Valuation of options for hedging against exchange rate exposure

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Declaration

I declare that

Valuation of options for hedging against exchange rate exposure is my own work, that it has not been submitted before for any degree or examination in any other university, and that all the sources I have used or quoted have been indicated and acknowledged as complete references.

Marlon Don May 2019

Signed:..

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Dedication

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Abstract

The risk associated with currency exposure is one of the main sources of risk in terms of internationally diversified portfolios. Controlling the risk is important for improving the performance of international investments. One approach to hedging against exchange rate exposure is by employing financial derivatives, particularly, foreign currency options. Currency options provide insurance against unfavorable exchange rate fluctuations, but also make provision to lock in a profit when the exchange rate fluctuations are favorable. However, these instruments cannot be traded or managed without the relevant valuation techniques.

In this dissertation we discuss one of the approaches to cover the risk associated with currency exposure. In particular, we focus on the partial differential equation (PDE) valuation of currency options by employing various finite difference schemes. We commence by introducing the mathematical tools required for the valuation of financial derivatives. Thereafter we study the valuation of European options. This involves deriving the famous Black-Scholes PDE for pricing options on stocks that do not yield dividends. Using the Black-Scholes PDE we derive the Black-Scholes formula for pricing European options. This derivation involves transforming the Black-Scholes PDE into the heat equation and by solving the heat equation we obtain the Black-Scholes formula. After completing the pricing of European options we now move to the pricing of American options. The early exercise facility associated with American options, leads to a free boundary problem which makes the pricing process of American options a challenging task. As in the case of the European options, we first derive the Black-Scholes inequality for American options and then transform this inequality for application to the heat equation to value American options. In the absence of an explicit formula for pricing American options we use numerical methods. Thus, we discuss the finite difference methods quite extensively with a focus on the implicit and Crank-Nicholson finite difference methods. We apply these techniques to the valuation of American and European options. In order to solve the American option pricing problem we first rewrite it as a linear complementarity problem. This formulation allows us to solve the American option pricing problem without explicitly depending on the free boundary. The linear complementarity problem is then solved using the implicit finite difference method and the Crank-Nicholson finite difference methods in conjunction with the Projected over relaxation method. The aforementioned approaches to price European and American options on non-dividend paying stocks are then used to price currency options. Currency options are used as the main application of our numerical results. We price both the European and American currency options numerically. Additionally, we compare the convergence rates of the finite difference methods and conclude that they are indeed consistent with the existing literature.

KEYWORDS: Black-Scholes model, Heat diffusion equation, Early exercise boundary, Free boundary problem, Linear complimentary problem, Explicit difference method, Implicit finite difference method, Crank-Nicholson finite difference method, Projected Over-Relaxation method

Contents

List of Acronyms

Partial differential equation (PDE)

Over the counter (OTC)

Intra-daily implied volatility (IDIV)

Stochastic differential equation (SDE)

Linear complementarity problem (LCP)

Implied volatility (IV)

British Pound (GBD)

U.S. Dollar (USD)

Explicit finite difference method (EFDM)

Implicit finite difference method (IFDM)

Crank-Nicholson finite difference method (CNFDM)

Successive over relaxation (SOR)

Projected successive over relaxation (PSOR)

London Interbank Offered Rate (LIBOR)

List of Notations

Chapter 3

K: Strike price;

t: Time;

S: Underlying stock price;

$S(t)$: Current price of the underlying asset at time, t;

σ: Volatility of the underlying asset;

r: Risk-free interest rate which is continuously compounded;

T: Expiration date;

μ: Constant drift of the underlying asset;

Π: Value of the portfolio;

$V(S,t)$: Option value;

dS: Change in the price underlying asset;

dt: Change in time;

τ: Remaining time to expiry and is denoted by $\tau = T - t$. At expiry, $\tau = 0$;

$\mathcal{N}$: Cumulative distribution function of a standard normal distribution;

$\mathcal{N}'(x)$: Standard normal probability density function;

$B(t)$: Standard Brownian motion;

$C(S,t)$: Price of a European call option at time t;

$P(S,t)$: Price of a European put option at time t;

$u(x,\tau)$: The function used to solve the option pricing problem;

x: Transforming variable that relates S and K;

τ: Transforming variable that relates t and T;

$v(x,\tau)$: Function related to the option price;

k: Transforming variable relating σ an r;

α and β: Constants relating the functions $v(x, \tau)$ and $u(x, \tau)$;

D_u: Domain of the function $u(x, \tau)$;

V^{AM}: Price of an American option;

V^{EU}: Price of an European option;

$S_f(t)$: Optimal exercise price at time t;

C^{AM}: Price of an American call option;

P^{AM}: Price of an American put option;

E: Domain where early exercise is recommended;

H: Domain where holding the option is recommended;

$u(x)$: Displacement of the string;

$g(x)$: Height of the obstacle;

$g(x; \tau)$: Transformed payoff function;

D_S: Transformed domain where early exercise is recommended;

D_C: Transformed domain where holding the option is recommended;

q: Continuous dividend rate;

r_f: Foreign interest rate;

Chapter 4

C_{OBS}: The European option price observed in the market;

σ_{imp}: The implied volatility;

C_{BS}: the Black-Scholes option price;

K: The strike price;

$\mathcal{N}(d)$: Cumulative distribution function of a normal distribution function;

$S(t)$: Current price of the underlying asset at time, t;

$C(S, t)$: Price of a European call option at time t;

S_0: Price of the underlying asset at time $t = 0$;

$S(t)$: Current price of the underlying asset at time, t;

$\mathbb{E}(x)$: Expected value of a variable x;

$g(S(t))$: The risk-neutral probability function of the underlying asset;

Chapter 5

$u_0(x)$: Initial condition of the heat equation;

Δx: Subinterval length of the x axis;

$\Delta \tau$: Subinterval length of the τ axis;

M: Number of subintervals on the τ axis;

N: Number of subintervals on the x axis;

x_{min}: Lower bound of the x axis after truncation;

x_{max}: Upper bound of the x axis after truncation;

u_n^m: The solution of the heat equation at node (n, m);

$O(\Delta)$: Truncation error;

w: Over relaxation parameter of both the SOR and the PSOR iterative methods;

g_n^m: Discretised payoff function;

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Chapter 1

Introduction

1.1 Background

Over the past years, exposure to foreign currency risk has been escalating. This is a result of currencies that are in floating regime along with the increase in the volume of the world trade. Even though investing in domestic assets can be risky, fluctuating exchange rates represent an additional risk factor for investors who want to diversify their portfolios internationally. Thus, there is a definitive need for hedging the exposure to currency risk. Investors are therefore faced with the daunting task of selecting the most appropriate hedging strategy to maximize the returns from international investments.

In today's economy, the exchange rate of major currencies has been permitted to float freely against each other. That is, the local currency value is determined by the Foreign Exchange market based on supply and demand [21]. Therefore, if travelers, importers, exporters and international investors demand more (or less) of a certain currency, that currency will increase (or decrease) in value. This is in contrast with the Bretton Woods system that was used in the 1960s, which was the first system used to control the value of currencies between different countries. The system simply fixed the expected rate to a level, which would remain fixed until time lapsed. However, due to its incompatibility issues and the inability to deal with the growing changes of the world economy, it became unsustainable and so the floating regime was born [35].

An indication of the market size can easily be observed from the average daily transaction cost. According to the Triennial Central bank survey (the triennial survey) of the Foreign Exchange and Over the counter (OTC) derivatives, this was estimated to be around 5,1 trillion dollars per day in April 2016. The triennial survey, on which the foreign exchange analysis is based, is obtained every alternating three years through a collaboration of the Bank of International Settlements and the national Central Banks of each country [42]. This turnover is composed out of spot transactions, outright forwards, foreign exchange swaps, currency swaps and foreign exchange options, where eighty percent of it is a result of foreign exchange swaps and spot transactions [7]. The above-mentioned activities suggest that the foreign exchange deals with a lot of participants, those making large transactions and those making small transactions. Due to this, the market is divided into different levels of access. The interbank market, where the largest participants of the market execute their transactions is on top. These participants are comprised of banks, central banks, commercial companies and large funds. Since large volumes are traded at the top level, the ask-bid spreads are smaller compared to the lower levels of access. These spreads increase as the levels decrease. The most traded currencies are the US Dollar, Euro, GBP and the Japanese Yen.

1.2 Currency exposure

Before any financial investment is pursued, investors normally familiarize themselves with the risks associated with the investment at hand. This makes risk to be a core component in investment management. For this reason, risk management is regarded as a highly specialized field. Over the past years, exposure to foreign currency risk became an important aspect of international investments. In particular, exchange rate fluctuation, this is a major source of risk to multinational corporations around the world. These movements affect expected future cash flows, and consequently affect the value of these institutions by changing the home currency value of foreign revenues.

One way to minimize risk, is to use insurance. Insurance is the premise on which financial derivatives were created. These financial instruments provide a certain level of protection against financial lost. Formally defined, a derivative is a financial instrument whose value is derived from the value of some underlying variable [13]. These variables underlying the derivative are simply referred to as the underlying. There are many different types of financial derivatives that are tailored to insure specific investments. A variety of approaches for hedging the exposure to currency risk has been adopted. These strategies include but are not limited to currency swaps, multi-currency diversification and hedging via forwards, futures and options. The question that remain is which one will be the most effective for hedging the risks associated with currency trading. There exists a lot of documented literature on this issue and by far currency forwards and futures have been proved to be the most favourable (see for example [36]). However, the effectiveness of currency options are becoming more and more popular.

1.3 Option theory

A forward contract gives one the ability to lock in an exchange rate for the future. It is an agreement to buy (or sell) a currency at a certain future time for a certain price [27]. Although, many institutions have extensive experience in the use of this type of derivative instrument, the desire for more flexibility often arises than what forward contracts provide. For example, suppose a manufacturer has sales priced in US dollars as well as Euros. Then depending on the relative strength of the two currencies, greater revenues may be realized in either one of the two currencies. In this case, utilizing a forward contract would not be appropriate, since there is no point in hedging something you might not own. What is called for is a foreign exchange option, i.e., the right but not the obligation to exchange a currency at a predetermined rate.

1.3.1 What is an option

This type of derivative is a legal contract between two parties that ensures the future delivery of a currency for another. The option contract is sold by one part (the option writer) and bought by another (the option holder). However, it differs from a forward contract in the sense that it gives the holder the right to buy or sell a currency at an agreed price (also referred to as the strike price or exercise price), but the holder of the option contract is not obligated to do so. There are two types of options: call and put options. A call option offers the holder the right, but not the obligation to buy a specific underlying for an agreed upon price at a predetermined future time. A put option on the other hand, offers the holder the right, but not the obligation to sell a specific underlying for an agreed upon price at a predetermined future time [56].

1.3.2 Classification of options

Generally, options are classified into two categories: plain vanilla options and exotic options. In the financial world the term plain vanilla is used to describe any tradable asset, or financial instrument, that is the simplest, most standard version of its kind. The two types of vanilla options are American and European options. It is important to take note that American and European does not refer to any geographical region, but are indeed names that refer to their different exercise rights. These options are actively traded at any exchange and their value can be determined from the market prices. American options can be exercised at any time from the commencement date to maturity whereas its European counterpart, is limited and can only be exercised upon maturity. Exotic options are options that are specifically designed based on the needs of the client. Therefore, if a vanilla option does not meet the necessary requirements one can use exotic options.

1.3.3 Mechanics of options

In order to use these types of derivatives one must have a complete understanding of how they work and also be familiar with the jargon associated with them. At time $t = 0$ or t_0 the buyer purchase the option at a certain price (option premium). The value of the underlying at this time is S_0. The option contract then binds the two parties (writer and the holder) for a specific time interval $[0, T]$, where T is the expiration date. Regardless of the underlying asset's actual price at time T and depending on the type of option bought, the holder now has the right to buy or sell the underlying asset at a predetermined price, called the strike price. Buying or selling the underlying option is referred to as exercising the option.

For an American option, the holder can choose to execute his right by exercising the option at any time before or on the experation date, T ([27, 54]). Therefore, premature exercising is possible with American options. Upon the expiration date, both the American or European option holder can choose whether or not to exercise the option. If the holder wishes to exercise the derivative, depending on the type of option, then the underlying is either purchased or sold at the predetermined price (the strike price), irrespective of what the market price of the asset is at maturity. In this case the option holder is protected from any fluctuations in the asset price. However, if the option holder does not wish to exercise his right, he loses his initial premium [27], [54].

1.3.4 Valuation of options

The famous Black-Scholes formula established an efficient way of evaluating the worth of options as a prerequisite to be recognised as a strong legitimate derivtive instrument. Building on the literature previously developed by market researchers and practitioners such as Louis Bachelier, Sheen Kassouf and Ed Thorp among others, Fisher Black and Myron Scholes developed a mathematical model of the financial market containing derivative investment instruments. Using this model one can find the theoretical estimate of the price of an European option [9]. The model is used extensively, but due to the models relaxed assumptions, it is often used with adjustments and corrections by option market participants [27]. The success of the Black-Scholes model is known to rely upon several assumptions. Assumptions that are to a certain extent counterfactual [19]. Among these are the continuity of the stock price process (it does not jump), the ability to hedge continuously without transaction costs, independent Gaussian returns and constant volatility. However, it is the insights of the model that makes it attractive to use as guidance for derivative valuation.

A European option can only be exercised at the expiration date while an American option can be exercised at any time up to the expiration date. Intuitively, it makes logical sense for the American option to be more expensive than the European option due to this additional privilege. Although this can be a very attractive feature for the buyer it is usually accompanied by a question whose answer is not so straight forward: when should one exercise the option [56]?

The valuation of American options is generally more complicated than European options, except in the case of an American call option with no dividends. In this case, the value of the call option is equal to that of the European call option, since it has been proven to exercise the option only at the expiration date [27]. The property of exercising only at a fixed time made it easier to find the theoretical value of a European option. With the assumption of an arbitrage free market one can derive the Black-Scholes equation. This is a partial differential equation (PDE) which describes the prices of derivative securities as a function consisting of a few parameters. Using this PDE, one can derive a closed form solution for the price of a European option.

The freedom of exercising the derivative prior to expiration makes it difficult to value American options. A choice of methods to approximate the price is available, but in general there is no analytical solution. The extra flexibility that American options offer represents a free boundary problem. This is because the maturity date is unknown at time $t = 0$. Intuitively the holder will only exercise the option on a date that will maximize the payout of the option, therefore the valuation of these types of derivatives are also known as optimal stopping problems. Obtaining an analytical formula for American options is still one of the unsolved problems of finance theory.

1.4 Literature Review

Errol et al. [17] studied the exchange rate exposure of real sector firms in emerging economies. They attempted to directly estimate the exposure of real sector firms from an emerging market. In order to do this the focus was mainly on the determinants of exposure. Exchange rate exposure can be defined as the sensitivity of a firm's market value to random variations in the exchange rate during some specific time interval ([17], [1]). This exposure occurs as the firms market value gets altered by unexpected changes in the exchange rate which in turn changes the expected cash flow in the domestic country. Their study revealed that firms in emerging economies are vulnerable to higher degrees of transaction costs and economic exposure. This is in contrast to those in advanced economies. Their results suggested that this is due to the absence of the derivative markets.

Allayannis and Weston [4] studied the use of foreign currency derivatives and how these financial instruments impacts a firms value. Using a sample of 720 large nonfinancial firms between 1990 and 1995, they examined whether firms using foreign currency derivatives obtained higher market values during currency exposure. Using Tobin's Q as an approximation for the firms market value the authors found significant evidence that there is a positive relation between the firms market value and the use of foreign currency derivatives. In paticular, they found that firms that use foreign exchange derivatives to hedge currency risk have a 4.78% higher market value than those who do not. Allayannis et al. [2] observed similar results. Investigating the management of financial currency exposure through the use of financial and operational hedges, they observed that firms that apply operational hedges do not reduce exchange rate exposure. On the contrary, firms that employ financial hedging strategies are related to lower exposure. Consistent with these findings, they found that firms that use operational hedging strategies are not associated with higher market value, however the use of financial hedging strategies does increase market value.

Allayannis and Ofek [3] examined whether foreign currency derivatives are used for hedging or speculating purposes. Executing their study using a sample of 200 non-financial firms from 1993, their research indicated that these instruments are used mainly for the former, since it reduces the exchange rate exposure firms get exposed to. Additionally their results also indicated that a firms exposure through foreign sales and trade gives an indication of the level of derivatives needed.

In his paper, Jorison [32] examined the exposure of American multinationals to foreign currency risk. In his research he found that multinationals, exporters and manufacturing industries in the United States of America are not significantly affected by exchange rate movements. He explains that possible explanation for this is the extensive use of currency derivatives by these corporations to protect themselves against the activity in the exchange rates. The result is consistent with [17].

Due to its attractive properties, the valuation of financial derivatives has gained a lot of importance in the financial and economic world. A variety of models have been developed to value financial instruments. In their pioneering work, Black and Scholes [9] introduced a model for option pricing. Their model suggests that by continuously adjusting the proportions of the underlying and the options in a portfolio, a risk-less portfolio can be created. Therefore, eliminating all market risks. They then argued that these types of portfolios must then have an expected rate of return equal to the risk free interest rate. This approach led to the parabolic partial differential equation known as the Black-Scholes equation. In physics this is also known as the heat equation. This PDE gave rise to the evolution of the European option pricing formula. It is reffered to as the Black-Scholes formula for pricing an European option when the underlying asset pays no dividends [9].

Subsequent modifications of the Black-Scholes model was introduced by numerous authors . In his 1973 paper, Merton [40] generalized the Black-Scholes model by changing various assumptions on the interest rate, dividend payments and other variables. In particular, he assumed that the interest rate behaved scholastically. However, his findings showed that the Black-Scholes model can be derived by using weaker assumptions than the ones in their original derivation. Therefore, proving the relaxed assumptions by the Black-Scholes model modifies the analysis in no signifivant way. In another paper Merton [40] derived a option formula under the assumption that the stock returns are generated by both continuous and jump process. Similar research was done by Cox and Ross [16]. They introduced several jump and diffusion processes which at that time were not introduced to other models. The authors of both these articles produced explicit formulas for the valuation of options under the assumed conditions. Morover, Merton's findings showed that this new formula contains most of the features of the original Black-Scholes formula. His findings showed that this new formula has most of the features of the original Black-Scholes formula. In addition to finding an explicit solution, the authors also includes solutions to problems involving the pricing of dividend paying stocks and potential bankruptcy. Thorpe [49] investigated dividends and the effects of restrictions against the use of proceeds short sales. He concluded that these two factors do not invalidate the Black-Scholes analysis. These results are consistent with [40].

Currency options have certain features that make them different from options on common stock. As a consequence models commonly used for pricing the latter, such as the Black-Scholes model are inadequate for pricing currency options [23]. Feiger and Jacquillat [18] attempted to price foreign currency options by first pricing a currency option bond. To do this they showed that a default free discounted domestic bond and a portfolio composed of a risk free discounted domestic bond and a currency call option have similar pay-offs. However, they were unable to obtain closed form solutions using this procedure. Stultz [47] also studied currency option bond pricing. He developed a series of analytical formulas for both European call and put options on the minimum or maximum of two risky assets which can be applied to price currency options. However, it is not easy to grasp the fundamentals of foreign currency option pricing, because his paper is primarily concerned with the question of default risk on part of a contract. In 1983 Biger and Hull [5], and Garman and Kohlhangen [20] did some pioneering work in this area. Using the methodology of Black and Scholes to derive their model, the authors of these papers derived explicit formulas for the valuation of a European put and call currency options.

Jorion [31] investigated the information content and predictive power of the implied volatility obtained from currency options. He focused on options traded on the Chicago Mercantile Exchange and examined the predictive power of the implied volatilities for the German Mark, the Japanese Yen, and the Swiss Franc against the American Dollar. Jorion's findings showed that in terms of information content and predictive power the implied volatility outperforms statistical time series models. However, implied volatility appears to be a bias estimate of future volatility. In their paper Xu and Taylor [58] examined the informational efficiency of the currency option market in the Philadelphia Stock Exchange. In their research they examined the same three currencies as the author of [31] and also compared it to the American dollar over the period ranging from January 1985 to January 1991. Making use of the likelihood ratio test, they concluded that the option prices contains incremental information about the future volatilities. Using Mincer-Zarnowitz regressions, Christoffersen and Mazzotta [14] found that over-the-counter implied volatility provides largely unbias and fairly accurate forecasts of one-month ahead and three-month ahead actual volatility. Tabak et al. [48] examined the information content of implied volatilities to see whether it contains information about the volatility over the remaining life of the option. Using GMM estimation their results indicated that implied volatility in option prices contain information not present in the past returns for the Brazilian exchange rate against the American dollar. Their results also indicated that implied volatility are bias estimators with respect to future volatilities, as found in [31]. In a study to price one-day-ahead currency option, Hogue and Kalev [24], studied the effectiveness of intra-daily implied volatility (IDIV) to price currency options. Unlike the IV which only contains information for a specific time of a trading day, the IDIV is modeled to accurately capture intra-daily trading day information for option pricing. Making use of the F-test and the Diebold-Mariano test to compare the effectiveness of IDIV, their results revealed that the IDIV outperforms the realized volatility when estimating one day ahead options.

Asset pricing models for the valuation of financial derivatives are often multi-dimensional, which makes the derivation of closed form solutions a challenging task. For this reason different numerical methods have been developed to solve these partial differential equations for option pricing. Examples of such methods are finite difference methods, Fourier methods and Monte Carlo simulations [37]. Brennan and Schwartz [11] in their pioneering paper were the first to apply the finite difference method to obtain the value of an option. They applied the finite difference method to price options for which there were no closed form solutions and considered an American option on a dividend paying stock. Courtadon [15] added to their idea by using a finite difference approximation that is more accurate than that of Schwartz. In [27], Hull solves the Black-Scholes PDE using some traditional numerical methods, particularly the finite difference methods. He discretized the Black Scholes equation using a set of difference equations and then solves the equation using the finite difference methods. To make the discretization process easier the authors of [38] and [56] first transformed the Black Scholes PDE into the heat equation. They then solved the heat equation numerically using the explicit, implicit and Crank-Nicholson Schemes. These solutions were used to find the values for both European and

American options. However, for American options they introduce the projected over relaxation method to deal with the problems associated with early exercising. In addition to this Wilmott et al. [56] also discussed the stability issues associated with the explicit finite difference method and how to overcome these non-stability issues. The Explicit Finite Difference Scheme is known to be conditionally stable. Hull and White [28] modified the explicit finite difference method to value financial derivatives. In their paper they suggested the use of smaller time intervals to insure that the value of the financial derivative converges to the solution of the underlying differential equation. Hu et al. [26] also indicated that the limitations of the Explicit finite difference method does not make it inadequate and argues that its flexibility and robustness makes it an attractive method. In their paper, they obtained an optimal convergence rate for an explicit finite difference scheme when dealing with a variational inequality problem.

The privilege of early exercise is what distinguishes American options from European options. This extra feature makes the process of pricing an American option a challenging task. McKean [39] suggested that the optimal stopping problem for valuing an American option can be transformed into a free boundary problem. Using this insight he derived rigorous valuation formulas for finite-lived and perpetual American options. His formula provides an explicit presentation of the option's price in terms of the optimal stopping boundary function, however analytical implementation and numerical examination is quite difficult. McKean's work was improved upon by Moerbeke [51] who studied the properties of the optimal stopping boundary. Later in 1973, Black and Scholes [9] in their pioneering paper developed an explicit formula for the price of a European option using arbitrage arguments. In that same year, Merton [40] showed that the Black-Scholes European option pricing methodology applied to American call options on a non-dividend paying stock and that these type of options are equivalent. He also pointed out that this does not apply to American put options and that McKean's results could be adapted to this end.

Today the prices of American options are attained through numerical methods since analytical solutions are not available. As mentioned earlier Brennan and Schwartz [11] were the first to take a numerical approach to value American options. To find the value of an American put option Hull [27], applied the finite difference methods directly to the stochastic differential equation of the American put option problem. The result of his computation was a surface of option prices. Wilmot et al. [56], first transformed the Black-Scholes inequality to the heat equation and then considered a discretized linear complementarity approach to price an American option.

1.5 Dissertation Overview

Chapter 1 provides an introduction to the research.

