2019

# Recovering a Random Variable from Conditional Expectations Using Reconstruction Algorithms for the Gauss Radon Transform

Jeremy Becnel
*Stephen F Austin State University*, becneljj@sfasu.edu

Daniel Riser-Espinoza
*Wester EcoSystems Technology, Inc.*

**Asian Journal of Probability and Statistics**

**3(1): 1-31, 2019; Article no.AJPAS.46373**

# Recovering a Random Variable from Conditional Expectations Using Reconstruction Algorithms for the Gauss Radon Transform

**Jeremy Becnel[1*] and Daniel Riser-Espinoza[2]**

[1]*Department of Mathematics, Stephen F. Austin State University, Nacogdoches, TX, 75962-3040, USA.*
[2]*Wester EcoSystems Technology, Inc., Fort Collins, Colorado, USA.*

***Authors' contributions***

*This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.*

***Article Information***

DOI: 10.9734/AJPAS/2019/v3i130081

*Editor(s):*

(1) Dr. Manuel Alberto M. Ferreira, Professor, Department of Mathematics, ISTA-School of Technology and Architecture, Lisbon University, Portugal.

(2) Dr. Mervat Mahdy Ramadan Mahdy, Associate Professor, Department of Statistics, Mathematics, Insurance, College of Commerce, Benha University, Egypt.

*Reviewers:*

(1) A. V. Narasimhadhan, National Institute of Technology Karnataka, India.

(2) R. Harikumar, Bannari Amman Institute of Technology, India.

(3) Mohamed Reda Ali, Benha University, Egypt.

Complete Peer review History: http://www.sdiarticle3.com/review-history/46373

***Received: 02 November 2018***

***Accepted: 08 January 2019***

***Published: 14 February 2019***

***Original Research Article***

## Abstract

The Radon transform maps a function on $n$-dimensional Euclidean space onto its integral over a hyperplane. The fields of modern computerized tomography and medical imaging are fundamentally based on the Radon transform and the computer implementation of the inversion, or reconstruction, techniques of the Radon transform. In this work we use the Radon transform with a Gaussian measure to recover random variables from their conditional expectations. We derive reconstruction algorithms for random variables of unbounded support from samples of conditional expectations and discuss the error inherent in each algorithm.

*Keywords: Radon transform; reconstruction.*

2010 Mathematics Subject Classification: 53C25; 83C05; 57N16

**Corresponding author: E-mail: becneljj@sfasu.edu*

# 1 Introduction

In 1917 Johann Radon showed that one could determine a function (with domain in $\mathbb{R}^2$) completely from knowledge of the density along all lines through the domain of the function; equivalently, if we know the integral along each line through a function we can recover the original function [1]. These integrals are collectively called the *Radon transform*. Radon's pioneering work was later generalized and extended to higher dimensions and remains an active area of research at the confluence of mathematical tomography, integral geometry, and inverse problems; see [2, 3, 4]. His work later led to some extraordinary technologies including CT scans.

The problem of reconstructing objects from a finite number of samples of the Radon transform is at the heart of modern tomographic imaging and a large focus of this manuscript. Knowing that it is possible to reconstruct an object from its Radon transforms is quite different from actually being able implement a process which accurately and reliably accomplishes this end from a finite data set.

In this paper we endeavor to build off of the work of Becnel and Chang in [5] and further develop an application of the Radon transform to probability theory: recovering a random variable from its conditional expectations. Hence, in Section 2 we begin with a brief introduction to the Radon transform, its properties and relevant results. We present an algorithm for recovering a function from samples of the function's Radon transform via a powerful result relating the Radon transform of a function to the Fourier transform of that function. Next we will discuss the geometric dual of the Radon transform, the backprojection operator, and its relationship to the filtered backprojection algorithm–one of the primary algorithms used to reconstruct functions from Radon transform samples. Following the description of the Fourier algorithm, we provide a brief remark on the sources of error; we conclude the chapter with a theorem giving the error components of the filtered backprojection algorithm, and a brief discussion of the sources of those error components.

In Section 3, we present results found primarily in [5] establishing a Gaussian measure on the hyperplane in $\mathbb{R}^n$ and defining the Gauss Radon transform. Again, we present most results without proof; the interested reader is directed to [5]. In particular, we present results that relate the Gauss Radon transform to the Radon transform, and a representation of the Gauss Radon transform as the conditional expectation of a random variable defined over an appropriate probability space.

In the final chapter, we adapt the Fourier and filtered backprojection algorithms to the recovery of a random variable from samples of its conditional expectation. We also develop results that allow us to establish error estimates when the random variable (function) of interest is not of compact support—a common assumption in Radon transform reconstruction problems. The chapter concludes with a full error analysis of the Gauss filtered backprojection algorithm, along with some heuristics for minimizing each error component in practice. We part with a few remarks on the future of this research.

We have also included an appendix which catalogs the necessary results outside the purview of the contents of this article. For the sake of brevity we have omitted proofs for most results; in lieu of proof for these results we have one or more references.

# 2 Radon Transform and Reconstruction

In this chapter, we begin by introducing the Radon transform and describing its action on functions of a suitable variety. We state some of the properties of the Radon transform that are integral to the analysis ahead. After defining the Radon transform, we will introduce two of the most prominent methods of reconstruction developed in the literature of mathematical tomography: Fourier Reconstruction and Filtered Backprojection. The mathematical results motivating each reconstruction method will be presented; the reader is referred to the appendix for several auxiliary results necessary to obtain the algorithms. In conclusion we will discuss the error incurred by the reconstruction algorithms and the dependence of that error on data filtering and sampling schemes.

## 2.1 The Radon Transform

We denote the set of hyperplanes in $\mathbb{R}^n$ as $\mathbb{P}^n$. That is,

$$\mathbb{P}^n = \{\alpha\theta + \theta^\perp \,;\, \alpha \in \mathbb{R}, \theta \in \mathbb{R}^n \text{ is a unit vector}\}.$$

where in the above $\theta^\perp$ is the orthogonal complement of the the singleton set $\{\theta\}$ containing the unit vector $\theta$. Observe that each hyperplane $\alpha\theta + \theta^\perp$ is specified by the parameters $\alpha$ and $\theta$. We may think of $\theta$ as the vector normal to the hyperplane, while $|\alpha|$ represents the distance from the hyperplane to the origin. At times we may find it convenient to use an alternative representation of the hyperplane $\alpha\theta + \theta^\perp$:

$$\alpha\theta + \theta^\perp = \{x \in \mathbb{R}^n \,;\, \langle x, \theta\rangle = \alpha\}.$$

From this representation we gain a conceptually nice interpretation of a hyperplane as the set of all vectors with "$\theta$ direction coordinate" equal to $\alpha$.

**Definition 2.1.** The *Radon transform* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a function $\mathcal{R}_f$ on the set $\mathbb{P}^n$ given by

$$\mathcal{R}_f(\alpha\theta + \theta^\perp) = \int_{\alpha\theta+\theta^\perp} f(x)\, dx \tag{2.1}$$

where $dx$ is the Lebesgue measure on the hyperplane $\alpha\theta + \theta^\perp$.

*Remark* 2.1. From the definition above, we can think of the Radon transform as a function of $\alpha$ and $\theta$. Henceforth we will typically write the Radon transform of $f$ as $\mathcal{R}_f(\alpha, \theta)$. There are several ways to express the Radon transform. At times we may find it more convenient to use one of the following equivalent expressions:

$$\mathcal{R}_f(\alpha, \theta) = \mathcal{R}_f(\alpha\theta + \theta^\perp) = \int_{\theta^\perp} f(\alpha\theta + y)dy = \int_{\langle x,\theta\rangle=\alpha} f(x)dx.$$

When we apply the Radon transform to a function, we are integrating that function over all of the points in a plane defined by the pair $(\alpha, \theta)$. Lacking restrictions on $f$ the function $\mathcal{R}_f$ may not exist for every element of $\mathbb{P}^n$. To ensure that the $\mathcal{R}_f$ is defined and finite for every element of $\mathbb{P}^n$, one typically assumes that $f$ is *rapidly decreasing*, i.e.

$$\sup_{x\in\mathbb{R}^n} |x|^k |f(x)| < \infty \qquad \text{for all } k > 0$$

or that $f$ is in the Schwartz space $\mathscr{S}(\mathbb{R}^n)$.

Alternatively, we can also assure the existence of $\mathcal{R}_f$ by assuming that the function $f$ is infinitely differentiable with compact support; that is $f \in C_0^\infty(\mathbb{R}^n)$. This is a commonly used assumption in much of the literature on reconstruction, justified for practical purposes. In such applications, the function $f$ sometimes represents the density of an organ such as the brain which, at least for most of us, is of finite extent. Thus for most practical purposes, those studying the Radon transform have been principally interested in deriving algorithms and error bounds for compactly supported functions. Foreshadowing to later chapters, we will endeavor to remove the assumption of compact support in the derivation of reconstruction algorithms and error bounds for the Gauss Radon transform.

## 2.2 Fourier Reconstruction

The Fourier reconstruction approach exploits a powerful result, the Fourier Slice Theorem. The algorithm to follow is essentially a numerical approximation to this algorithm broken down into four distinct steps. Before delving into the algorithm, let us state the two main results we will need. For proofs the interested reader may consult [5, 6, 2].

**Theorem 2.1.** *(Fourier Slice Theorem). The n-dimensional Fourier transform of a function $f \in \mathscr{S}(\mathbb{R}^n)$ is equivalent to the one-dimensional Fourier transform of $\mathcal{R}_f(\alpha, \theta)$. That is,*

$$\widehat{\mathcal{R}}_f(\alpha, \theta) = (2\pi)^{\frac{n-1}{2}} \hat{f}(\alpha\theta), \tag{2.2}$$

*where $\widehat{\mathcal{R}}_f(\alpha,\theta)$ is understood to the be the Fourier transform with respect to the first argument $\alpha \in \mathbb{R}$, treating $\theta$ as a fixed constant.*

The Fourier Slice Theorem, along with the inverse Fourier transform (see appendix), can be used to derive an inversion formula for the Radon transform. See [5, 6, 7]

**Theorem 2.2.** *The inversion formula for a function $f \in \mathscr{S}(\mathbb{R}^n)$ in terms of the Radon transform is*

$$f(x) = (2\pi)^{(\frac{1}{2}-n)} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{R}_f(\alpha,\theta) e^{-i\beta(\alpha-\langle\theta,x\rangle)} \beta^{n-1} d\alpha d\beta d\sigma(\theta). \tag{2.3}$$

## 2.3 Fourier Reconstruction Algorithm

The idea behind this algorithm is to (1) gather Radon transform samples from a polar grid covering the support of the function, (2) approximate the Fourier transform of the Radon data, (3) use Fourier transform data from (2) to approximate the Fourier transform of the original function, and (4) then perform a discrete Fourier inversion to recover the original function.

*Algorithm—Fourier Reconstruction*: Let $f \in C_0^\infty(\Omega_r^n)$, where $\Omega_r^n$ is the closed ball of radius $r$ in $\mathbb{R}^n$, centered at the origin.

*Step 1:* Sample the Radon transform of $f$ at $(s_l,\theta_j)$ for $j = 1,...,p$ ; $l = -q,...,q$, obtaining samples $\mathcal{R}_f(s_l,\theta_j)$. Here $\theta_j \in S^{n-1}$, and $s_l = hl, h = r/q$.

Note that in this case we are following a "naive" sampling scheme sampling at points evenly spaced $r/q$ units apart in each direction $\theta_j$.

*Step 2:* Approximate $\hat{f}$. This is accomplished by a discrete approximation to the Fourier transform of $\widehat{\mathcal{R}}_f$ based on the sampled points above, followed by application of the Fourier Slice Theorem (2.1). For $j = 1,...,p$, approximate $\widehat{\mathcal{R}}_f(r\pi,\theta_j)$ by $\widehat{\mathcal{R}}_f^{(j,r)}$, where

$$\widehat{\mathcal{R}}_f^{(j,r)} = (2\pi)^{-\frac{1}{2}}\ h \sum_{l=-q}^{q-1} e^{i\pi rl/q} \mathcal{R}_f(s_l,\theta_j)\ ,\quad r = -q,...,0,...,q-1. \tag{2.4}$$

Note that (2.4) is simply the discrete Fourier transform of $\mathcal{R}_f$ in $\mathbb{R}$. Also notice that we took Radon transform samples from points in $\Omega_r^n$ and now have an approximation to the Fourier transform of $f$ on the polar grid

$$G_{p,q} = \{\pi r\theta_j :\ \ r = -q,...,q-1\ ;\ j = 1,...,p\}.$$

Hence

$$\widehat{f}(r\pi\theta_j) = (2\pi)^{\frac{1-n}{2}} \widehat{\mathcal{R}}_f^{(j,r)}, \tag{2.5}$$

up to discretization errors.

*Step 3 (Interpolation)*: This step is necessary in order to make use of a fast Fourier transform (FFT) algorithm; otherwise the Fourier algorithm cannot compete in computational efficiency with other reconstruction algorithms. The FFT cannot be used on the polar grid $G_{p,q}$ [3]. Thus we must interpolate to a suitable Cartesian coordinate grid. For now we will only consider nearest neighbor interpolation.

For each $k \in \mathbb{Z}^n$, $|k| < q$, find the point $\xi_k = \pi r\theta_j$ in the polar grid closest to $\pi k$ and set

$$\hat{f}_k = (2\pi)^{\frac{1-n}{2}} \widehat{\mathcal{R}}_f^{(j,r)} \tag{2.6}$$

*Step 4*: approximate $f(hm), m \in \mathbb{Z}^n$, using a discrete inverse Fourier transform,

$$f_m = \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \hat{f}_k\ , |m| < q, \tag{2.7}$$

where 2.7 is the discrete inverse Fourier transform in $\mathbb{R}^n$.

## 2.4 Filtered Backprojection

An alternative to the Fourier algorithm, the filtered backprojection algorithm exploits a key relationship between the Radon transform, backprojection operator, and convolution. First, we define the backprojection operator, the dual of the Radon transform. We will go on to characterize the backprojection operator in terms of integral geometry in relation to the Radon transform.

**Definition 2.2.** Let $g : S^{n-1} \times \mathbb{R} \to$. We define the *backprojection operator* to be the function $\mathcal{R}^{\#}$, $\mathcal{R}^{\#}g : \mathbb{R}^n \to$ defined by

$$\mathcal{R}^{\#}g(x,\theta) = \int_{S^{n-1}} g(\langle x,\theta\rangle,\theta)d\theta \tag{2.8}$$

Recall that $\mathcal{R}$ acts on a function $f$ by integrating over all points in a hyperplane. The backprojection operator $\mathcal{R}^{\#}$ forms a natural dual to $\mathcal{R}$ in terms of integral geometry since it integrates over all planes through a point. Furthermore, it turns out that the backprojection operator is the *formal adjoint* of the Radon transform [2].

