ weakly,}$$

i.e., in the topology  $\sigma(\ell^1(X), \ell^\infty(X))$ . But then Lemma A.0.5 implies that there is a net  $(h_\gamma)$  in  $\mathcal{F}_X$  such that

$$\|h_\gamma - \lambda(s)h_\gamma\|_1 \rightarrow 0,$$

for all  $s \in G$ .

To prove that the Følner's condition holds, we need to show that given  $\epsilon > 0$  and  $s_1, \dots, s_n \in G$ , there exists a finite set  $F \subset X$  such that

$$\frac{|(s_i \cdot F \Delta F)|}{|F|} < \epsilon \quad \text{for all } i = 1, \dots, n.$$

Let  $\gamma$  be fixed. Since  $h_\gamma \in \mathcal{F}_X$ , it follows that  $h$  has finite support and

$$\|h_\gamma\|_1 = \sum_{x \in X} h_\gamma(x) = 1.$$

Let

$$a_0 = 0 < a_1 < \dots < a_m,$$

be the distinct values of  $h_\gamma$ . For all  $1 \leq j \leq m$ , define

$$A_j = \{x \in X : a_{m-j+1} \leq h_\gamma(x)\}.$$

Then  $A_j \subset A_{j+1}$  for all  $j = 1, \dots, m-1$ , and

$$h_\gamma = a_1 1_{A_m} + (a_2 - a_1) 1_{A_{m-1}} + \dots + (a_m - a_{m-1}) 1_{A_1}.$$

By putting

$$\lambda_j = |A_j|(a_{m-j+1} - a_{m-j}),$$

for  $j = 1, \dots, m$ , we can express  $h_\gamma$  as

$$h_\gamma = \sum_{j=1}^m \frac{\lambda_j}{|A_j|} 1_{A_j},$$

where

$$\sum_{j=1}^m \lambda_j = 1,$$

since

$$\sum_{x \in X} h_\gamma(x) = 1.$$

Fix some  $s \in G$ . Let

$$B = \bigcup_{j=1}^m (A_j - s \cdot A_j).$$

Since  $A_j \subset A_{j+1}$  for all  $j = 1, \dots, m-1$ , we have

$$(A.0.27) \quad s \cdot A_j - A_j \subset X - B,$$

for all  $j = 1, \dots, m$ . To see this, fix  $1 \leq j \leq m$ . If  $x \in s \cdot A_j - A_j$ , then  $x \in s \cdot A_j$  and  $x \notin A_j$ , so  $x \in X - B$  provided we show  $x \notin B$ . However, since  $x \notin A_j$  we have

$$x \notin \bigcup_{i=1}^j (A_i - s \cdot A_i),$$

because  $A_i \subset A_j$  for all  $i \leq j$ ; and similarly, since  $x \in s \cdot A_j$  it follows that  $x \notin A_i - s \cdot A_i$  for  $i \geq j$ . Hence

$$x \notin \bigcup_{i=1}^m (A_i - s \cdot A_i) = B,$$

which proves A.0.27.

Since  $\lambda(s)1_{A_i} = 1_{s \cdot A_i}$ , we can write

$$\|\lambda(s)h_\gamma - h_\gamma\|_1 \geq \sum_{x \in X-B} \left| \sum_{i=1}^m \frac{\lambda_i}{|A_i|} (1_{s \cdot A_i}(x) - 1_{A_i}(x)) \right|.$$

Since each  $1_{s \cdot A_i} - 1_{A_i} \geq 0$  on  $X - B$  (in fact, for  $1_{s \cdot A_i}(x) - 1_{A_i}(x) < 0$ , we need  $x \in A_i - s \cdot A_i$  for some  $x \in X - B$  which is not possible since  $A_i - s \cdot A_i \subset B$ ), we get

$$\|\lambda(s)h_\gamma - h_\gamma\|_1 \geq \sum_{x \in X-B} \sum_{i=1}^m \frac{\lambda_i}{|A_i|} (1_{s \cdot A_i}(x) - 1_{A_i}(x))$$

$$= \sum_{i=1}^m \frac{\lambda_i}{|A_i|} \sum_{x \in X-B} (1_{s \cdot A_i}(x) - 1_{A_i}(x)).$$