In Chapter 2 we introduce some of the financial and mathematical preliminaries involved in option pricing. In addition to this, we describe the concept of a geometric brownian motion and how it is related to option pricing. Finally, we end this chapter with the necessary theory on the heat diffusion equation that will be used later for deriving the Black-Scholes formula for pricing European options.

We give a detailed overview of the valuation of the two types of vanilla options in Chapter 3. In particular, we discuss the challenging task of pricing an American option. Since there exist no analytical method to value these types of options, we have to make use of numerical methods. We begin by deriving the the Black-Scholes partial differential inequality for American options. This adjustment enables us to define the American option pricing problem as a free boundary problem. After a formal discussion of the free boundary problem, we present the American option pricing formula as a Linear compelentarity problem which allows us to price the option without explicitly mentioning the free boundary.

Chapter 4 investigates the concept of non-constant volatility. We show that the volatility of currency options are not constant but in fact varies across different strike prices. The resulting graph of the volatility against the different strike prices is observed to be ∪-shaped as described in various literature. Following the work of Hull [27], we show how to derive the risk-neutral probability distribution of an asset and show how it corresponds to the implied volatility. This new distribution is referred to as the implied volatility distribution. Finally, we discuss how to obtain the volatilities to price options whose information is not available in the market.

We introduce the finite difference methods that can be employed for option pricing in Chapter 5. This includes three finite difference schemes. We use these methods to price both European and American options. These methods are easily implemented for European options but the boundary issues of the American option makes the employment of finite difference a challenging task. For this reason we employ the iterative Projected over relaxation (PSOR) method.

In Chapter 6 we compare the different results obtained from the finite difference methods explained in Chapter 5. Considering the factors affecting the convergence rate of the numerical methods, we run simulations accordingly. Furthermore we make inference on which method converge the fastest to the theoretical values of the options.

Chapter 7 concludes the study and makes some recommendations.

Chapter 2

Technical Preliminaries

This chapter is devoted to introducing some basic concepts from finance, probability and measure theory that will be used in the chapters that follow. These concepts provides us with the relevant definitions and theorems to reduce any possible ambiguity and hence familiarize the reader with the concept.

2.1 Mathematical and Financial Preliminaries

Definition 2.1.1. ***Sigma-algebra*** [45]

Let X be a non-empty set. A collection $\mathbb{F}$ of subsets of X is said to be a *sigma (σ) algebra* on X, provided the following conditions are satisfied:

i. $\emptyset \in \mathbb{F}$.

ii. If $\lambda \in \mathbb{F}$, then $\lambda^{\complement} \in \mathbb{F}$ (That is, closed under complementation).

iii. If $\lambda_1, \lambda_2, ..., \lambda_n, ... \in \mathbb{F}$, then $\bigcup_{i=1}^{\infty} \lambda_i \in \mathbb{F}$ and $\bigcap_{i=1}^{\infty} \lambda_i \in \mathbb{F}$.

We shall often refer to an *index set*, by which we mean either a finite set $\{1, 2, ..., n\}$, N, or an interval $[0, T]$ or $[0, \infty)$.

Definition 2.1.2. ***Probability space*** [45]

A *probability space* is a triple $(X, \mathbb{F}, \mathbb{P})$, where X is a set, $\mathbb{F}$ is a $\sigma-$algebra on X and $\mathbb{P}$ is a probability measure.

Definition 2.1.3. ***Filtration*** [45]

Given an indexing set I, a *filtration* of a sigma algebra $(X, \mathbb{F})$ is an increasing family $\{\mathcal{F}_{t\in I}\}$ of sub-sigma algebras.

Definition 2.1.4. ***Filtered Probability space*** [45]

A *filtered probability space* is a probability space $(X, \mathbb{F}, \mathbb{P})$ together with a filtration $\mathcal{F}_t$ of $\mathbb{F}$. That is:

$$\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}_T \subset \mathbb{F}\,, \quad \text{for all} \quad 0 < s < t < T\,.$$

Definition 2.1.5. ***Adapted stochastic process*** [12]

A stochastic process is a collection of random variables $\{X(t)|t \in I\}$ for $I = N$ (discrete time process) or I an interval (continuous time process), defined on a given filtered probability space. We consider a continuous time process and we take $I = [0, T)$ or $I = R_+$. A collection of $X(t)$ is said to be an *adapted stochastic process* if the random variable $X(t)$ is $\mathcal{F}_t$-measurable.

Definition 2.1.6. ***Brownian motion*** [52]

A standard *Brownian motion* $\{B(t) : t \in [0, \infty)\}$ is a $\mathbb{R}$-valued random process, having the following properties :

- The process has continuous sample paths.
- At the initial time $t = 0$, the process is at rest. That is, $B(0) = 0$.
- The process has independent increments over non-overlapping time intervals. That is, $B(t) - B(s)$ is independent of $\{B(k) : k \leq s\}$, whenever $s < t$.
- The increment $B(t) - B(s)$, where $s < t$, has normal probability distribution with zero mean and with variance, the length of the increment. That is, $B(t) - B(s) \sim N(0, t - s)$.

Definition 2.1.7. ***Stochastic differential equation*** [12]

Let $S(t)$ be a stochastic process. A *stochastic differential equation* (SDE) is an equation which consists of combination of deterministic term and a stochastic term (also referred to as white noise). Such a process is represented in the form

$$dS(t) = \mu(t, S(t))\, dt + \sigma(t, S(t))dB(t)\,, \tag{2.1.1}$$

where $\mu(t, S(t))$ and $\sigma(t, S(t))$ are adapted processes. In the integral form, it is represented as

$$S(t) = S(0) + \int_0^t \mu(u, S(u))du\, + \int_0^t \sigma(u, S(u))dB(u)\,.$$

Definition 2.1.8. ***Itô's Lemma*** **(in one dimension)** [8]

Let $C^{1,2}$ be the class of . Let $S(t)$ be a stochastic process which follows an Itô process defined in equation (2.1.1). Suppose there exists a function $f \in C^{1,2}$ and define the process Z by $Z(t) = f(t, S(t))$. *Itô's lemma* shows that Z follows the stochastic differential equation given by

$$\mathrm{d}f(t,S(t)) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right)\mathrm{d}t + \frac{\partial f}{\partial S}\sigma\,\mathrm{d}B(t)\,.$$

According to Hull [27], the Brownian motion underlying f and Z are the same.

2.2 Geometric Brownian Motion

It should be noted that since a Brownian Motion can take on negative values it cannot be used directly for modeling stock prices. Therefore, we introduce a non-negative variation of the Brownian Motion called a Geometric Brownian Motion. It is an exponentiated form of the Brownian motion. A Geometric Brownian Motion is a model that measures the changes in the random process $dS(t)$ to the current underlying value $(S(t))$. It is defined by the stochastic differential equation

$$dS(t) = \mu S(t)dt + \sigma S(t)dB(t) \tag{2.2.1}$$

where μ and σ are the drift and variance parameter respectively. This model was used by Black and Scholes to describe the behavior of a stock price when they derived the Black-Scholes model. The models assumes that the percentage change in a short period of time is normally distributed. That is,

$$\frac{dS(t)}{S(t)} \sim \mathcal{N}(\mu dt, \sigma^2 dt) \tag{2.2.2}$$

where $dS(t)$ is the change in the stock price $S(t)$ in a short period of time dt and $\mathcal{N}(m, v)$ denotes a normal distribution with mean m and variance v.

To solve equation 2.2.1 we apply Itô's lemma. Let $f \in \mathcal{C}^{1,2}$ be a function defined by

$$f(S(t),t) = \log(S(t))$$

where $S(t)$ follows the path described in equation 2.2.1. Itô's lemma asserts that f follows the process

$$\mathrm{d}f(t,S(t)) = \left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial S}\mu + \frac{1}{2}\frac{\partial^2 f}{\partial S^2}\sigma^2\right)\mathrm{d}t + \frac{\partial f}{\partial S}\sigma\,\mathrm{d}B(t). \tag{2.2.3}$$

The first and second order derivatives of f are given by:

$$\frac{\partial f}{\partial S} = \frac{1}{S}, \quad \frac{\partial^2 f}{\partial S^2} = -\frac{1}{S^2} \quad \text{and} \quad \frac{\partial f}{\partial t} = 0\,.$$

Substituting these derivatives in equation 2.2.3 we observe that f follows the path

$$d\log(S(t)) = (\mu - \frac{\sigma^2}{2})dt + \sigma dB(t). \tag{2.2.4}$$

Equation 2.2.4 indicates that $d(f(S,t))$ follows a geometric Brownian motion. Therefore, it is normally distributed with mean $(\mu - \frac{\sigma^2}{2})$ and variance σ^2. We obtain the solution of equation 2.2.4 by integrating both sides of equation 2.2.4 from 0 to t. That is,

$$\int_0^t d\log(S(t)) = \int_0^t (\mu - \frac{\sigma^2}{2})ds + \sigma \int_0^t dB(s),$$

which simplifies to

$$\log(S(t)) = \log(S(0)) + (\mu - \frac{\sigma^2}{2})t + \sigma B(t). \tag{2.2.5}$$

Therefore the solution for $S(t)$ is given by

$$S(t) = S(0)e^{(\mu - \frac{\sigma^2}{2})t + \sigma B(t)}.$$

Properties of geometric Brownian motion

Consider equation 2.2.5, it can be rearranged as

$$\log(S(t)) - \log(S(0)) = (\mu - \frac{\sigma^2}{2})t + \sigma B(t). \tag{2.2.6}$$

From equation 2.2.6 we observe that the change in $\ln(S)$ over a time interval $[0, T]$, is normally distributed with mean $(\mu - \frac{\sigma^2}{2})T$ and variance $\sigma^2 T$. Meaning

$$\log(S(T)) - \log(S(0)) \sim \mathcal{N}\Big((\mu - \frac{\sigma^2}{2})T, \sigma^2 T\Big)$$

which implies that

$$\log(S(T)) \sim \mathcal{N}\Big(\log(S(0)) + (\mu - \frac{\sigma^2}{2})T, \sigma^2 T\Big). \tag{2.2.7}$$

Equation 2.2.7 shows that $f(S(t))$ is normally distributed. So it follows that $S(t)$ is log normally distributed with probability density function

$$f(S:\mu,\sigma,t) = \frac{1}{S\sigma\sqrt{2\pi t}}\exp\left(-\frac{\Big(\log(S(t)) - \log(S(0)) - (\mu - \frac{\sigma^2}{2})t\Big)^2}{2\sigma^2 t}\right),$$

expected value

$$\mathbb{E}[S(t)] = S(0)e^{\mu t}$$

and variance

$$\mathrm{Var}[S(t)] = (S(0))^2 e^{2\mu t}(e^{\sigma^2 t} - 1)\,.$$

Figure 2.1 depicts a computer simulation of an underlying asset price of which the value is considered a geometric Brownian motion.

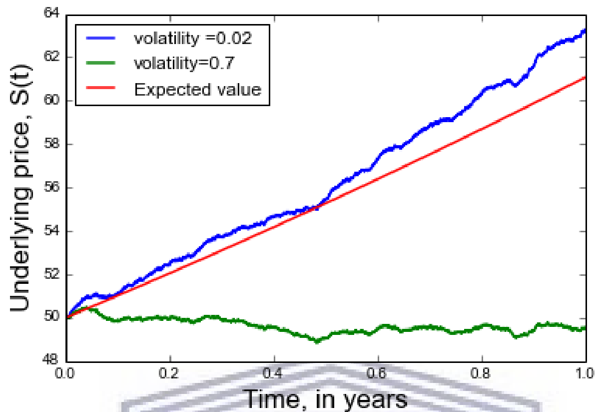

Figure 2.1: Geometric Brownian motion simulation

We used the parameters $S(0) = 50, \quad r = 0.2, \quad \sigma_1 = 0.02, \quad \sigma_2 = 0.7$ and $T = 1$ to obtain the graph in Figure 1. Here the smooth curved line represents the expected value of the underlying. If σ remains small we notice that the asset path remains close to the expected value and larger values for σ forces the trajectory to move away from it.

2.3 The Heat equation

The derivation of the Black-Scholes-Merton partial differential equation was an enormous step towards finding the value of an option. This equation gave rise to the Black-Scholes formula for pricing European options. One way of deriving this formula is by first transforming the Black-Scholes PDE to the well known heat equation. This equation is given by [55]

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{2.3.1}$$

for $u = u(x, \tau)$ and x and τ in the domain

$$D_u = \{(x, \tau) : -\infty < x < \infty; \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T\}. \tag{2.3.2}$$

The heat equation is a homogeneous, one-dimensional, second order, linear forward parabolic equation. This equation models the flow of heat in a continuous medium and is very common in the field of mathematics [56]. The heat equation describes the diffusion of heat in a long, thin, uniform bar of length L. Additionally, the bar is perfectly insulated so that no heat can escape. Therefore, the heat flows only from a hot to a cooler area, canceling out the temperature differences over a certain time period, T. The function $u(x,\tau)$ represents the temperature in the bar which varies with distance x and time τ [56]. In the next section we discuss the solution of the heat equation. We will also make use of the simple form of this equation to find the value of American options.

Solution to the Heat diffusion equation

Consider the heat equation described by equations 2.3.1 to 2.3.2 and let us assume that the solution and initial conditions $u_x(0)$ satisfy the following requirements:

i. $u_x(0)$ is sufficiently well behaved.

ii. $\lim_{|x|\to\pm\infty} u_0(x)e^{-\alpha x^2} = 0$ for any $a > 0$

iii. $\lim_{|x|\to\pm\infty} u(x,\tau)e^{-\alpha x^2} = 0$ for any $a > 0$

Then there exist a solution for $u(x,\tau)$ (at all times), given by [55]

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{\frac{-(x-s)^2}{4\tau}} ds. \tag{2.3.3}$$

We will evaluate this existing solution of heat diffusion equation in the following chapter to obtain the the Black-Scholes formulas to price European call and put options.

The reason for transforming the Black-Scholes equation to the heat equation is because the heat equation has been studied thoroughly and its solution is well-known. This solution can then be evaluated and by reversing the transformation we obtain the solution of the Black-Scholes equation. Also in the case where no analytical solution is available, like in the case of American options, numerical techniques can be implemented to estimate the solution to the heat equation.

Chapter 3

The valuation of European and American Options

This chapter addresses the topic of mathematical models for pricing European and American options. We discuss the option payoff structure which plays an important role in the valuation process. Thereafter we derive the Black-Scholes PDE and discuss how similar this PDE is to the heat equation. Later on in this chapter we transform the Black-Scholes PDE into the heat equation, which makes the derivation of the Black-Scholes pricing formulas for European options easier. Upon completing the transformation we use the existing solution of the heat equation to derive the famous Black-Scholes formulas. These concepts are also used to derive the Black-Scholes inequality for American options which can also be transformed to the one dimensional heat equation. Due to the early exercise facility offered by American options, the American option pricing problem is also referred to as a free boundary problem for which no explicit solution exists. To aid understanding of the free boundary problem we introduce the obstacle problem and show how these problems are related. Finally, we construct the free boundary problem as a linear complementarity problem (LCP). This formulation allows us to solve the American option pricing problem without explicitly mentioning the free boundary. Upon completing the valuation of European and American options on non-dividend paying stocks, we continue to pricing currency options. Currency options can be priced in a similar manner to options on non-dividend paying stocks, however there are some differences we first have to take into account.

3.1 European options

This section is dedicated to the valuation of European options. Following the work of [27] and [56], we begin by introducing the option's payoff structure. This will play an important role in the valuation process. The second part of this section is allocated to the derivation of the Black-Scholes PDE, which will be used to obtain the pricing formulas for European options. To do the latter we first transform the Black-Scholes PDE into the heat diffusion equation. This transformation makes the derivation of the pricing formula easier. This section is mainly based on the work of Wilmott et al. [56] and Seydel [44].

As mentioned earlier, there are two types of options namely, the call option the and the put option. An European call option offers its holder the right to buy a specific underlying asset for a specified price, K, called the strike price or the exercise price, at a specified time, T, which is referred to as the expiration date. On the other hand, an European put option gives its holder the right to sell a specific underlying asset for a specified price at a specified time in the future. However, in both cases the holder of the contract is not obliged to exercise this right. It is for this reason that options are becoming more popular among investors. European options can only be exercised on the date of expiration, whereas American options can be exercised at any time on the interval $[0, T]$, i.e., the time from the purchase of the option to its maturity [27].

3.1.1 The Option's payoff structure

As mentioned in the previous section, an option's payoff structure plays an important role in the valuation process. Therefore, it is neccessary to have a clear understanding of the option's payoff structure. The option's payoff is dependent on two factors, that is, the type of option considered and the position taken on that particular option. An option contract has two potential positions [27]:

(1) A long position

This position is taken by the investor, who becomes the holder. This investor purchases the option and waits for the reward.

(2) A short position

A position taken by the investor. The investor sells or writes the option, therefore becomes the option writer. Being the option writer, this investor receives payment upfront and therefore faces potential liabilities in the future.

An investor can therefore consider four different positions in terms of option trading [27].

- A long position in a call option.
- A long position in a put option.
- A short position in a call option.
- A short position in a put option.

Payoff from a long position in an option contract.

For European options the payoff which describes the value of the option depends on what the contract is worth at expiry. This can be explained through a simple arbitrage argument. Suppose that an investor takes a long position in an European call option and that the price of the underlying asset is given by $S(t)$ at time t. Now at expiry, i.e., $t = T$, if $S(t) > K$, it would make financial sense for the holder to exercise the option. The holder will then hand over an amount K, in return for the underlying asset worth $S(T)$. The payoff from this transaction is then $S(T) - K$. On the other hand, at expiry, if $S(t) < K$, the holder will cease to exercise the option, because the market price is more favorable. He will then purchase the underlying asset in the open market. In doing so the option will be worthless at expiration. Therefore, the payoff to the holder of an European call option which is also the value of the option at the expiration date, can be written as [44]

$$\text{payoff} = (S(T) - K)^+ = \max[S(T) - K), 0)]. \tag{3.1.1}$$

A similiar argument implies that the payoff to the holder of an European put option is given by [44]

$$\text{payoff} = (K - S(T))^+ = \max[K - S(T), 0]. \tag{3.1.2}$$

This can be seen in Figure 3.1 and Figure 3.2.

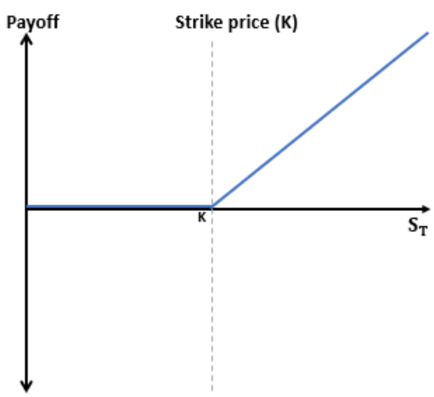

Figure 3.1: Long position on call option

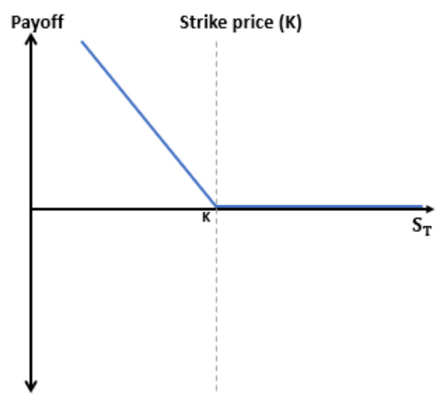

Figure 3.2: Long position on put option

Payoff from a short position in an option contract.

At the time of agreement, $t = 0$, the option writer receives the option premium. Based on the type of option contract he is now obliged to buy or sell at the time of maturity. If the option is a call option, then at time $t = T$, if $S(t) > K$ the option holder will most likely exercise the option. The option writer must therefore pay out the amount $S(T) - K$. If $S(t) < K$, the option holder will not exercise the option and the option writer will have the option premium as profit (it is important to note that the underlying asset need not ever be in the possession of the option writer). Therefore, the payoff to the writer of an European call option at the time of expiry, i.e., time $t = T$ is [27]

$$\text{payoff} = \text{-max}(S(T) - K), 0) = \min[K - S(T), 0]. \tag{3.1.3}$$

A similiar argument implies that the payoff to the writer of an European put option is given by [27]

$$\text{payoff} = \text{-max}(K - S(T)), 0) = \min[S(T) - K, 0]. \tag{3.1.4}$$

This can be seen in Figure 3.3 and Figure 3.4.

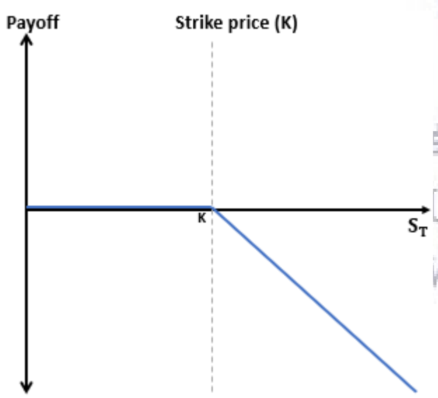

Figure 3.3: Short position on call option

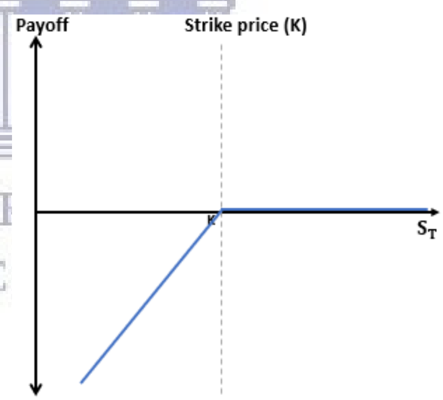

Figure 3.4: Short position on put option

3.2 The Black-Scholes Pricing formulas for European Options

For better understanding of the Black-Scholes pricing model, we derive the Black-Scholes PDE. There exist many different methods to derive this PDE. Wilmott [53], for example, has a full chapter dedicated to this subject. In his book he provides numerous methods to derive the Black-Scholes PDE, however in this section we only use the method that involves the application of Itô's lemma. What follows are the assumptions that led to the discovery of this model [9]:

i. The price of the underlying asset corresponds to a model with constant drift and volatility.

ii. The market is frictionless, i.e., transactions are incurred at no cost,

iii. The short-term risk-free rate is constant and known,

iv. The stock pays no dividends during the life of the options,

v. Short shelling is permitted,

vi. Investors can borrow or lend at the same risk-free interest rate,

vii. No risk-less arbitrage opportunities exist.

Based on these assumptions the mathematical model was developed and plays an important role in the valuation of financial derivatives.

Notations

What follows are the notations used to derive the Black-Scholes PDE.

- $S(t)-$ the current price of the stock at time, t.
- $\sigma-$ the volatility or the standard deviation of the underlying's return.
- $K-$ the strike price.
- $r-$ the risk-free interest rate which is continuously compounded.
- $T-$ the time to expiry.
- $\mu-$ the drift term on the stock.
- $\Pi-$ the value of the portfolio.
- $V(S,t)-$ the option value.
- $\tau-$ the remaining time to expiry and is denoted by $\tau = T - t$. At expiry, $\tau = 0$.
- $\mathcal{N}()-$ the cumulative distribution function of a standard normal distribution and it is defined by

$$\mathcal{N}(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{1}{2}z^2} \mathrm{d}z\,.$$

- $\mathcal{N}'(x)-$ refers to the standard normal probability density function which is defined as

$$\mathcal{N}'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}.$$

- $B(t)$ is a standard Brownian motion defined on a real world filtered probability space.

3.2.1 Derivation of the Black-Scholes PDE

Using the assumptions mentioned in the previous section and mathematical tools described in Chapter 2, we are now able to derive the Black-Scholes PDE for the valuation of an European option on a non-dividend paying stock.