**Theorem 2.3.** *Let $\langle f,g\rangle_{L^2} = \int_{\mathbb{R}^n} f(x)g(x)dx$ and let $f(x), g(x) \in L^2(\mathbb{R}^n)$, where $f$ and $g$ are assumed to be nonnegative and measurable functions. Then*

$$\langle \mathcal{R}_f, g\rangle_{L^2} = \langle f, \mathcal{R}^{\#}g\rangle_{L^2}. \tag{2.9}$$

At this point we might ask if it is possible to recover a function from its Radon transform using the backprojection operator. The next theorem, which can be found in [2, 3], suggests that any attempt to reconstruct $f$ using this strategy will yield poor results.

**Theorem 2.4.** *(Backprojection Theorem). If $f$ is a nonnegative Lebesgue measurable function on $\mathbb{R}^n$ then*

$$\mathcal{R}^{\#}\mathcal{R}_f = T \star f, \tag{2.10}$$

*where*

$$T(x) = \left|S^{n-2}\right| \frac{1}{|x|},$$

*and $\left|S^d\right|$ is the "surface area" of the unit sphere in $\mathbb{R}^d$.*

While it is perhaps unfortunate that $\mathcal{R}^{\#}$ alone failed to achieve reasonable reconstruction, the following theorem leads us to a very successful approach known as filtered backprojection.

**Theorem 2.5.** *Let $f \in \mathscr{S}(\mathbb{R}^n)$ and $w \in \mathscr{S}(\mathbb{R} \times S^{n-1})$, where $w$ is bounded and measurable. Then*

$$(\mathcal{R}^{\#}w) \star f = \mathcal{R}^{\#}(w \star \mathcal{R}_f). \tag{2.11}$$

*Proof.* Using the definition of convolution (see 6.1) and the backprojection operator we have

$$\begin{aligned}(\mathcal{R}^{\#}w \star f)(x) &= \int_{\mathbb{R}^n} \mathcal{R}^{\#}w(x-y)f(y)dy \\ &= \int_{\mathbb{R}^n}\int_{S^{n-1}} w(\langle (x-y),\theta\rangle,\theta)d\theta f(y)dy \\ &= \int_{S^{n-1}}\int_{\mathbb{R}^n} w(\langle (x-y),\theta\rangle,\theta)f(y)dyd\theta,\end{aligned}$$

where in the last step we used the Fubini Theorem to change the order of integration. To disintegrate the integral over $\mathbb{R}^n$ to the integral of hyperplanes moving through $\mathbb{R}$, let $y = s\theta + z$, $z \in \theta^\perp$. Hence

$$\begin{aligned}\int_{S^{n-1}}\int_{\mathbb{R}^n} w(\langle(x-y),\theta\rangle,\theta)f(y)dyd\theta &= \int_{S^{n-1}}\int_{\mathbb{R}}\int_{\theta^\perp} w(\langle x,\theta\rangle - s,\theta)f(s\theta+z)dzdsd\theta \\ &= \int_{S^{n-1}}\int_{\mathbb{R}} w(\langle x,\theta\rangle - s,\theta)\mathcal{R}_f(s,\theta)dsd\theta \\ &= \int_{S^{n-1}} (w \star \mathcal{R}_f)(\langle x,\theta\rangle,\theta)d\theta \\ &= \mathcal{R}^\#(w \star \mathcal{R}_f)(x),\end{aligned}$$

which is the desired result. □

We can think of the term $\mathcal{R}^\# w$ as a "filter" for the backprojection defined above. In the next subsection, we will investigate some of the desirable properties for this filter, and how we might practically use such a filter to reconstruct a function from its Radon transform.

## 2.5 Filtered Backprojection Algorithm

The filtered backprojection algorithm is a numerical implementation of

$$W_b(\xi) \star f = \mathcal{R}^\#(w_b \star \mathcal{R}_g)\ ,\ \ W_b = \mathcal{R}^\# w_b. \tag{2.12}$$

In order to use Theorem 2.5 to accurately reconstruct a function $f$, we choose $w_b$ so that $W_b$ approximates the $\delta$-distribution. More specifically, in most literature $W_b$ is called a low-pass filter with cutoff frequency $b$. Accordingly, we can represent $W_b$ as

$$\widehat{W_b}(\xi) = (2\pi)^{-\frac{n}{2}}\hat{\Phi}\left(\frac{|\xi|}{b}\right), \tag{2.13}$$

where:

1. $0 \le \hat{\Phi} \le 1$
2. $\hat{\Phi}\left(\frac{|\xi|}{b}\right) = 0\ for\ \frac{|\xi|}{b} \ge 1.$

Observe that as $b$ gets arbitrarily large, $\hat{\Phi}$ approaches a constant, which is the Fourier transform of $\delta_0$ up to a constant multiple $(2\pi)^{-\frac{n}{2}}$.

*Algorithm—Filtered Backprojection*: Let $f \in C_0^\infty(\Omega_r^n)$.

*Step 1*: We first sample the Radon transform of $f$ at points $(s_l,\theta_j)$ , $j = 1,2,...,p$ , $l = -q,...,q$ , $\theta_j \in S^{n-1}$ , $s_l = hl = \frac{r}{q}l.$

*Step 2*: Perform discrete convolution of $w_b \star \mathcal{R}_f(s_l,\theta_j)$. That is

$$w_b \overset{h}{\star} \mathcal{R}_f(\sigma,\theta_j) = h\sum_{l=-q}^{q} w_b(\sigma - s_l)\mathcal{R}_f(s_l,\theta_j). \tag{2.14}$$

In this step we are performing a 1-dimensional convolution of $w_b$ and $\mathcal{R}_f(s_l,\theta_j)$ for each direction $\theta_j$.

*Step 3 (Discrete Backprojection)*: We now require a quadrature rule on $S^{n-1}$, based on the choice of $\theta_1,\theta_2,...,\theta_j$, with positive weights $\alpha_{pj}$. Furthermore we assume that this quadrature rule is exact in $H'_{2m}$, the even spherical harmonics of degree $2m$, for some $m$ (see [3]). Hence

$$\int_{S^{n-1}} \nu(\theta)d\theta = \sum_{j=1}^{p}\alpha_{pj}\nu(\theta_j),\ \ for\ \nu \in H'_{2m} \tag{2.15}$$

Thus we may interchange the backprojection on the right side of 2.5 for the discrete backprojection defined by $\mathcal{R}_p^\#$

$$\mathcal{R}_p^\# v(x) = \sum_{j=1}^{p} \alpha_{pj}\, v(\theta_j, x\cdot\theta_j). \tag{2.16}$$

Therefore

$$f_{FB} = R_p^\#(w_b \overset{h}{\star} \mathcal{R}_f(\sigma,\theta_j)) = \sum_{j=1}^{p} \alpha_{pj}\, h \sum_{l=-q}^{q} w_b(\sigma - s_l)\mathcal{R}_f(s_l,\theta_j) \tag{2.17}$$

where $f_{FB}$ is the filtered backprojection algorithm approximation of $f$.

## 2.6 Error Analysis

### 2.6.1 Fourier Error

Following the development of these reconstruction algorithms, it is natural to ask what kind of error each algorithm incurs. For the sake of brevity we omit a detailed analysis of the error in the Fourier algorithm; however we will provide a summary of the relevant sources of error. As delineated in [3], the Fourier algorithm only has one major source of error: the interpolation step. Assuming that the Radon transform is properly sampled, according the Nyquist–Shannon sampling theorem (Theorem 6.6), steps 1, 2, and 4 are effectively error free. Interpolating from the polar grid $G_{p,q}$ to the Cartesian grid $\mathbb{Z}^n$ is necessary in order to use a FFT algorithm–otherwise the Fourier algorithm cannot compete in computational efficiency with other algorithms. In the algorithm above we took a naive approach, using nearest neighbor interpolation. It turns out that this method produces large artifacts in the recovered function. In Chapter 5 of [3], the author derives the necessary sampling geometry to minimize the error incurred by interpolating to the Cartesian grid, and provides examples in $\mathbb{R}^2$.

## 2.7 Filtered Backprojection Error

We have established the Filtered Backprojection Algorithm for recovering the function $f$ from its Radon transform sampled at discrete points in $\Omega_r^n$. It is natural to investigate the error incurred by this numerical routine and establish an upper bound on this error. The following theorems adapted from [3] provides an error bound for the filtered backprojection algorithm applied to compactly supported functions on the ball of radius $r$.

**Theorem 2.6.** *Let $w(\theta,\alpha) \in \mathscr{S}(S^{n-1}\times\mathbb{R})$. Then*

$$\widehat{\mathcal{R}^\# w}(\xi) = (2\pi)^{\frac{n-1}{2}}\, |\xi|^{1-n}\left(\hat{w}\left(\frac{\xi}{|\xi|}\,,\ |\xi|\right) + \hat{w}\left(\frac{-\xi}{|\xi|}\,,\ -|\xi|\right)\right). \tag{2.18}$$

**Theorem 2.7.** *Let $f \in C_0^\infty(\Omega_r^n)$. Assume that* (2.15) *holds on $H'_{2m}$ and that, for some $\vartheta$ with $0<\vartheta<1$,*

$$b \le \vartheta m,\ \ b \le \pi/h. \tag{2.19}$$

*Define the approximate reconstruction of $f$ as*

$$f_{FB} = W_b \star f + e_1 + e_2 = \mathcal{R}^\# w_b \star \mathcal{R}_f + e_1 + e_2, \tag{2.20}$$

*where*

$$e_1 = \mathcal{R}_p^\#(w_b \overset{h}{\star} \mathcal{R}_f - w_b \star \mathcal{R}_f)$$

*and*

$$e_2 = (\mathcal{R}_p^\# - \mathcal{R}^\#)(w_b \star \mathcal{R}_f).$$

*Then,*

$$|e_1| \le \frac{1}{2}(2\pi)^{\frac{-(n+1)}{2}}|S^{n-1}| \sup_{\theta\in S^{n-1}} \int_{|\alpha|\ge b} |\alpha|^{n-1}|\hat{f}(\alpha\theta)|d\alpha \tag{2.21}$$

*and*

$$\begin{aligned}|e_2| &\le 2\left|S^{n-1}\right|(2\pi)^{\frac{-(n)}{2}}(r)^{\frac{-n}{2}}\|f\|_{L_\infty(\Omega_r^n)} \sum_{l>m} \frac{N(n,l)}{c(n,l)} l^{\frac{n}{2}} \eta_1(\vartheta, 2l+k) \\ &\le \left|S^{n-1}\right|(2\pi)^{\frac{-(n)}{2}}(r)^{\frac{-n}{2}}\|f\|_{L_\infty(\Omega_r^n)}\eta(\vartheta, m)\end{aligned}$$

This Theorem 2.7 suggests several sources of error in the reconstruction of $f$ from its Radon transform in practice. The term $e_1$ includes the error contributed by using a discrete approximation to the convolution on the right side of Theorem 2.5. Observe that $e_1$ is a function of $f$ and $b$; note that in practice we only have control over $b$ through the sampling distance $h$. If $f$ is $b$-band limited or *essentially* $b$-band limited (see section 6.2) then the integral bounding $e_1$ will be negligible. In practice we do not know $f$ but if we choose $b$ large enough (or $h$ small enough) then we make $e_1$ small. The assumption $b \le \frac{\pi}{h}$ ensures that the discrete convolutions and Fourier transforms are essentially error-free.

The $e_2$ error term results from discretizing the backprojection in 2.5. The interpretation of this term is not readily apparent from the statement of the theorem. The assumption that $b \le \vartheta m$ ensures that the discrete backprojection is indeed exact on $H'_{2m}$. The derivation of $e_2$ involves expanding $w_b \star \mathcal{R}_f$ in terms of a series involving *spherical harmonics*. The series must be truncated after finitely many terms. It turns out that the tail of the series being truncated is bounded by the exponentially decaying term $\eta(\vartheta, m)$. The interested reader can refer to the proof of 2.7 found in [3] for an unbowdlerized account of the error terms.

# 3 Gauss Radon Transform

In this chapter we will construct the Gauss Radon transform and discuss its relationship to the Radon transform. Motivation for the Gauss Radon transform will be provided along with illustrative examples. For this chapter we have borrowed heavily from [5]. The interested reader is directed to this paper for further details.

## 3.1 Gaussian Measure

To construct the Gauss Radon transform, we must first construct a Gaussian measure on a hyperplane $\alpha\theta + \theta^\perp$. We will denote such a measure by $\mu_{\alpha\theta+\theta^\perp}$. Suppose $d\mu_{\alpha\theta+\theta^\perp}(x) = c_\alpha e^{-|x|^2/2}\,dx$, where $dx$ is the Lebesgue measure on the hyperplane. We need to determine the constant $c_\alpha$ such that $\int_{\alpha\theta+\theta^\perp} d\mu_{\alpha\theta+\theta^\perp} = 1$. To this end, note that by a change of variables we have

$$\begin{aligned}\int_{\alpha\theta+\theta^\perp} e^{-|x|^2/2}\,dx &= \int_{\theta^\perp} e^{-|\alpha\theta+x|^2/2}\,dx \\ &= \int_{\theta^\perp} e^{-\frac{|x|^2+\alpha^2}{2}}\,dx \qquad \text{since } |\theta| = 1 \text{ and } \langle\theta, x\rangle = 0 \\ &= e^{-\alpha^2/2}\int_{\theta^\perp} e^{-\frac{|x|^2}{2}} \\ &= e^{-\alpha^2}(2\pi)^{(n-1)/2}.\end{aligned}$$

Using this calculation we define the Gaussian measure on $\alpha\theta + \theta^\perp$ as follows:

**Definition 3.1.** The Gaussian measure $\mu_{\alpha\theta+\theta^\perp}$ on the hyperplane $\alpha\theta + \theta^\perp$ is defined by

$$d\mu_{\alpha\theta+\theta^\perp}(x) = \frac{e^{\alpha^2/2}}{(2\pi)^{(n-1)/2}} e^{-|x|^2/2}\,dx \tag{3.1}$$

where $dx$ is the Lebesgue measure on the hyperplane $\alpha\theta + \theta^\perp$.

*Remark* 3.1. If $x \in \alpha\theta + \theta^{\perp}$, we have that

$$|x - \alpha\theta|^2 = |x|^2 - 2\langle x, \alpha\theta\rangle + |\alpha\theta|^2 = |x|^2 - 2\alpha^2 + \alpha^2 = |x|^2 - \alpha^2$$

allows us also to express the $d\mu_{\alpha\theta+\theta^{\perp}}$ as

$$d\mu_{\alpha\theta+\theta^{\perp}}(x) = e^{-|x-\alpha\theta|^2/2}\frac{dx}{(2\pi)^{(n-1)/2}}.$$

## 3.2 Properties of Gaussian measure on the hyperplane

We now describe some properties for this measure. We begin by computing the measures characteristic function.

