Sobolev homeomorphisms are dense in volume preserving automorphisms

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Abstract

In this paper we prove a weak version of Lusin's theorem for the space of Sobolev-$(1,p)$ volume preserving homeomorphisms on closed and connected n-dimensional manifolds, $n \geq 3$, for $p < n-1$. We also prove that if $p > n$ this result is not true. More precisely, we obtain the density of Sobolev-$(1,p)$ homeomorphisms in the space of volume preserving automorphisms, for the weak topology. Furthermore, the regularization of an automorphism in a uniform ball centered at the identity can be done in a Sobolev-$(1,p)$ ball with the same radius centered at the identity.

Keywords: Lusin theorem, volume preserving, Sobolev homeomorphism

1. Introduction

J. E. Littlewood formulated the classic Lusin theorem by saying that *'every measurable function is nearly continuous'*. In the spirit of this formulation, we prove that *'every measurable volume preserving map is nearly a Sobolev-$(1,p)$ volume preserving homeomorphism'*. More precisely, we prove a weak version of Lusin's theorem (see Theorem A in Section 4) for the

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space of Sobolev-$(1,p)$ volume preserving homeomorphisms on closed and connected n-dimensional manifolds, $n \geq 3$, for $p < n-1$.

This theorem generalizes for the Sobolev setting previous continuous versions proved by Oxtoby [10], White [12], Alpern [1] and Alpern and Edwards [2].

The proof of Theorem A is based on a key perturbation result (Theorem 3.1). This perturbation theorem is the Sobolev version of a classical result proved by Oxtoby and Ulam [9] which, in rough terms, says that given any $\varepsilon > 0$ and any two sets of distinct N points $\{P_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ in $\mathbb{R}^n$ such that P_i is ε-close to Q_i for all i we can construct a volume preserving homeomorphism h ε-close to the identity which maps some neighbourhood of P_i by simple translation onto a neighbourhood of Q_i. An explicit construction of h can be seen in [2, 3]. Yet, the strategy used in [2, 3] cannot be applied in our Sobolev setting because we need to control the L^p-norm of the partial derivatives of h. To obtain this control we define h as the composition of $2N$ simpler perturbations, which locally is a composition of just two perturbations.

Besides its intrinsic importance in Analysis, the weak Lusin theorem in the volume preserving class was crucial in a proof given in [3] of the result of Oxtoby and Ulam, that ergodicity is generic for measure preserving homeomorphisms of compact manifolds. Therefore, a Sobolev version of this theorem could be useful in the study of dynamical properties in the Sobolev class. Indeed, homeomorphisms on the Sobolev class gain significance presently in applications to certain type of PDE's in nonlinear elasticity (see the Ball-Evans Problem in [7]), in ergodic theory (genericity of infinite topological entropy in [5, 4] and also subjects correlated with the closing lemma (see again [5]). We believe that building bridges connecting these two areas could be of utmost interest both in applications and fundamental mathematics.

This paper is organized as follows. In Section 2 we introduce the space of automorphisms and the space of Sobolev-$(1,p)$ volume preserving homeomorphisms. In Section 3 we prove the perturbation result (Theorem 3.1). Finally, in Section 4, we prove a weak version of Lusin's theorem (Theorem A) for the space of Sobolev-$(1,p)$ volume preserving homeomorphisms, for $p < n-1$ and we present a counterexample (see Example 4.1) for $p > n$.

2. Preliminaries

Throughout the article X is a smooth closed connected Riemannian manifold of dimension n and d is the geodesic distance on X. We denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. We shall denote by λ the volume measure on both X and $\mathbb{R}^n$.

2.1. Automorphisms and homeomorphisms of (X, λ)

An *automorphism* of the underlying Borel measure space (X, λ) is a bijection $g\colon X \to X$ such that both g and g^{-1} are measurable functions and $\lambda(B) = \lambda(g(B)) = \lambda(g^{-1}(B))$ for all measurable sets B. Automorphisms which differ on a set of measure zero will be identified. We denote by $\mathcal{G}(X)$ the space of automorphisms of (X, λ). We shall consider two topologies on $\mathcal{G}(X)$: the *weak topology* given by the metric $\rho(f, g) = \inf\{\delta : \lambda\{x : d(f(x), g(x)) \geq \delta\} < \delta\}$, and the *uniform topology* defined by the metric $\|f - g\|_\infty \equiv \sup \text{ess}_{x \in X} d(f(x), g(x))$. The space $\mathcal{G}(X)$ is topologically complete with the weak topology (see [6]) and complete with the uniform topology. Thus, with each of these topologies, $\mathcal{G}(X)$ is a Baire space. We denote by $\mathcal{M}(X)$ the subspace of all homeomorphisms in $\mathcal{G}(X)$, endowed with the uniform topology. This space is topologically complete (see [11]). We shall call *volume preserving homeomorphisms* of X the elements in $\mathcal{M}(X)$.