We assume that the stock price follows a geometric Brownian motion described in Chapter 2, equation 2.2.1. That is,

$$dS = \mu S dt + \sigma S dB(t) \tag{3.2.1}$$

where $S = S(t)$ is the price of the stock [27]. Let $V(S,t)$ be the value of an option on the non-dividend paying stock. At this stage it is not necessary to specify whether $V(S,t)$ is the value of a call or a put option. According to Itô's lemma $V(S,t)$ follows the process [27]

$$dV(S,t) = \left(\frac{\partial V(S,t)}{\partial t} + \mu S \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2}\right) dt + \sigma S \frac{\partial V(S,t)}{\partial S} dB(t). \tag{3.2.2}$$

Also from the definition of Itô's lemma discussed in Chapter 2, we know that the Brownian motion underlying $V(S,t)$ and $S(t)$ are identical. Therefore, the random component can be eliminated by constructing a portfolio consisting of the aforementioned stock and the derivative. Let Π be a risk-less self financing portfolio consisting of one derivative and $-\Delta$ units of the underlying stock. That is,

$$\Pi = V(S,t) - \Delta S. \tag{3.2.3}$$

The change in the portfolio value at time dt is given by

$$d\Pi = dV(S,t) - \Delta dS. \tag{3.2.4}$$

Substituting equations 3.2.1 and 3.2.2 in 3.2.4 shows that $d\Pi$ follows the random walk

$$d\Pi = \sigma S\left(\frac{\partial V(S,t)}{\partial S} - \Delta\right) dB(t) + \left(\mu S \frac{\partial V(S,t)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + \frac{\partial V(S,t)}{\partial S} - \mu \Delta S\right) dt.$$

If we select $\Delta = \frac{\partial V(S,t)}{\partial S}$, the random component is eliminated and this gives rise to a riskless portfolio

$$d\Pi = \left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} \right) dt. \tag{3.2.5}$$

Now suppose an amount Π is invested in a risk-less asset with constant interest rate r. Then the capital growth of this amount would be $r\Pi dt$ at time dt. Therefore, to avoid arbitrage the change in the value of the portfolio must replicate the change in the amount Π invested in the risk-less asset. That is,

$$d\Pi = r\Pi dt$$

which after substituting equations (3.2.3), (3.2.5) and Δ becomes

$$\left(\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} \right) dt = r \left(V(S,t) - S \frac{\partial V(S,t)}{\partial S} \right) dt.$$

Rearranging the terms in the equation above we obtain

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V(S,t)}{\partial S^2} + rS \frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0. \tag{3.2.6}$$

This is known as the Black-Scholes PDE.

The Black-Scholes PDE is a backward parabolic PDE that can be used to obtain the present value of the option contract. At expiry the value of the option is known and thus, the solution of the PDE can be found either analatically or numerically using the terminal and boundary conditions given below. For call options the following terminal and boundary conditions hold [56] :

$$C(S,T) = \max(S(T) - K, 0), \tag{3.2.7}$$

$$C(0,t) = 0, \tag{3.2.8}$$

$$C(S,t) \sim S \quad \text{as} \quad S \to \infty, \tag{3.2.9}$$

and for put options we have [56]:

$$P(S,T) = \max(K - S(T), 0), \tag{3.2.10}$$

$$P(0,t) = Ke^{-r(T-t)}, \tag{3.2.11}$$

$$P(S,t) \sim 0 \quad \text{as} \quad S \to \infty. \tag{3.2.12}$$

3.2.2 Transformation of the Black-Scholes PDE into heat equation

The Black-Scholes PDE is satisfied for any option whose value can be expressed as some smooth function $V(S,t)$. As we mentioned in the introduction this equation can be transformed into the heat diffusion equation from which the pricing formulas for both the European put and call options can be obtained.

In order to make the transformation, it is important to notice that the Black-Scholes PDE

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rV(S,t) = 0, \tag{3.2.13}$$

with domain

$$D_V = \{(S,t) : S > 0, \quad 0 \leq t \leq T\}, \tag{3.2.14}$$

is similar to the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{3.2.15}$$

for x and τ in the domain

$$D_u = \{(x,\tau) : -\infty < x < \infty, \quad 0 \leq \tau \leq \frac{1}{2}\sigma^2 T\}, \tag{3.2.16}$$

in that it has a first derivative of the unknown with respect to time and the second derivative of the unknown with respect to the space variable. However, we have to take into account the differences in order to make the transformation. Notice that:

(i) The Black-Scholes equation is not a constant coefficient equation,

(ii) There is a first derivative of V with respect to S in the Black-Scholes equation,

(iii) There is a term that contains only the function V,

(iv) The Black-Scholes equation is a backward parabolic equation while the heat equation is a forward parabolic equation [55].

We can eliminate these differences by employing a suitable change in variables. In doing so we reduce the Black-Scholes equation into the heat equation of which the solution is known and can be used to obtain the Black-Scholes formula [56].

Equation (3.2.13) denotes the value of the option as $V(S,t)$ to show that the Black-Scholes

PDE can be used to price any derivative. However, to price an option one has to specify the type of option being referred to. This will enable us to employ the necessary conditions. Suppose we want to calculate the value of a call option. We denote the value of the option as $C(S,t)$. Therefore, equation (3.2.13) becomes

$$\frac{\partial C(S,t)}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 C(S,t)}{\partial S^2}+rS\frac{\partial C(S,t)}{\partial S}-rC(S,t)=0. \tag{3.2.17}$$

The Black-Scholes equation can be converted to a constant coefficient parabolic linear equation of the form:

$$\frac{\partial v}{\partial t}=a\frac{\partial^2 v}{\partial x^2}+b\frac{\partial v}{\partial x}+cv \tag{3.2.18}$$

where $v=v(x,t)$, $b,c\in\mathbb{R}$ and $a=1$.

What follows is a transformation of the Black-Scholes partial differential equation to the heat equation as described in [50] and [56]. We achieve this transformation by using various substitutions. To reduce equation (3.2.17) to the dimensionless form of equation (3.2.18) we make the change of variables:

$$S=Ke^x,\quad t=T-\frac{\tau}{\frac{1}{2}\sigma^2}\quad\text{and}\quad C=Kv(x,\tau).$$

Using the chain rule, we obtain the following derivatives:

$$\frac{\partial C(S,t)}{\partial t}=K\left(\frac{\partial v}{\partial \tau}\frac{\partial \tau}{\partial t}\right)=-\frac{1}{2}K\sigma^2\frac{\partial v}{\partial \tau}, \tag{3.2.19}$$

$$\frac{\partial C(S,t)}{\partial S}=K\left(\frac{\partial x}{\partial S}\frac{\partial v}{\partial x}\right)=\frac{K}{S}\frac{\partial v}{\partial x}, \tag{3.2.20}$$

and

$$\frac{\partial^2 C(S,t)}{\partial S^2}=K\left(\frac{\partial}{\partial S}\frac{\partial C(S,t)}{\partial S}\right)=-\frac{K}{S^2}\frac{\partial v}{\partial x}+\frac{K}{S^2}\frac{\partial v^2}{\partial x^2}\,. \tag{3.2.21}$$

Substituting these derivatives into equation (3.2.17), the Black-Scholes equation reduces to the constant coefficient parabolic linear equation described in equation 3.2.18. That is,

$$\frac{\partial v}{\partial \tau}=\frac{\partial^2 v}{\partial x^2}+(k-1)\frac{\partial v}{\partial x}-kv \tag{3.2.22}$$

where $k=\dfrac{2r}{\sigma^2}$. Notice that we are approaching the diffusion equation and can complete the transformation by considering another change of variables. Let

$$v(x,\tau)=e^{\alpha x+\beta\tau}u(x,\tau).$$

Then the derivatives of $v(x,\tau)$ with respect to τ and x is given by

$$\frac{\partial v}{\partial \tau} = e^{\alpha x+\beta\tau}\left(\frac{\partial u}{\partial \tau} + \beta u(x,\tau)\right),$$
$$\frac{\partial v}{\partial x} = e^{\alpha x+\beta\tau}\left(\frac{\partial u}{\partial x} + \alpha u(x,\tau)\right),$$

and

$$\frac{\partial^2 v}{\partial x^2} = e^{\alpha x+\beta\tau}\left(\frac{\partial^2 u}{\partial x^2} + 2\alpha\frac{\partial u}{\partial x} + \alpha^2 u(x,\tau)\right).$$

Substituting these derivatives in equation (3.2.22) we obtain:

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (2\alpha + k - 1)\frac{\partial u}{\partial x} + (\alpha^2 + \alpha(k-1) - \beta - k)u(x,\tau). \tag{3.2.23}$$

In order to obtain the diffusion equation defined in equation 3.2.15 we have to choose α and β in such a way that the coefficients of $\frac{\partial u}{\partial x}$ and $u(x,\tau)$ equate to zero, therefore eliminating these two terms. To accomplish this we set

$$2\alpha + k - 1 = 0$$

and

$$\alpha^2 + \alpha(k-1) - \beta - k = 0.$$

Solving the system of equations we have

$$\alpha = \frac{1-k}{2} \quad \text{and} \quad \beta = -\frac{1}{4}(1+k)^2.$$

With these solutions for α and β we obtain:

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) \tag{3.2.24}$$

and equation (3.2.23) reduces to the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{3.2.25}$$

where $(x,\tau) \in D_u$ and $u(x,\tau)$ is given by

$$u(x,\tau) = e^{\frac{1}{2}(k-1)x+\frac{1}{4}(k+1)^2\tau}v(x,\tau). \tag{3.2.26}$$

Boundary and initial conditions of the transformed Black-Scholes PDE

Working with the transformed Black-Scholes PDE we have to derive the initial and boundary conditions of the function $u(x,\tau)$ in equation (3.2.25). To do this we make use of the initial and boundary conditions of the original Black-Scholes equation. The solution of equation (3.2.25) is subjected to these conditions.

We obtain the initial condition of $u(x,\tau)$ by making use of the transformed variables. Recall the transformation $t = T - \frac{\tau}{\frac{1}{2}\sigma^2}$, therefore

$$\tau = \frac{1}{2}\sigma^2(T-t). \tag{3.2.27}$$

Notice that in this case τ is the transformed time variable and represents the time left to maturity of the contract. For example at $t=0$, τ equates to the full life of the derivative contract.

Consider a call option. To find the initial conditions of $u(x,\tau)$, recall that at maturity we have

$$C(S,T) = \max((S-K),0). \tag{3.2.28}$$

Making use of the transformations $C = Kv(x,\tau)$ and $S = Ke^x$ we obtain

$$Kv(x,0) = \max((Ke^x - K),0), \tag{3.2.29}$$

therefore

$$v(x,0) = \max((e^x - 1),0). \tag{3.2.30}$$

From equation (3.2.24) we know that $v(x,0)$ is given by

$$v(x,0) = e^{-\frac{1}{2}(k-1)x}u(x,0). \tag{3.2.31}$$

Substituting $v(x,0)$ into equation 3.2.30 we have

$$e^{-\frac{1}{2}(k-1)x}u(x,0) = \max((e^x-1),0) \tag{3.2.32}$$

which simplifies to

$$u(x,0) = \max(e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x},0). \tag{3.2.33}$$

A similar derivation as above shows that the initial condition for a put is given by

$$u(x,0) = \max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x},0). \tag{3.2.34}$$

To find the boundary conditions note that for a call option $C(S,t) = 0$ when $S = 0$. However, our transformation $S = Ke^x$ does not make provision for $S = 0$ due to the domain of the logarithmic function. Therefore, we look at the limit as S tends to zero. In doing this we notice that if $S \to 0$, $x \to -\infty$. Therefore, the condition

$$C(S,t) \to 0 \quad \text{as} \quad S \to 0 \tag{3.2.35}$$

transforms to

$$Kv(x,\tau) \to 0 \quad \text{as} \quad x \to -\infty.$$

Substituting for $v(x,\tau)$ in the equation above we obtain

$$Ke^{\alpha x + \beta \tau} u(x,\tau) \to 0.$$

Therefore, for a call option

$$u(x,\tau) \to 0 \quad \text{as} \quad x \to -\infty. \tag{3.2.36}$$

Similarly for a put option the condition

$$P(S,t) \to 0 \quad \text{as} \quad S \to \infty \tag{3.2.37}$$

becomes

$$u(x,\tau) \to 0 \quad \text{as} \quad x \to \infty. \tag{3.2.38}$$

What remains is to look at the behavior of $u(x,\tau)$ when $x \to \infty$ for a call option and when $x \to -\infty$ for a put option. For this we make use of the Put-Call parity,

$$C(S,t) - P(S,t) = S(t) - Ke^{-r(T-t)}. \tag{3.2.39}$$

We know that as $S \to 0$, the value of the call become negligible, thus

$$P(S,t) \to Ke^{-r(T-t)} - S(t). \tag{3.2.40}$$

Also as $S \to 0$ we know that $x \to -\infty$, therefore we have

$$u(x,\tau) \to Ke^{\frac{1}{2}(k-1)x + \frac{1}{4}(1+k)^2\tau}(Ke^{-k\tau} - Ke^{x}) \quad \text{as} \quad x \to -\infty \tag{3.2.41}$$

for a put option. Notice that equation (3.2.41) was obtained by substituting the transformed variables in equation (3.2.40). A similar argument can be used for a call option. When $S \to \infty$ the value of the put option becomes negligible, and so

$$C(S,t) \to S(t) - Ke^{-r(T-t)}. \tag{3.2.42}$$

We know that as $S \to \infty$, $x \to \infty$, therefore we have

$$u(x,\tau) \to Ke^{\frac{1}{2}(k-1)x+\frac{1}{4}(1+k)^2\tau}(Ke^{x} - Ke^{-k\tau}) \quad \text{as} \quad x \to \infty \tag{3.2.43}$$

for call option.

Derivation of the Black-Scholes pricing formulas

Now that we derived the initial and boundary conditions for $u(x,\tau)$ the next step is to solve the diffusion equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}$$

with initial condition $u(x,0) = u_x(0)$ and $(x,\tau) \in D_u$. The solution to this equation is given in Chapter 1 by

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) e^{\frac{-(x-s)^2}{4\tau}} ds. \tag{3.2.44}$$

To find the Black-Scholes formula for European options we have to evaluate equation 3.2.44. We do this by considering another change of variable, that is, let

$$y = \frac{s-x}{\sqrt{2\tau}} \quad \text{so that} \quad s = x + y\sqrt{s\tau} \quad \text{and} \quad ds = \sqrt{2\tau}dy.$$

Thus, equation (3.2.44) becomes

$$\begin{aligned} u(x,\tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(x + y\sqrt{2\tau}) e^{-\frac{1}{2}y^2} \sqrt{2\tau} dy\,, \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(x + y\sqrt{2\tau}) e^{-\frac{1}{2}y^2} dy\,. \end{aligned} \tag{3.2.45}$$

For a call options the initial condition of the heat equation given by equation (3.2.33) allows us to evaluate the integral only for

$$e^{\frac{1}{2}(k+1)x} - e^{\frac{1}{2}(k-1)x} > 0 \tag{3.2.46}$$

or

$$e^{\frac{1}{2}(k+1)x} > e^{\frac{1}{2}(k-1)x}, \tag{3.2.47}$$

otherwise it equates to 0. Taking logarithms at both sides we notice that

$$\frac{1}{2}(k+1)x > \frac{1}{2}(k-1)x, \tag{3.2.48}$$

which after further simplification implies that $x > -x$. This is true for all $x > 0$. So to evaluate equation (3.2.45) when it is not equal to zero we must have

$$x + y\sqrt{2\tau} > 0 \tag{3.2.49}$$

therefore,

$$y > -\frac{x}{\sqrt{2\tau}}. \tag{3.2.50}$$

The integral in equation (3.2.45) now becomes

$$u(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} u_0(x + y\sqrt{2\tau}) e^{-\frac{1}{2}y^2} dy. \tag{3.2.51}$$

Substituting $u_x(0)$ in equation (3.2.51) we obtain

$$\begin{aligned} u(x,\tau) &= \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} \left(e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} - e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} \right) e^{-\frac{1}{2}y^2} \mathrm{d}y \\ &= \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} \, \mathrm{d}y - \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} \, \mathrm{d}y. \end{aligned} \tag{3.2.52}$$

Notice that these two integrals are similar. The only difference is the $k+1$ and the $k-1$ terms in the exponential expressions. Therefore, we can evaluate them using the same approach. We write equation (3.2.52) as:

$$u(x,\tau) = I_\alpha - I_\beta \tag{3.2.53}$$

where

$$I_\alpha = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} \, \mathrm{d}y \tag{3.2.54}$$

and

$$I_\beta = \frac{1}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k-1)(x+y\sqrt{2\tau})} e^{-\frac{1}{2}y^2} \, \mathrm{d}y. \tag{3.2.55}$$

To avoid repetition we only evaluate I_α. The second integral, I_β can be calculated in exactly the same manner, but with $k+1$ replaced with $k-1$. We have

$$\begin{aligned} I_\alpha &= \frac{e^{\frac{x}{2}(k+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{\frac{1}{2}(k+1)(y\sqrt{2\tau}) - \frac{1}{2}y^2} \, \mathrm{d}y \\ &= \frac{e^{\frac{x}{2}(k+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}[y^2 - \sqrt{2\tau}(k+1)y]} \, \mathrm{d}y. \end{aligned} \tag{3.2.56}$$

Completing the square in the exponent of the exponential expression, equation (3.2.56) becomes

$$I_\alpha = \frac{e^{\frac{x}{2}(k+1)}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}[y - \frac{1}{2}(k+1)\sqrt{2\tau}]^2 + \frac{\tau}{4}(k+1)^2} \, \mathrm{d}y \tag{3.2.57}$$

which after simplification becomes

$$I_\alpha = \frac{e^{\frac{x}{2}(k+1) + \frac{\tau}{4}(k+1)^2}}{\sqrt{2\pi}} \int_{\frac{-x}{\sqrt{2\tau}}}^{\infty} e^{-\frac{1}{2}[y - \frac{1}{2}(k+1)\sqrt{2\tau}]^2} \, \mathrm{d}y. \tag{3.2.58}$$

Now let

$$k_1 = y - \frac{1}{2}(k+1)\sqrt{2\tau}$$

this implies that $dk_1 = dy$. Substituting this into equation (3.2.58) we get

$$I_\alpha = \frac{1}{\sqrt{2\pi}} e^{\frac{x}{2}(k+1)+\frac{\tau}{4}(k+1)^2} \int_{\frac{-x}{\sqrt{2\tau}}-\frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\frac{1}{2}{k_1}^2}\,\mathrm{d}y. \tag{3.2.59}$$

Equation (3.2.59) contains the probability density function of the standard normal distribution. Therefore, using the cumulative normal distribution function,

$$\mathcal{N}(d) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d} e^{\frac{1}{2}s^2} ds, \tag{3.2.60}$$

I_α can be written as

$$I_\alpha = e^{\frac{x}{2}(k+1)+\frac{\tau}{4}(k+1)^2} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\right).$$

Similarly,

$$I_\beta = e^{\frac{x}{2}(k-1)+\frac{\tau}{4}(k-1)^2} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right). \tag{3.2.61}$$

Therefore, equation (3.2.52) can be written as:

$$\begin{aligned} u(x,\tau) &= e^{\frac{x}{2}(k+1)+\frac{\tau}{4}(k+1)^2} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}\right) \\ &\quad - e^{\frac{x}{2}(k-1)+\frac{\tau}{4}(k-1)^2} \mathcal{N}\left(\frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\right). \end{aligned} \tag{3.2.62}$$

With this solution for $u(x,\tau)$ we can systematically extract the solution of the Black-Scholes equation by reversing the transformations. We start by computing the function $v(x,\tau)$. That is

$$v(x,\tau) = e^{\alpha x + \beta\tau} u(x,\tau).$$

Substituting for α, β and $u(x,t)$ we obtain

$$v(x,\tau) = e^{-\frac{x}{2}(k-1)-\frac{\tau}{4}(k+1)^2}\left(e^{\frac{x}{2}(k+1)+\frac{\tau}{4}(k+1)^2}\mathcal{N}(d_1) - e^{\frac{x}{2}(k-1)+\frac{\tau}{4}(k-1)^2}\mathcal{N}(d_2)\right)$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}, \tag{3.2.63}$$

$$d_2 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}\,. \tag{3.2.64}$$

Now observe that

$$e^{\frac{x}{2}(k+1)+\frac{\tau}{4}(k+1)^2} \times e^{\frac{-x}{2}(k-1)-\frac{\tau}{4}(k+1)^2} = e^{x}$$
$$e^{\frac{x}{2}(k-1)+\frac{\tau}{4}(k-1)^2} \times e^{\frac{-x}{2}(k-1)-\frac{\tau}{4}(k+1)^2} = e^{-k\tau}.$$

So $v(x,\tau)$ simplifies to

$$v(x,\tau) = e^{x}\mathcal{N}(d_1) - e^{-k\tau}\mathcal{N}(d_2). \tag{3.2.65}$$

Finally, we can calculate the price of the call option by reversing each of the changes of variables. Recall that

$$S = Ke^{x}, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C = Kv(x,t) \quad \text{and} \quad k = \frac{r}{\frac{1}{2}\sigma^2}.$$

Reversing our transformations in equation (3.2.65) we obtain

$$\frac{C(S,t)}{K} = \frac{S}{K}\mathcal{N}(d_1) - e^{-r(T-t)}\mathcal{N}(d_2). \tag{3.2.66}$$

Therefore, the price of an European call option is given by

$$C(S,t) = S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2). \tag{3.2.67}$$

In addition to this, d_1 and d_2 described in equations (3.2.63) and (3.2.64) now become

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{3.2.68}$$

$$d_2 = \frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} = d_1 - \sigma\sqrt{T-t}. \tag{3.2.69}$$

To find an explicit formula for the value of the European put option we can follow the same procedure. However, since we possess the analytical expression for a European call option, we can make use of the put-call parity formula

$$C(S,t) - P(S,t) = S - Ke^{-r(T-t)} \tag{3.2.70}$$

to find the value of $P(S,t)$. Substituting for $C(S,t)$ and solving for $P(S,t)$ in equation (3.2.70) we have

$$\begin{aligned} P(S,t) &= S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2) - S + Ke^{-r(T-t)} \\ &= Ke^{-r(T-t)}(1-\mathcal{N}(d_2)) + S(\mathcal{N}(d_1) - 1) \\ &= Ke^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1). \end{aligned}$$

where $\mathcal{N}(-d)$ is defined by the identity

$$\mathcal{N}(d) + \mathcal{N}(-d) = 1. \tag{3.2.71}$$

We now possess the explicit formula to find the theoretical price of an European option at any time, t, prior to expiry. The formula expresses the option price in the original variables and the cumulative normal distribution function $\mathcal{N}(d)$, for which tables to find the values of $\mathcal{N}(d)$ are readily available.

The Black-Scholes model for pricing derivative instruments was a major breakthrough in finance. Using this model one can deduce the Black-Scholes formula which provides the theoretical estimate of the price of an European option. This formula led to a boom in option trading and also provided mathematical legitimacy to activities of the option markets. In the next section we discuss the American option pricing problem.

3.3 American options

This section addresses the American option pricing problem. The absence of an analytical formula for American options makes the valuation process a challenging task. This is due to the early exercise privilege offered by these type of derivatives. We begin this section by introducing the early exercise condition of American options. This is followed by a derivation of the Black-Scholes inequality for American options. The American option pricing problem is then discussed as a free boundary problem. To aid understanding of the free boundary problem we introduce the obstacle problem. Following the work of Wilmott et al. [56], we show that a unique solution to the American option pricing problem can be obtained by making use of a similar set of constraints as in the case of the obstacle problem. Finally, we formulate the American option pricing problem as a linear complementarity problem that will enable us to solve the problem without explicit dependence on the unknown free boundary.

3.3.1 Lower boundary of American options

The main difference between an American option and its European counterpart is that an American option gives the holder the privilege of inhibiting early exercise before the expiration of the contract. Due to this extra feature it is expected of the American option to be worth more than its European counterpart. Hence,

$$V^{AM} \geq V^{EU} \tag{3.3.1}$$

where $V^{AM} = V^{AM}(S,t)$ and $V^{EU} = V^{EU}(S,t)$ are the prices of American and European options respectively. This is referred to as the early exercise constraint [44].