**Proposition 3.1.** *The characteristics function of the Gaussian measure on the hyperplane $\alpha\theta + \theta^{\perp}$ is*

$$\int_{\alpha\theta+\theta^{\perp}} e^{i\langle x,y\rangle} d\mu_{\alpha\theta+\theta^{\perp}}(x) = e^{i\alpha\langle\theta,y\rangle - \frac{1}{2}|y_{\theta^{\perp}}|^2} \qquad \text{for any } y \in \mathbb{R}^n$$

*where $y_{\theta^{\perp}}$ denotes the orthogonal projection of $y$ onto the subspace $\theta^{\perp}$.*

A useful and fundamental property of the probability measure $\mu_{\alpha\theta+\theta^{\perp}}$ is that every vector in $\mathbb{R}^n$ can be understood as a normally (or Dirac) distributed random variable in its probability space, as illustrated by the following proposition.

**Proposition 3.2.** *There is a mapping given by*

$$\mathbb{R}^n \to L^2(\mu_{\alpha\theta+\theta^{\perp}})$$
$$u \mapsto \tilde{u}(x) = \langle u, x\rangle$$

*such that $\tilde{u}$ is normally (or Dirac) distributed with mean $\alpha\langle\theta, u\rangle$ and variance $|u_{\theta^{\perp}}|^2$.*

Note that if $u_{\theta^{\perp}} = 0$ the random variable $\tilde{u}$, as stated above, takes on the Dirac distribution. In all other cases, $\tilde{u} \sim N(\alpha\langle\theta, u\rangle, |u_{\theta^{\perp}}|^2)$.

As we continue, Proposition 3.2 will provide us with a convenient way of studying many functions in terms of the Gauss Radon transform. Before we begin, we make one more observation about the random variable $\tilde{u}$.

**Proposition 3.3.** *If $u, v \in \mathbb{R}^n$ satisfy $\langle u_{\theta^{\perp}}, v_{\theta^{\perp}}\rangle = 0$, then the random variables $\tilde{u}, \tilde{v}$ are independent with respect to $\mu_{\alpha\theta+\theta^{\perp}}$.*

## 3.3 Gauss Radon Transform Definition

With the definition of the Gaussian measure on the hyperplane securely behind us, we can turn our attention to defining the Gauss Radon transform. Just as the Radon transform finds the integral of a function over a hyperplane using the Lebesgue measure for the hyperplane, the Gauss Radon transform will output the integral of a function over a hyperplane using the Gaussian measure for the hyperplane.

**Definition 3.2.** The *Gauss Radon transform* of a function $f : \mathbb{R}^n \to \mathbb{R}$ is a function $\mathcal{G}_f$ on the set $\mathbb{P}^n$ given by

$$\mathcal{G}_f(\alpha\theta + \theta^{\perp}) = \int_{\alpha\theta+\theta^{\perp}} f(x)\, d\mu_{\alpha\theta+\theta^{\perp}}(x) \tag{3.2}$$

where $\mu_{\alpha\theta+\theta^{\perp}}$ is the Gaussian measure on the hyperplane $\alpha\theta + \theta^{\perp}$.

Again, with no restrictions on $f$, the function $\mathcal{G}_f$ may not exist for every element of $\mathbb{P}^n$. For example, it is fairly easy to justify that $\mathcal{G}_f$ does not exist when $f(x) = e^{|x|^2}$. That said, it is important to note that using the Gaussian measure in place of the Lebesgue measure ensures the existence of $\mathcal{G}_f$ for a much broader class of function, including polynomials. We will now compute some examples of the Gauss Radon transform for common functions.

## 3.4 Gauss Radon Transform as a Conditional Expectation

In this section we provide an interesting result relating the the Gauss Radon transform to a conditional expectation. We will exploit this relationship heavily in the final chapter. In the following, $\mu$ represents the standard Gaussian measure on $\mathbb{R}^n$.

**Theorem 3.1.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a integrable function on the probability space $(\mathbb{R}^n, \mu)$ such that $G_f$ exists for all hyperplanes in $\mathbb{P}^n$. Then the Gauss Radon transform of $f : \mathbb{R}^n \to \mathbb{R}$ on a hyperplane $\alpha\theta + \theta^\perp$ is equal to the conditional expectation of $f$ under the condition $\tilde{\theta} = \alpha$. i.e.*

$$\mathcal{G}_f(\alpha\theta + \theta^\perp) = E[f \mid \tilde{\theta} = \alpha]. \tag{3.3}$$

## 3.5 Gaussian Bounded Functions

The rapidly decreasing functions and Schwartz space functions used in the Radon transform seem too restrictive for the Gauss Radon transform. A class of functions suitable for the Gauss Radon transform were defined in [5] as follows:

**Definition 3.3.** A function $f \in C^\infty(\mathbb{R}^n)$ is *$\delta$-Gaussian bounded* if for any polynomial function $p$, there exist constants $M_p > 0$ and $0 < k_p < \delta$ such that

$$|p(\partial_1, \partial_2, \ldots, \partial_n) f(x)| \leq M_p e^{k_p |x|^2} \qquad \text{for all } x \in \mathbb{R}^n.$$

*Remark* 3.2. From the definition it is clear that if $f$ is $\frac{1}{2}$-Gaussian bounded, then $f(x)e^{-\frac{|x|^2}{2}} \in \mathscr{S}(\mathbb{R}^n)$.

Notice that the class of $\delta$-Gaussian bounded functions holds a much broader class of functions than the usual Schwartz space. In particular, polynomial functions are $\delta$-Gaussian bounded for all $\delta > 0$.

## 3.6 Relationship between Radon and Gauss Radon transform

The relationship between the Radon transform and the Gauss Radon transform is really the key to developing inversion formulas for the Gauss Radon transform. For our results we extensively use the following proposition from [5].

**Proposition 3.4.** *Suppose $f$ is an $\frac{1}{2}$-Gaussian bounded function. Then*

$$\mathcal{G}_f(\alpha\theta + \theta^\perp) = e^{\frac{\alpha^2}{2}} \mathcal{R}_g(\alpha\theta + \theta^\perp), \tag{3.4}$$

*or equivalently by Theorem 3.1*

$$E[f \mid \tilde{\theta} = \alpha] = e^{\frac{\alpha^2}{2}} \mathcal{R}_g(\alpha\theta + \theta^\perp), \tag{3.5}$$

*where $g(x) = f(x)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{(n-1)/2}}$.*

## 3.7 Inversion Formulas for the Gauss Radon Transform

**Theorem 3.2.** *The inversion formula for a $\frac{1}{2}$-Gaussian bounded function $f$ in terms of its Gauss Radon transform is*

$$f(x) = (2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \mathcal{G}_f(\alpha\theta + \theta^\perp) e^{-i\beta(\alpha - \langle\theta, x\rangle) - \frac{\alpha^2}{2}} \beta^{n-1} \, d\alpha \, d\beta \, d\sigma(\theta).$$

*for any $x \in \mathbb{R}^n$.*

*Proof.* Simply replace $f$ in Theorem 2.3 with $g(x) = f(x)\frac{e^{-|x|^2/2}}{(2\pi)^{(n-1)/2}}$. Then use Proposition 3.4 to replace $\mathcal{R}_g(\alpha\theta + \theta^\perp)$ with $e^{-\alpha^2/2}\mathcal{G}_f(\alpha\theta + \theta^\perp)$. □

We now recast the above theorem in terms of a conditional expectation using Proposition 3.4.

**Corollary 3.1.** *Let X be a $\frac{1}{2}$-Gaussian bounded random variable on the probability space $(\mathbb{R}, \mu)$ where $\mu$ is the standard Gaussian measure. Then X can be expressed in terms of the conditional expectations $E[X \,|\, \tilde{\theta} = \alpha]$ as follows:*

$$X(y) = (2\pi)^{-\frac{n}{2}} e^{\frac{|y|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} E[X \,|\, \tilde{\theta} = \alpha] e^{-i\beta(\alpha - \langle\theta, y\rangle) - \frac{\alpha^2}{2}} \beta^{n-1} \, d\alpha \, d\beta \, d\sigma(\theta).$$

*for any $y \in \mathbb{R}^n$.*

# 4 Reconstruction from Conditional Expectation

We have explored the reconstruction of functions from discretely sampled Radon transforms and the construction of a Gaussian measure over hyperplanes to motivate the Gauss Radon transform. Furthermore, the Gauss Radon transform can also be interpreted as the conditional expectation of a random variable in the probability space $(\mathbb{R}^n, \mu)$. We use this representation of the Gauss Radon transform as we explore the reconstruction of a random variable $f$ from its conditional expectations by adapting reconstruction methods for the Radon transform to the Gauss Radon transform. In parallel to Section 2, we begin the main results of this work by briefly exploring a Fourier reconstruction algorithm before developing the Gauss Radon analog of the filtered backprojection algorithm. With the algorithm in hand, we present a rigorous analysis of the error incurred by the Gauss Radon analog of the filtered backprojection algorithm. We conclude by discussing the sources of error—in particular, the error derived from relaxing the assumption that the function (random variable) of interest is of compact support—and how to approach mitigating those sources of error.

## 4.1 Gauss Fourier Reconstruction from $E[f \mid \tilde{\theta} = \alpha]$

As in Section 2 this numerical approximation is a discretization of the Fourier Slice Theorem and inverse Fourier transform. This algorithm differs with respect to the samples, now conditional expectations $E[f \mid \tilde{\theta} = \alpha]$, and we need to add a step where we use (3.4) to transform each conditional expectation into the corresponding Radon transform.

The idea behind this algorithm is to 1) gather conditional expectations of the random variable $f$ from a polar grid covering $\Omega_r^n$, 2) transform the conditional expectations from 1) into Radon transforms via (3.1) and (3.4), 3) approximate the Fourier transform of the Radon transform samples, 4) use Fourier transform data from 3) to approximate the Fourier transform of $g(x) = f(x)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{(n-1)/2}}$, 5) interpolation, 6) perform a discrete Fourier inversion to recover $g$, and 7) multiply by $(2\pi)^{(n-1)/2} e^{\frac{|x|^2}{2}}$ to recover $f$.

*Algorithm-Gauss Fourier Reconstruction*: Let $f$ be a $\frac{1}{2}$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$.

*Step1*: Collect the conditional expectations $E[f \mid \tilde{\theta}_j = s_l]$ of $f$ sampled at $(\theta_j, s_l)$ for $j = 1, ..., p$ ; $l = -q, ..., q$ where $\theta_j \in S^{n-1}$, and $s_l = hl, h = r/q$.

*Remark* 4.1. In this case we are once again following a "naive" sampling scheme sampling at points spaced $r/q$ units apart in the direction $\theta$. Note that $f$ no longer needs to be of compact support. Later in this chapter we will discuss how to choose $r$ in order to achieve a reasonable reconstruction.

*Step 2:* Multiply each sample by the appropriate Gaussian term as per Proposition 3.4; that is

$$\mathcal{R}_g(s_l, \theta_j) = e^{\frac{-s_l^2}{2}} E[f \mid \tilde{\theta}_j = s_l] \tag{4.1}$$

*Step 3:* Approximate $\hat{g}$. We accomplish this by a discrete approximation to the Fourier transform of $\mathcal{R}_g$ based on the sampled points above, followed by application of the Fourier Slice Theorem (Theorem 2.1).

For $j = 1, ..., p$, approximate $\widehat{\mathcal{R}}_g(\theta_j, r\pi)$ by $\widehat{\mathcal{R}}_g^{(j,r)}$, where

$$\widehat{\mathcal{R}}_g^{(j,r)} = (2\pi)^{-\frac{1}{2}}\ h \sum_{l=-q}^{q-1} e^{i\pi rl/q} \mathcal{R}_g(\theta_j, s_l)\ ,\quad r = -q, ..., q-1. \tag{4.2}$$

Note that (4.2) is simply the discrete Fourier transform in $\mathbb{R}$.

*Step 4*: Applying Theorem 2.1 to the Fourier Radon samples of $g$ we obtain

$$\hat{g}(r\pi\theta_j) = (2\pi)^{\frac{1-n}{2}} \widehat{\mathcal{R}}_g^{(j,r)}, \tag{4.3}$$

up to discretization errors. Note that we took conditional expectation samples from points in $\Omega_r^n$ and now have an approximation to the Fourier transform of $g$ on the polar grid

$$G_{p,q} = \{\pi r\theta_j :\ \ r = -q, ..., q-1\ ;\ j = 1, ..., p\}.$$

*Step 5 (Interpolation)*: It is now necessary to make use of a fast Fourier transform (FFT) algorithm in order to efficiently calculate the Fourier transforms, as in step 3 of the Section 2 Fourier algorithm; but first we must interpolate to a suitable Cartesian coordinate grid.

For each $k \in \mathbb{Z}^n, |k| < q$, find the point $\xi_k = \pi r\theta_j$ in the polar grid closest to $\pi k$ and set

$$\hat{g}_k = (2\pi)^{\frac{1-n}{2}} \widehat{\mathcal{R}}_g^{(j,r)} \tag{4.4}$$

*Step 6*: We now approximate $g(hm), m \in \mathbb{Z}^n$, using the discrete inverse Fourier transform,

$$g_m = \left(\frac{\pi}{2}\right)^{\frac{n}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \hat{g}_k\ , |m| < q, \tag{4.5}$$

where 4.5 is the discrete inverse Fourier transform in $\mathbb{R}^n$.

*Step 7*: By Proposition 3.4

$$g(hm) = f(hm) \frac{e^{\frac{-|hm|^2}{2}}}{(2\pi)^{\frac{n-1}{2}}}, \tag{4.6}$$

which implies

$$\begin{aligned}
f(hm) &\approx \frac{\pi^n}{\sqrt{2\pi}} e^{\frac{|hm|^2}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \cdot \hat{g}_k && \text{by (4.6)},\\
&= \frac{\pi^n}{\sqrt{2\pi}} e^{\frac{|hm|^2}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \cdot (2\pi)^{\frac{1-n}{2}} \widehat{\mathcal{R}}_g^{(j,r)} && \text{by (4.4)},\\
&= \pi^n (2\pi)^{\frac{-(n+1)}{2}} e^{\frac{|hm|^2}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \cdot\ h \sum_{l=-q}^{q-1} e^{i\pi lr/q} \mathcal{R}_g(\theta_j, s_l) && \text{by (4.2)},\\
&= \pi^n (2\pi)^{\frac{-(n+1)}{2}} e^{\frac{|hm|^2 - s_l^2}{2}} \sum_{|k|<q} e^{i\pi m\cdot k/q} \cdot\ h \sum_{l=-q}^{q-1} e^{i\pi lr/q} E[f \mid \tilde{\theta}_j = s_l],
\end{aligned}$$

by Proposition 3.4.

## 4.2 Gauss Convolution

We also develop a filtered backprojection based reconstruction algorithm for the Gauss Radon transform. This not only provides another way to reconstruct the random variable $f$ from its conditional expectations; we also have a promising method of attack for deriving the error incurred by such an algorithm. In Chapter 2 we saw that Theorem 2.5 was instrumental in the filtered backprojetion algorithm for the Radon transform. Unfortunately the same theorem does not hold when we exchange $\mathcal{R}_f(\theta, \alpha)$ for $\mathcal{G}_f(\theta, \alpha)$ and use the Gaussian measure $\mu_{\alpha\theta+\theta^\perp}$ in place of the Lebesgue measure. In light of this obstacle, we define a new convolution type operator, $\diamond$ , and use it to prove the Gauss Radon analog to Theorem 2.5.