Since each  $s \cdot A_i - A_i \subset X - B$ , and since as we saw above on  $X - B$ ,  $1_{s \cdot A_i} - 1_{A_i}(x) \geq 0$ , it follows that

$$\sum_{x \in X-B} (1_{s \cdot A_i}(x) - 1_{A_i}(x)),$$

has terms which are either 0 or 1, and the 1 occurs exactly when  $x \in s \cdot A_i - A_i$ , hence

$$\sum_{x \in X-B} (1_{s \cdot A_i} - 1_{A_i}(x)) = |s \cdot A_i - A_i|.$$

Thus

$$\|\lambda(s)h_\gamma - h_\gamma\|_1 \geq \sum_{i=1}^m \frac{\lambda_i}{|A_i|} |s \cdot A_i - A_i|.$$

Similarly,

$$\begin{aligned} \|\lambda(s^{-1})h_\gamma - h_\gamma\|_1 &= \sum_{x \in X} |\lambda(s^{-1})h_\gamma(x) - h_\gamma(x)| \\ &= \sum_{x \in X} |h_\gamma(s \cdot x) - h_\gamma(x)| \\ &= \sum_{x \in X} |h_\gamma(x) - h_\gamma(s^{-1} \cdot x)| \\ &= \sum_{x \in X} |\lambda(s)h_\gamma(x) - h_\alpha(x)|. \end{aligned}$$

As we argued before, we have

$$\begin{aligned} \|\lambda(s^{-1})h_\gamma - h_\gamma\|_1 &= \sum_{x \in X} |\lambda(s)h_\gamma(x) - h_\gamma(x)| \\ &\geq \sum_{x \in X-B} \sum_{i=1}^m \frac{\lambda_i}{|A_i|} (1_{s \cdot A_i}(x) - 1_{A_i}(x)) \\ &= \sum_{i=1}^m \frac{\lambda_i}{|A_i|} |s \cdot A_i - A_i| \\ &= \sum_{i=1}^m \frac{\lambda_i}{|A_i|} |A_i - s^{-1} \cdot A_i|. \end{aligned}$$

Choose  $s_1, \dots, s_n \in G$  and  $\epsilon > 0$ . By enlarging the set  $s_1, \dots, s_n$  if necessary, we may assume (without loss of generality) that this set contains the inverses of its elements.

Let  $\gamma$  be large enough so that:

$$\epsilon > \sum_{i=1}^n (\|\lambda(s_i)h_\gamma - h_\gamma\|_1 + \|\lambda(s_i^{-1})h_\gamma - h_\gamma\|_1).$$

Then by the estimates above:

$$\begin{aligned} \epsilon &\geq \sum_{i=1}^n \sum_{j=1}^m \frac{\lambda_j |s_i \cdot A_j - A_j|}{|A_j|} + \sum_{i=1}^n \sum_{j=1}^m \frac{\lambda_j |A_j - s_i^{-1} \cdot A_j|}{|A_j|} \\ &= \sum_{j=1}^m \frac{\lambda_j}{|A_j|} \left( \sum_{i=1}^n |s_i \cdot A_j - A_j| + \sum_{i=1}^n |A_j - s_i^{-1} \cdot A_j| \right) \\ &= \sum_{j=1}^m \frac{\lambda_j}{|A_j|} \sum_{i=1}^n (|s_i \cdot A_j \Delta A_j|) \\ &= \sum_{j=1}^m \lambda_j \sum_{i=1}^n \frac{|s_i \cdot A_j \Delta A_j|}{|A_j|}, \end{aligned}$$

where the penultimate equality follows since  $\{s_1, \dots, s_n\}$  is symmetric.

Since  $\sum_{j=1}^m \lambda_j = 1$ , there must exist at least one  $1 \leq j \leq m$  such that

$$\epsilon > \sum_{i=1}^n \frac{|s_i \cdot A_j \Delta A_j|}{|A_j|}.$$

Letting  $F = A_j$ , we get

$$\frac{|s_i \cdot F \Delta F|}{|F|} < \epsilon \text{ for all } i = 1, 2, \dots, n,$$

which is the required Følner's condition.  $\square$



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