Additionally, an American option must have at least the value of its payoff. This follows from a simple arbitrage argument. Suppose that an American put option has a value less than its payoff, that is

$$P^{AM} < \max(K - S, 0).$$

Then there exist an arbitrage argument. An investor can buy the asset in the market for $S(t)$ and at the same time buy the option for P^{AM}. By immediately exercising the option the investor can make an instantaneous profit of $K - P^{AM} - S(t) > 0$ [56]. A similar argument for an American call implies that for

$$C^{AM} < \max(S(t) - K),$$

arbitrage opportunities exists. Therefore, we have the following elementary lower bounds for the value of these type of options [44]:

$$P^{AM}(S,t) \geq \max(K - S, 0) \quad \text{for all} \quad (S,t), \tag{3.3.2}$$

$$C^{AM}(S,t) \geq \max(S - K, 0) \quad \text{for all} \quad (S,t). \tag{3.3.3}$$

3.3.2 Black-Scholes inequality for American options

To accommodate the early exercise facility related to American options, the Black-Scholes equation can be adapted to cover American options [44]. As in the case of the European option, we construct a portfolio consisting of one American option and $-\Delta$ units of the underlying asset. Suppose the option is an American put option. Then we have

$$\Pi = P^{AM} - \Delta S(t). \tag{3.3.4}$$

Following the same steps as in Section 2, with $\Delta = \frac{\partial P^{AM}}{\partial S}$, at time dt the value of the portfolio changes by the amount

$$d\Pi = \left(\frac{\partial P^{AM}}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P^{AM}}{\partial S^2} \right) dt. \tag{3.3.5}$$

Recall that in the European case, in order to avoid arbitrage, we concluded that this expression must be the same as the riskless return of an amount Π invested in a riskless asset with constant interest rate r. However, when the option contained in the portfolio is of American style, the holder of the option controls the early exercise privilege, hence to avoid arbitrage the return cannot be greater then the risk-free rate on the portfolio . So

$$\begin{aligned} d\Pi &\leq r\Pi dt \\ &= r\left(P^{AM} - S(t)\frac{\partial P^{AM}}{\partial S(t)}\right)dt. \end{aligned} \tag{3.3.6}$$

Therefore, if the investor fails to optimally exercise the option, the change of the portfolio might be less than the riskless return [57]. Substituting equation (3.3.5) in equation (3.3.6) gives

$$\left(\frac{\partial P^{AM}}{\partial t} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P^{AM}}{\partial S^2}\right)dt \leq r\left(P^{AM} - S(t)\frac{\partial P^{AM}}{\partial S(t)}\right)dt. \tag{3.3.7}$$

Rearranging the terms gives us the Black-Scholes inequality for an American put option. That is

$$\frac{\partial P^{AM}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P^{AM}}{\partial S^2} + rS\frac{\partial P^{AM}}{\partial S} - rP^{AM} \leq 0. \tag{3.3.8}$$

Using the same approach as in Section 3.2.2, the Black-Scholes inequality can also be transformed into the heat equation. One can then solve the heat equation to obtain a unique value for an American put option. However, solving this equation can be demanding since the boundary needed cannot be specified beforehand. This problem originates from the early exercise privilege associated with American options and is also known as the free boundary problem.

3.3.3 The free boundary problem

The additional feature accompanied by American options that permits early exercise at any time during the life of an option complicates the Black-Scholes analysis for American options. This extra feature prevents us from using the explicit formula available for pricing European options and often causes difficulties in the numerical methods used to price these options. See for example [25] and [56].

As mentioned earlier, the American option pricing problem is also reffered to as a free boundary problem [55]. Wilmott et al. [56] describes the free boundary problem as follows: at each time, t, there exist a stock price, $S = S_f(t)$, marking the boundary between two regions, i.e., the exercise region and the holding region. Therefore, the free boundary problem can be viewed as finding the stock price $S_f(t)$, at each point $0 < t < T$, such that it is optimal to exercise the American option.

In an attempt to explain the free boundary problem in more detail we make use of the option's payoff function. Considering an American put option, the payoff function is given by [27]

$$\text{payoff} = \max[K - S(T), 0]. \tag{3.3.9}$$

Notice that if the stock price, $S(t)$, is large, the option's payoff would be zero, therefore, exercising the American put option would not be worthwhile. On the other hand, as the stock price approaches zero, the payoff from exercising approaches the maximum value, K, hence, exercising the option would be optimal. Interpolating between these two extreme scenarios, produces the optimal exercise boundary [25]. Thus, at each time point t, on the time interval $0 \le t \le T$, there is a critical asset price $S_f(t)$, dividing the S-axis into two distinct regions. The one side of this value represents early exercising while the other side represents holding the option [56], see Figure 3.5. This exercising boundary plays an important role towards finding the options price.

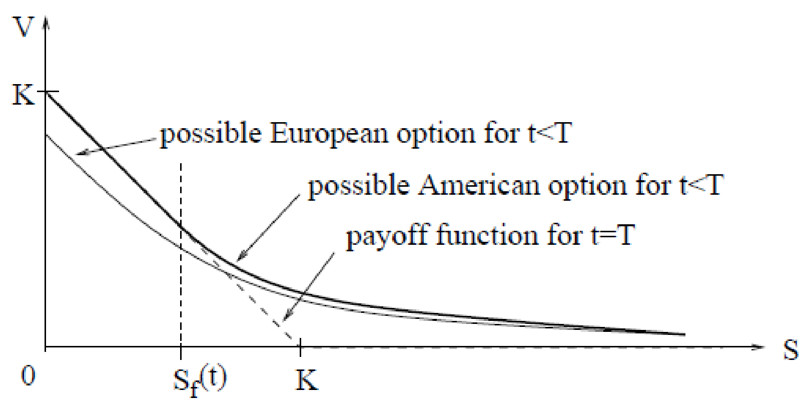

Figure 3.5: European and American put option prices [44]

In the case of American put options [25]:

- if $S < S_f(t)$, it is optimal to exercise the option, therefore $P^{AM}(S,t) = \max(K - S(t), 0)$,
- if $S > S_f(t)$, it is optimal to hold the option, therefore $P^{AM}(S,t) > \max(K - S(t), 0)$.

A similar argument holds for an American call, therefore we have

- if $S > S_f(t)$, it is optimal to exercise the option, therefore $C^{AM}(S,t) = \max(S(t) - K, 0)$,
- if $S < S_f(t)$, it is optimal to hold the option, therefore $C^{AM}(S,t) > \max(S(t) - K, 0)$.

It is important to know the critical asset prices, $S_f(t)$, at each point t on the interval $0 < t < T$, because these prices form the optimal exercise boundary. This is the second boundary condition needed to find a unique solution for the Black-Scholes inequality. The process of obtaining this optimal exercise boundary is often referred to as the free boundary problem.

The obstacle problem as a free boundary problem

The free boundary concept has been the subject of numerous studies and is not unique to the American option pricing problem [56]. McKean [39] and Moerbeke [51] were the first to relate the American option pricing problem to the obstacle problem. To promote better understanding of the topic we follow the authors of [44] and [56] by first discussing the obstacle problem.

Suppose an elastic string is stretched over a smooth obstacle and tied up between points x_0 and x_1, see Figure 3.6. The obstacle and string are defined by the functions $g(x)$ and $u(x)$ respectively. Initially, the region of contact between the string and the obstacle is unknown. Therefore, the string is either in contact with the obstacle (the position is known) or it is not (the string must be straight). Additionally, the slope of the string have to be continuous. In summary [57]:

- the string must lie either above or on the obstacle,
- the string must have negative or zero curvature,
- the string must be continuous,
- the string's slope must be continuous.

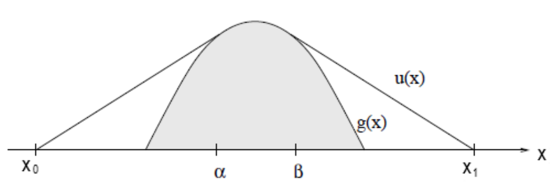

Figure 3.6: The obstacle problem [44]

Under these constraints, a unique solution to the obstacle problem can be obtained [55].

The American option as a free boundary problem

According to Wilmott et al. [56], making use of a similar set of constraints, the American option pricing problem can be stipulated as a free boundary problem from which a unique solution to the American option pricing problem can be obtained. These conditions are:

(i) The option value must be greater then or equal to the option payoff,

(ii) One must have the Black-Scholes inequality,

(iii) The option value must be a continuous function of S,

(iv) The option delta (its slope) must be continuous.

What follows is a discussion of all these constraints individually to make sure that they are valid. We consider an American put option, therefore we must have

(i) $P^{AM} \geq \max[K - S, 0] \quad \text{for} \quad 0 < t < T, \quad S > 0.$

This constraint is also known as the lower boundary of the American put option. It indicates that early exercise is possible, but without arbitrage opportunities [56].

(ii) $\dfrac{\partial P^{AM}}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 P^{AM}}{\partial S^2} + rS\dfrac{\partial P^{AM}}{\partial S} - rP^{AM} \leq 0$

These two constraints have been established in Sections 3.3.1 and 3.3.2.

(iii) $P^{AM}(S, t)$ must be a continuous function of the price of the underlying asset, S. A discontinuity in the option value as a function of S for any small interval of time, would allow for arbitrage opportunities to arise. In this case a portfolio consisting only of options would make a guarenteed risk-free profit if the underlying asset price ever reaches the value at which the discontinuity occurred. Although this argument holds, it does not eliminate such an event entirely since discontinuities in option prices do occur at some occasions. These are refered to as jumps [55].

(iv) To prove constraint number four we make use of the payoff function of the option. Recall that the payoff of a put option is given by

$$\text{Payoff} = \max[K - S, 0]. \tag{3.3.10}$$

The slope (Δ) of this payoff function is -1. This can be observed in Figure 3.5. As previously discussed, the American put option has an exercise boundary at $S_f(t)$ and should be exercised if $S < S_f(t)$ and held otherwise [56]. What follows is an investigation of the slope at this critical asset price, $S_f(t)$. Wilmott et al. [56] suggested that the slope at $S_f(t)$ can have three possible values. These are,

$$\frac{\partial P^{AM}}{\partial S_f(t)} < -1, \tag{3.3.11}$$

$$\frac{\partial P^{AM}}{\partial S_f(t)} > -1, \tag{3.3.12}$$

$$\frac{\partial P^{AM}}{\partial S_f(t)} = -1. \tag{3.3.13}$$

In order to show that constraint (iv) is upheld, the first two equations above must be incorrect. To prove this, we begin by examining

$$\frac{\partial P^{AM}}{\partial S_f(t)} < -1. \tag{3.3.14}$$

Suppose that $S_f(t) < K$ and the inequality 3.3.14 holds. Then as S increases from $S_f(t)$, the value of the put option will drop below the payoff value. This is due to the slope being negative. Therefore, we have $P^{AM}(S,t) < \max[K - S, 0]$, as shown in Figure 3.7. This contradicts the arbitrage argument that $P^{AM}(S,t) \geq \max[K - S, 0]$ and so $\frac{\partial p^{AM}}{\partial S_f(t)} < -1$ is ruled out.

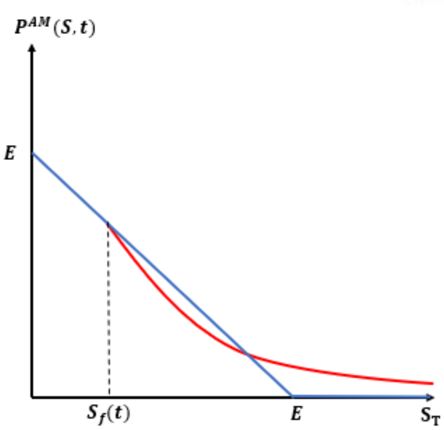

Figure 3.7: $\frac{\partial p^{AM}}{\partial S_f(t)} < -1$

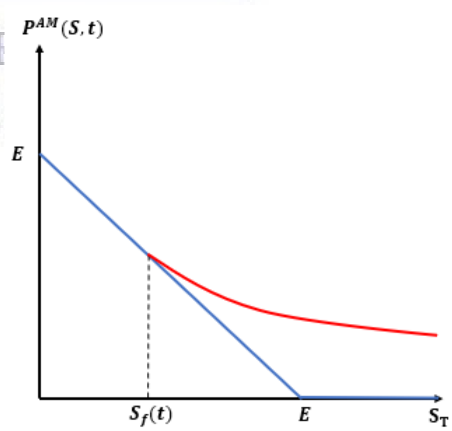

Figure 3.8: $\frac{\partial p^{AM}}{\partial S_f(t)} > -1$

Next, we have to rule out the case where $\frac{\partial p^{AM}}{\partial S_f(t)} > -1$. If $\frac{\partial p^{AM}}{\partial S_f(t)} > -1$ at $S = S_f(t)$, the value of the option can be increased by choosing a smaller value for $S_f(t)$. This leads to an increase in the option payoff and consequently the option value. Additionally, decreasing $S_f(t)$ implies

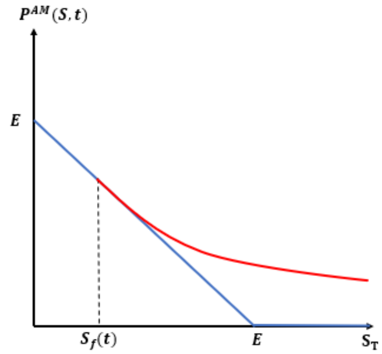

Figure 3.9: $\frac{\partial p^{AM}}{\partial S_f(t)} = -1$

that the slope also decreases, as it becomes more steep. By decreasing $S_f(t)$ sufficiently, we will arrive at the asset value where the inequality given by equation 3.3.12 no longer holds. This can be seen in Figure 3.8. The option is therefore miss-valued [56]. Hence, equation 3.3.12 is ruled out.

Since neither of the of the two inequalities at the critical asset value, $S_f(t)$ holds, we can conclude that

$$\frac{\partial p^{AM}}{\partial S_f(t)} = -1, \tag{3.3.15}$$

as seen in Figure 3.9. Together constraints (iii) and (iv) are collectively known as the smooth pasting condition or the tangency condition [46], since it represents the smooth joining of the option value function P^{AM} to its payoff function [53].

Making use of the aforementioned constraints we can now formulate the American option pricing problem as a free boundary problem. Consider an American put option with value $P^{AM}(S,t)$. At each time t, there exists a boundary which divides the $S(t)$ axis into two regions. Firstly, the region where early exercise is optimal:

$$E = \{(S,t) : 0 \leq S < S_f(t), t > 0\}, \tag{3.3.16}$$

$$P^{AM}(S,t) = K - S(t), \tag{3.3.17}$$

and

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rS < 0. \tag{3.3.18}$$

Secondly, there is the holding region, where early exercise is not optimal

$$H = \{(S,t) : S_f(t) < S < \infty, t > 0\}, \tag{3.3.19}$$

$$P^{AM}(S,t) > K - S(t), \tag{3.3.20}$$

and

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V(S,t)}{\partial S^2} + rS\frac{\partial V(S,t)}{\partial S} - rS = 0. \tag{3.3.21}$$

At the boundary we have the following conditions

$$\lim_{S\to\infty} P^{AM}(S,t) = 0,$$
$$\frac{\partial p^{AM}}{\partial S_f(t)} = -1,$$
$$p^{AM}(S_f(t),t) = \max[K - S_f(t), 0].$$

And the final condition

$$P^{AM}(S,t) > \max[K - S(t), 0]. \tag{3.3.22}$$

In the next section we re-formulate the American option pricing problem as a fixed boundary problem. We accomplish this by formulating the free boundary problem as a LCP. This approach allows us to solve the American option pricing problem and obtain the free boundary afterwords.

3.3.4 The linear complementarity problem

In order to formulate the free boundary problem as a fixed boundary problem we first discuss the linear complementarity problem (LCP). Wilmott et al. [56], explains that by transforming the free boundary problem to a LCP, we reduce the American option pricing problem to a fixed boundary problem that does not explicitly depend on the unknown free boundary. The LCP is then solved numerically. Once the LCP has been solved, the location of the free boundary can subsequently be obtained.

The linear complementarity problem can be described as follows:
Let $\mathbf{M} \in \mathbb{R}^n \times \mathbb{R}^n$ be a known matrix and $\mathbf{q} \in \mathbb{R}^n$ be a known vector. Suppose that both the vectors $\mathbf{w}$ and $\mathbf{z}$ are also in $\mathbb{R}^n$. The linear complementarity problem is to find $\mathbf{w} = (w_1, ..., w_n)$

and $\mathbf{z} = (z_1, ..., z_n)$ such that [41] and [43]:

$$\mathbf{w} - \mathbf{M}\mathbf{z} = \mathbf{q}$$
$$\mathbf{w} \geq 0$$
$$\mathbf{z} \geq 0$$
$$w_i z_i = 0 \quad \text{for} \quad i = 1, ..., n.$$

The obstacle problem as a linear complementarity problem

We discuss the obstacle problem as a LCP. This simplistic model will guide us to better understand the American option pricing problem as a LCP, which will be discussed later.

What follows is a summary of the obstacle problem as described in [44] and [56]. Consider Figure 3.6, in which we take the end of the string to be at x_0 and x_1. Next let $g(x)$ be the height of the obstacle where $x \in \mathbb{R}$, $g \in C^2$ and $g''(x) < 0$. Also assume that $g(x_0) < 0$ and $g(x_1) < 0$. Let $u(x)$ be the function representing the displacement of the string where $u \in C^1[x_0, x_1]$ and $u(x_0) = u(x_1) = 0$. On the interval $[\alpha, \beta]$ the string clings to the boundary of the obstacle, therefore $u(x) = g(x)$. In this contact region the string is bent, therefore $u''(x) < 0$. When the string is not in contact with the obstacle then $u(x) > g(x)$ holds and the string is straight, implying that $u''(x) = 0$. Initially α and β are unknown. This obstacle problem represents a simple free boundary problem. Wilmott et al. [56] defines the obstacle problem as a free boundary problem as follows:

$$x = x_0 : \qquad u(x_0) = 0, \tag{3.3.23}$$
$$x_0 < x < \alpha : \qquad u(x) > g(x) \quad \text{and} \quad u''(x) = 0 \tag{3.3.24}$$
$$x = \alpha : \qquad u(\alpha) = g(\alpha) \quad \text{and} \quad u'(\alpha) = g'(\alpha) \tag{3.3.25}$$
$$\alpha < x < \beta : \qquad u(x) = g(x) \quad \text{and} \quad u''(x) = g''(x) < 0 \tag{3.3.26}$$
$$x = \beta : \qquad u(\beta) = g(\beta) \quad \text{and} \quad u'(\beta) = g'(\beta) \tag{3.3.27}$$
$$\beta < x < x_1 : \qquad u(x) > g(x) \quad \text{and} \quad u''(x) = 0 \tag{3.3.28}$$
$$x = x_1 : \qquad u(x_1) = 0. \tag{3.3.29}$$

The problem consists of finding $u(x)$ and the points α and β such that the requirements given above are fulfilled. However, it can also be noticed that the characteristics of the outer intervals (equations 3.3.24 and 3.3.28) are identical. This demonstrates a complementarity in the sense [44]:

$$\text{for} \quad u(x) > g(x) \quad \text{and} \quad u''(x) = 0 \tag{3.3.30}$$

$$\text{for} \quad u(x) = g(x) \quad \text{and} \quad u''(x) < 0. \tag{3.3.31}$$

Therefore, the obstacle problem can be reformulated as a linear complimentarity problem. That is, find $u(x)$ such that [44]:

$$u''(u-g) = 0 \tag{3.3.32}$$

$$-u'' \geq 0 \tag{3.3.33}$$

$$u - g \geq 0, \tag{3.3.34}$$

subject to the conditions that $u(x_0) = u(x_1) = 0$ and $u \in C^1(x_0, x_1)$.

The American option pricing problem as a linear complementarity problem

Using the same approach we now formulate the American option pricing problem as a LCP. Before presenting the American option pricing problem as a LCP we first transform the Black-Scholes inequality into the heat equation. To do this we consider the Black-Scholes inequality derived for the American put option in Section 3.3.2

$$\frac{\partial P^{AM}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 P^{AM}}{\partial S^2} + rS\frac{\partial P^{AM})}{\partial S} - rP^{AM} \leq 0. \tag{3.3.35}$$

Applying the same arguments as in Section 3.2.2 we can transform equation (3.3.35) to the following inequality for the heat diffusion equation [55]

$$\frac{\partial u}{\partial \tau} \geq \frac{\partial^2 u}{\partial x^2}. \tag{3.3.36}$$

Working with the (x, τ) variables instead of the (S, t) variables we have to make provision for a few changes. First we write the exercise boundary $S_f(t)$ as $x_f(\tau)$. The transformation on $S_f(t)$ was given by $S_f(t) = Ke^{x_f(t)}$ and since $S < K$, we deduce that $x_f(\tau) < 0$ [56]. Secondly, we have to find the transformed payoff function. Now the payoff function for an American put option is given by [27]

$$P^{AM}(S,t) = \max(K - S, 0). \tag{3.3.37}$$

Substituting the parameters $P(S,t) = Kv(x,\tau)$ and $S = Ke^x$ into equation (3.3.37) we get

$$Kv(x,\tau) = \max(K - Ke^x, 0). \tag{3.3.38}$$

The function $v(x,t)$ is defined in terms of the function $u(x,t)$ in Section 3.2.2 as

$$v(x,\tau) = e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau).$$

Therefore, we can re-write equation (3.3.38) as

$$\begin{aligned} Ke^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) &= \max(K - Ke^x, 0) \\ e^{-\frac{1}{2}(k-1)x-\frac{1}{4}(k+1)^2\tau}u(x,\tau) &= \max(1 - e^x, 0) \\ u(x,\tau) &= e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0). \end{aligned} \tag{3.3.39}$$

Equation (3.3.39) represents the new payoff function. We define this function as $g(x,t)$. Thus, the transformed payoff function is given by

$$g(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0). \tag{3.3.40}$$

As previously mentioned, the privilege of early exercising associated with American options imposes the constraint

$$P^{AM}(S,t) \geq \max(K - S(t), 0). \tag{3.3.41}$$

Using the same arguments as above, equation (3.3.41) can now be written in terms of the transformed variables as

$$g(x,\tau) \geq e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0). \tag{3.3.42}$$

Taking these changes into account we can now rewrite the American put option pricing problem as a free boundary problem in terms of the heat equation. Hence, the exercise region given by equations (3.3.16) to (3.3.18) now becomes

$$D_S = \{(x,\tau) : x \leq x_f(t); \quad 0 \leq \tau \leq \frac{\sigma^2}{2}T\} \tag{3.3.43}$$

where

$$\frac{\partial u}{\partial \tau} > \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad u(x,\tau) = g(x,\tau) \tag{3.3.44}$$

and for the continuation region given by equations (3.3.19) to (3.3.21) we have

$$D_C = \{(x,\tau) : x > x_f(t); \quad 0 \leq \tau \leq \frac{\sigma^2}{2}T\} \tag{3.3.45}$$

where

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \quad \text{and} \quad u(x,\tau) > g(x,\tau). \tag{3.3.46}$$

To complete these changes we have to stipulate the initial and boundary conditions. The initial condition as mentioned in Section 3.2.2, is given by equation (3.2.34)

$$u(x,0) = g(x,0) = \max(e^{\frac{1}{2}(k-1)x-\frac{1}{2}(k+1)x}, 0). \tag{3.3.47}$$

For the boundary conditions, observe that $g(x,\tau) \to 0$ when $x \to \infty$ since it vanishes for all $x \geq 0$. Also from equation (3.2.38) we know that $u(x,\tau) \to 0$ when $x \to \infty$. Therefore,

$$\lim_{x\to\infty} u(x,\tau) = \lim_{x\to\infty} g(x,\tau). \tag{3.3.48}$$

Also when $x \to -\infty$, we are in the exercising region and so $u(x,\tau) = g(x,\tau)$. Thus,

$$\lim_{x\to-\infty} u(x,\tau) = \lim_{x\to-\infty} g(x,\tau). \tag{3.3.49}$$

Consider how the American put option pricing problem resembles the obstacle problem. Therefore an analogy similiar to that of the obstacle problem can now be used to formulate the option pricing problem as a linear complimentarity problem [56].