**Definition 4.1.** Let $f$ and $g$ be $\frac{1}{2}$-Gaussian bounded functions on $\mathbb{R}^n$. Define the *Gauss convolution* of $f$ and $g$ to be $f \diamond g$, where

$$(f \diamond g)(x) = \int_{\mathbb{R}^n} f(x-y)g(y)e^{\frac{-|y|^2}{2}} \frac{dy}{(2\pi)^{\frac{n}{2}}}. \tag{4.7}$$

From this definition we immediately derive the relationship between Gauss convolution and standard convolution.

**Theorem 4.1.** *Let $f$ and $g$ be $\frac{1}{2}$-Gaussian bounded functions on $\mathbb{R}^n$. Then*

$$(f \diamond g)(x) = (f \star h)(x), \tag{4.8}$$

*where*

$$h(x) = g(x)\frac{e^{\frac{-|x|^2}{2}}}{(2\pi)^{\frac{n}{2}}}. \tag{4.9}$$

We are now ready to show filtered backprojection for the Gauss Radon transform using the Gauss convolution operator.

**Theorem 4.2.** *Let $f$ and $g$ be $\frac{1}{2}$-Gaussian bounded functions on $\mathbb{R}^n$. Then*

$$\left(\mathcal{R}^{\#}g\right) \diamond f = \mathcal{R}^{\#}\left(g \diamond \mathcal{G}_f\right). \tag{4.10}$$

*Proof.* Using the the Gauss convolution operator defined above we have

$$\begin{aligned}
\left((\mathcal{R}^{\#}g) \diamond f\right)(x) &= \int_{\mathbb{R}^n} (\mathcal{R}^{\#}g(x-y))f(y)e^{\frac{-|y|^2}{2}} \frac{dy}{(2\pi)^{\frac{n}{2}}} \\
&= \int_{\mathbb{R}^n}\int_{S^{n-1}} g(\theta, \langle (x-y), \theta\rangle)d\theta f(y)e^{\frac{-|y|^2}{2}} \frac{dy}{(2\pi)^{\frac{n}{2}}} \\
&= \int_{S^{n-1}}\int_{\mathbb{R}^n} g(\theta, \langle (x-y), \theta\rangle)f(y)e^{\frac{-|y|^2}{2}} \frac{dy}{(2\pi)^{\frac{n}{2}}} d\theta,
\end{aligned}$$

where in the last step we used the Fubini theorem to change the order of integration. Decomposing the integral over $\mathbb{R}^n$ to an integral over the hyperplane parallel to $\theta^\perp$ at distance $|s|$ from the origin, and an integral over all $s \in \mathbb{R}$, for $y \in \mathbb{R}^n$ let $y = s\theta + z$, $z \in \theta^\perp$. Hence

$$\begin{aligned}
((\mathcal{R}^{\#}g) \diamond f)(x) &= \int_{S^{n-1}}\int_{\mathbb{R}}\int_{\theta^\perp} g(\theta, \langle x, \theta\rangle - s)f(s\theta + z)e^{\frac{-|s\theta+z|^2}{2}} \frac{dz ds d\theta}{(2\pi)^{\frac{n}{2}}} \\
&= \int_{S^{n-1}}\int_{\mathbb{R}} g(\theta, \langle x, \theta\rangle - s)\left[\int_{\theta^\perp} f(s\theta + z)e^{\frac{-|z|^2}{2}} \frac{dz}{(2\pi)^{\frac{n-1}{2}}}\right] e^{\frac{-s^2}{2}} \frac{ds d\theta}{(2\pi)^{\frac{1}{2}}} \\
&= \int_{S^{n-1}}\int_{\mathbb{R}} g(\theta, \langle x, \theta\rangle - s)\, \mathcal{G}_f(s, \theta)e^{\frac{-s^2}{2}} \frac{ds d\theta}{(2\pi)^{\frac{1}{2}}} \\
&= \int_{S^{n-1}} g(\theta, \langle x, \theta\rangle) \diamond \mathcal{G}_f(\theta, \langle x, \theta\rangle)d\theta \\
&= \mathcal{R}^{\#}(g \diamond \mathcal{G}_f)(x),
\end{aligned}$$

which is the desired result.

□

Since ultimately we are interested in reconstructing the random variable $f$ from its conditional expectations, we implement the filtered backprojection algorithm with the following corollary.

**Corollary 4.1.** *Let $f$ be a $\frac{1}{2}$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$ and $w$ a $\frac{1}{2}$-Gaussian bounded function on $\mathbb{R}^n$. Then*

$$\left(\mathcal{R}^{\#}w\right) \diamond f = \mathcal{R}^{\#}\left(w \diamond E[f \mid \tilde{\theta} = \alpha]\right). \tag{4.11}$$

*Proof.* The result follows directly from applying Theorem 3.1 to Theorem 4.2. □

### 4.3 Gauss Filtered Backprojection from $E[f \mid \tilde{\theta} = \alpha]$

In practice, we are forced to reconstruct a random variable $f$ from a finite set of conditional expectations. Thus we need to discretize Theorem 4.2. In the filtered backprojection algorithm of Section 2 we defined the discrete backprojection operator $\mathcal{R}^{\#}_p$ and used discrete convolution. We now define the discrete analog of the Gauss convolution $\diamond$.

**Definition 4.2.** Let $f$ and $g$ be $\frac{1}{2}$-Gaussian bounded functions on $\mathbb{R}^n$. Define the *discrete Gauss convolution* by $\overset{h}{\diamond}$, where

$$f \overset{h}{\diamond} g = \frac{h^n}{(2\pi)^{\frac{n}{2}}} \sum_l f(x - hl) g(hl) e^{\frac{-\|hl\|^2}{2}}. \tag{4.12}$$

In parallel to the filtered backprojection algorithm for the Radon transform we now implement Corollary 4.1 numerically to recover $f$ from its conditional expectations, appropriately sampled over the support of $f$:

$$W_b \diamond f = \mathcal{R}^{\#}(w_b \diamond E[f \mid \tilde{\theta} = \alpha]), \quad W_b = \mathcal{R}^{\#} w_b. \tag{4.13}$$

As before, in order to accurately reconstruct $f$ we choose $w_b$ so that $W_b$ approximates the $\delta$-distribution and satisfies the properties of 2.13.

*Algorithm—Gauss Filtered Backprojection*: Let $f$ be a $\frac{1}{2}$-Gaussian bounded random variable on the probability space $(\mathbb{R}^n, \mu)$.

*Step 1*: Sample the conditional expectation of $f$, $E[f \mid \tilde{\theta}_j = s_l]$, at points $(s_l, \theta_j)$, $j = 1, 2, ..., p$, $l = -q, ..., q$, $\theta_j \in S^{n-1}$, $s_l = hl = \frac{r}{q} l$.

*Step 2*: Perform discrete Gauss convolution of $w_b \diamond E[f \mid \tilde{\theta}_j = s_l]$. That is

$$w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_j = s_l] = h \sum_{l=-q}^{q} w_b(\sigma - s_l) E[f \mid \tilde{\theta}_j = hl]\, e^{\frac{-\|hl\|^2}{2}}. \tag{4.14}$$

In this step we are performing a 1-dimensional Gauss convolution of $w_b$ and $E[f \mid \tilde{\theta}_j = s_l]$ for each direction $\theta_j$.

*Step 3 (Discrete Backprojection)*: We use the same quadrature rule on $S^{n-1}$ defined in (2.15).

$$\int_{S^{n-1}} v(\theta) d\theta = \sum_{j=1}^{p} \alpha_{pj} v(\theta_j), \quad for\ v \in H'_{2m} \tag{4.15}$$

Thus we may interchange the backprojection on the right side of Theorem 4.2 for the discrete backprojection defined by $\mathcal{R}^{\#}_p$,

$$f_{GFB} = \mathcal{R}^{\#}_p(w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_j = s_l]) = \sum_{j=1}^{p} \alpha_{pj} h \sum_{l=-q}^{q} w_b(\sigma - s_l) E[f \mid \tilde{\theta}_j = hl]\, e^{\frac{-\|hl\|^2}{2}} \tag{4.16}$$

where $f_{GFB}$ is the approximation of $f$ given by the filtered backprojection algorithm via Corollary 4.1.

## 5 Error Analysis for Gauss Radon Algorithms

### 5.1 Gauss Fourier Error

The Fourier algorithm for the conditional expectation (Gauss Radon transform) is effectively the same as that for the Radon transform. After using Proposition 3.4 to convert the conditional expectation samples to Radon transforms the algorithm is exactly the same as the Fourier algorithm of Section 2 2. Hence this algorithm will be subject to the same error estimates summarized in Section 2.6.1.

## 5.2 Gauss Filtered Backprojection Error

We are now ready to assess the sources and magnitudes of error incurred by using the Gauss-filtered backprojection algorithm to reconstruct $f$ from $E[f \mid \tilde{\theta} = \alpha]$. The proof of Theorem 2.7, found in [3], motivates much of the proceeding derivation of error. Recall that in Theorem 2.7 we assumed that $f \in C_0^\infty(\Omega_r^n)$. We will now relax that assumption and allow $f$ to be defined over all of $\mathbb{R}^n$. To accommodate this change we will introduce a *bump function* and prove two auxiliary results that give us insight into how best to sample when $f$ is no longer of compact support, and a convenient way to put bounds on the filter $w_b$.

**Theorem 5.1.** *Let $f, h$ be $\frac{1}{2}$-Gaussian bounded functions, $f, h : \mathbb{R}^n \to \mathbb{R}$. Let $\varepsilon > 0$. Then*

$$|f(x) - h(x)| \leq \varepsilon(2\pi)^{-\frac{n}{2}} \tag{5.1}$$

*if and only if*

$$\left|\mathcal{G}_f(\alpha, \theta) - \mathcal{G}_h(\alpha, \theta)\right| \leq \varepsilon, \tag{5.2}$$

*where $x = \alpha\theta$.*

*Proof.* (⇒) By Theorem 3.2

$$\begin{aligned}
|f(x) - h(x)| &\leq (2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \left|\mathcal{G}_f(\alpha, \theta) - \mathcal{G}_h(\alpha, \theta)\right| [e^{-2\beta\alpha} e^{\frac{-\alpha^2}{2}} d\alpha] e^{i\beta\langle\theta,x\rangle} \beta^{n-1} d\beta d\sigma(\theta) \\
&\leq (2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_0^\infty \int_{\mathbb{R}} \varepsilon\, [e^{-\beta\alpha} e^{\frac{-\alpha^2}{2}} d\alpha] e^{i\beta\langle\theta,x\rangle} \beta^{n-1} d\beta d\sigma(\theta) \\
&= \varepsilon(2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} \int_{S^{n-1}} \int_0^\infty e^{\frac{-|\beta|^2}{2}} e^{i\beta\langle\theta,x\rangle} \beta^{n-1} d\beta d\sigma(\theta) \\
&= \varepsilon(2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} \int_{\mathbb{R}^n} e^{\frac{-|y|^2}{2}} e^{i\langle y,x,\rangle} dy \\
&= \varepsilon(2\pi)^{-\frac{n}{2}} e^{\frac{|x|^2}{2}} e^{\frac{-|x|^2}{2}}.
\end{aligned}$$

Thus

$$|f(x) - h(x)| \leq \varepsilon(2\pi)^{-\frac{n}{2}}.$$

(⇐) Using the definition of the Gauss Radon transform (3.2) we have

$$\begin{aligned}
\left|\mathcal{G}_f(\alpha, \theta) - \mathcal{G}_h(\alpha, \theta)\right| &\leq \int_{\alpha\theta+\theta^\perp} |f(x) - h(x)|\, d\mu_{\alpha\theta+\theta^\perp} \\
&\leq \int_{\alpha\theta+\theta^\perp} \varepsilon(2\pi)^{-\frac{n}{2}} d\mu_{\alpha\theta+\theta^\perp} \\
&= \varepsilon(2\pi)^{-\frac{n}{2}} \int_{\alpha\theta+\theta^\perp} d\mu_{\alpha\theta+\theta^\perp} \\
&= \varepsilon(2\pi)^{-\frac{n}{2}} * 1 \\
&< \varepsilon,
\end{aligned}$$

by definition of the Gaussian measure. □

This theorem is significant in that, replacing $h$ with 0 above, it tells us that $\mathcal{G}_f$ and $f$ decay at the same rate, up to a constant. Alternatively, suppose we hypothesize that the unknown function $f$ may actually be a function $h$ with known Gauss Radon transform $\mathcal{G}_h$. We can take the difference $\left|\mathcal{G}_f(\alpha, \theta) - \mathcal{G}_h(\alpha, \theta)\right|$ between samples of each Gauss Radon transform and note whether or not the differences are getting small within some region of interest.