2.2. Sobolev maps

Let Ω be an open bounded subset of $\mathbb{R}^n$ with Lipschitz boundary and let $1 \leq p \leq \infty$. Given a set $A \subseteq \mathbb{R}^n$ and $\delta > 0$ we denote by $V_\delta(A)$ the set $\{x \in \mathbb{R}^n : \inf_{a \in A} |x - a| < \delta\}$.

Recall that a measurable map $f = (f_1, \ldots, f_n)\colon \Omega \to \mathbb{R}^n$ is in the Sobolev class $W^{1,p}(\Omega, \mathbb{R}^n)$ if, for all $i = 1, \ldots, n$, f_i and all its distributional partial derivatives $\partial f_i / \partial x_j$ are in $L^p(\Omega)$.

We endow $W^{1,p}(\Omega, \mathbb{R}^n)$ with the norm defined by

$$\|f\|_{1,p} = \|f\|_p + \|Df\|_p, \quad \forall f \in W^{1,p}(\Omega, \mathbb{R}^n)\,,$$

where $\|f\|_p = \max_i \|f_i\|_p$ and $\|Df\|_p = \max_{i,j} \left\| \frac{\partial f_i}{\partial x_j} \right\|_p$.

We shall be interested only on Sobolev maps that are *continuous* up to the boundary. More precisely, we will consider the space

$$W^{1,p}(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)\,.$$

The (natural) norm in this space is equivalent to the one defined by

$$\|f\|_\infty + \|Df\|_p \,,$$

since $C^0(\overline{\Omega}, \mathbb{R})$ is compactly included in $L^p(\Omega)$.

Remark 2.1. *If $p > n$ then $W^{1,p}(\Omega, \mathbb{R}^n) \subseteq C^0(\overline{\Omega}, \mathbb{R}^n)$ and the norms on $W^{1,p}(\Omega, \mathbb{R}^n)$ defined above are equivalent.*

Finally we define the Sobolev space we are going to work with.

Definition 2.1. *We define $\mathbb{W}^{1,p}_\lambda(\Omega)$ as the set of all volume preserving homeomorphisms $f \colon \Omega \to \Omega$ such that $f \in W^{1,p}(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$. In this space we consider the natural metric defined by*

$$d_{\mathbb{W}^{1,p}_\lambda(\Omega)}(f,g) = \|f-g\|_\infty + \|D(f-g)\|_p,$$

for $f, g \in \mathbb{W}^{1,p}_\lambda(\Omega)$. We shall call Sobolev-$(1,p)$ volume preserving homeomorphisms of Ω the elements in $\mathbb{W}^{1,p}_\lambda(\Omega)$.

For simplicity we will denote $d_{\mathbb{W}^{1,p}_\lambda(\Omega)}(f,g)$ by $\|f-g\|_{\infty;1,p}$.

Since $\mathcal{M}_\lambda(\overline{\Omega})$ is topologically complete and $W^{1,p}(\Omega, \mathbb{R}^n) \cap C^0(\overline{\Omega}, \mathbb{R}^n)$ is complete, we have the following.

Proposition 2.1. *$\mathbb{W}^{1,p}_\lambda(\Omega)$ is a Baire space.*

Finally, we define a similar space for the manifold X. We denote by $\mathbb{W}^{1,p}_\lambda(X)$ the space of volume preserving homeomorphisms on X which in all local charts are Sobolev-$(1,p)$ maps.

3. A key perturbation theorem: ellipsoids

In this section we prove a key perturbation result which is the main ingredient to prove a volume preserving Sobolev weak Lusin theorem. Let $I^n = [0,1]^n$ stand for the n-dimensional unit cube.

Theorem 3.1. *Let $n \geq 3$, $\varepsilon > 0$ and $N \in \mathbb{N}$. Let $\{P_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$ be two sets of N distinct interior points of I^n such that $|P_i - Q_i| < \varepsilon$, for all i.*

If $p < n-1$, then, for all $k \in \mathbb{N}$ there is $F_k \in \mathbb{W}_\lambda^{1,p}(I^n)$ such that, for all i, $F_k(P_i) = Q_i$, F_k sends a neighbourhood of P_i by translation onto a neighbourhood of Q_i, and

$$\sup_k \|F_k - Id\|_\infty < \varepsilon, \qquad \lim_{k\to\infty} \|F_k - Id\|_{1,p} = 0.$$

As a consequence, for k large enough, $\|F_k - Id\|_{\infty;1,p} < \varepsilon$.