That is, find a function $u(x,t)$ such that:

$$\begin{gathered} \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)\Big(u(x,\tau) - g(x,\tau\Big) = 0 \\ \left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right) \geq 0 \\ u(x,\tau) - g(x,\tau) \geq 0, \end{gathered} \tag{3.3.50}$$

where

$$g(x,\tau) = e^{\frac{1}{4}(k+1)^2\tau}\max(e^{\frac{1}{2}(k-1)x} - e^{\frac{1}{2}(k+1)x}, 0), \tag{3.3.51}$$

subject to the initial conditions

$$u(x,0) = g(x,0) \quad \text{for} \quad -\infty < x < \infty, \tag{3.3.52}$$

and the boundary conditions at $x = \pm\infty$

$$u(x,\tau) = g(x,\tau) \quad \text{for} \quad 0 \leq t \leq \frac{1}{2}\sigma^2 T. \tag{3.3.53}$$

and the constraints that

$$u(x,\tau) \quad \text{and} \quad \frac{\partial u(x,\tau)}{\partial x} \tag{3.3.54}$$

are continuous.

Note that this is similar to the LCP of the obstacle problem. The two different scenarios in equation (3.3.50) corresponds to the different exercise scenarios explained in equations (3.3.43) to (3.3.46). When early exercise is optimal, $u = g$, and when holding the option $u > g$. Lastly it is important to take note that there is no explicit mention of the free boundary, which makes solving the American option problem less complicated [56].

The next section is dedicated to currency options. According to existing literature these type of options are similar to options on dividend paying stocks. Hence we study the effects of dividends on the Black-Scholes PDE and also derive the Black-Scholes PDE for currency options.

3.4 Currency options

This section is dedicated to the valuation of currency options. As mentioned in the introduction, these types of derivatives are very attractive to investors with internationally diversified portfolios. Currency options enables them to hedge against exchange rate fluctuations that might have a negative influence on their future cash flows, but at the same time profit from desirable exchanges rates.

In order to use currency options for hedging, one needs to have an idea of how to price these derivatives. Although the mechanics of these types of options are similar to that of options on non-divident paying stocks there are a few differences. In particular, we have to take into account that we are now dealing with two different currencies and consequently, two different interest rates. Hull [27] and Willmot et al. [55], among many others explain that these types of options are similar to options on dividend paying stocks. Therefore, we start this section by explaining the theory of dividends and the effects that these dividends have on the price of the underlying asset. As a result of the latter the Black-Scholes PDE is also affected. We derive the Black-Scholes PDE for European currency options and show that this PDE can be transformed to the original Black-Scholes PDE with interest rate $r - r_f$, where r is the domestic interest rate and r_f is the foreign interest rate. Hence, the Black-Scholes PDE for European currency options can be solved using the same approach as in Section 3.2. That is, by first transforming the Black-Scholes PDE with interest rate, $r - r_f$, into the heat equation. Additionally, since the Black-Scholes PDE for currency options can be transformed into the original Black-Scholes PDE we can use the same arguments as in Section 3.2 to price American currency options.

3.4.1 Options on dividend-paying assets

As mentioned above, currency options are similar to options on dividend paying stocks. Hence, we first study the concept of dividends. Dividends are the distribution of a portion of a company's earnings to the shareholders and the effect of this future dividend stream is reflected in today's share price. As a result, the value of the option which is a function of the asset price and time is also affected, since it is derived from the Black-Scholes equation whose derivation includes shorting ΔS shares of the underlying asset. In the next section we discuss how dividends affect the price of the underlying asset.

3.4.2 Effects of dividends on asset price

Suppose that a stock pays out a continuous dividend at a rate of q. That is, at time $t + dt$ an investor holding one unit of stock will receive a cash payment of

$$qSdt$$

where q is the continuous dividend yield. Taking arbitrage into account it can be shown that in each time step dt, the asset price must fall by the amount of the dividend payment, qdt [56]. Hence the path followed by the asset price now becomes

$$dS = \mu Sdt - qSdt + \sigma SdB(t), \tag{3.4.1}$$

$$= (\mu - q)Sdt + \sigma SdB(t). \tag{3.4.2}$$

An example of an asset which can be considered to be paying out a continuous dividend is a foreign currency earning, the foreign interest rate. In this case $q = r_f$, where r_f is the foreign interest rate. Therefore equation 3.4.2 becomes

$$dS = (\mu - r_f)Sdt + \sigma SdB(t). \tag{3.4.3}$$

In the next section we extend the Black-Scholes model to accommodate this change in the price of the underlying asset.

3.4.3 Black-Scholes model for currency options

In 1983, Garman and Kohlhagen [20] extended the Black-Scholes model to accommodate the presence of an additional interest rate. In this section we will derive this result. Proceeding in the same fashion as in Section 3.2.1, we construct a portfolio

$$\Pi = V(S,t) - \Delta S \tag{3.4.4}$$

where $V(S,t)$ is the price of the currency option and Δ is chosen so that the value of the portfolio is deterministic. However, taking into account the dividends (in this case the foreign interest rate) for holding the currency , the value of the portfolio changes to

$$\Pi = V(S,t) - \Delta S - \Delta r_f S dt. \tag{3.4.5}$$

Hence, the change in the portfolio value in the time interval dt is given by

$$d\Pi = dV(S,t) - \Delta dS - r_f S \Delta dt. \tag{3.4.6}$$

Following the same procedure as in Section 3.2.1 we can find the Black-Scholes model for currency options. In doing this we find that as before delta must be defined as

$$\Delta = \frac{\partial V(S,t)}{\partial S},$$

in order to eliminate the random component. Also with the addition of the new term, $-r_f S \Delta dt$, we find that $V(S,t)$ has to satisfy the modified Black-Scholes equation

$$\frac{\partial V(S,t)}{\partial t} + \frac{1}{2} S^2 \sigma^2 \frac{\partial^2 V(S,t)}{\partial S^2} + (r - r_f) S \frac{\partial V(S,t)}{\partial S} - rS = 0. \tag{3.4.7}$$

Equation (3.4.7) looks similar to the original Black-Scholes PDE. However, there are two differences that we have to take into account. In the original Black-Scholes PDE the coefficients of $S\frac{\partial V(S,t)}{\partial S}$ and $V(S,t)$ are equal whereas in equation 3.4.7 they differ. Therefore, to find a solution to equation (3.4.7) we need to adjust our approach or look at an alternative approach.

3.4.4 Black-Scholes formula for currency options

Solving the Black-Scholes PDE for currency options can be done in a similar manner to Section 3.2.2. That is, by introducing the appropriate parameters and reducing equation (3.4.7) to the diffusion equation from which the solution can be obtained. However, a faster approach would be to make the coefficients of $S\frac{\partial V(S,t)}{\partial S}$ and $V(S,t)$ equal. This will lead us to the original

Black-Scholes equation for which we already derived the Black-Scholes formula.

To transform equation (3.4.7) to the original Black-Scholes PDE, we introduce a change in the variable $V(S,t)$. That is, we let

$$V(S,t) = e^{-r_f(T-t)}V_1(S,t). \tag{3.4.8}$$

Taking the derivatives of $V(S,t)$ with respect to t and S we have

$$\begin{aligned}
\frac{\partial V(S,t)}{\partial t} &= r_f e^{-r_f(T-t)}V_1 + e^{-r_f(T-t)}\frac{\partial V_1}{\partial t},\\
\frac{\partial V(S,t)}{\partial S} &= e^{-r_f(T-t)}\frac{\partial V_1(S,t)}{S},\\
\frac{\partial^2 V(S,t)}{\partial S^2} &= e^{-r_f(T-t)}\frac{\partial^2 V_1(S,t)}{S^2}.
\end{aligned}$$

Substituting these derivatives into the modified Black-Scholes equation (3.4.7) we obtain

$$\begin{aligned}
r_f e^{-r_f(T-t)}V_1(S,t) + e^{-r_f(T-t)}\frac{\partial V_1(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2 e^{-r_f(T-t)}\frac{\partial^2 V_1(S,t)}{\partial S^2}\\
+(r-r_f)e^{-r_f(T-t)}S\frac{\partial V_1(S,t)}{\partial S} - re^{-r_f(T-t)}V_1(S,t) = 0
\end{aligned} \tag{3.4.9}$$

which simplifies to

$$\frac{\partial V_1(S,t)}{\partial t} + \frac{1}{2}S^2\sigma^2\frac{\partial^2 V_1(S,t)}{\partial S^2} + (r-r_f)S\frac{\partial V_1(S,t)}{\partial S} - (r-r_f)V_1(S,t) = 0 \tag{3.4.10}$$

when multiplying equation (3.4.9) by $e^{r_f(T-t)}$. This is the same as the Black-Scholes equation for an European option with price $V_1(S,t)$ and interest rate of $r-r_f$.

Now that we have the original Black-Scholes equation for an European option, we can make use of the results obtained in Section 3.2.2 to find the value of the option. Suppose the option considered is an European call option with price $C(S,t)$. Therefore, equation (3.4.8) becomes

$$C(S,t) = e^{-r_f(T-t)}C_1(S,t) \tag{3.4.11}$$

where $C_1(S,t)$ satisfies the original Black-Scholes PDE with interest rate $r-r_f$. From Section 3.2.2 we know that the explicit formula for $C_1(S,t)$ is given by:

$$C_1(S,t) = S\mathcal{N}(d_1) - Ke^{-(r-r_f)(T-t)}\mathcal{N}(d_2). \tag{3.4.12}$$

To find the value of the European currency call option we substitute this value for $C_1(S,t)$ back into equation (3.4.11). Thus we have

$$\begin{aligned} C(S,t) &= e^{-r_f(T-t)}(S\mathcal{N}(d_1) - Ke^{-(r-r_f)(T-t)}\mathcal{N}(d_2)) \\ &= e^{-r_f(T-t)}S\mathcal{N}(d_1) - Ke^{-(r-r_f)(T-t)-r_f(T-t)}\mathcal{N}(d_2) \\ &= e^{-r_f(T-t)}S\mathcal{N}(d_1) - Ke^{-r(T-t)}\mathcal{N}(d_2)\,. \end{aligned}$$

With the interest rate equal to $r - r_f$, the value for d_1 and d_2 now becomes

$$d_1 = \frac{\log\left(\frac{S}{K}\right) + \left(r - r_f + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} \tag{3.4.13}$$

and

$$\begin{aligned} d_2 &= \frac{\log\left(\frac{S}{K}\right) - \left(r - r_f - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}} && (3.4.14) \\ &= d_1 - \sigma\sqrt{T-t}\,. && (3.4.15) \end{aligned}$$

To find the value of the currency put option we follow a similar approach. That is we set

$$P(S,t) = e^{-r_f(T-t)}P_1(S,t). \tag{3.4.16}$$

In doing this we obtain the Black-Scholes PDE for a put option $P_1(S,t)$ with interest rate $r-r_f$. Hence, the explicit formula to value a put option as

$$P_1(S,t) = Ke^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1). \tag{3.4.17}$$

Substituting $P_1(S,t)$ into equation (3.4.16) we get the explicit formula for the currency put option. That is,

$$\begin{aligned} P(S,t) &= e^{-r_f(T-t)}(Ke^{-r(T-t)}\mathcal{N}(-d_2) - S\mathcal{N}(-d_1)) \\ &= Ke^{-r(T-t)}\mathcal{N}(-d_2) - Se^{-r_f(T-t)}\mathcal{N}(-d_1) \end{aligned}$$

where d_1 and d_2 is defined by equations (3.4.13) and (3.4.14) respectively and $\mathcal{N}(-d)$ is defined by the identity

$$\mathcal{N}(d) + \mathcal{N}(-d) = 1. \tag{3.4.18}$$

Pricing options on assets that pays out dividends or in our case foreign currency options are almost similar to pricing options on non-dividend paying assets. However, it is important for one to take into account the differences in the two PDE's before calculating the value of the option.

We observed that these differences can be eliminated using a suitable change in the variable $V(S,t)$. This change allowed us to transform the modified Black-Scholes PDE to the original Black-Scholes PDE, which enabled us to use a similar approach as in the previous sections to price European currency options. Additionally, we can also use this result in conjunction with the results of Section 3.3 to price American currency options.

This concludes Chapter 3. We started off with the concept of European options. Firstly, we derived the famous Black-Scholes PDE and the transformed Black-Scholes PDE. This, together with the boundary and initial conditions allowed us to derive the Black-Scholes formula to price European options. What followed was a formal definition of the American option pricing problem. Applying some of the concepts used to price European options we derived the Black-Scholes inequality for American options. Next, we define the American option pricing problem as a free boundary problem. Finally we stipulated the American pricing problem as a LCP, which make solving the problem easier. This chapter concluded with the valuation of currency options. Firstly, we observed how the additional interest rate influenced the currency value and derived the Black-Scholes PDE for currency options accordingly. By making a suitable change in the variable, $V(S,t)$, which denotes the option price, we transformed the Black-Scholes PDE for currency options into the original Black-Scholes PDE. This achievement allowed us to find the values of European and American currency options using the same methods as in Sections 3.1 and 3.3 respectively.

In the next chapter we address the concept of implied volatility. The volatility is the only parameter that is not publicly available at the time the option contract is agreed upon. Therefor finding a method to accurately predict this unknown parameter is very important to an investor.

Chapter 4

Volatility

The valuation of option contracts requires one to have the necessary knowledge of the parameters defining the option's value. All the parameters are publicly observable at the time the contract is agreed upon except for one, the volatility. This parameter of the underlying asset measures the uncertainty of the returns provided by the underlying asset [27]. The volatility is the only non-observable parameter of the underlying asset during the lifetime of the option contract. Therefore, it is important for the investor to use the method that most accurately predicts the value of this unknown variable.

One approach towards finding the best prediction for the volatility is through known market information. That is, by calculating the volatilities of the options whose prices can be observed in the market. The prices of the options can then be calculated using this known volatility, also referred to as the implied volatility (IV). This is the volatility used by traders to price currency options. The IV is easily obtainable and has shown to be very strong in information content and predictive power See [31] and [58].

This chapter is dedicated to familiarize ourselves with the concept of IV. We start off by formally defining the IV and then continue to show why this particular method of finding the volatility provides better results when pricing currency options. Most of the literature in this chapter is desribed in [27], in particular the implied volatilities of currency options.

4.1 Implied volatility

Definition 4.1.1. ***Implied volatility***

Given an observed European call option price, C_{OBS}, for a contract with strike price K and expiration date T, the *implied volatility*, σ_{imp}, is defined to be the value of the volatility parameter that must go into the Black-Scholes formula to match [19]

$$C_{BS}(\sigma_{imp}, S, K, t, r) = C_{OBS}. \tag{4.1.1}$$

The *implied volatility* can be considered as the volatility that we have to insert into the Black-Scholes formula to obtain the observed market price of an option. Therefore, we can write IV as

$$\sigma_{imp} = BS^{-1}(C_{OBS}, S, K, T, r), \tag{4.1.2}$$

where BS^{-1} denotes the inverse Black-Scholes option pricing function [37]. Definition 4.1.1 refers to solving for σ_{imp} in equation (4.1.1) or (4.1.2). Unfortunately it is not possible to do this analytically. However, there exist procedures that can be applied to find the value of σ_{imp} [27]. Using these procedures to find the IV, it is important to know how many possible solutions exist for σ_{imp}. We resolve this issue in the following proposition.

Preposition 4.1.1. There exist atmost one solution to equation (4.1.1), and if

$$C_{OBS} > C_{BS}(0, S, K, t, r) \tag{4.1.3}$$

then there exists exactly one strictly positive solution [19].

Proof. According to [6], we have

$$\frac{\partial C}{\partial \sigma} = S(t)\sqrt{T-t}\mathcal{N}(d_1) > 0 \tag{4.1.4}$$

where d_1 is defined as in equation (3.4.13). Equation (4.1.4) implies that $C(S,t)$ is a strictly increasing function of σ. Hence, there will always exists exactly one strictly positive solution to equation (4.1.1) as long as $C_{OBS} > C_{BS}(0, S, K, t, r)$, and none otherwise.

Remark 4.1.1. The IV of a European call option is the same as that of a European put option provided that they have the same strike price and time to maturity. Therefore, when we refer to the IV of an option, we do not have to specify whether it is a call or a put option [27].

4.2 Volatility smile

Suppose that the Black-Scholes model was accurate, this implies that the market's IV would be equal to the volatility used in the Black-Scholes model. Therefore, the IV as a function of the strike price should be a flat curve. However, empirical results indicate that this is not true. It shows that the graph of the IV versus the strike price for fixed t and S obtained from market option prices are $\cup$-shaped (smile curve), see Figure 4.1.

4.2.1 Interpretation of the smile curve

Unlike the Black-Scholes assumption of constant volatility, it can be seen from the smile curve that the IV changes as the underlying asset moves in-the-money or out-of-the-money. The volatility is relatively low for at-the-money options and increases as the volatility moves into or out-of-the-money, see Figure 4.1. Based on the Black-Scholes model the IV is expected to be the same for all options expiring on the same date regardless of the strike price. However, empirical results indicate otherwise. The smile curve obtained from these results shows that the premium charged for out-of-the-money put options and in-the-money call options is greater than the Black-Scholes price computed with at-the-money IV. A similar pattern is observed for in-the-money put options and out-of-the-money call options. A graphical presentation of this phenomenon can be seen in Figure 4.1.

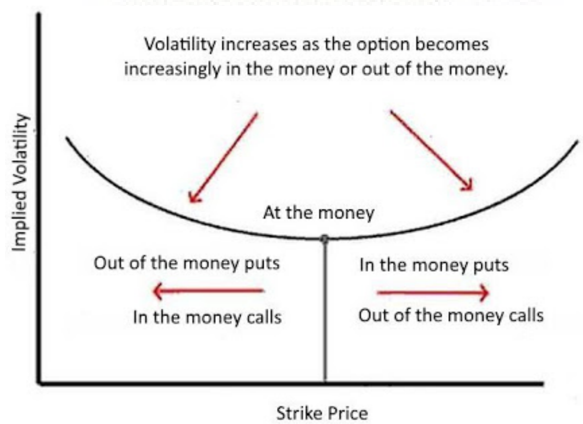

Figure 4.1: Volatility smile interpretation [30]

Figure 4.2 is an example of a volatility smile obtained from real life data [29]. Observe how

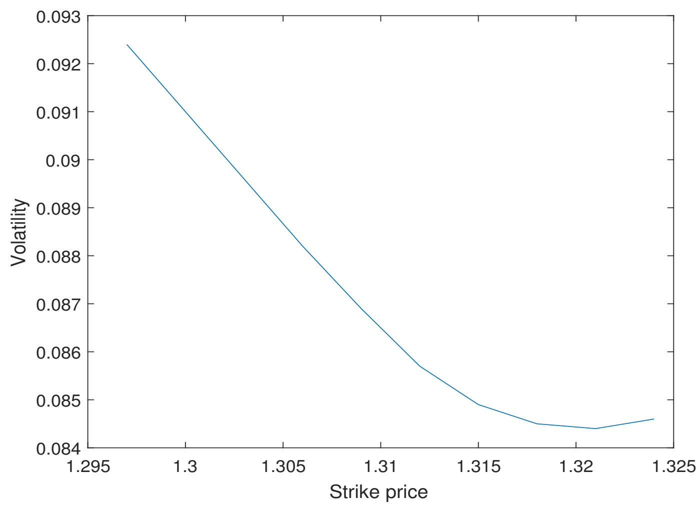

Figure 4.2: Volatility smile

the volatility for a deep in the money put option is greater then the volatility for at-the-money options. This observation is consistent with the known literature. The curve is traditionally referred to as a smile curve, however depending on the market it maybe skewed as in Figure 4.2.

4.2.2 An alternative approach to characterize the volatility smile

The relationship between the IV and the strike price depends on the current price of the asset. To understand this, consider Figure 4.1. The lowest point of the volatility smile is very close to the current exchange rate. However, as the exchange rate increase or decrease, the IV increases. Consequently, traders often calculate the volatility smile as the relationship between the IV and the ratio K/S_0, rather than the relationship between the IV and K. The smile is then much more stable [27].

4.3 The implied volatility distribution

In Chapter 2 we have shown that when the volatility of the underlying asset is constant then the probability distribution of that asset at any future time is log-normal. This is not the assumption made by foreign currency traders [27]. Since it has been observed that the difference in the Black-Scholes price and the market price can be accounted for by IV it is obvious that the probability distribution of the asset price would change. We refer to this risk neutral probability distribution as the implied distribution and can be determined from the volatility smile.

To determine the implied distribution, consider a European call option on an asset with strike price K, and maturity T. In the absence of arbitrage, the option value must be equal to the present value of the terminal payoff, i.e.,

$$C(S,T) = e^{-rt}\mathbb{E}\Big[(S(t)-K)^+\Big]$$
$$= e^{-rt}\int_{S(t)=K}^{\infty}(S(t)-K)g(S(t))dS(t) \quad (4.3.1)$$

where r is the interest rate (assumed constant) and $g(S(t))$ is the risk-neutral probability density function of the underlying asset. We can find $g(S(t))$ by differentiating equation 4.3.1 twice with respect to K. That is,

$$\frac{\partial C}{\partial K} = e^{-rt}\int_{S(t)=K}^{\infty} g(S(t))dS(t) \quad (4.3.2)$$

and therefore

$$\frac{\partial C^2}{\partial K^2} = e^{-rT}g(K). \quad (4.3.3)$$

Solving for $g(K)$ in equation (4.3.3) yields

$$g(K) = e^{rT}\frac{\partial C^2}{\partial K^2}. \quad (4.3.4)$$

This result was first discovered by Breeden and Litzenberg [10]. It enables us to estimate the risk-neutral probabilities from the volatility smiles. According to Hull [27], an estimate of $g(K)$ is given by

$$g(K) = e^{rt}\frac{C_1+C_3-2C_2}{\delta^2}, \quad (4.3.5)$$

where C_1, C_2 and C_3 are the prices of T year European call options with strike prices $K-\delta, K$ and $K+\delta$ respectively.

An example of the implied distribution is shown by the solid line in Figure 4.3. The dashed line in Figure 4.3 is a lognormal distribution obtained from the assumption that the volatility is constant. These two distributions have the same mean and standard deviation. However, the IV distribution seem to have heavier tails then the lognormal distribution.

Since the volatility distribution is determined from the volatility smile, consistency should exist among Figures 4.1 and 4.3. To prove that this is indeed so, consider the following two cases.

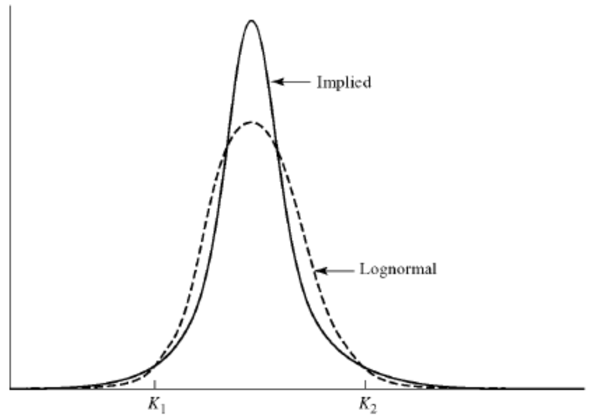

Figure 4.3: Implied and lognormal distribution for foreign currency options [27]

Case 1: A deep out of the money call option with high strike price of K_2. This option will be exercised only if the exchange rate at maturity is greater than K_2, i.e., when $S(t) > K_2$. Looking at Figure 4.3 we notice that the probability of the option being exercised is higher for the implied probability distribution than for the lognormal distribution. This suggests that the implied distribution must give a relatively high price for the option which in turn leads to a relatively high IV. This is exactly what the volatility smile in Figure 4.1 predicts.
Case 2: A deep out of the money put option with low strike price of K_1. A similiar argument as above shows that the two graphs are consistent with each other.

This shows us that the volatility smile used by currency option traders implies that the assumption that the stock price has a lognormal distribution understates the probability of extreme movements in the exchange rates and may lead to major losses. In the next section we study the case where information needed to price a specific option are not currently available in the market.

4.4 Volatility surface

As mentioned earlier the volatility can be estimated using the market option values along with the Black-Scholes pricing formula. However, it might sometimes occur that the information required to price a particular option are not available. This information can then be extracted from what is known as a volatility surface. A volatility surface is a combination of volatility smiles and the volatility term structure (the relationship between the IV and maturity for a particular strike price) to tabulate the volatilities appropriate for pricing an option with any strike price and any maturity.