**Corollary 5.1.** *Let $f, h$ be $\frac{1}{2}$-Gaussian bounded random variables on the probability space $(\mathbb{R}^n, \mu)$ where $\mu$ is the standard Gaussian measure. Let $\varepsilon > 0$. Then*

$$|f(x) - h(x)| \leq \varepsilon(2\pi)^{-\frac{n}{2}}$$

*if and only if*

$$\left|E[f \mid \tilde{\theta} = \alpha] - E[h \mid \tilde{\theta} = \alpha]\right| \leq \varepsilon,$$

*where $x = \alpha\theta$.*

*Proof.* This follows directly from Theorem 3.1. □

**Lemma 5.2.** *Let $W_b$ and $w_b$ be defined as in 2.5 and 2.13. Then*

$$\hat{w}_b(\sigma) = \frac{1}{2}(2\pi)^{\frac{1}{2}-n} |\sigma|^{n-1} \hat{\Phi}(|\sigma|/b), \tag{5.3}$$

*which implies*

$$0 \leq \hat{w}_b(\sigma) \leq \frac{1}{2}(2\pi)^{\frac{1}{2}-n} |\sigma|^{n-1}. \tag{5.4}$$

*Proof.* Observe that by Theorem 2.6, we may express $W_b$ in terms of $w_b$ via their Fourier transforms:

$$\widehat{W}_b(\xi) = (2\pi)^{\frac{n-1}{2}} |\xi|^{1-n} \left(\hat{w}_b(\tfrac{\xi}{|\xi|}, |\xi|) + \hat{w}_b(-\tfrac{\xi}{|\xi|}, -|\xi|)\right).$$

Note that by 2.13, $\widehat{W}_b$ is a radial function, thus the first argument of $\hat{w}_b$ is dropped. For $\hat{w}_b$ even we obtain

$$\hat{w}_b(\sigma) = \frac{1}{2}(2\pi)^{\frac{1}{2}-n} |\sigma|^{n-1} \hat{\Phi}(\tfrac{|\sigma|}{b}).$$

From the assumptions on $\hat{\Phi}$ below (2.13), we arrive at the desired result. □

**Proposition 5.1.** *Let $w_b$ be defined as in 2.13. Then*

$$|w_b(s)| \leq (2\pi)^{-n} \frac{b^n}{n} \tag{5.5}$$

*Proof.* We can write $w_b(s)$ in terms of its inverse Fourier transform and use Lemma 5.2 to obtain

$$\begin{aligned}
|w_b(s)| &\leq \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left|e^{y\langle i,s\rangle}\right| |\hat{w}_b(y)|\, dy \\
&= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} |\hat{w}_b(y)|\, dy \\
&\leq \frac{1}{\sqrt{2\pi}} \frac{1}{2}(2\pi)^{\frac{1-2n}{2}} \int_{-b}^{b} |\sigma|^{n-1}\, d\sigma \\
&= \frac{1}{2}(2\pi)^{-n} 2 \int_0^b |\sigma|^{n-1}\, d\sigma \\
&\leq (2\pi)^{-n} \frac{b^n}{n}.
\end{aligned}$$

□

## 5.3 Bump Functions and Bounds

In the typical applications of the Radon transform and filtered backprojection (i.e. medical imaging), it is assumed the function of interest $f$ is of compact support. Now considering the reconstruction of a random variable $f$ from its conditional expectations, we would like to relax this assumption. In practice we will have to limit the collection of conditional expectations within some ball of radius $r$. This allows us to use the Gauss-filtered backprojection algorithm above; however we incur additional error in the information lost outside of this sampling region. We will assess this error with the help of *bump functions*. The bump functions will enable us to represent $f$ as compactly supported function on $\Omega_r^n$ smoothly connected to $(\Omega_r^n)^c$ via the bump function. The notion of a $O\left(\frac{1}{|x|}\right)$ function provides a way to bound the error outside the sampling region while invoking a relatively weak assumptions on $f$.

**Definition 5.1.** A function $\phi : \mathbb{R}^n \to \mathbb{R}$ is called a *bump function* if $\phi$ is a smooth function, satisfying $\phi(x) \leq 1$, with compact support; that is $\phi \in C_0^\infty(\mathbb{R}^n)$.

**Proposition 5.2.** *Define* $h : \mathbb{R}^n \to \mathbb{R}$

$$h(|x|) = \begin{cases} e^{-\frac{1}{|x|}} & |x| > 0 \\ 0 & x = 0 \end{cases}. \tag{5.6}$$

*Then* $h \in C^\infty(\mathbb{R}^n)$ *and all of its derivatives are* 0 *at* $x = 0$.

We can now construct a bump function $\phi_{r,\Delta}(|x|)$ from the function $h$ defined above, which is 1 on $\Omega_r^n$ and 0 outside of $\Omega_{r+\Delta}^n$.

**Proposition 5.3.** *Let h be defined as in Proposition 5.2. Define*

$$b(x) = \frac{h(r + \Delta - |x|)}{h(r + \Delta - |x|) + h(|x| - r)} = \frac{e^{\frac{-1}{r+\Delta-|x|}}}{e^{\frac{-1}{r+\Delta-|x|}} + e^{\frac{-1}{|x|-r}}}. \tag{5.7}$$

*Then*

$$\phi_{r,\Delta}(x) = \begin{cases} 0 & |x| \geq r + \Delta \\ b(x) & r < |x| < r + \Delta \\ 1 & |x| \leq r \end{cases} \tag{5.8}$$

*is a bump function.*

**Definition 5.2.** Let $f$ be a function on $\mathbb{R}^n$. We say $f$ is $O\left(\frac{1}{|x|}\right)$ if there exists $M, R > 0$ such that $|f(x)| \leq \frac{M}{|x|}$ for all $x$ with $|x| > R$.

**Proposition 5.4.** *Suppose* $f$ *is* $O\left(\frac{1}{|x|}\right)$. *Then for every* $\varepsilon > 0$ *there exists r such that* $|f(x)| < \varepsilon$ *for every x with* $|x| > r$.

*Proof.* Let $\varepsilon > 0$. Since $f$ is $O\left(\frac{1}{|x|}\right)$ we know there exists $M, R > 0$ such that

$$|f(x)| \leq \frac{M}{|x|}.$$

for all $x$ with $|x| > R$. Choose $r > R$ so that

$$\frac{M}{|x|} \leq \varepsilon.$$

for all $x$ with $|x| > r$. Hence

$$|f(x)| \leq \frac{M}{|x|} < \varepsilon$$

for every $x$ with $|x| > r$, which is the desired result. □

The following lemma will be instrumental in the error analysis to follow.

**Lemma 5.3.** *Suppose $f$ is $O\left(\frac{1}{|x|}\right)$. Let $\varepsilon > 0$. Then there exists $r$ and $\Delta$ such that*

$$\left|f(x) - f(x)\phi_{r,\Delta}(x)\right| \leq \varepsilon, \tag{5.9}$$

*where $\phi_{r,\Delta}(x)$ is as defined in Proposition 5.3.*

*Proof.* Using Proposition 5.4 with $\phi_{r,\Delta}(x)$ from (5.7) we have

$$\left|f(x) - f(x)\phi_{r,\Delta}(x)\right| = |f(x)|\left|1 - \phi_{r,\Delta}(x)\right| \leq \varepsilon$$

since $0 \leq \phi_{r,\Delta}(x) \leq 1$. □

## 5.4 Error Estimates: $e_1, e_2, e_3$

We now have all the tools necessary to derive the error estimate for using the Gauss-filtered backprojection algorithm to recover a random variable from samples of its conditional expectations.

**Theorem 5.2.** *Let $f \in C^\infty$ be a $O\left(\frac{1}{|x|}\right)$ random variable on the probability space $(\mathbb{R}^n, \mu)$ and let $g(x) = f(x)\frac{e^{-\frac{|x|^2}{2}}}{(2\pi)^{(n-1)/2}}$. Assume that (2.15) holds on $H'_{2m}$ and that, for some $\vartheta$ with $0 < \vartheta < 1$,*

$$b \leq \vartheta m, \quad b \leq \pi/h. \tag{5.10}$$

*Let $\varepsilon$ and $r$ be as in Lemma 5.3. Define the approximate reconstruction of $f$ as*

$$\begin{aligned} f_{FB} &= W_b \diamond f + e_1 + e_2 + e_3 \\ &= \mathcal{R}^\# w_b \diamond E[f \mid \tilde{\theta} = \alpha] + e_1 + e_2 + e_3, \end{aligned}$$

*where*

$$e_1 = \mathcal{R}^\# \left(w_b \diamond E[f \mid \tilde{\theta} = \alpha]\right) - \mathcal{R}^\# \left(w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta} = \alpha]\right),$$

$$e_2 = \mathcal{R}^\# \left(w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta} = \alpha]\right) - \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l]\right),$$

*and*

$$e_3 = \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l]\right) - \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_p = s_l]\right)$$

*Then,*

$$|e_1| \leq \varepsilon \left|S^{n-1}\right| \frac{1}{n(2h)^n},$$

$$\begin{aligned} |e_2| &= |e_2 a| + |e_2 b|, \\ |e_2 a| &\leq \frac{1}{2}(2\pi)^{\frac{-(n+1)}{2}} |S^{n-1}| \sup_{\theta \in S^{n-1}} \int_{|\alpha| \geq b} |\alpha|^{n-1} \left|\widehat{g\phi_{r,\Delta}}(\alpha\theta)\right| d\alpha \\ |e_2 b| &\leq 2\left|S^{n-1}\right| (2\pi)^{\frac{-(n+1)}{2}} (r + \Delta)^{\frac{-n}{2}} \|g\|_{L_\infty(\Omega^n_{r+\Delta})}\, \eta(\vartheta, 2l + k)\, |e_3| \qquad \leq \varepsilon \left|S^{n-1}\right| \frac{1}{n(2h)^n}. \end{aligned}$$

*Proof.* Observe that

$$\begin{aligned} &\left|\mathcal{R}^\# \left(w_b \diamond E[f \mid \tilde{\theta} = \alpha]\right) - \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_p = s_l]\right)\right| \\ &\leq \left|\mathcal{R}^\# \left(w_b \diamond E[f \mid \tilde{\theta} = \alpha]\right) - \mathcal{R}^\# \left(w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta} = \alpha]\right)\right| \\ &+ \left|\mathcal{R}^\# \left(w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta} = \alpha]\right) - \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l]\right)\right| \\ &+ \left|\mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l]\right) - \mathcal{R}_p^\# \left(w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_p = s_l]\right)\right|. \end{aligned}$$

We will proceed by looking at each error term individually.

$e_1$: Using linearity in the backprojection and convolution operators,

$$\begin{aligned}&\left|\mathcal{R}^{\#}\left(w_b \diamond E[f \mid \tilde{\theta}=\alpha]\right)-\mathcal{R}^{\#}\left(w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right)\right|\\ &=\left|\mathcal{R}^{\#}\left\{w_b \diamond \left(E[f \mid \tilde{\theta}=\alpha]-E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right)\right\}\right|\\ &\leq \mathcal{R}^{\#}\left(w_b \diamond \left|E[f \mid \tilde{\theta}=\alpha]-E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right|\right)\end{aligned}$$

We now invoke Lemma 5.3, with $\left|f(x)-f(x)\phi_{r,\Delta}(x)\right| \leq \varepsilon(2\pi)^{-\frac{n}{2}}$, and Corollary 5.1 to establish the bound

$$\left|E[f \mid \tilde{\theta}=\alpha]-E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right| \leq \varepsilon.$$

Hence

$$\begin{aligned}&\mathcal{R}^{\#}\left(w_b \diamond \left|E[f \mid \tilde{\theta}=\alpha]-E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right|\right)\\ &\leq \mathcal{R}^{\#}(|w_b| \diamond \varepsilon)\\ &\leq \int_{S^{n-1}}\int_{\mathbb{R}}|w_b(s-t)|\ \varepsilon\, e^{\frac{-t^2}{2}}\ \frac{dt}{\sqrt{2\pi}}\\ &\leq \varepsilon\left|S^{n-1}\right|\int_{\mathbb{R}}|w_b(s-t)|\, e^{\frac{-t^2}{2}}\ \frac{dt}{\sqrt{2\pi}}.\end{aligned}$$

Using Proposition 5.1 and the assumption $b \leq \pi/h$ we find

$$\begin{aligned}&\leq \varepsilon\left|S^{n-1}\right|\left[(2\pi)^{-n}\ \frac{b^n}{n}\right]\int_{\mathbb{R}} e^{\frac{-t^2}{2}}\ \frac{dt}{\sqrt{2\pi}}\\ &\leq \varepsilon\left|S^{n-1}\right|(2\pi)^{-n}\ \frac{b^n}{n}\\ &\leq \varepsilon\left|S^{n-1}\right|\frac{1}{n}\ \left(\frac{1}{2h}\right)^n.\end{aligned}$$

First, $e_2$: This error term is essentially the Gauss Radon analog of Theorem 2.7. We will proceed by breaking up $e_2$ into two parts similar to Theorem 2.7.

$e_2a$: The error incurred by using the discrete Gauss convolution on the compactly supported part of $f$ is

$$\mathcal{R}_p^{\#}\left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p=s_l]-w_b \diamond E[f\phi_{r,\Delta} \mid \tilde{\theta}=\alpha]\right).$$

Using Proposition 3.4 we obtain

$$\begin{aligned}&\mathcal{R}_p^{\#}\left(w_b \overset{h}{\diamond} e^{\frac{\alpha^2}{2}}\mathcal{R}_{g\phi_{r,\Delta}}-w_b \star e^{\frac{\alpha^2}{2}}\mathcal{R}_{g\phi_{r,\Delta}}\right)\\ &=\mathcal{R}_p^{\#}\left(w_b \overset{h}{\star} e^{\frac{\alpha^2}{2}}\mathcal{R}_{g\phi_{r,\Delta}}\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{2\pi}}-w_b \star e^{\frac{\alpha^2}{2}}\mathcal{R}_{g\phi_{r,\Delta}}\frac{e^{\frac{-\alpha^2}{2}}}{\sqrt{2\pi}}\right)\\ &=\frac{1}{\sqrt{2\pi}}\mathcal{R}_p^{\#}\left(w_b \overset{h}{\star}\mathcal{R}_{g\phi_{r,\Delta}}-w_b \star \mathcal{R}_{g\phi_{r,\Delta}}\right).\end{aligned}$$

Using Theorem 6.7, we express the term in parentheses in terms of its Fourier transform,

$$(w_b \overset{h}{\star}\mathcal{R}_{g\phi_{r,\Delta}}-w_b \star \mathcal{R}_{g\phi_{r,\Delta}}\widehat{)}(\theta,\alpha)=(2\pi)^{1/2}\hat{w}_b(\alpha)\sum_{l\neq 0}\widehat{\mathcal{R}}_{g\phi_{r,\Delta}}\left(\theta,\alpha-\frac{2\pi}{h}l\right).$$

By taking the inverse Fourier transform and absolute value we obtain

$$\left|w_b \overset{h}{\star} \mathcal{R}_{g\phi_{r,\Delta}} - w_b \star \mathcal{R}_{g\phi_{r,\Delta}}\right|(\theta, s) = \int_{-b}^{b} |\hat{w}_b(\alpha)| \sum_{l\neq 0} \left|\widehat{\mathcal{R}}_{g\phi_{r,\Delta}}(w, \alpha - \frac{2\pi}{h} l)\right| \left|e^{is\cdot\alpha}\right| d\alpha$$
$$\leq \int_{-b}^{b} |\hat{w}_b(\alpha)| \sum_{l\neq 0} \left|\widehat{\mathcal{R}}_{g\phi_{r,\Delta}}(w, \alpha - \frac{2\pi}{h} l)\right| d\alpha.$$

Since $b \leq \pi/h$ we may employ Lemma 5.2 to obtain

$$\left|w_b \overset{h}{\star} \mathcal{R}_{g\phi_{r,\Delta}} - w_b \star \mathcal{R}_{g\phi_{r,\Delta}}\right|(\theta, s) \leq \frac{1}{2}(2\pi)^{\frac{1}{2}-n} \int_{|\alpha|\geq b} |\alpha|^{n-1} \left|\widehat{\mathcal{R}}_{g\phi_{r,\Delta}}(\theta, \alpha)\right| d\alpha.$$

Using the Fourier Slice Theorem (Theorem 2.1) we now have

$$\left|w_b \overset{h}{\star} \mathcal{R}_{g\phi_{r,\Delta}} - w_b \star \mathcal{R}_{g\phi_{r,\Delta}}\right|(\theta, s) \leq \frac{1}{2}(2\pi)^{\frac{-n}{2}} \int_{|\alpha|\geq b} |\alpha|^{n-1} \left|\widehat{g\phi}_{r,\Delta}(\alpha\theta)\right| d\alpha.$$

Now, applying the backprojection operator and quadrature rule (2.15) we obtain

$$\frac{1}{\sqrt{2\pi}} \mathcal{R}_p^{\#}(w_b \overset{h}{\star} \mathcal{R}_{g\phi_{r,\Delta}} - w_b \star \mathcal{R}_{g\phi_{r,\Delta}})$$
$$\leq \frac{1}{\sqrt{2\pi}} \mathcal{R}_p^{\#}\left[\frac{1}{2}(2\pi)^{\frac{-n}{2}} \int_{|\alpha|\geq b} |\alpha|^{n-1} \left|\widehat{g\phi}_{r,\Delta}(\alpha\theta)\right| d\alpha\right]$$
$$\leq \sum_{j=1}^{p} \alpha_{pj} \sup_{\theta\in S^{n-1}} \left[\frac{1}{2}(2\pi)^{\frac{-(n+1)}{2}} \int_{|\alpha|\geq b} |\alpha|^{n-1} \left|\widehat{g\phi}_{r,\Delta}(\alpha\theta)\right| d\alpha\right].$$

Observe that after applying the supremum the expression in brackets above is no longer a function of $\theta_j$ . Therefore $\nu(\theta_j,\ \langle x, \theta\rangle = \alpha) = 1$ in the expression for the quadrature rule. Furthermore

$$\sum_{j=1}^{p} \alpha_{pj} = \left|S^{n-1}\right|.$$

Thus we obtain the desired result for $e_2 a$.