In addition, the functions F_k are $\mathbb{C}^\infty$ diffeomorphisms and equal to the identity in a neighbourhood of the boundary of I^n.

Remark 3.1. *Notice that, if $p > n$, the conclusion of this theorem is false unless $P_i = Q_i$, for all i. In fact, taking in mind Remark 2.1, if $\lim_{k\to\infty} \|F_k - Id\|_{1,p} = 0$ then also $\lim_{k\to\infty} \|F_k - Id\|_\infty = 0$. Hence, $\max_{i=1,\dots,N} |Q_i - P_i| = 0$.*

The rest of this section is devoted to the proof of Theorem 3.1.

Let $a, b, \mu > 0$ with $a \geq b$ and $0 < \mu \leq 1$ and consider the ellipsoids

$$\Sigma_{a,b} = \left\{ x \in \mathbb{R}^n : \left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^n \left(\frac{x_i}{b}\right)^2 \leq 1 \right\}, \quad \Sigma_{a,b,\mu} = \Sigma_{(1-\mu)a,(1-\mu)b}.$$

Notice that $\lambda(\Sigma_{a,b}) = w_n\, a\, b^{n-1}$, where w_n is the volume of the unitary sphere in $\mathbb{R}^n$, and $\mu\,\lambda(\Sigma_{a,b}) \leq \lambda(\Sigma_{a,b} \setminus \Sigma_{a,b,\mu}) \leq n\,\mu\,\lambda(\Sigma_{a,b})$.

For each a, b, μ, we will define a volume preserving $\mathbb{C}^\infty$ diffeomorphism of $\mathbb{R}^n$, $F_{a,b,\mu}$, that is rigid in $\Sigma_{a,b,\mu}$ and is equal to the identity outside $\Sigma_{a,b}$. In order to do that, we first consider a function $h_\mu \in C^\infty(\mathbb{R})$, strictly decreasing in $]1-\mu, 1[$ and constant in $]-\infty, 1-\mu]$ and in $[1, +\infty[$. We let h_μ be defined by $h_\mu(t) = \frac{\pi\, e^{\frac{\mu}{t-1}}}{e^{\frac{\mu}{t-1}} + e^{\frac{\mu}{1-t-\mu}}}$, if $1-\mu < t < 1$, $h_\mu(t) = 0$, if $t \geq 1$ and $h_\mu(t) = \pi$, if $t \leq 1-\mu$.

Lemma 3.2. *In the above conditions, $h_\mu \in C^\infty(\mathbb{R})$ and $\left|h'_\mu(t)\right| \leq \frac{2\pi}{\mu}$, for all $t \in \mathbb{R}$.*

Proof. Since $h_1(\frac{1}{\mu}(t-(1-\mu))) = h_\mu(t)$, $t \in \mathbb{R}$, it is enough to show the result for $\mu = 1$. But

$$\begin{aligned}
\tfrac{1}{\pi}\,|h_1'(t)| &= \frac{e^{-\frac{1}{t}} \cdot e^{\frac{1}{t-1}}}{\left(e^{-\frac{1}{t}} + e^{\frac{1}{t-1}}\right)^2} \left(\frac{1}{t^2} + \frac{1}{(t-1)^2}\right) \\
&= \underbrace{\frac{e^{-\frac{1}{t}}}{t^2}}_{\leq 4e^{-2}} \left(\frac{e^{\frac{1}{t-1}}}{\left(e^{-\frac{1}{t}} + e^{\frac{1}{t-1}}\right)^2}\right) + \underbrace{\frac{e^{\frac{1}{t-1}}}{t^2}}_{\leq 4e^{-2}} \left(\frac{e^{-\frac{1}{t}}}{\left(e^{-\frac{1}{t}} + e^{\frac{1}{t-1}}\right)^2}\right) \\
&\leq \frac{4e^{-2}}{e^{-\frac{1}{t}} + e^{\frac{1}{t-1}}}
\end{aligned}$$

and the conclusion follows, since $e^{-\frac{1}{t}} + e^{\frac{1}{t-1}} \geq 2e^{-\frac{1}{2t} + \frac{1}{2(t-1)}} \geq 2e^{-2}$, remembering that the geometric mean of two numbers is less than or equal to their arithmetic mean. □

Consider now the function $F_{a,b,\mu} : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$F_{a,b,\mu}(x) = \left(x_1 \cos(\alpha(x)) - \tfrac{a}{b}\, x_2 \sin(\alpha(x)), \tfrac{b}{a}\, x_1 \sin(\alpha(x)) + x_2 \cos(\alpha(x)), \bar{x}\right)$$

where $\bar{x} = (x_3, \ldots, x_n)$ and $\begin{array}{rccc} \alpha : & \mathbb{R}^n & \longrightarrow & \mathbb{R}\,. \\ & x & \mapsto & h_\mu\left(\sqrt{\left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^n \left(\frac{x_i}{b}\right)^2}\right) \end{array}$

To be precise, we should write $\alpha_{a,b,\mu}$, but we choose to drop the subscripts as it will be clear in the context.