Table 4.1 is an example of a volatility surface for foreign currency options (GBP/USD) where one dimension of Table 4.1 is K/S_0 and the other dimension is allocated to the time to maturity. The entries of the main body of the table shows the implied volatilities calculated from reliable market data using the Black-Scholes equation. When a new option has to be priced, traders or financial engineers will look up the appropriate volatility in the table. However, when the volatility is not provided by the table then it can be obtained through interpolation. To understand the aforementioned, consider the following example. Consider a financial engineer who wants to value an option with a K/S_0 ratio of 1.0228 and expiration time of nine months. The financial engineer would then have to interpolate between the values 0.0945 and 0.0970 in Table 4.1. In doing this, the financial engineer would obtain a volatility value of 0.09575. This volatility is then substituted into the Black-Scholes formula derived in Chapter 3 to calculate the price of an European currency option. The same volatility is used to value an American currency option.

Time to maturity	K/S_0			
	0.9922	1.0076	1.0228	1.0380
1 Month	0.0985	0.0916	0.0874	0.0863
2 Months	0.1000	0.0937	0.0896	0.0880
3 Months	0.1013	0.0959	0.0918	0.0896
6 Months	0.1022	0.0980	0.0945	0.0921
1 Year	0.1038	0.0981	0.0970	0.0949

Table 4.1: Volatility surface for the GBP/USD currency pair.

In Chapter 3 we observed that one of the assumptions made by the Black-Scholes model and its extensions is that the volatility is constant. This is not the assumption made by traders. Empirical results indicate that the volatility is not constant and is in fact ∪-shaped when plotted against the strike price. This ∪-shape is often referred to as the volatility smile. These results also show that the lognormal distribution understates the probability of extreme movements in exchange rates. Traders found these results to be more attractive than the aforementioned Black-Scholes assumption of constant volatilty. Therefore, they often use the IV obtained from market data to price currency options.

In the next chapter we address the finite difference numerical methods. Before attempting the complex task of solving an option pricing problem using these numerical methods, it is essential to have a clear understanding of the finite difference techniques. This is particularly when solving the American option pricing problem since no closed form solution for finding the price of these type of options exists.

Chapter 5

Numerical Approximations

In Chapter 3, we formally defined both the European and American option pricing problems. An analytical solution was derived for the European options using the heat diffusion equation. However, due to the American option pricing problem's complexity, no generic closed form solution exists. Therefore, one needs to employ numerical methods to obtain the option's value. Different numerical methods have been developed to solve this particular type of option pricing problem. These methods include, but are not limited to, finite difference methods, Fourier methods and Monte Carlo simulations [37]. In this chapter, we present the finite difference methods. Following the work of [44] and [55], we use these methods to find the values of both European and American options. The goal is to find the values of American options, but the European option values are also computed using the finite difference methods for comparison purposes.

5.1 Finite Difference methods for European options

As mentioned in the previous section, the finite difference methods for obtaining numerical solutions for the Black-Scholes PDE is one of several techniques that can be used. These methods involve solving the associated PDE on a discrete space-time grid. In [27], Hull discretized the original Black-Scholes equation to find the value of an American put option. Using two variables, the result of his computation was a surface of option prices. This surface presents all the option values, P^{AM}, on the intervals $S > 0$ and $0 \leq t \leq T$ [44]. However, since we transformed the Black-Scholes equation into the heat equation, these intervals are now transformed to, $-\infty < x < \infty$ and $0 \leq \tau \leq \frac{1}{2}\sigma^2 T$, respectively [44].

In this chapter finite difference method is used to solve the transformed Black-Scholes equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{5.1.1}$$

with initial condition $u(x,0) = u_0(x)$ and $(x,\tau) \in D_u$. The solution is then converted back to the original variables, S and t to obtain the value of the option.

The finite difference mesh and approximation

To apply the finite difference approximation to the transformed Black-Scholes PDE the first step would be to discretize the x and τ axes. That is, we divide both the x and τ axes into equally spaced subintervals of length Δx and $\Delta\tau$ respectively. Before doing this we take into account that the discretization process is normally done on a finite interval and the domain of our space variable x is defined on the infinite interval $-\infty < x < \infty$. We circumvent this problem by replacing the unbounded interval by the bounded interval, $x_{\text{min}} < x < x_{\text{max}}$, where $x_{\text{min}} < 0$ and $x_{\text{max}} > 0$ are sufficiently large integers so that no significant errors are introduced. What follows is a summary of the discretization process as described in [55] and [56].

1. We discretize our τ axis as follows:

   - $\tau = 0, \Delta\tau, 2\Delta\tau, ..., (M-1)\Delta\tau, M\Delta\tau = \frac{1}{2}\sigma^2 T$,
     where M is the number of intervals on the τ axis. The length $\Delta\tau$ of the subintervals is defined as:

$$\Delta\tau = \frac{\frac{1}{2}\sigma^2 T}{M}. \tag{5.1.2}$$

2. We discretize our x axis as follows:

   - $N^-\Delta x, (N^-+1)\Delta x, ..., -\Delta x, 0, \Delta x, ..., (N^+-1)\Delta x, N^+\Delta x$
     where $N^-\Delta x = x_{\text{min}}$ and $N^+\Delta x = x_{\text{max}}$. The length Δx, of the subintervals is defined as:

$$\Delta x = \frac{x_{\text{max}} - x_{\text{min}}}{N} \tag{5.1.3}$$

where N is the number of subintervals on the x axis. At each node (n,m) we make use of the notation

$$u_n^m = u(n\Delta x, m\Delta\tau) \quad \text{for} \quad N^- < n < N^+ \quad \text{and} \quad 0 < m < M, \tag{5.1.4}$$

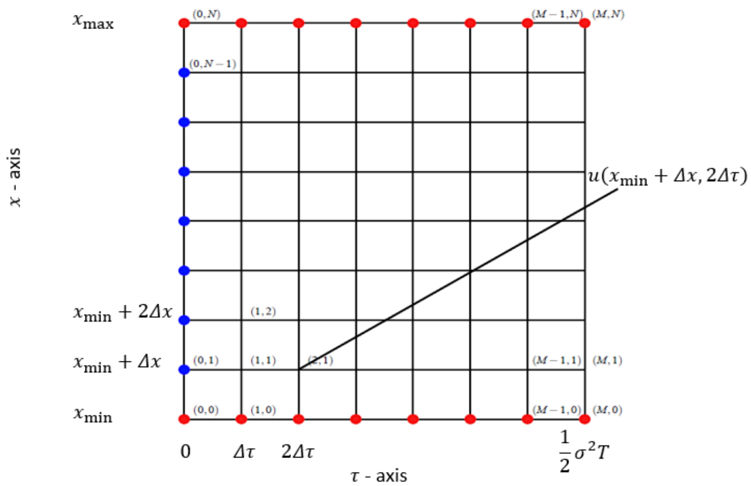

Figure 5.1: $x - \tau$ discretization

to denote the exact solution of the heat equation. A representation of the $x - \tau$ grid can be seen in Figure 5.1. The blue dots in this figure represent the known values at the initial time, $\tau = 0$, whereas the red dots represents the values known at the boundaries, $x_{\min}$ and $x_{\max}$. Furthermore, the values at nodes $(0, 0)$ and $(0, N)$ are both initial and boundary values. The values of the rest of the grid are unknown and needs to be determined.

5.2 Explicit finite difference method

Consider the transformed Black-Scholes model,

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \tag{5.2.1}$$

for the valuation of European options, discussed in Chapter 3 . This model has initial condition

$$u(x, 0) = u_0(x)$$

and boundary conditions

$$u(x, \tau) = u_{-\infty}(x, \tau) \quad \text{and} \quad u(x, \tau) = u_{\infty}(x, \tau) \quad \text{as} \quad x \to \pm\infty.$$

To find the values of $u(x, \tau)$ at each mesh point we make use of a forward difference Taylor approximation to approximate the left hand side of equation (5.2.1) and the symmetric central difference Taylor approximation to approximate the right hand side of equation (5.2.1). That is, we use

$$\frac{\partial u}{\partial \tau} = \frac{u_n^{m+1} - u_n^m}{\Delta \tau} + O(\Delta \tau) \tag{5.2.2}$$

and

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + O((\Delta x)^2) \tag{5.2.3}$$

respectively. Substituting this into equation 5.2.1 and ignoring the terms of $O(\Delta\tau)$ and $O((\Delta x)^2)$, the transformed Black-Scholes equation approximated by a Taylor expansion is [56]

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}. \tag{5.2.4}$$

By separating the terms containing u^m and u^{m+1} we rearrange equation 5.2.4, as follows:

$$u_n^{m+1} = u_n^m + \alpha(u_{n+1}^m - 2u_n^m + u_{n-1}^m) \tag{5.2.5}$$

where,

$$\alpha = \frac{\Delta\tau}{(\Delta x)^2}. \tag{5.2.6}$$

This can be re-written as:

$$u_n^{m+1} = \alpha u_{n+1}^m + (1-2\alpha)u_n^m + \alpha u_{n-1}^m, \quad \text{for} \quad N^- < n < N^+, \quad 0 < m \leq M. \tag{5.2.7}$$

Subject to the initial condition

$$u_n^0 = u_0(n\Delta x), \quad N^- < n < N^+ \tag{5.2.8}$$

and boundary conditions

$$u_{N^-}^m = g(N^-\Delta x, m\Delta\tau), \quad 0 < m \leq M \tag{5.2.9}$$

and

$$u_{N^+}^m = g(N^+\Delta x, m\Delta\tau), \quad 0 < m \leq M \tag{5.2.10}$$

where $g(x,\tau)$ is the transformed payoff function defined by equation 3.3.40 in Chapter 3.

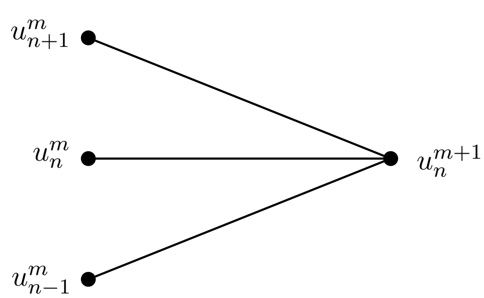

Figure 5.2: Explicit finite difference discretisation

Equation (5.2.7) implies that if at time step m, the values of u_n^m are known for all $N^- < n < N^+$, we can find u_n^{m+1} explicitly [56], see Figure 5.2. This is made possible by the initial condition and boundary conditions, since at time $\tau = 0$ we have all the values of u_n^m and therefore, we can continue to find u_n^{m+1} explicitly. We then continue with this forward iteration until we find the approximate values of $u(x, \tau)$ at time $M\Delta\tau$. The algorithm is straight forward, however the stability of the explicit finite difference method (EFDM) is conditional. It can be shown that the system is [56]:

1. stable if $0 < \alpha \leq \frac{1}{2}$;

2. unstable if $\alpha > \frac{1}{2}$.

Therefore, it is important to pay special attention when choosing the interval lengths Δx and $\Delta\tau$.

The EFDM is a very attractive method due to its simple algorithm. However due to the aforementioned restriction we consider the implicit finite difference method (IFDM) and the Crank-Nicholson finite difference method (CNFDM) which are more flexible.

5.3 Implicit finite difference method

For the IFDM, we consider a backward difference Taylor approximation in time and the symmetrical central difference Taylor approximation for the second derivative with respect to the space variable x. Hence, we have the following numerical approximations:

$$\frac{\partial u}{\partial \tau} = \frac{u_n^m - u_n^{m-1}}{\Delta\tau} + O(\Delta\tau)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + O(\Delta x)^2 \, .$$

Making use of these Taylor expansions, and ignoring the error terms $O(\Delta\tau)$ and $O((\Delta x)^2)$, the heat equation now becomes:

$$\frac{u_n^m - u_n^{m-1}}{\Delta\tau} = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2}. \tag{5.3.1}$$

By separating the terms containing u^m and u^{m-1} we rearrange equation (5.3.1), as follows:

$$-\alpha u_{n-1}^m + (1 + 2\alpha)u_n^m - \alpha u_{n+1}^m = u_n^{m-1} \tag{5.3.2}$$

where α is defined by equation 5.2.6. Equation 5.3.2 is subjected to the initial condition:

$$u_n^0 = u_0(n\Delta x), \quad N^- < n < N^+ \tag{5.3.3}$$

and boundary conditions

$$u_{N^-}^m = g(N^-\Delta x, m\Delta\tau), \quad 0 < m \leq M \tag{5.3.4}$$

$$u_{N^+}^m = g(N^+\Delta x, m\Delta\tau), \quad 0 < m \leq M. \tag{5.3.5}$$

Equation 5.3.2 indicates that u_n^m, u_{n-1}^m and u_{n+1}^m are all implicitly dependent on u_n^{m-1}. Unlike in the EFDM the new values cannot be solved explicitly in terms of the old values and hence an alternative approach must be considered. Figure 5.4.2 below describes the IFDM [27].

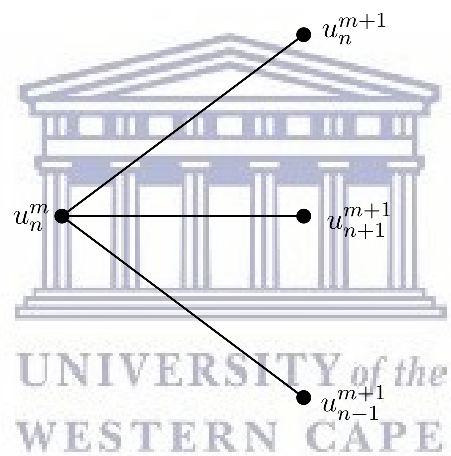

Figure 5.3: Implicit Finite difference discretisation

Equation 5.3.2 when expanded yields

$$\begin{aligned}
-\alpha u_{N^+-2}^m + (1+2\alpha)u_{N^+-1}^m - \alpha u_{N^+}^m &= u_{N^+-1}^{m-1} \\
-\alpha u_{N^+-3}^m + (1+2\alpha)u_{N^+-2}^m - \alpha u_{N^+-1}^m &= u_{N^+-2}^{m-1} \\
-\alpha u_{N^+-4}^m + (1+2\alpha)u_{N^+-3}^m - \alpha u_{N^+-2}^m &= u_{N^+-3}^{m-1} \\
&\vdots \\
-\alpha u_{N^-+1}^m + (1+2\alpha)u_{N^-+2}^m - \alpha u_{N^-+3}^m &= u_{N^-+2}^{m-1} \\
-\alpha u_{N^-}^m + (1+2\alpha)u_{N^-+1}^m - \alpha u_{N^-+2}^m &= u_{N^-+1}^{m-1}
\end{aligned}$$

which can be expressed further in matrix form notation as

$$\begin{bmatrix} 1+2\alpha & -\alpha & 0 & \cdots & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & & & 0 \\ 0 & -\alpha & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -\alpha \\ 0 & 0 & & & -\alpha & 1+2\alpha \end{bmatrix} \begin{bmatrix} u^m_{N^+-1} \\ u^m_{N^+-2} \\ \vdots \\ u^m_0 \\ \vdots \\ u^m_{N^-+1} \end{bmatrix} = \begin{bmatrix} u^{m-1}_{N^+-1} \\ u^{m-1}_{N^+-2} \\ \vdots \\ u^{m-1}_0 \\ \vdots \\ u^{m-1}_{N^-+1} \end{bmatrix} + \alpha \begin{bmatrix} u^m_{N^+} \\ \\ \vdots \\ 0 \\ \vdots \\ u_{N^-} \end{bmatrix}. \qquad (5.3.6)$$

We can write this system of equations in compact form as

$$\mathbf{A}\mathbf{u}^m = \mathbf{b}^m \qquad (5.3.7)$$

where $\mathbf{A}$ is the square matrix and $\mathbf{u}^m$ and $\mathbf{b}^m$ are the vectors. In particular,

$$\mathbf{b}^m = \begin{bmatrix} u^{m-1}_{N^+-1} \\ u^{m-1}_{N^+-2} \\ \vdots \\ u^{m-1}_0 \\ \vdots \\ u^{m-1}_{N^-+1} \end{bmatrix} + \alpha \begin{bmatrix} u^m_{N^+} \\ \\ \vdots \\ 0 \\ \vdots \\ u_{N^-} \end{bmatrix}. \qquad (5.3.8)$$

The matrix $\mathbf{A}$ has a tridiagonal structure. According to Wilmott et al. [56], it can be shown that for $\alpha \geq 0$, this matrix is invertible and thus the solution to the matrix equation is simply $\mathbf{u}^m = \mathbf{A}^{-1}\mathbf{b}^m$. Therefore, we can find the values of $\mathbf{u}^m$ provided that we have entries of $\mathbf{b}^m$. These values can be obtained from $\mathbf{u}^{m-1}$ and the boundary conditions. For if $m = 1$, then we have

$$\mathbf{b}^m = \begin{bmatrix} u^0_{N^-+1} \\ u^0_{N^-+2} \\ \vdots \\ u^0_0 \\ \vdots \\ u^0_{N^+-1} \end{bmatrix} + \alpha \begin{bmatrix} u^1_{N^-} \\ 0 \\ \vdots \\ 0 \\ \vdots \\ u^1_{N^+} \end{bmatrix} \qquad (5.3.9)$$

and as the initial conditions determines $\mathbf{u}^0$, we can find each $\mathbf{u}^m$ sequentially.

There are numerous approaches towards finding the inverse of matrix of $\mathbf{A}$, however the idea is to have a fine discretisation grid and as a result the dimensions of the matrix might be very large. Consequently, these methods may be too time consuming for a high dimensional problem matrix similar to **A**. We overcome this problem by taking advantage of the simplified structure of the matrix **A** which allows us to use alternative highly efficient algorithms to solve equation 5.3.7.

In the next section we discuss two different approaches that can be applied towards solving the tridiagonal matrix. First we discuss the LU-decomposition method, which is a direct elimination method. This method analytically solves the system of equations. The second is the use of iterative methods. Particularly we will make use of the Successive over relaxation method, because it will be useful when solving American options. In addition iterative methods are often used in practice to solve large tridiagonal systems.

5.3.1 LU Decomposition

As previously mentioned, this is a direct method that can be used to solve a system of equations directly. To apply the method of LU-decomposition we decompose matrix **A** into lower and upper triangular matrices denoted by **L** and **U** respectively. As a result our matrix **A** takes on the form

$$\mathbf{A} = \mathbf{LU}, \tag{5.3.10}$$

that is,

$$\begin{bmatrix} 1+2\alpha & -\alpha & 0 & \cdots & 0 & 0 \\ -\alpha & 1+2\alpha & -\alpha & & & 0 \\ 0 & -\alpha & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -\alpha \\ 0 & 0 & & & -\alpha & 1+2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ w_{N^-+1} & 1 & 0 & & & 0 \\ 0 & w_{N^-+1} & 1 & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & 0 \\ 0 & 0 & & & w_{N^+-1} & 1 \end{bmatrix} \begin{bmatrix} y_{N^-+1} & z_{N^-+1} & 0 & \cdots & 0 & 0 \\ 0 & y_{N^-+2} & z_{N^-+1} & & & 0 \\ 0 & 0 & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & z_{N^+-2} \\ 0 & 0 & & & 0 & y_{N^+-1} \end{bmatrix}$$

where the quantities w_n, y_n and z_n are calculated by multiplying matrices **L** and **U** and equating the entries in this new matrix to the entries in matrix **A**. In doing this we obtain the following:

i. When multiplying row one of matrix **L** by column n of matrix **U** where $N^- < n < N^+$

$$\begin{aligned} 1+2\alpha &= (1\times y_{N^-+1}) + (0\times 0) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ -\alpha &= (1\times z_{N^-+1}) + (0\times 0) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ &\vdots \\ 0 &= (0\times 0) + \cdots + (0\times 0) + (0\times 0) + (0\times z_{N^+-2}) + (0\times y_{N^+-1}) \end{aligned}$$

ii. When multiplying row two of matrix **L** by column n of matrix **U** where $N^- < n < N^+$

$$\begin{aligned} -\alpha &= (w_{N^-+1}\times y_{N^-+1}) + (1\times 0) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ 1+2\alpha &= (w_{N^-+1}\times z_{N^-+1}) + (1\times y_{N^-+2}) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ -\alpha &= (w_{N^-+1}\times 0) + (1\times z_{N^-+2}) + (0\times y_{N^-+3}) + (0\times 0) + \cdots + (0\times 0) \\ &\vdots \\ 0 &= (w_{N^-+1}\times 0) + (1\times 0) + (0\times 0) + \cdots + (0\times z_{N^+-2}) + (0\times y_{N^+-1}) \end{aligned}$$

iii. When multiplying row three of matrix **L** by column n of matrix **U** where $N^- < n < N^+$

$$\begin{aligned} -\alpha &= (w_{N^-+1}\times y_{N^-+1}) + (1\times 0) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ 1+2\alpha &= (w_{N^-+1}\times z_{N^-+1}) + (1\times y_{N^-+2}) + (0\times 0) + (0\times 0) + \cdots + (0\times 0) \\ -\alpha &= (w_{N^-+1}\times 0) + (1\times z_{N^-+2}) + (0\times y_{N^-+3}) + (0\times 0) + \cdots + (0\times 0) \\ &\vdots \\ 0 &= (w_{N^-+1}\times 0) + (1\times 0) + (0\times 0) + \cdots + (0\times z_{N^+-2}) + (0\times y_{N^+-1}). \end{aligned}$$

Simplifying these equations we find that

$$y_{N^-+1} = (1+2\alpha) \tag{5.3.11}$$

$$z_n = -\alpha, \quad n = N^-+1, \cdots, N^+-2 \tag{5.3.12}$$

$$y_n = (1+2\alpha) - \frac{\alpha^2}{y_{n-1}}, \quad n = N^-+2, \cdots, N^+-1 \tag{5.3.13}$$

and

$$w_n = \frac{-\alpha}{y_n}, \quad n = N^-+1, \cdots, N^+-2. \tag{5.3.14}$$

Equation 5.3.7 can now be written as

$$(\mathbf{LU})\mathbf{u}^m = \mathbf{b}^m \tag{5.3.15}$$

and by the associative property of matrices we have

$$\mathbf{L}(\mathbf{U}\mathbf{u}^m) = \mathbf{b}^m. \tag{5.3.16}$$

Now let $\mathbf{U}\mathbf{u}^m = \mathbf{q}^m$, then equation 5.3.16 becomes

$$\mathbf{L}\mathbf{q}^m = \mathbf{b}^m \tag{5.3.17}$$

where $\mathbf{q}^m$ is an intermediate vector. By doing this we decompose equation 5.3.16 into two manageable sub-problems. That is, we have to solve

$$\begin{bmatrix} y_{N^-+1} & -\alpha & 0 & \cdots & 0 & 0 \\ 0 & y_{N^-+2} & -\alpha & & & 0 \\ 0 & 0 & y_{N^-+3} & -\alpha & & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & y_{N^+-2} & -\alpha \\ 0 & \cdots & & & 0 & y_{N^+-1} \end{bmatrix} \begin{bmatrix} u^m_{N^-+1} \\ u^m_{N^-+2} \\ \vdots \\ u^m_0 \\ \vdots \\ u^m_{N^+-1} \end{bmatrix} = \begin{bmatrix} q^m_{N^-+1} \\ q^m_{N^-+2} \\ \vdots \\ q^{m-1}_0 \\ \vdots \\ q^m_{N^+-1} \end{bmatrix} \tag{5.3.18}$$

and

$$\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ -\frac{\alpha}{y_{N^-+1}} & 1 & 0 & & & 0 \\ 0 & -\frac{\alpha}{y_{N^-+2}} & 1 & 0 & & \ddots \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & & \\ 0 & \cdots & & & -\frac{\alpha}{y_{N^+-2}} & 1 \end{bmatrix} \begin{bmatrix} q^m_{N^-+1} \\ q^m_{N^-+2} \\ \vdots \\ q^{m-1}_0 \\ \vdots \\ q^m_{N^+-1} \end{bmatrix} = \begin{bmatrix} b^m_{N^-+1} \\ b^m_{N^-+2} \\ \vdots \\ b^m_0 \\ \vdots \\ b^m_{N^+-1}. \end{bmatrix} \tag{5.3.19}$$

To find the values of $\mathbf{u}^m$ in equation 5.3.18 we start by finding the intermediate quantities $\mathbf{q}^m$ in equation 5.3.19. These quantities can be obtained by multiplying the lower triangular matrix

$\mathbf{L}$ by the intermediate vector $\mathbf{q}^m$ and equating it to the entries in $\mathbf{b}^m$. Doing this we obtain

$$\begin{aligned} q^m_{N^-+1} &= b^m_{N^-+1} \\ -\frac{\alpha}{y_{N^-+1}} \cdot q^m_{N^-+1} + q^m_{N^-+2} &= b^m_{N^-+2} \\ -\frac{\alpha}{y_{N^-+2}} \cdot q^m_{N^-+2} + q^m_{N^-+3} &= b^m_{N^-+3} \\ &\vdots \\ -\frac{\alpha}{y_{N^+-2}} \cdot q^m_{N^+-2} + q^m_{N^+-1} &= b^m_{N^+-1}. \end{aligned} \tag{5.3.20}$$

Rearranging these equations we notice that

$$q^m_n = b^m_n + \frac{\alpha q^m_{n-1}}{y_{n-1}} \quad \text{for} \quad n = N^- + 2, \cdots, N^+ - 1. \tag{5.3.21}$$

Equation 5.3.20 allows us to read the value of $q^m_{N^-+1}$ directly, since the values of $\mathbf{b}^m$ are known. Additionally, from equation 5.3.21 we observe that q^m_n is defined in terms of q^m_{n-1}. Therefore, if we solve the system in n-indicial increasing order, then we have q^m_{n-1} available at the time we have to solve for q^m_n.