Now $e_2 b$: Expand $(w_b \star \mathcal{R}_{g\phi_{r,\Delta}})$ in terms of spherical harmonics (Corollary 6.2) noting that all spherical harmonics of odd degree drop out since $(w_b \star \mathcal{R}_{g\phi_{r,\Delta}})$ is an even function of $\theta$:

$$(w_b \star \mathcal{R}_{g\phi_{r,\Delta}})(\theta, \langle x, \theta\rangle = \alpha) = \sum_{l=0}^{\infty} \frac{1}{c(n,l)} \sum_{k=1}^{N(n,l)} Y_k^{2l}(\theta) \nu_{l,k}(x),$$

where

$$\nu_{l,k}(x) = \int_{S^{n-1}} Y_k^{2l}(\theta)(w_b \star \mathcal{R}_{g\phi_{r,\Delta}})(\theta, \langle x, \theta\rangle = \alpha) d\theta$$
$$\left(i.e\ \sum_{l=0}^{\infty} \sum_{k=1}^{N(n,l)} \langle (w_b \star \mathcal{R}_{g\phi_{r,\Delta}}), Y_k^{2l}(\theta)\rangle Y_k^{2l}(\theta)\right).$$

Note that, using the Funk-Hecke Theorem (Theorem 6.10) it can be shown,

$$\int_{S^{n-1}} \left|Y_k^{2l}(\theta)\right|^2 d\theta = c(n,l), \tag{5.11}$$

and, using (6.17)

$$\left|Y_k^{2l}\right| \leq 1. \tag{5.12}$$

Let us express $(w_b \star \mathcal{R}_{g\phi_{r,\Delta}})$ in terms of its Fourier transform, yielding

$$\nu_{l,k}(x) = \int_{S^{n-1}} Y_k^{2l}(\theta)\left[\int_{\mathbb{R}} e^{i\alpha x\cdot\theta}(\widehat{w_b \star \mathcal{R}_{g\phi_{r,\Delta}}})(\theta,\alpha)d\alpha\right]d\theta.$$

Using Fourier inversion and the Fourier Slice Theorem 2.1

$$\begin{aligned}\nu_{l,k}(x) &= (2\pi)^{\frac{n-1}{2}}\int_{S^{n-1}} Y_k^{2l}(\theta)\int_{-b}^{b} e^{i\alpha\langle x,\theta\rangle}\hat{w}_b(\alpha)\widehat{g\phi}_{(r,\Delta)}(\alpha\theta)d\alpha\, d\theta\\ &= (2\pi)^{\frac{n-1}{2}}\int_{S^{n-1}} Y_k^{2l}(\theta)\int_{-b}^{b} e^{i\alpha\langle x,\theta\rangle}\hat{w}_b(\alpha)(2\pi)^{\frac{-n}{2}}\int_{\mathbb{R}^n} g\phi_{(r,\Delta)}(y)dyd\alpha\, d\theta.\\ &= (2\pi)^{\frac{1}{2}}\int_{\mathbb{R}} g(y)\phi_{(r,\Delta)}(y)\int_{-b}^{b}\hat{w_b}(\alpha)\int_{S^{n-1}} Y_k^{2l}(\theta)e^{i\alpha\langle(x-y),\theta\rangle}d\theta\, d\alpha\, dy.\end{aligned}$$

Using (6.23), the innermost integral yields

$$(2\pi)^{\frac{n}{2}} i^{2l}(\alpha|x-y|)^{\frac{(2-n)}{2}} J_{2l+\frac{n-2}{2}}(\alpha|x-y|)Y_k^{2l}\left(\frac{x-y}{|x-y|}\right).$$

Also note that

$$\frac{x-y}{|x-y|}\in S^{n-1}.$$

Employing Lemma 5.2 again and recalling that $\left|Y_k^{2l}\right|\le 1$,

$$\left|\nu_{l,k}(x)\right| \le \frac{1}{2}(2\pi)^{\frac{-n}{2}}\cdot|-1|\int_{\Omega^n_{r+\Delta}}\left|g(y)\phi_{(r,\Delta)}(y)\right|\cdot|x-y|^{\frac{2-n}{2}}\int_{-b}^{b}\alpha^{\frac{n}{2}}J_{2l+\frac{n-2}{2}}(\alpha|x-y|)d\alpha\, dy. \tag{5.13}$$

Now let $l > m$ and set $k = \frac{n-2}{2}$. Also note that $b \le \vartheta m < \vartheta l$ and by construction of $\phi_{r,\Delta}$ we have $|x-y| \le 2(r+\Delta)$. Then, using Proposition 6.3, we obtain

$$\begin{aligned}\int_{-b}^{b}\alpha^{\frac{n}{2}}J_{2l+k}(\alpha|x-y|)d\alpha &\le \int_{-b}^{b}|\alpha|^{\frac{n}{2}}\,|J_{2l+k}(\alpha|x-y|)|\,d\alpha\\ &\le b^{\frac{n}{2}}\int_{-b}^{b}|J_{2l+k}(\alpha|x-y|)|\,d\alpha\\ &\le l^{\frac{n}{2}}\int_{-\vartheta l}^{\vartheta l}|J_{2l+k}(\alpha|x-y|)|\,d\alpha,\ \text{since } \vartheta<1,\\ &\le \sup_{|s|\le 1} l^{\frac{n}{2}}\int_{-\vartheta l}^{\vartheta l}|J_{2l+k}(2(r+\Delta)s\alpha)|\,d\alpha\\ &= \frac{1}{2(r+\Delta)}l^{\frac{n}{2}}\sup_{|s|\le 1}\int_{-(r+\Delta)\vartheta l}^{(r+\Delta)\vartheta l}|J_{2l+k}(ts)|\,dt\\ &\le \frac{1}{2(r+\Delta)}l^{\frac{n}{2}}\sup_{|s|\le 1}\int_{-\vartheta((r+\Delta)l+k)}^{\vartheta((r+\Delta)l+k)}|J_{2l+k}(ts)|\,dt\\ &\le \frac{1}{2(r+\Delta)}l^{\frac{n}{2}}\eta_1(\vartheta,2l+k).\end{aligned}$$

Hence,

$$\begin{aligned}|\nu_{l,k}| &\le \frac{1}{2}(2\pi)^{\frac{-n}{2}}\int_{|y|\le 2(r+\Delta)}\left|g(y)\phi_{r,\Delta}(y)\right|\cdot|x-y|^{\frac{2-n}{2}}\left[\frac{1}{2(r+\Delta)}l^{\frac{n}{2}}\eta_1(\vartheta,2l+k)\right]dy\\ &\le \frac{1}{2}(2\pi)^{\frac{-n}{2}}\int_{\Omega^n_{r+\Delta}}\left|g(y)\phi_{r,\Delta}(y)\right|(2(r+\Delta))^{\frac{-n}{2}}\,l^{\frac{n}{2}}\eta_1(\vartheta,2l+k)dy\\ &\le \frac{1}{2}(4\pi(r+\Delta))^{\frac{-n}{2}}\|g\phi_{r,\Delta}\|_{L_\infty(\Omega^n_{r+\Delta})}l^{\frac{n}{2}}\eta_1(\vartheta,2l+k).\end{aligned}$$

Let $(w_b \star g)_m$ represent the spherical harmonic series expansion for $w_b \star g$, truncated after the $m$th term. It follows that

$$\begin{aligned}
&\|w_b \star g\phi_{r,\Delta} - (w_b \star g\phi_{r,\Delta})_m\|_{L_\infty(Z)} \\
&\leq \sum_{l>m} \frac{1}{c(n,l)} \sum_k \|\nu_{l,k}\|_{L_\infty(\Omega^n_{r+\Delta})} \\
&\leq \frac{1}{2}(4\pi(r+\Delta))^{\frac{-n}{2}} \|g\|_{L_\infty(\Omega^n_{r+\Delta})} \sum_{l>m} \frac{N(n,l)}{c(n,l)} l^{\frac{n}{2}} \eta_1(\vartheta, 2l+k).
\end{aligned}$$

Note that

$$\begin{aligned}
|e_2 b| &= \left|(\mathcal{R}^\#_p - \mathcal{R}^\#)(w_b \star g\phi_{r,\Delta} - (w_b \star g\phi_{r,\Delta})_m)\right| \\
&= \left|\mathcal{R}^\#_p(w_b \star g\phi_{r,\Delta}) - \mathcal{R}^\#(w_b \star g\phi_{r,\Delta}) - \mathcal{R}^\#_p(w_b \star g\phi_{r,\Delta})_m + \mathcal{R}^\#(w_b \star g\phi_{r,\Delta})_m\right| \\
&= \left|2\mathcal{R}^\#_p(w_b \star g\phi_{r,\Delta} - (w_b \star g\phi_{r,\Delta})_m)\right|.
\end{aligned}$$

Employing the quadrature rule on $H_2'm$ (2.15), we obtain

$$\begin{aligned}
|e_2 b| &\leq 2\left|S^{n-1}\right| \|w_b \star g\phi_{r,\Delta} - (w_b \star g\phi_{r,\Delta})_m\|_{L_\infty(Z)} \\
&\leq 2\left|S^{n-1}\right| \frac{1}{2}(4\pi(r+\Delta))^{\frac{-n}{2}} \|g\phi_{r,\Delta}\|_{L_\infty(\Omega^n_{r+\Delta})} \sum_{l>m} \frac{N(n,l)}{c(n,l)} l^{\frac{n}{2}} \eta_1(\vartheta, 2l+k) \\
&\leq \left|S^{n-1}\right| (4\pi(r+\Delta))^{\frac{-n}{2}} \|g\phi_{r,\Delta}\|_{L_\infty(\Omega^n_{r+\Delta})} \eta(\vartheta, m),
\end{aligned}$$

where

$$\begin{aligned}
\eta(\vartheta, m) &= \sum_{l>m} \frac{N(n,l)}{c(n,l)} l^{\frac{n}{2}} \eta_1(\vartheta, 2l+k) \\
&\leq \sum_{l>m} \frac{N(n,l)}{c(n,l)} l^{\frac{n}{2}} \frac{\sqrt{2}\vartheta}{(1-\vartheta)^{\frac{1}{4}}} e^{-\frac{2l+\frac{n-2}{2}}{4}(1-\vartheta^2)^{\frac{3}{2}}},
\end{aligned}$$

which converges for all $n, m \in \mathbb{N}$ (see Section 6.4; for details see pages 65-66 of [3]). This completes the $e_2$ error term.

Lastly, $e_3$: This error term accounts for the discretization of the backprojection operator and Gauss convolution with respect to the first error term, $e_1$. Using linearity in the discrete backprojection and convolution operators,

$$\begin{aligned}
&\left|\mathcal{R}^\#_p\left(w_b \overset{h}{\diamond} E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l]\right) - \mathcal{R}^\#_p\left(w_b \overset{h}{\diamond} E[f \mid \tilde{\theta}_p = s_l]\right)\right| \\
&= \mathcal{R}^\#_p \left| w_b \overset{h}{\diamond} \left(E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l] - E[f \mid \tilde{\theta}_p = s_l]\right)\right| \\
&\leq \sum_{j=1}^{p} \alpha_{pj} \frac{h}{(2\pi)^{\frac{1}{2}}} \sum_l w_b(s-hl) \left|E[f\phi_{r,\Delta} \mid \tilde{\theta}_p = s_l] - E[f \mid \tilde{\theta}_p = s_l]\right| e^{\frac{-|hl|^2}{2}}.
\end{aligned}$$

Using Lemma 5.3 with $\left|f(x) - f(x)\phi_{r,\Delta}(x)\right| \leq \varepsilon(2\pi)^{-\frac{n}{2}}$ and Corollary 5.1, we obtain

$$\begin{aligned}
&\leq \sum_{j=1}^{p} \alpha_{pj} \frac{h}{(2\pi)^{\frac{1}{2}}} \sum_l |w_b(s-hl)| \; \varepsilon \, e^{\frac{-|hl|^2}{2}} \\
&\leq \varepsilon \left|S^{n-1}\right| \frac{h}{(2\pi)^{\frac{1}{2}+n}} \frac{b^n}{n} \sum_l e^{\frac{-|hl|^2}{2}},
\end{aligned}$$

by Proposition 5.1 and (2.15).

We now use the first assumption on $b$ and bound the infinite sum over $l$ by its integral giving us

$$\begin{aligned}
&\leq \varepsilon \left|S^{n-1}\right| \frac{h}{(2\pi)^{\frac{1}{2}+n}} \frac{\left(\frac{\pi}{h}\right)^n}{n} \sum_l e^{\frac{-\|hl\|^2}{2}} \\
&\leq \varepsilon \left|S^{n-1}\right| \frac{1}{n\sqrt{2\pi}} \frac{1}{2^n h^{n-1}} \int_{-\infty}^{\infty} e^{\frac{-\|hl\|^2}{2}} dl \\
&\leq \varepsilon \left|S^{n-1}\right| \frac{1}{n\sqrt{2\pi}} \frac{1}{2^n h^{n-1}} \frac{\sqrt{2\pi}}{h} \\
&\leq \varepsilon \left|S^{n-1}\right| \frac{1}{n(2h)^n}.
\end{aligned}$$

□

Several comments are in order with respect to the sources of error. Each error component of the Gauss filtered backprojection algorithm gives us some insight into how to reduce or eliminate that component. The first error term $e_1$ is primarily a function of $r + \Delta$. Recall that for any $\varepsilon$ we could construct a bump function $\phi_{r,\Delta}$ with parameters $r$ and $\Delta$ such that $\left|f(x) - f(x)\phi_{r,\Delta}(x)\right| \leq \varepsilon$. Thus by choosing $r + \Delta$ large we can make $\varepsilon$ arbitrarily small and offset the remaining constants; however, we also want to choose $b$ to be large to minimize some of the other error terms. Although the parameter $b$ is not directly within our control, a related quantity—the sampling distance $h$—is chosen during sampling. We choose $h \leq \frac{\pi}{b}$, satisfying the Nyquist–Shannon Theorem 6.6. Taking $h$ small implies $b$ is large. Since $b^n$ appears in $e_1$ we must increase $r + \Delta$ to accommodate the choice of b. In practical terms, this translates to sampling conditional expectations out to a greater radius.

The second error term is entirely analogous to the two error terms in the original filtered backprojection algorithm (2.7). If $g$ is $b$-band limited or essentially $b$-band limited then the integral in $e_2a$ is negligible or zero. The reduction of this error term can be achieved by taking the sampling distance $h$ to be small, inflating $b$ to include the largest frequency component of $g$. The term $e_2b$ is inversely proportional to $(r + \Delta)^{\frac{n}{2}}$ so as we increase $r$ by sampling a larger ball this term goes to zero since the remaining terms are essentially constants.

The third error term is the discrete analog of $e_1$. This term is similarly reduced by choosing $r + \Delta$ to make $\varepsilon$ small and offset $b$ (or equivalently $h$).

In summary, the choice of sampling interval $h$ and the parameters $r$ and $\Delta$ determine the accuracy of the Gauss filtered backprojection algorithm. The reader will likely have noticed that, while the Gauss Radon transform can be performed on $\frac{1}{2}$-Gaussian bounded functions, the error analysis required a slightly stronger assumption, $O\left(\frac{1}{|x|}\right)$, on the function (random variable) of interest. It is fairly easy to see that it was not necessary for $f$ to be in $\mathscr{S}(\mathbb{R}^n)$ for the error analysis to hold; however, we found that it was necessary to assume that the function was "eventually" bounded by some decreasing curve outside the ball of radius $r$. A future research topic would be to investigate the error bounds for reconstruction of $\frac{1}{2}$-Gaussian bounded functions, or the weakest assumptions necessary to bound the error for reconstruction of $\frac{1}{2}$-Gaussian bounded functions.

## 5.5 Future Work, and Observations

The last chapter of the Radon/Gauss Radon transform has certainly not been written. There are many subtopics in mathematical tomography that we have not touched upon in relation to the Gauss Radon transform; see [3, 2, 6, 4]. We did not investigate sampling geometry for either algorithm, although we already know from the comments following the Fourier algorithm, that sampling geometry can be a deciding factor in the utility of that particular algorithm.

# 6 Appendix

## 6.1 Convolution

**Definition 6.1.** Let $f$ and $g \in L_1(\mathbb{R}^n)$. Define the *convolution* of $f$ and $g$, $f \star g$, by

$$f \star g(x) = \int_{y \in \mathbb{R}^n} f(x-y)g(y)dy \tag{6.1}$$

*Remark* 6.1. If $F$ and $H \in L_1(\mathbb{R} \times S^{n-1})$ then we define the convolution of $F \star H$ in terms of the the first variable only. That is

$$(F \star H)(s) = \int_{t \in \mathbb{R}} F(s-t, \theta)H(t, \theta)dt \tag{6.2}$$

Here we present several basic properties of the convolution operation.

**Proposition 6.1.** *Let $f$, $g$, and $h \in L_1(\mathbb{R}^n)$. Then the following hold:*

1. *(commutativity)* $f \star g = g \star f$
2. *(associativity)* $f \star (g \star h) = (f \star g) \star h$
3. *(distributivity)* $f \star (g + h) = f \star g + f \star h$

The space of Lebesgue integrable functions $L_1(\mathbb{R}^n)$ can be considered a commutative algebra, defining multiplication as convolution; however, the algebra is not without issue [2]. This algebra does not have a multiplicative identity. That is, there exists no element $\phi \in L_1(\mathbb{R}^n)$ such that $f \star \phi = f$. Fortunately we can define a collection of functions called *approximate identities* that, when convoluted with a function $f$, return $f$ approximately to any degree of accuracy.

**Definition 6.2.** Let $f \in L_1(\mathbb{R}^n)$. The function $\phi_\epsilon(x) \in L_1(\mathbb{R}^n)$ is an *approximate identity* if

$$\lim_{\epsilon \to 0} f \star \phi_\epsilon(x) = f(x)$$

for all $x \in \mathbb{R}^n$.

The next theorem guarantees that such functions exist and provides some insight into how we might construct them. A proof for the following theorem can be found in [2]. For more general results consult [8].

**Theorem 6.1.** *(Approximate Identity Theorem). Let $\phi$ be a nonnegative, integrable function defined on $\mathbb{R}^n$ such that*

$$\int_{\mathbb{R}^n} \phi(x)dx = 1.$$

*For any $\epsilon > 0$ define*

$$\phi_\epsilon(x) = \frac{1}{\epsilon}\phi\left(\frac{x}{\epsilon}\right). \tag{6.3}$$

*Then for any function $f$ such that $f$ is continuous and bounded,*

$$\lim_{\epsilon \to 0} f \star \phi_\epsilon(x) = f(x) \tag{6.4}$$

*for all $x \in \mathbb{R}^n$.*

## 6.2 Fourier Transform Results

There are several different conventions commonly used to define the Fourier transform of a function [7, 8]. These conventions typically differ by the location of the $2\pi$ factor. For this manuscript we will adopt the definition to follow. We will define the Fourier transform, several key results, and present the inversion formula based on our definition. The interested reader may consult the sources above for a more complete exposition on the subject.

**Definition 6.3.** Let $f \in L_1(\mathbb{R}^n)$. The *Fourier transform* of $f$ is defined by

$$\widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\langle x,\xi\rangle} f(x)dx, \tag{6.5}$$

where $dx$ is the Lebesgue measure on $\mathbb{R}^n$.

**Theorem 6.2.** *Let $f \in L_1(\mathbb{R}^n)$. Then $\widehat{f} \in \mathscr{S}(\mathbb{R}^n)$, and*

$$\|\hat{f}\|_{L_\infty} \leq \|f\|_{L_1} \tag{6.6}$$

Thus the Fourier transform is a mapping from $L_1(\mathbb{R}^n)$ into $\mathscr{S}(\mathbb{R}^n)$, the sup-normed Banach space of complex continuous functions on $\mathbb{R}^n$ that vanish at infinity. The next two theorems tells how the Fourier transform acts on $f \in \mathscr{S}(\mathbb{R}^n)$ and $L_2(\mathbb{R}^n)$.

**Theorem 6.3.** *The Fourier transform is a continuous linear homeomorphism from $\mathscr{S}(\mathbb{R}^n)$ onto $\mathscr{S}(\mathbb{R}^n)$.*

**Theorem 6.4.** *(Plancherel Theorem). Let $f \in \mathscr{S}(\mathbb{R}^n)$. Then the Fourier transform $f \to \widehat{f}$ extends a linear isometry of $L_2(\mathbb{R}^n)$ onto $L_2(\mathbb{R}^n)$ and*

$$\|f\|_{L_2} = \|\hat{f}\|_{L_2} \tag{6.7}$$

**Theorem 6.5.** *(*Fourier Inversion Theorem*). Let $f \in L_1(\mathbb{R}^n)$. Then*

$$f(x) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{i\langle x,\xi\rangle} \widehat{f}(\xi)d\xi. \tag{6.8}$$

The following are several useful results relating convolution operator to the Fourier transform.

**Proposition 6.2.** *Let $f, g \in \mathscr{S}(\mathbb{R}^n)$. Then the following hold:*

1. $(\widehat{f \star g}) = (2\pi)^{\frac{n}{2}} \widehat{f}\widehat{g}$
2. $\widehat{(fg)} = (2\pi)^{-\frac{n}{2}} \widehat{f} \star \widehat{g}$
3. $f \star g \in \mathscr{S}(\mathbb{R}^n)$

In numerous applications we are interested in studying the Fourier transform or inverting the Fourier transform using discrete samples. We are typically unable to evaluate the integrals explicitly from sampled data so we require discrete analogs to Fourier transforms and inverse transforms.

**Definition 6.4.** Let $f(x) \in L_1(\mathbb{R}^n)$. Let $f$ be sampled at intervals of length $h$. Define the *discrete Fourier transform* as

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} h^n \sum_k f(hk)e^{-ih\langle\xi,k\rangle}\ ,\ k \in \mathbb{Z}^n. \tag{6.9}$$

Similarly, define the *discrete inverse Fourier transform* as

$$(2\pi)^{-\frac{n}{2}} \sum_k \hat{f}(hk)e^{ih\langle x,k\rangle}\ ,\ k \in \mathbb{Z}^n. \tag{6.10}$$

The next natural step is to establish the accuracy of such discrete approximations and determine how we must sample in order to reliably approximate a Fourier or inverse Fourier transform. First we must define a class of functions whose Fourier transforms disappear (or nearly so) far enough from the origin. The Nyquist-Shannon Sampling theorem then tells us exactly the conditions that must be satisfied to accurately approximate a Fourier transform from this class of functions. We present the following results in the vernacular of [3]. The interested reader should consult [9] for a more in depth treatment.

**Definition 6.5.** Let $f \in L_1(\mathbb{R}^n)$ and $b > 0$. The function $f$ is called *b-band-limited* or *band-limited with bandwidth b* if $\hat{f}$ is locally integrable and $\hat{f}(\xi) = 0$ a.e. for all $\xi$ such that $|\xi| > b$.

*Remark* 6.2. We call a function $f$ *essentially b-band-limited* if for every $\varepsilon > 0$ there exists $b$ such that

$$\int_{|\xi|\geq b} |\xi|^{n-1} \left|\hat{f}(\xi)\right| d\xi \leq \epsilon. \tag{6.11}$$

**Theorem 6.6.** *(*Nyquist-Shannon sampling theorem*). Let $f \in L_1(\mathbb{R}^n)$ be b-band-limited and let $h \leq \frac{\pi}{b}$. Then the following hold in $L_2(\mathbb{R}^n)$*

$$\hat{f}(\xi) = (2\pi)^{-\frac{n}{2}} h^n \sum_k f(hk) e^{-ih\langle \xi,k\rangle} \ , \quad k \in \mathbb{Z}^n \ , \tag{6.12}$$

*and*

$$f(x) = \sum_k f(hk) \operatorname{sinc}\left(\tfrac{\pi}{2}(x - hk)\right) \tag{6.13}$$

*where* $\operatorname{sinc}(z) = \frac{sin(z)}{z}$.

The significance of the sampling theorem cannot be understated. If we can sample a function $f$ such that the distance between sampled points is at most half of the smallest "frequency" component of $f$ then we can reconstruct the function or its Fourier transform exactly. Put another way, we can compute Fourier transforms (and inner products) exactly by a quadrature rule (e.g. trapezoidal) in $L_2(\mathbb{R}^n)$. This theorem plays a major role in the reconstruction of a function from its Radon and Gauss Radon transforms due to their intrinsic relationship to the Fourier transform via the Fourier slice theorem.

We conclude this section with a result from [3] giving the difference between the Fourier transform of the discrete and continuous convolution of two functions from the Schwartz space. We need this result for the derivation of error components for the filtered backprojection algorithm.

**Theorem 6.7.** *Let $f,\ g \in \mathscr{S}(\mathbb{R}^n)$. Then*

$$(\widehat{f \overset{h}{\star} g})(\xi) - (\widehat{f \star g})(\xi) = (2\pi)^{\frac{n}{2}} \hat{f}(\xi) \sum_{l\neq 0} \hat{g}\left(\xi - \tfrac{2\pi}{h} l\right) \tag{6.14}$$

## 6.3 Spherical Harmonics

**Definition 6.1.** Consider the following sets of polynomials:

1. Let $P_k$ be the set of polynomials of degree less than or equal to $k$.
2. Let $P_k^h$ be the set of homogeneous polynomials of degree $k$, i.e. $p$ is degree $k$ and $p(\alpha x) = \alpha^k p(x)$.
3. Let $H_k = \{p \in P_k^k\,;\, \Delta p = 0\}$, where $\Delta$ is the *Laplacian Operator*, i.e. $\Delta p = \sum_{i=1}^n \frac{\partial^2 p}{\partial x_i^2}$.
4. Let $H_k'$ be the the restriction to $S^{n-1}$ of $H_k$. That is $|x| = 1$.

The set $H_k'$ is the space of spherical harmonics of degree $k$.

**Theorem 6.8.** *The dimension of $H_k'$ is given by*

$$N(n,l) = \dim(H_k') = \frac{(2k+n-2)(n+k-3)!}{k!(n-2)!} = O(k^{n-2}).$$

See [10, 11]

The following results are presented in [11] on page 141 as Corollaries 2.3 and 2.4, and the remarks following Corollary 2.4. This result is powerful in that it allows us to express any function restricted to $S^{n-1}$ as a linear combination of spherical harmonics. Furthermore, spherical harmonics of different degree are orthogonal on $S^{n-1}$. Thus after normalizing we can construct an orthonormal basis for $L^2(S^{n-1}, d\theta)$ from $\bigcup_{k=0}^{\infty} H_k'$.

**Theorem 6.9.** *The collection of all finite linear combinations of elements of* $\bigcup_{k=0}^{\infty} H_k'$ *is*

1. *dense in the space of all continuous function on* $S^{n-1}$
2. *dense in* $L^2(S^{n-1}, d\theta)$

**Corollary 6.1.** *If* $Y^j$ *and* $Y^l$ *are are spherical harmonics of degree* $j$ *and* $l$*, respectively, with* $j \neq l$*, then*

$$\int_{S^{n-1}} Y^j(\theta)Y^l(\theta)d\theta = 0 \tag{6.15}$$

**Corollary 6.2.** *The collection* $\bigcup_{l=0}^{\infty}\bigcup_{k=1}^{N(n,l)}\left\{\frac{Y_k^l}{c(n,l)}\right\}$ *of normalized spherical harmonics form an orthonormal basis for* $L^2(S^{n-1}, d\theta)$*. That is if* $f \in L^2(S^{n-1}, d\theta)$*, then there exists a unique representation*

$$f(\theta) = \sum_{l=0}^{\infty} \frac{1}{c(n,l)} \sum_{k=1}^{N(n,l)} Y_k^l(\theta)\nu_{l,k}, \tag{6.16}$$

*where*

$$\nu_{l,k} = \int_{S^{n-1}} Y_k^l(\theta)f(\theta)d\theta$$

*and* $c(n,l) = \int_{S^{n-1}} \left|Y_k^l(\theta)\right|^2 d\theta$.

*Remark* 6.3. Using the Funk-Hecke Theorem, it can be shown that

$$\left|Y^l(\theta)\right|^2 \leq \frac{1}{c(n,l)} \int_{S^{n-1}} Y^{l^2}(\omega)d\omega. \tag{6.17}$$

see [3].