After obtaining the values for $\mathbf{q}^m$, we can proceed towards finding the values of $\mathbf{u}^m$. We obtain these values by following the same procedure used to find $\mathbf{q}^m$. By multiplying the upper triangular matrix $\mathbf{U}$ by the $\mathbf{u}^m$ vector we obtain

$$u^m_n = \frac{q^m_n + \alpha u^m_{n+1}}{y_n} \quad \text{for} \quad n = N^+ - 2, \cdots, N^- + 1, \tag{5.3.22}$$

where

$$u^m_{N^+-1} = \frac{q^m_{N^+-1}}{y_{N^+-1}}. \tag{5.3.23}$$

Notice that this time we can read off the value $u^m_{N^+-1}$ directly. Also the values of u^m_n is defined in terms of u^m_{n+1}, hence if we can solve the system in decreasing n-indicial order we can find all of the u^m_n values.

The LU decomposition algorithm can be summarized as follows [56]:

- find the values of y_n using equation 5.3.13;
- find the vector $\mathbf{b}^m$ and use equation 5.3.21 to find the vector $\mathbf{q}^m$;
- lastly, use equation 5.3.22 to find the required values of $\mathbf{u}^m$.

In this section, we made use of the method of LU decomposition which is a direct method to solve the system of equations. This method seeks to find the unknown entries u_n^m of the vector $\mathbf{u}^m$ in one pass. However, direct methods are known to have certain disadvantages [33]. In particular, the method of LU decomposition is not immediately applicable to American options [57]. For this reason we employ iterative methods. An advantage of iterative methods over direct methods is they can easily be applied to the American option pricing problem. In the next section we introduce the Successive over-relaxation method which is an example of an iterative method that can be used to solve the American option pricing problem.

5.3.2 Successive Over Relaxation Method

Iterative methods differ from direct methods in that one starts with an initial approximation and then generate successively improved approximations until it converges to the exact solution [33]. An advantage of these methods over direct methods is that they can be employed to solve the American option pricing problem as well as models involving transaction costs [56]. Also iterative methods are easier to program [57].

In this dissertation we will make use of an iterative method called the Successive Over Relaxation (SOR) method. This is an indirect method which originated from the Gauss-Seidel method, which in turn was developed from the Jacobi Method [56]. Therefore, in order to apply the SOR method, we first give a brief description of the Jacobi and the Gauss-Seidel methods.

To apply these methods we rewrite the system of equations given by equation (5.3.7) as

$$u_n^m = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^m + u_{n+1}^m)\right). \tag{5.3.24}$$

We do this to isolate the diagonal terms in the problem [57].

Jacobi Method

What follows is a summary of the Jacobi method as described in [56]. We modify equation (5.3.24) as follows:

$$u_n^{m,k+1} = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^{m,k} + u_{n+1}^{m,k})\right) \quad \text{for} \quad N^- < n < N^+ \tag{5.3.25}$$

where $u_n^{m,k}$ is the k-th iterative for u_n^m. The Jacobi method start by taking an initial guess for u_n^m. We denote our initial guess by $u_n^{m,0}$, where $k = 0$. These initial values for u_n^m are then substituted in the right hand side of equation (5.3.25) to obtain a new approximation for u_n^m on the left hand side of equation (5.3.25). The new approximation is denoted by $u_n^{m,1}$. This process is repeated and as $k \to \infty$ we expect that $u_n^{m,k} \to u_n^m$. The process is stopped when a measure such as

$$\left\| u_n^{m,k+1} - u_n^{m,k} \right\|^2 = \sum_n (u_n^{m,k+1} - u_n^{m,k})^2 \tag{5.3.26}$$

becomes so small it enables us to regard any further iterations as unnecessary. This method is known to converge to the correct solution for any value of $\alpha > 0$.

It is important to notice that the order in which the equations are examined is not important, since the Jacobi method treats each equation independently. Due to this indepencence, the Jacobi method is also referred to as the method of simultaneous displacements [33].

Gauss-Seidel Method

As mentioned before, the Gauss-Seidel method is a replica of the Jacobi method, but uses the new information immediately upon availability. In particular, we notice in our problem that $u_{n-1}^{m,k+1}$ is already available when we decide to calculate $u_n^{m,k+1}$. Therefore, we formulate equation (5.3.25) as follow:

$$u_n^{m,k+1} = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^{m,k+1} + u_{n+1}^{m,k})\right) \quad \text{for} \quad N^- < n < N^+. \tag{5.3.27}$$

Unlike in the case of the Jacobi method, each new approximation is used immediately when it becomes available. Therefore, order plays an important role in the Gauss-Seidel method, because each new approximation depends on the previously computed one. For this reason the Gauss-Seidel method is sometimes also referred to as the method of successive displacements [33]. An advantage of using the most recent information is that this method converges faster than the Jacobi method. Another advantage of the Gauss-Seidel method is that when the new iterates are achieved they overwrite the old ones, in the process canceling any storage problems that might occur during simulations. According to the authors of [56], this method converges for any positive value of α.

Successive Over Relaxation Method

The SOR method is an improvement of the Gauss-Seidel method [56]. It was formulated to accelerate the convergence of the Gauss-Seidel method by introducing a new parameter, ω [34].

This new parameter is referred to as the relaxation parameter. The idea behind introducing the parameter, ω, is to choose a value for ω that will accelerate the rate of convergence of the iterates to the solution [33].

To make use of this method we write $u_n^{m,k+1}$ as

$$u_n^{m,k+1} = u_n^{m,k} + (u_n^{m,k+1} - u_n^{m,k}). \tag{5.3.28}$$

This simple expression will allow the sequence to converge faster than both the Gauss-Seidel method and its predecessor. We then solve the $u_n^{m,k+1}$ on the right hand side of equation (5.3.28) using the Gauss-Seidel method, but to avoid confusion we first write

$$y_n^{m,k+1} = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^{m,k+1} + u_{n+1}^{m,k})\right). \tag{5.3.29}$$

Thus equation (5.3.28) becomes

$$u_n^{m,k+1} = u_n^{m,k} + (y_n^{m,k+1} - u_n^{m,k}). \tag{5.3.30}$$

We regard $y_n^{m,k+1} - u_n^{m,k}$ as a correction term to be added to $u_n^{m,k}$ in order to obtain $u_n^{m,k+1}$. In addition to this we can get a faster rate of convergence by multiplying the correction term by a over-relaxation parameter. That is we modify equation (5.3.30) as follows

$$u_n^{m,k+1} = u_n^{m,k} + w(y_n^{m,k+1} - u_n^{m,k}) \tag{5.3.31}$$

where w is the over-relaxation parameter.

We notice that when $\omega = 1$, then the SOR method simplifies to the Gauss-Seidel method. When $1 < \omega < 2$ and $\alpha > 0$, the SOR method converges to the correct solution [56]. To find the optimal value of ω one has to look at the characteristics of the matrix involved. This is one of the difficulties associated with the SOR method [34]. However, we can bypass this problem by changing ω at each time step until an optimal value is found that minimizes the number of iterations of the SOR method [56].

5.4 Crank-Nicholson finite difference method

The Crank-Nicolson finite difference method (CNFDM) is used to overcome the limitations imposed by the stability and convergence restrictions of the explicit finite difference method.

Another advantage of this method is that for the heat equation and many other equations, it can be shown that the Crank-Nicolson scheme is unconditionally stable and has a faster convergence rate than the IFDM [56].

This method uses the average of the explicit and implicit finite difference methods. Using a forward difference approximation at node (n, m) to approximate the time partial derivative we obtain the explicit scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} + O(\Delta\tau) = \frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + O((\Delta x)^2). \tag{5.4.1}$$

Similarly a backward difference approximation at node $(n, m+1)$ provides the implicit scheme

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} + O(\Delta\tau) = \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\Delta x)^2} + O((\Delta x)^2). \tag{5.4.2}$$

Taking the average of equations (5.4.1) and (5.4.2) and ignoring the error terms we have

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} = \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\Delta x)^2}\right). \tag{5.4.3}$$

Rearranging equation (5.4.3) by separating the terms u^{m+1} and u^m we obtain

$$u_n^{m+1} - \frac{1}{2}\alpha\big(u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\big) = u_n^m + \frac{1}{2}\alpha\big(u_{n-1}^m - 2u_n^m + u_{n+1}^m\big), \quad N^- < n < N^+, \quad 0 < m < M, \tag{5.4.4}$$

where α is defined as in equation (5.2.6). This is known as the Crank-Nicholson scheme. Equation (5.4.4) is subjected to the initial condition [56]

$$u_n^0 = u_0(n\Delta x), \quad N^- < n < N^+. \tag{5.4.5}$$

where u_n^0 is defined in equation (5.3.3). In addition to this initial condition, equation (5.4.4) is also subjected to the boundary conditions [56]:

$$u_{N^-}^m = g(N^-\Delta x, m\Delta\tau), \quad 0 < m \leq M \tag{5.4.6}$$

and

$$u_{N^+}^m = g(N^+\Delta x, m\Delta\tau), \quad 0 < m \leq M. \tag{5.4.7}$$

where $g(x, \tau)$ is defined by equation (3.3.40) in Chapter 3.

Notice that the right hand side of equation (5.4.4) can be evaluated explicitly if the u_n^m values are known. By setting $m = 0$, we can obtain these values using the initial and boundary conditions. Next, if we equate the right hand side of equation (5.4.4) to K_n^m then we have:

$$K_n^m = u_n^m + \frac{1}{2}\alpha\big(u_{n-1}^m - 2u_n^m + u_{n+1}^m\big)$$

which can also be rearranged as

$$K_n^m = \big(1-\alpha\big)u_n^m + \frac{1}{2}\alpha\big(u_{n-1}^m + u_{n+1}^m\big)\,. \tag{5.4.8}$$

Therefore equation (5.4.4) now reduce to

$$u_n^{m+1} - \frac{1}{2}\alpha\big(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}\big) = K_n^m$$

which can also be written as

$$-\frac{1}{2}\alpha u_{n-1}^{m+1} + (1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha u_{n+1}^{m+1} = K_n^m\,. \tag{5.4.9}$$

Expanding equation (5.4.9) yields

$$\begin{aligned}
-\frac{1}{2}\alpha u_{N^+-2}^{m+1} + (1+\alpha)u_{N^+-1}^{m+1} - \frac{1}{2}\alpha u_{N^+}^{m+1} &= K_{N^+-1}^m \\
-\frac{1}{2}\alpha u_{N^+-3}^{m+1} + (1+\alpha)u_{N^+-2}^{m+1} - \frac{1}{2}\alpha u_{N^+-1}^{m+1} &= K_{N^+-2}^m \\
-\frac{1}{2}\alpha u_{N^+-4}^{m+1} + (1+\alpha)u_{N^+-3}^{m+1} - \frac{1}{2}\alpha u_{N^+-2}^{m+1} &= K_{N^+-3}^m \\
&\vdots \\
-\frac{1}{2}\alpha u_{N^-+1}^{m+1} + (1+\alpha)u_{N^-+2}^{m+1} - \frac{1}{2}\alpha u_{N^-+3}^{m+1} &= K_{N^-+2}^m \\
-\frac{1}{2}\alpha u_{N^-}^{m+1} \quad + (1+\alpha)u_{N^-+1}^{m+1} - \frac{1}{2}\alpha u_{N^-+2}^{m+1} &= K_{N^-+1}^m
\end{aligned}$$

which can be expressed further in matrix form notation as

$$\begin{bmatrix} 1+\alpha & -\frac{1}{2}\alpha & 0 & \cdots & 0 & 0 \\ -\frac{1}{2}\alpha & 1+\alpha & -\frac{1}{2}\alpha & & & 0 \\ 0 & -\frac{1}{2}\alpha & \ddots & \ddots & & \\ \vdots & & \ddots & \ddots & \ddots & \\ & & & \ddots & \ddots & -\frac{1}{2}\alpha \\ 0 & 0 & & & -\frac{1}{2}\alpha & 1+\alpha \end{bmatrix} \begin{bmatrix} u_{N^+-1}^{m+1} \\ u_{N^+-2}^{m+1} \\ \vdots \\ u_0^{m+1} \\ \vdots \\ u_{N^-+1}^{m+1} \end{bmatrix} = \begin{bmatrix} K_{N^+-1}^m \\ K_{N^+-2}^m \\ \vdots \\ K_0^m \\ \vdots \\ K_{N^-+1}^m \end{bmatrix} + \frac{1}{2}\alpha \begin{bmatrix} u_{N^+}^{m+1} \\ \\ \vdots \\ 0 \\ \vdots \\ u_{N^-}^{m+1} \end{bmatrix}. \tag{5.4.10}$$

Equation (5.4.10) can be written in compact form as

$$\mathbf{A}\mathbf{u}^m = \mathbf{b}^m. \tag{5.4.11}$$

In Equation (5.4.11), $\mathbf{A}\mathbf{u}^{m+1}$ is the product of the square matrix and the row vector on the left hand side and $\mathbf{b}^m$ is the sum of the row vectors on the right hand side.

Notice that in equation (5.4.9) the values of u_{n+1}^{m+1}, u_n^{m+1} and u_{n-1}^{m+1} all depend on K_n^m in an implicit manner as in the IFDM. Additionally, equation (5.4.10) is similar to equation (5.3.6) in Section 5.3. Therefore equation (5.4.10) can be solved using either the LU decomposition or the SOR method described in the previous section.

Applying the finite difference methods to price European options lays a good foundation towards understanding the characteristics of these methods, since these values can be compared to the theoretical values obtained from the Black-Scholes formula. The next section is dedicated to applying these finite difference schemes to the American option pricing problem.

5.5 Finite Difference Methods for American options

For American options the possibility of early exercise makes the use of finite difference methods not as straightforward as in the case of European options. Although this property of American options is beneficial to the holder of the contract, it gives rise to the free boundary problem. Since we do not know where these free boundaries are, we cannot impose the free boundary conditions and as a result this makes the numerical analysis more challenging.

There are different strategies towards finding numerical solutions for the free boundary problem. However, in this chapter we will only look at one approach, that is the the finite difference formulation of the LCP. In Section 3.3.4 we observed that the pricing of an American option can be reduced to solving a LCP. This formulation allows us to reduce the problem to a fixed boundary problem which in turn enables us to solve the problem and retrieve the free boundaries afterwards.

5.5.1 Finite difference formulation for solving the American pricing problem

In this section we explain the finite difference formulation for the LCP, used for solving the American option pricing problem. We consider only the IFDM and the CNFDM. However, instead of repeating ourselves by explaining both methods again, we only consider the Crank-Nicholson scheme. Also the discussion will not be as detailed as before, since the main idea is to apply the finite difference methods to the LCP.

Recall the Black-Scholes inequality

$$\frac{\partial u}{\partial \tau} \geq \frac{\partial^2 u}{\partial x^2}. \tag{5.5.1}$$

From Section 5.4, we observed that if we use the CNFDM to approximate equation (5.5.1), we obtain

$$\frac{u_n^{m+1} - u_n^m}{\Delta\tau} \geq \frac{1}{2}\left(\frac{u_{n+1}^m - 2u_n^m + u_{n-1}^m}{(\Delta x)^2} + \frac{u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}}{(\Delta x)^2}\right), \tag{5.5.2}$$

which when rearranged gives

$$u_n^{m+1} - \frac{1}{2}\alpha\big(u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\big) \geq u_n^m + \frac{1}{2}\alpha\big(u_{n-1}^m - 2u_n^m + u_{n+1}^m\big), \tag{5.5.3}$$

where α is defined in Section 5.2 by equation (5.2.6) and u_n^m is defined by $u_m^n = u(n\Delta x, m\Delta\tau)$. We define the discretised payoff function by

$$g_n^m = g(n\Delta x, m\Delta\tau) \quad \text{for} \quad N^- < n < N^+, \quad 0 < m < M. \tag{5.5.4}$$

Therefore, the condition $u(x,\tau) \geq g(x,\tau)$ is approximated by

$$u_n^m \geq g_n^m \quad \text{for} \quad m \geq 1. \tag{5.5.5}$$

Following the same procedure as in Section 5.4, let

$$\begin{aligned} Z_n^m &= u_n^m + \frac{1}{2}\alpha\big(u_{n+1}^m - 2u_n^m + u_{n-1}^m\big) \\ &= \big(1-\alpha\big)u_n^m + \frac{1}{2}\alpha\big(u_{n-1}^m + u_{n+1}^m\big)\,. \end{aligned} \tag{5.5.6}$$

Note that at time step $(m+1)\Delta\tau$ we can find Z_n^m explicitly since we have the values for u_n^m. Therefore, equation (5.5.3) now becomes

$$u_n^{m+1} - \frac{1}{2}\alpha\big(u_{n-1}^{m+1} - 2u_n^{m+1} + u_{n+1}^{m+1}\big) \geq Z_n^m. \tag{5.5.7}$$

We can now write the finite difference formulation for the LCP described in Chapter 3 by equation (3.3.50) as

$$
\begin{aligned}
((1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha(u_{n+1}^{m+1} + u_{n-1}^{m+1}) - Z_n^m)(u_n^{m+1} - g_n^{m+1}) &= 0 \\
u_n^{m+1} - \frac{1}{2}\alpha\left(u_{n+1}^{m+1} - 2u_n^{m+1} + u_{n-1}^{m+1}\right) - Z_n^m &\geq 0 \qquad (5.5.8) \\
u_n^m &\geq g_n^m
\end{aligned}
$$

where the last inequality holds for $N^- < n < N^+$ and $0 < m < M$. The formulation described by equation 5.5.8 is subject to the initial and boundary conditions

$$
u_n^0 = g_n^0, \quad u_{N^+}^m = g_{N^+}^m, \quad u_{N^-}^m = g_{N^-}^m. \qquad (5.5.9)
$$

If we define the payoff vector as

$$
\mathbf{g}^m = \begin{bmatrix} g_{N^-+1}^m \\ g_{N^-+2}^m \\ \vdots \\ g_{N^+-2}^m \\ g_{N^+-1}^m \end{bmatrix}, \qquad (5.5.10)
$$

then we can rewrite the finite difference formulation in equation (5.5.8) in constraint matrix notation as:

$$
\begin{aligned}
(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m)(\mathbf{u}^{m+1} - \mathbf{g}^{m+1}) &= 0, \\
(\mathbf{A}\mathbf{u}^{m+1} - \mathbf{b}^m) &\geq 0, \qquad (5.5.11) \\
(\mathbf{u}^{m+1} - \mathbf{g}^{m+1}) &\geq 0, \quad \text{for} \quad m = 0, ..., M,
\end{aligned}
$$

where the matrix $\mathbf{A}$ and the vectors $\mathbf{u}^m$ and $\mathbf{b}^m$ are defined in equation (5.4.10). The next step is to solve the constrained matrix problem. We do this by implementing the Projected successive over relaxation (PSOR) method described below.

5.5.2 Projecter Successive Over Relaxation method for American options

The PSOR is used when solving constraint matrix problems as described in equation (5.5.11). This method is a minor modification of the SOR method. It allows for one additional feature when compared to the SOR method. The PSOR method ensures that the constraint $u_n^{m+1} \geq g_n^{m+1}$ is also satisfied.

Implicit finite difference method

As mentioned above the PSOR is a minor modification of the SOR method. Recall that using an implicit finite difference formulation for the SOR method in Section 5.3.2 equation (5.3.31) we obtained the following equation

$$u_n^{m,k+1} = u_n^{m,k} + w(y_n^{m,k+1} - u_n^{m,k}) \tag{5.5.12}$$

where

$$y_n^{m,k+1} = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^{m,k+1} + u_{n+1}^{m,k})\right). \tag{5.5.13}$$

This method allowed us to solve the matrix problem

$$\mathbf{A}\mathbf{u}^m = \mathbf{b}^m. \tag{5.5.14}$$

In solving the above matrix problem, we satisfy one of the constraints of the LCP. To satisfy the constraint that $\mathbf{u}^m \geq \mathbf{g}^m$ we make use of the PSOR method. This method is obtained by taking the maximum value of equation (5.5.12) and the payoff function. That is, write equation (5.5.12) as

$$u_n^{m,k+1} = \max\left(u_n^{m,k} + w(y_n^{m,k+1} - u_n^{m,k}), g_n^m\right). \tag{5.5.15}$$

Therefore, the PSOR method is used to iterate the equations

$$y_n^{m,k+1} = \frac{1}{1+2\alpha}\left(b_n^m + \alpha(u_{n-1}^{m,k+1} + u_{n+1}^{m,k})\right) \tag{5.5.16}$$

$$u_n^{m,k+1} = \max\left(u_n^{m,k} + w(y_n^{m,k+1} - u_n^{m,k}), g_n^m\right) \tag{5.5.17}$$

until the increment

$$\left\|\mathbf{u}^{m,k+1} - \mathbf{u}^{m,k}\right\| \tag{5.5.18}$$

is so small it can be ignored. We then set $\mathbf{u}^m = \mathbf{u}^{m,k+1}$.

Crank-Nicholson method

Consider the Crank-Nicholson finite difference formulation for the transformed Black-Scholes equation obtained in Section 5.4, equation (5.4.9). That is,

$$-\frac{1}{2}\alpha u_{n-1}^{m+1} + (1+\alpha)u_n^{m+1} - \frac{1}{2}\alpha u_{n+1}^{m+1} = K_n^m. \tag{5.5.19}$$

As mentioned before, this equation is similar to the equation obtained using the Implicit finite difference scheme in Section 5.3, equation (5.3.2), therefore we can use the aforementioned arguments to formulate a PSOR algorithm for it. We do this by first adopting the SOR algorithm obtained in equations (5.5.12) for the Crank-Nicholson finite difference formulation, i.e.,

$$y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left(b_n^m + \frac{1}{2}\alpha(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right) \tag{5.5.20}$$

$$u_n^{m+1,k+1} = u_n^{m+1,k} + w(y_n^{m+1,k+1} - u_n^{m+1,k}). \tag{5.5.21}$$

We formulate a PSOR algorithm for the Crank-Nicholson scheme as follows:

$$y_n^{m+1,k+1} = \frac{1}{1+\alpha}\left(b_n^m + \frac{1}{2}\alpha(u_{n-1}^{m+1,k+1} + u_{n+1}^{m+1,k})\right) \tag{5.5.22}$$

$$u_n^{m+1,k+1} = \max\left(u_n^{m+1,k} + w(y_n^{m+1,k+1} - u_n^{m+1,k}), g_n^{m+1}\right). \tag{5.5.23}$$

We then iterate these equations until the increment described in equation (5.5.18) becomes so small it can be regarded as negligible.

Being an iterative method, the PSOR generates solutions that are self-consistent, because it does not violate any of the conditions. Any solution generated by the PSOR method has the property that either $u_n^{m+1} = g_n^{m+1}$ or the n-th component of $\mathbf{C}\mathbf{u}^{m+1} - \mathbf{b}^m$ vanishes. Therefore, guaranteeing that both the conditions $\mathbf{u}^{m+1} \geq \mathbf{g}^{m+1}$ and $(\mathbf{C}\mathbf{u}^{m+1} - \mathbf{b}^m)(\mathbf{u}^{m+1} - \mathbf{b}^m) = 0$ are met. Lastly the condition that $\mathbf{C}\mathbf{u}^{m+1} \geq \mathbf{b}^m$ follows as a consequence of the structure of C [56].