Before presenting the Funk-Hecke Theorem, a powerful result instrumental in deriving the error bounds in both Radon and Gauss-Radon Reconstruction, we must define the *Gegenbauer Polynomials*.

Consider the weight function $G^\lambda(x) = (1-x^2)^{\lambda-\frac{1}{2}}$ and the corresponding space $L^2([-1,1], G^\lambda(x)dx)$ with inner product given by

$$\langle f, g\rangle_\lambda = \int_{-1}^{1} f(x)g(x)G^\lambda(x)\,dx$$

By performing the Gram-Schmidt procedure on the set $\{1, x, x^2, \dots\}$, we obtain an orthogonal set of polynomials $\{C_k^\lambda\}_{k=0}^{\infty}$ where $k$ is the degree of the polynomial. That is $\langle C_k^\lambda, C_l^\lambda\rangle = 0$ when $l \neq k$. The polynomials are often normalized so that $C_k^\lambda(1) = 1$, in which case $C_0^\lambda \equiv 1$.

**Theorem 6.10** (Funk-Hecke Theorem)**.** *Suppose that* $F : \mathbb{R} \to$ *satisfies* $\int_{-1}^{1} |F(t)|\,(1-t^2)^{(n-3)/2}\,dt < \infty$ *and* $S_m \in H_m'$*, then*

$$\int_{S^{n-1}} F(\langle\theta,\omega\rangle)S_m(\theta)\,d\theta = S_m(\omega)|S^{n-2}| \int_{-1}^{1} F(t)C_m^\lambda(t)(1-t^2)^{(n-3)/2}\,dt$$

*where* $\lambda = \frac{1}{2}n - 1$.

Note that the above theorem assumes the normalized version of $C_m^\lambda$ where $C_m^\lambda(1) = 1$.

*Remark* 6.4. Note that the assumption $\int_{-1}^{1} |F(t)|\,(1-t^2)^{(n-3)/2}\,dt < \infty$ is equivalent to stating that $F \in L^1([-1,1], G^{\frac{1}{2}n-1}(t)dt)$.

## 6.4 Bessel Functions

For an extensive survey of Bessel functions the reader is directed to [12]. Bessel functions of the first kind, $J_k(x)$, are defined as the solutions to the differential equation

$$x^2\frac{d^2w}{dx^2}+x\frac{dw}{dx}+\left(x^2-k^2\right)w=0. \tag{6.18}$$

Bessel functions of the first kind can be defined by the generating function

$$e^{x(z-\frac{1}{z})/2}=\sum_k z^k J_k(x)$$

where $J_k(x)=\frac{1}{2\pi i}\int_{S^1}e^{x(z-\frac{1}{z})/2}z^{-k-1}\,dz$

*Fact* 6.1. The Bessel functions of the first kind satisfy the following growth bound for $0<\alpha<1$

$$0\le J_\nu(\alpha\nu)\le\frac{1}{\sqrt{2\pi\nu}}\frac{1}{(1-\alpha^2)^{1/4}}e^{-\frac{\nu}{3}(1-\alpha^2)^{3/2}}.$$

See [3] for derivation.

**Proposition 6.3.** *For $0<\alpha<1$ and $m\ge 0$ define*

$$\eta_1(\alpha,m)=\sup_{-1<r<1}\int_{-\alpha m}^{\alpha m}|J_m(r\sigma)|\,d\sigma.$$

*Then there exists nonnegative $C_\alpha,\lambda_\alpha$, and $M_\alpha$ such that*

$$\eta_1(\alpha,m)\le C_\alpha e^{-\lambda_\alpha m}$$

*for all $m>M_\alpha$.*

*Proof.* Observe

$$\begin{aligned}\eta_1(\alpha,m)&=\sup_{-1<r<1}\int_{-\alpha m}^{\alpha m}|J_m(r\sigma)|\,d\sigma\\&=\sup_{-1<r<1}\int_{-\alpha m}^{\alpha m}|J_m(r\frac{\sigma}{m}m)|\,d\sigma\end{aligned}$$

let $t=\frac{\sigma}{m}$ and $dt=\frac{d\sigma}{m}$ to get

$$=\sup_{-1<r<1}m\int_{-\alpha}^{\alpha}|J_m(rtm)|\,dt$$

apply Fact 6.1 to get

$$\begin{aligned}&\le\sup_{-1<r<1}m\int_{-\alpha}^{\alpha}\frac{1}{\sqrt{2\pi m}}\frac{1}{(1-(rt)^2)^{1/4}}e^{-\frac{m}{3}(1-(rt)^2)^{3/2}}\,dt\\&\le\sup_{0\le r<1}2m\int_{0}^{\alpha}\frac{1}{\sqrt{2\pi m}}\frac{1}{(1-(rt)^2)^{1/4}}e^{-\frac{m}{3}(1-(rt)^2)^{3/2}}\,dt\end{aligned}$$

Since $\frac{1}{(1-x^2)^{1/4}}$ and $e^{-\frac{m}{3}(1-x^2)^{3/2}}$ are increasing functions on $(0,1)$

$$\begin{aligned}&\le 2m\int_{0}^{\alpha}\frac{1}{\sqrt{2\pi m}}\frac{1}{(1-\alpha^2)^{1/4}}e^{-\frac{m}{3}(1-\alpha^2)^{3/2}}\,dt\\&=(2m)^{1/2}\frac{\alpha}{(1-\alpha^2)^{1/4}}e^{-\frac{m}{3}(1-\alpha^2)^{3/2}}.\end{aligned}$$

Now we use the following: for $0 < \beta < \delta$ and $k \in \mathbb{R}$ there exists a constant $M$ such that $x^k e^{-\delta x} \le e^{-\beta x}$ for all $x > M$. So for the above there exists an $M_\alpha$ such that

$$\eta_1(\alpha, m) \le 2^{1/2} \frac{\alpha}{(1-\alpha^2)^{1/4}} e^{-\frac{m}{4}(1-\alpha^2)^{3/2}}$$

for all $m \ge M_\alpha$. Thus we arrive at the desired result, taking $C_\alpha = \frac{\sqrt{2}\alpha}{(1-\alpha^2)^{1/4}}$ and $\lambda_\alpha = \frac{(1-\alpha^2)^{3/2}}{4}$ □

*Fact* 6.2. The Bessel functions and the Gegenbauer polynomials are related via the Fourier transform as follows

$$(\widehat{G^\lambda C_m^\lambda})(\sigma) = \frac{\Gamma(2\lambda)}{\Gamma(\lambda)} (2\pi)^{\frac{1}{2}}\ 2^{-\lambda} i^{-m} \sigma^{-\lambda} J_{m+\lambda}(\sigma).$$

See [13].

**Example 6.3.** *It can be shown that using the Funk-Hecke Theorem and 6.2 we can compute:*

$$\int_{S^{n-1}} e^{-i\sigma y\cdot\theta} S_m(\theta)\, d\theta$$

*as*

$$(2\pi)^{n/2} i^m (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|) S_m(y/|y|), \tag{6.19}$$

*where $y \in \mathbb{R}^n$, $\sigma \in \mathbb{R}$, and $S_m \in H'_m$.*

*We begin by rewriting the above integral as:*

$$\int_{S^{n-1}} e^{-i\sigma|y|\frac{y}{|y|}\cdot\theta} S_m(\theta)\, d\theta$$

*We can use the Funk-Hecke Theorem with $F(t) = e^{-i\sigma|y|t}$ and get that the above is equivalent to*

$$S_m(y/|y|)|S^{n-2}| \sqrt{2\pi} \int_{-1}^{1} e^{-i\sigma|y|t} C_m^\lambda(t)(1-t^2)^{(n-3)/2} \frac{dt}{\sqrt{2\pi}},$$

*where we have multiplied and divided by $\sqrt{2\pi}$ in anticipation for the next step. Temporarily ignoring the constants in front, the integral above is nothing more than the Fourier transform of $G^\lambda C_m^\lambda$ evaluated at $\sigma|y|$ (with $\lambda = \frac{1}{2}n - 1$). Note here that the function $G^\lambda$ is only defined on the interval $[-1, 1]$ and thus the Fourier Transform integral is only taken over $[-1, 1]$ and not the entire real line. Thus using Fact 6.2, the previous integral becomes*

$$(\widehat{G^\lambda C_m^\lambda})(\sigma|y|) = \frac{\Gamma(2\lambda)}{\Gamma(\lambda)} (2\pi) 2^{-\lambda} i^{-m} (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|)$$

*where $J_{m+\lambda}$ is a Bessel function. In summary*

$$\int_{S^{n-1}} e^{-i\sigma y\cdot\theta} S_m(\theta)\, d\theta = S_m(y/|y|)|S^{n-2}| \frac{\Gamma(2\lambda)}{\Gamma(\lambda)} (2\pi) 2^{-\lambda} i^{-m} (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|) \tag{6.20}$$

*where $\lambda = \frac{1}{2}n - 1$.*

*Using the substitution $y = -y$ in* (6.20) *we obtain*

$$\int_{S^{n-1}} e^{i\sigma y\cdot\theta} S_m(\theta)\, d\theta = S_m(-y/|y|)|S^{n-2}| \frac{\Gamma(2\lambda)}{\Gamma(\lambda)} (2\pi) 2^{-\lambda} i^{-m} (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|).$$

*Since $S_m$ is homogeneous of degree m we have $S_m(-1\theta) = (-1)^m S_m(\theta)$ and thus*

$$\int_{S^{n-1}} e^{i\sigma y\cdot\theta} S_m(\theta)\, d\theta = (-1)^m S_m(y/|y|)|S^{n-2}| \frac{\Gamma(2\lambda)}{\Gamma(\lambda)} (2\pi) 2^{-\lambda} i^{-m} (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|)$$

*combining the* $(-1)^m$ *and* $i^{-m}$ *we get simply* $i^m$ *and thus*

$$\int_{S^{n-1}} e^{i\sigma y\cdot\theta} S_m(\theta)\, d\theta = S_m(y/|y|)|S^{n-2}|\frac{\Gamma(2\lambda)}{\Gamma(\lambda)}(2\pi)2^{-\lambda} i^m (\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|). \tag{6.21}$$

*We now simplify* (6.20) *and* (6.21) *by using that* $\lambda = \frac{1}{2}n - 1$ *and* $|S^{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma(\frac{n-1}{2})}$ *[3] to get that the above is equivalent to*

$$|S^{n-2}|\frac{\Gamma(2\lambda)}{\Gamma(\lambda)}(2\pi)2^{-\lambda} = 2\pi^{\frac{n-1}{2}} \frac{\Gamma(n-2)}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2}n-1)}(2\pi)2^{-(\frac{1}{2}n-1)}. \tag{6.22}$$

*We can then apply the duplication formula (multiplication theorem) for the gamma function*

$$\Gamma(z+\tfrac{1}{2})\Gamma(z) = 2^{(1-2z)}\sqrt{\pi}\,\Gamma(2z),$$

*found on page 256 of [12]. Using* $z = \frac{1}{2}n - 1 = \frac{n-2}{2}$ *in the above yields*

$$\Gamma(\tfrac{1}{2}n - \tfrac{1}{2})\Gamma(\tfrac{1}{2}n - 1) = 2^{(-n+3)}\sqrt{\pi}\,\Gamma(n-2).$$

*or equivalently*

$$\frac{\Gamma(n-2)}{\Gamma(\frac{n-1}{2})\Gamma(\frac{1}{2}n-1)} = 2^{(n-3)}\pi^{-1/2}$$

*Using the above in* (6.22) *yields*

$$|S^{n-2}|\frac{\Gamma(2\lambda)}{\Gamma(\lambda)}(2\pi)2^{-\lambda} = 2\pi^{\frac{n-1}{2}} 2^{(n-3)}\pi^{-1/2}(2\pi)2^{-(\frac{1}{2}n-1)} = (2\pi)^{n/2}$$

*Therefore* (6.20) *and* (6.21) *become*

$$\int_{S^{n-1}} e^{-i\sigma y\cdot\theta} S_m(\theta)\, d\theta = (2\pi)^{n/2} i^{-m}(\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|) S_m(y/|y|) \tag{6.23}$$

*and*

$$\int_{S^{n-1}} e^{i\sigma y\cdot\theta} S_m(\theta)\, d\theta = (2\pi)^{n/2} i^{m}(\sigma|y|)^{-\lambda} J_{m+\lambda}(\sigma|y|) S_m(y/|y|) \tag{6.24}$$

*respectively.*

**Example 6.4.** *In the previous examples* (6.20) *and* (6.21)*, we take the special case* $S_0 = 1 \in H_0'$ *we have*

$$\int_{S^{n-1}} e^{-i\sigma y\cdot\theta}\, d\theta = \int_{S^{n-1}} e^{i\sigma y\cdot\theta}\, d\theta = (2\pi)^{n/2}(\sigma|y|)^{-(\frac{1}{2}n-1)} J_{\frac{1}{2}n-1}(\sigma|y|). \tag{6.25}$$

# Competing Interests

Authors have declared that no competing interests exist.

# References


[1] Radon J. Über die Bestimmung von Funktionen durch ihre Integralwerte l angs gewisser mannigfaltigkeiten. Berichte Šachsische Akademie der Wissenschaften Leipzig Math. Nat. Kl. 1917;69:262-277.

[2] Markoe A. Analytic Tomography, volume 106 of Encyclopedia of Mathematics and its Applications. Cambridge University Press, New York, NY; 2006.

[3] Natterer F. The Mathematics of Computerized Tomography. John Wiley & Sons, B.G. Teubner, Stuttgart, Germany; 1986.

[4] Kak AC, Slaney M. Principles of Computerized Tomographic Imaging, volume 33 of Classics in Applied Mathematics. SIAM, New York, NY; 2001.

[5] Becnel JJ, Chang C. The Gauss Radon Transform. Pioneer Journal of Mathematics and Mathematical Sciences. 2012;5:55-78.

[6] Ramm AG, Katsevich AI. The Radon Transform and Local Tomography. CRC Press, Boca Raton, Florida; 1996.

[7] Helgason S. The Radon transform. Birkhäuser, Boston, MA, 2 edition; 1999.

[8] Rudin W. Functional Analysis. McGraw-Hill, New York, New York; 1987.

[9] Folland WB. Fourier Analysis and its Applications. Brooks/Cole Publishing Company, Belmont, California; 1992.

[10] Seeley R. Spherical Harmonics. American Mathematical Monthly. 1966;73:115-121.

[11] Stein EM, Weiss G. Introduction to Fourier Analysis on Euclidean Spaces. Princeton University Press, Princeton, New Jersey; 1971.

[12] Abramowitz M, Stegun IA. Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables. Dover Publications, New York, New York; 1965.

[13] Gradshteyn IS, Ryzhik IM. Table of Integrals, Series, and Products. Academic Press, New York, New York; 1965.

---



***Peer-review history:***
*The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)*
*http://www.sdiarticle3.com/review-history/46373*