An understanding of the finite difference methods is essential before attempting the complex task of applying this technique to solving an option pricing problem, especially in the case of the American option pricing problem for which no analytical formula exists. Therefore, applying the finite difference methods to value European options is a good start towards understanding these methods, since the values obtained using the finite difference methods can be compared to the results found using the Black-Scholes formula. In the next chapter we will present the numerical simulations obtained from the methods presented in this chapter. We present the numerical results for all three finite difference schemes and discuss the different factors affecting the convergence rate of these methods.

Chapter 6

Computational Results and Analysis

In this chapter we consider some of the findings observed during the implementation of the methodologies described in Chapter 5. We first consider the different factors that might have an effect on the convergence rate of the numerical methods, and then run simulations accordingly. The factors that affect the convergence rate of these methods are the space-time discretization steps, as well as the artificial limit for S, which we describe here as $S_{\max}$. The former introduced us to the limitations of the Explicit finite difference method. We notice that for different ratios of M and N the program produced errors (denoted as NaN in Table 6.1). This is because of the restriction of the EFDM. For the IFDM and the CNFDM, we only declare the results obtained from the SOR method, because both the SOR method and the method of LU decomposition produced similar results. For American options, we considered the PSOR method which is a modification of the SOR method to accommodate one of the conditions of the American option pricing problem.

For both the European and American currency options we considered the results obtained from an online trading site called Investing.com [29]. We chose the GBP/USD currency pair. This site provided all the parameters needed to price a European currency option, except for the interest rate of the different currencies. This was provided by another online website called Global-rates.com [22]. The website offers a wide range of information consisting of interest rates and economic indicators. To find the domestic and foreign interest rate we made use of the London Interbank Offered Rate (LIBOR) provided by Global-rates.com [22]. LIBOR is a benchmark rate that represents the average interbank interest rates at which banks on the London market offer to lend funds to one another. Using these interest rates we were able to obtain the approximate values of the options. The values obtained for the European currency options differs slightly from the values of the trading site. However, it coincides with the value obtained from the Black-Scholes formula for pricing an European currency option. This difference is a result of Investing.com using different interest rates for the domestic and foreign currency.

We begin this chapter by finding the price of an European currency options. Although these values can be obtained using the Black-Scholes formula, applying the finite difference methods is a good approach to acquaint ourselves with the mechanics of these methods. This is because the values obtained from these methods can be compared to the Black-Scholes theoretical values. After familiarizing ourselves with the finite difference methods we use the IFDM and CNFDM in conjuction with the PSOR method to find the values of American currency options. In this chapter we only consider the valuation of European and American put options as the values of the call options can be obtain using the same methods.

6.1 European options

To find the value of the European put option we consider the parameters obtained from Investing.com [29] and Global-rates.com [22]. That is, $S = 1.3101$, $K = 1.18$, $T = 1$, $r_d = 0.0262288$, $r_f = 0.0090388$ and $\sigma = 0.1258$, where r_d and r_f are the domestic and foreign interest rates respectively. First we used the Black-Scholes pricing formula to obtain the exact value of a put option. Thereafter we continued to find the price of the put option using the finite difference methods. These results were then compared to each other as well as the value provided by the trading site. The exact value of the European put option obtained from the Black-Scholes equation in Chapter 3 is **0.013586** and the value of the put option according to the trading site [29] is **0.01345**. As mentioned previously, the difference in these two values is a result of the trading site using different values for the interest rates.

Tables 6.1 to 6.3 illustrates the effects caused by an increase in the time-space discretization steps of the EFDM, IFDM and the CNFD respectively. When $N = M$, $N = 2M$ and $N = 3M$ we observed that the rate of convergence for the finite difference methods are very slow. Also the output obtained through these methods are very similar. However, it seems like the IFDM is converging a little faster than the CNFDM and the EFDM. For $N = 5M$, the CNFDM

Explicit finite difference method						
M	$N = M$	$N = 2M$	$N = 3M$	$N = 5M$	$N = 10M$	$N = 100M$
50	0.070144	0.016411	0.017241	0.016134	0.013905	NaN
100	0.016390	0.013891	0.014306	0.013881	0.013754	NaN
200	0.013884	0.014131	0.013975	0.013742	0.013611	NaN
300	0.014291	0.013972	0.013667	0.013634	NaN	NaN
400	0.014125	0.013835	0.013663	0.013605	NaN	NaN
500	0.013863	0.013735	0.013630	0.013599	NaN	NaN
1000	0.013733	0.013601	0.013593	0.013588	NaN	NaN
1500	0.013627	0.013592	0.013593	NaN	NaN	NaN
2000	0.013600	0.013594	0.013589	NaN	NaN	NaN

Table 6.1: The effect of increasing the discretisation steps on the Explicit finite difference values.

Implicit finite difference method						
M	$N = M$	$N = 2M$	$N = 3M$	$N = 5M$	$N = 10M$	$N = 100M$
50	0.070137	0.016326	0.017148	0.016050	0.013813	0.013540
100	0.016347	0.013864	0.014261	0.013835	0.013707	0.013563
150	0.013871	0.014109	0.013952	0.013719	0.013587	0.013573
300	0.014276	0.013956	0.013651	0.013618	0.013582	0.013577
500	0.013854	0.013726	0.013621	0.013590	0.013581	0.013580
1000	0.013728	0.013597	0.013588	0.013584	0.013584	0.013582
1500	0.013624	0.013589	0.013590	0.013585	0.013584	0.013584
2000	0.013598	0.013592	0.013586	0.013585	0.013584	0.013584

Table 6.2: The effect of increasing the discretisation steps on the Implicit finite difference values.

converges faster than the IFDM and the EFDM. Additionally, the difference in the convergence rate of the CNFDM and the IFDM is not very significant. When $N = 5M$, we also notice the rising of the stability issue associated with the EFDM. This is due to the restriction on alpha established in Chapter 5, Section 5.2. According to Wilmott et al. [56], the EFDM is stable for $0 < \alpha < \frac{1}{2}$, however, in this case $\alpha > \frac{1}{2}$. When $N = 10M$, the difference in the convergence rate of the IFDM and CNFDM are becoming more significant. In the beginning we observed a faster rate of convergence from the IFDM, but as we increased the value of M, the convergence rate

Crank-Nicholson finite difference method						
M	$N = M$	$N = 2M$	$N = 3M$	$N = 5M$	$N = 10M$	$N = 100M$
50	0.070140	0.016368	0.017194	0.016091	0.013858	0.013586
100	0.016368	0.013877	0.014283	0.013858	0.013731	0.013586
150	0.013877	0.014120	0.013964	0.013731	0.013599	0.013585
300	0.014283	0.013964	0.013659	0.013626	0.013590	0.013585
500	0.013858	0.013731	0.013626	0.013594	0.013586	0.013585
1000	0.013731	0.013599	0.013590	0.013586	0.013586	0.013585
1500	0.013626	0.013590	0.013592	0.013586	0.013585	0.013585
2000	0.013599	0.013593	0.013588	0.013586	0.013585	0.013585

Table 6.3: The effect of increasing the discretisation steps on the Crank-Nicholson finite difference values.

of the CNFDM became faster than that of the IFDM. For $N = 100M$, the CNFDM converges significantly faster than the IFDM. The method already converges when $M = 50$, whereas the IFDM only starts to show convergence when $M = 1500$. However, the CNFDM converges to 0.01385 instead of the exact value 0.01386. This might be a result of the oscillation problem associated with the CNFDM. In both cases where $N = 10M$ and $N = 100M$ the stability problem of the EFDM is evident. Again this is a result of α being greater than $\alpha > \frac{1}{2}$.

Crank-Nicholson finite difference method						
M	$S_{\max} = 2S$	$S_{\max} = 3S$	$S_{\max} = 4S$	$S_{\max} = 5S$	$S_{\max} = 10S$	$S_{\max} = 20s$
50	0.056119	0.075772	0.042024	0.068281	0.009315	0.079556
100	0.027928	0.016046	0.028958	0.025939	0.016443	0.023677
150	0.017876	0.017602	0.015304	0.018567	0.017828	0.018459
300	0.013950	0.014029	0.013911	0.015269	0.013660	0.015471
500	0.013684	0.013746	0.013639	0.013862	0.014304	0.014383
1000	0.013587	0.013723	0.013635	0.013652	0.013676	0.013635
1500	0.013582	0.013651	0.013569	0.013623	0.013584	0.013628
2000	0.013601	0.013602	0.013596	0.013626	0.013624	0.013624

Table 6.4: The effect that the different values of $S_{\max}$ have on the Crank-Nicholson values.

Table 6.4 illustrates how a change in the value of $S_{\max}$ affects the behaviour of an option's value. All the finite difference methods produced similar results. Therefore, we only published the results of the CNFDM. We chose the M and N ratio in Table 6.3 which converged the slowest ($M = N$) to analyse how these different values of $S_{\max}$ affected the output. We observed that only when $S_{\max} = 4S$ there was consistent and significant improvement in the convergence rate. When $S_{\max} = 2S$ we also observed improvement in the convergence rate, however, the value increased again at the last iteration. The convergence rate for the other values of $S_{\max}$ were very slow. Finally, inconsistency in the iteration values were observed when $S_{\max}$ was chosen to be $10S$.

6.2 American Options

For American currency options we also consider the parameters obtained from Investing.com [29] and Global-rates.com [22]. That is $S = 1.3101$, $K = 1.4$, $T = 0.5$, $r_d = 0.0262288$, $r_f = 0.0090388$ and $\sigma = 0.09$, where r_d and r_f are the domestic and foreign interest rates respectively. Unfortunately Investing.com does not provide the values for American put options to compare our results to. For this reason we used different methods to obtain the approximate value of the American put option. In particular, we made use of the IFDM and the CNFDM. These results were then compared to each other to see, approximately, what the theoretical price of the American put option is. Additionally, to confirm the price of the American option we used the same parameters to calculate the price of an European currency option. This was to confirm that the American currency option's price is indeed greater than that of its European counterpart. The exact value of the European option obtained from the Black-Scholes equation in Chapter 3 is **0.0860** and the value of the option according to the trading site [29] is **0.08571**.

In Tables 6.5 and 6.6 we illustrate the effects caused by an increase in the time-space discretization steps. These results reaffirms that the CNFDM converge faster than the IFDM which is consistent with what we observed in the case of the European option. For both cases we observed that when the space-time discretization steps are equal, the convergence rate is the slowest and it increases as N increases by multiples of M. Both, the IFDM and the CNFDM converged to the value 0.091035. Therefore, we can conclude that it should be the theoretical price of the American option. The results are consistent with the existent literature which suggests that the CNFDM should converge faster than the other two finite difference methods and that an

Implicit finite difference method					
M	$N = M$	$N = 2M$	$N = 3M$	$N = 5M$	$N = 10M$
50	0.184178	0.099963	0.096251	0.095154	0.095436
100	0.099963	0.095154	0.092935	0.092681	0.091952
150	0.096251	0.092935	0.091497	0.091730	0.090965
300	0.092935	0.091730	0.091328	0.091153	0.091047
500	0.091952	0.091244	0.091047	0.091048	0.091076
1000	0.091244	0.091048	0.091051	0.091043	0.091039
1500	0.091047	0.091051	0.091047	0.091040	0.091037
2000	0.091048	0.091043	0.091040	0.091039	0.091036
3000	0.091051	0.091040	0.091035	0.091035	0.091035
5000	0.091039	0.091036	0.091036	0.091035	0.091035

Table 6.5: The effect that increasing the discretisation steps have on Implicit finite difference values.

Crank-Nicholson finite difference method					
M	$N = M$	$N = 2M$	$N = 3M$	$N = 5M$	$N = 10M$
50	0.184178	0.099963	0.096239	0.095150	0.095424
100	0.099963	0.095152	0.092928	0.092675	0.091945
150	0.096247	0.092930	0.091493	0.091723	0.090958
300	0.092933	0.091727	0.091324	0.091149	0.091042
500	0.091950	0.091242	0.091044	0.091045	0.091073
1000	0.091243	0.091046	0.091049	0.091041	0.091037
1500	0.091046	0.091050	0.091046	0.091039	0.091036
2000	0.091047	0.091042	0.091039	0.091038	0.091035
3000	0.091050	0.091040	0.091034	0.091035	0.091035
5000	0.091038	0.091035	0.091036	0.091035	0.091035

Table 6.6: The effect that increasing the discretisation steps have on Crank-Nicholson finite difference values

American option's price should be greater than that of its European counterpart. The latter is due to the early exercise privilege provided by an American option.

Crank-Nicholson finite difference method							
M	$S_{\max} = S \times 1.1$	$S_{\max} = S \times 1.2$	$S_{\max} = S \times 1.3$	$S_{\max} = S \times 1.4$	$S_{\max} = S \times 1.5$	$S_{\max} = S \times 1.6$	$S_{\max} = S \times 1.7$
50	0.119608	0.178126	0.190170	0.166105	0.113474	0.090820	0.126105
100	0.093126	0.116842	0.094235	0.124203	0.107553	0.093778	0.117258
150	0.093015	0.095558	0.101532	0.096120	0.102180	0.095605	0.106166
300	0.092415	0.095147	0.094495	0.094222	0.094676	0.095252	0.094778
500	0.090586	0.091872	0.091376	0.091147	0.092085	0.090790	0.091911
1000	0.090571	0.091251	0.091226	0.091173	0.090952	0.091027	0.091244
1500	0.090390	0.091009	0.091148	0.091107	0.091143	0.091062	0.090972
2000	0.090311	0.091053	0.091081	0.091090	0.091034	0.091062	0.091078
3000	0.090377	0.091043	0.091024	0.091048	0.091042	0.091055	0.091024
5000	0.090364	0.091032	0.091039	0.091035	0.091036	0.091041	0.091040

Table 6.7: The effect that increasing the discretisation steps have on Crank-Nicholson finite difference values.

Table 6.7 illustrates how a change in the value of $S_{\max}$ effects the convergence rate of the CNFDM. As in the case of the European options, we chose the M and N ratio with the slowest convergence rate to analyse how a change in $S_{\max}$ will influence the rate of convergence. Initially, we chose $S_{\max}$ to be multiples of S, but this method did not increase the convergence rate. We then increased $S_{\max}$ by the amounts shown in Table 6.7. This approach gave better results. For all the different values of $S_{\max}$ the option values maintained a regular pattern of decreasing towards the theoretical option value, except for when $S_{\max} = S \times 1.1$. For the latter we observed no convergence to the theoretical option price. This approach for choosing the value of $S_{\max}$ did not provide very significant improvement in the convergence rate, except for the case where $S_{\max} = S \times 1.5$. When $S_{\max} = S \times 1.5$ we saw a significant improvement in the convergence rate.

Figure 6.1 illustrates the option prices of both the American and European put option prices when the values of the underlying asset are between $S = 0$ and $S = 1.5$. We used the same parameters used to calculate the value of the American put option. Noticed that the form of

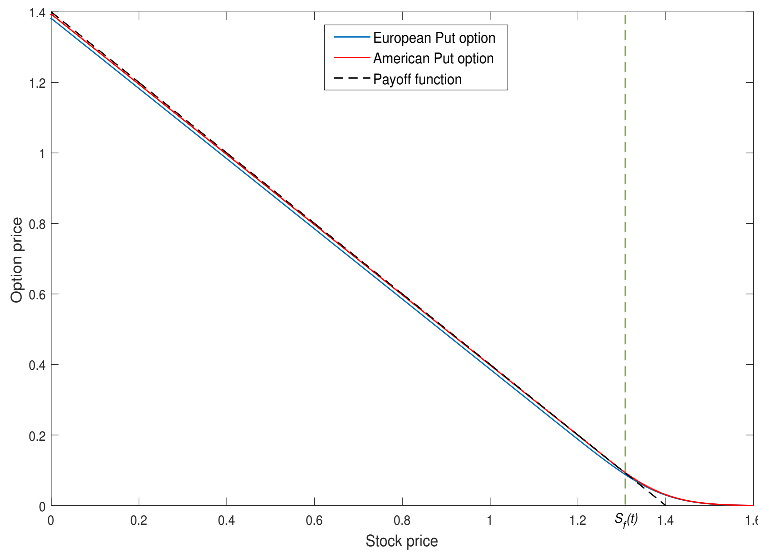

Figure 6.1: American and European put option prices.

these graphs follows the traditional form of the payoff function, which in the case of a put option decreases gradually as the underlying's price increases. Additionally, it can be seen graphically that the American put option's value is greater than that of its European counter part. We also noticed that the graph of the American put options joins smoothly with the payoff function at the free boundary, $S_f(t)$. As explained in Chapter 3, this point marks the boundary between the exercise and the holding region. This also confirms constraint (iv) of the American option described in Chapter 3, Section 3.3.3, i.e., the option delta must be continuous.

In this chapter we used different examples to find the values of European and American currency options. We considered the factors that might influence the convergence rate of the finite difference methods. The results obtained when we considered the space-time discretization steps were consistent with the existing literature for both European and American options. We saw that the EFDM is conditionally stable and in order to use this method we have to choose our discretization steps in such a way that $0 < \alpha < \frac{1}{2}$. Additionally, we noticed that the IFDM and the CNFDM are unconditionally stable and that the CNFDM does indeed converge faster than both the EFDM and the IFDM. Next we considered how the chosen value for $S_{\max}$ affects the convergence rate. In this case we notice that the numerical analysis for both the European and American options did not change significantly.

Chapter 7

Conclusion

With the increase in the volatility of the global markets, the management of the risk associated with currency exposure is very important. One approach for hedging the risk is through the use of foreign currency options. Currency options provide insurance against undesirable exchange rate fluctuations and it even allows for opportunities to make a profit. However, in order to use these types of financial derivatives it is imperative to be familiar with the valuation process. After extensive study of the relevant literature, the objective of this dissertation is to introduce different methods that can be employed for the valuation of currency options.

Chapter 1 provides an introduction to the research. In Chapter 2, we devoted our efforts towards wrapping our minds around the mathematics involved in option pricing. This was followed by the valuation of European and American options in Chapter 3. Chapter 3 is mainly based on the work of [56] and [44]. For European options, we derived the famous Black-Scholes partial differential equation (PDE), which was followed by the derivation of the Black-Scholes formula for valuing European options. We accomplished the latter by first transforming the Black-Scholes PDE to the heat equation described in Chapter 2. This transformed PDE simplified the derivation process. In addition to this, it also played a pivotal role towards the valuation of American options.

The second part of Chapter 3 was devoted to American options. We set off by deriving the Black-Scholes inequality for American options using a no arbitrage argument, similar to the case of European options. This was followed by introducing the free boundary problem associated with American options. Following the work of Wilmott et al. [56] and Seidel [44], we introduced the obstacle problem to promote a better understanding of the free boundary problem. We then illustrate that a unique solution to the American option valuation problem can be obtained by specifying a similar set of constraints to those of the obstacle problem. At this juncture, all the necessary tools were gathered to fully formulate the American option as a free boundary problem. Due to the unknown location of the optimal exercise boundary, we had to introduce an additional exercise boundary. What followed was a reconstruction of the American option pricing problem to reduce it to a fixed boundary problem. This was executed by using a linear complementarity formulation. Using the linear compementarity formulation we were able to effect the reduction of the American option pricing problem to a fixed boundary problem. The ascendancy of this approach is that the fixed boundary problem does not mention the free boundary explicitly and therefore the free boundary could be obtained afterwards. After completing the valuation of European and American options on non-dividend paying stocks, we continue to pricing currency options. We showed that these types of options can be priced in a similar manner to options on non-dividend paying stocks.

In Chapter 4 we discussed the work of Hull [27], who studies the problem of non-constant volatility. The Black-Scholes model used to value European options assumes that the volatility used to price these options are constant, whereas empirical results indicate otherwise. Using real life data obtained from investing.com [29], we showed that the volatility of currency options are indeed not constant but a graph of the implied volatility against different strike prices was observed to be ∪-shaped. This is in contrast with the Black-Scholes assumption of constant volatility and consistent with the results of [27]. Continuing, we showed how the risk-neutral probability distribution for an asset can be determined at a future time from the volatility smile. This is referred to as the implied distribution and it has heavier tails than the lognormal distribution proposed in Chapter 2. Additionally, we explained how the implied volatility corresponds to the implied distribution. Finally, we discussed how to construct a volatility surface from the market implied volatilities and how to use this to obtain the volatilities to price options whose information are not yet available in the market.

Chapter 5 was devoted to discussing the numerical methods employed to find the values of both European and American options. This chapter is mainly based on the work of [56], [55] and [44]. We commenced by discussing the finite difference methods for European options. Firstly, we dealt with the boundary issues that came with transforming the Black-Scholes equation. Upon transforming the Black-Scholes PDE we observed that the space variable now became unbounded. So we had to introduce a bounded interval. We continued to discuss the various types of finite difference methods. The finite difference methods consist of three schemes, namely the explicit, implicit and Crank-Nicholson scheme. Among these three methods the explicit method is the easiest to work with, however it has one drawback. This method generates unstable results unless stability constraints are imposed. The Implicit and the Crank-Nicholson schemes are more demanding to work with, but these two methods produce unconditionally stable results. To solve the transformed Black-Scholes equation using these two schemes we dealt with a large system of linear equations. Different methods to solve these equations had to be employed. For both the IFDM and CNFDM we made use of LU decomposition, which is a direct method and the Successive over-relaxation (SOR) method, which is an iterative method. The advantage of using the iterative method is that it can also be employed to find the value of American options.

The second part of Chapter 5 was dedicated to American options. The early exercise privilege accompanied by American options leads to a free boundary problem. This makes using finite difference methods not as straight forward as in the case of European options. The main problem associated with free boundaries in numerical methods is that the location of the free boundaries are unknown beforehand. As discussed in Chapter 3, we circumvent this problem by transforming it into a linear complementarity problem (LCP) from which the free boundary can be obtained afterwards. We then solve the LCP by making a minor modification to the SOR method. This new method is referred to as the Projected successive over relaxation method. The modification makes provision for the condition that the value of the option should always be greater or equal to the option's payoff.

In Chapter 6, we presented some of the findings observed during the implementation of the methodologies described in Chapter 5. The parameters used to find the values of the options in this chapter was obtained from a real life trading site called Investing.com [29]. However, the latter source does not provide the different interest rates used to price the currency options. This data was obtained from Global-rates.com [22].

Initially we considered the valuation of a European option. It appeared to be an effective approach to analyse whether the finite difference methods are capable of producing accurate results. This is due to the fact that the values obtained from these methods can be compared to the values produced by the Black-Scholes equation. For all three different schemes the estimated values were compared to the exact values and little discrepancies were observed. We also observed that for these finite difference methods the rate of convergence is dependent on the space-time discretization steps, as well as the artificial limit of S. However, the space-time discretization produced a faster convergence rate. Additionally, the latter also played an important role in the limitations of the EFDM. We noticed that for different M and N ratios the program produced either errors or inconsistent values.

For the valuation of American options, we employed the PSOR method, which is a modification of the SOR method. Since there exists no analytical formula for these types of options, we applied the PSOR method using both the IFDM and the CNFDM and compared the ouput. These two schemes produced similar results and we concluded that the value obtained should be the theoretical value for the American option. To reinforce our conclusion about the option's value we also calculated the price of the European option using the same parameters. We observed that the value of the American option is indeed greater than that of the European option. We also presented the previous observation graphically.

In terms of convergence, observations for the American and European options were similar. The rate of convergence for American options also depends on the space-time discretization steps and the $S_{\max}$ value. In both cases the CNFDM converged to the exact value the fastest. In the European option case, we noticed for the initial values of M and N the IFDM had a faster convergence rate. However, the CNFDM converged faster than the IFDM as the value of N was increased. This is consistent with the existing literature. We also noticed that the choice of $S_{\max}$ which increases the convergence rate, differs for American and European options.

Future research could focus on alternative ways to model the volatility of an option. Also, since the values obtained for European options using the finite difference methods showed little discrepancies when compared to the exact values, numerical approaches can equally successfully be used to approximate the prices of options. Therefore, methods to ensure accuracy and higher convergence rates to the true solution should be studied in future.

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