Note that $\left|\frac{\partial \alpha}{\partial x_1}\right| \leq \frac{2\pi}{\mu\, a}$ and $\left|\frac{\partial \alpha}{\partial x_i}\right| \leq \frac{2\pi}{\mu\, b}$, if $i \geq 2$.

Lemma 3.3. *In the above conditions, $F_{a,b,\mu}$ is a volume preserving $\mathbb{C}^\infty$ diffeomorphism, is equal to the identity in $\mathbb{R}^n \setminus \Sigma_{a,b}$ and $F_{a,b,\mu}(x_1, x_2, \ldots, x_n) = (-x_1, -x_2, x_3, \ldots, x_n)$ in $\Sigma_{a,b,\mu}$.*

In addition, denoting $F_{a,b,\mu}$ by $(F_1, \ldots, F_n)$, we have

$$\forall x \in \mathbb{R}^n \quad \left|\frac{\partial F_i}{\partial x_j}(x)\right| \leq \frac{a}{b} \cdot \frac{6\pi}{\mu}. \tag{1}$$

Proof. If $x \in \Sigma_{a,b}$ then $\alpha(x) = \alpha(F_{a,b,\mu}(x))$ since

$$\frac{\left(x_1 \cos(\alpha(x)) - \frac{a}{b}\, x_2 \sin(\alpha(x))\right)^2}{a^2} + \frac{\left(\frac{b}{a}\, x_1 \sin(\alpha(x)) + x_2 \cos(\alpha(x))\right)^2}{b^2} = \frac{x_1^2}{a^2} + \frac{x_2^2}{b^2}.$$

Therefore, $F_{a,b,\mu}^{-1}$ is obtained by replacing $\alpha(x)$ by $-\alpha(x)$ in $F_{a,b,\mu}$.

The function $F_{a,b,\mu}$ preserves the volume because the determinant of its jacobian matrix is equal to the determinant of the jacobian matrix of (F_1, F_2), which can easily be seen to be equal to 1. Indeed, for this calculation, we do not need to use the expression of α but only that it is a function of $\left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^n \left(\frac{x_i}{b}\right)^2$.

The conditions on the partial derivatives of $F_{a,b,\mu}$ are simply a consequence of the hypotheses on a, b, μ and of the inequalities $|\cos(\alpha(x))|, |\sin(\alpha(x))| \leq 1$, $|x_1| \leq a$, $|x_i| \leq b$, for $i \geq 2$. $\square$

A similar result can be obtained for ellipsoids in general position in the space.

Corollary 3.4. *Let $a, b, \mu \in \mathbb{R}$, with $a \geq b > 0$, $0 < \mu < 1$. If $F_{a,b,\mu,T} = T \circ F_{a,b,\mu} \circ T^{-1}$, where $T : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation, there exists a constant C_0, independent of a, b, μ, such that*

$$\forall x \in \mathbb{R}^n \ \forall i, j = 1, \dots, n \quad \left| \frac{\partial \left(F_{a,b,\mu,T}\right)_i}{\partial x_j}(x) \right| \leq C_0 \frac{a}{b\mu}. \tag{2}$$

Therefore, if $a' \geq b' > 0$, $0 < \mu' < 1$, $T' : \mathbb{R}^n \to \mathbb{R}^n$ is a rotation and $G = F_{a',b',\mu',T'} \circ F_{a,b,\mu,T}$ then

$$\forall x \in \mathbb{R}^n \ \forall i, j = 1, \dots, n \quad \left| \frac{\partial G_i}{\partial x_j}(x) \right| \leq C_1 \frac{a}{b\mu} \frac{a'}{b'\mu'}, \tag{3}$$

where $C_1 = n\, C_0^2$.

The following results will provide important ingredients for the proof of Theorem 3.1.

Proposition 3.5. *Let $n \geq 3$, $\varepsilon > 0$ and P, Q be two points in the interior of I^n such that $|P - Q| < \varepsilon$. Let $\delta > 0$ be such that $U = V_\delta(\overline{PQ})$ is compactly included in the interior of I^n. If $p < n - 1$, then, for all $k \in \mathbb{N}$ there exists $F_k = F_{k,P,Q} \in \mathbb{W}_\lambda^{1,p}(I^n)$ such that $F_k(P) = Q$, F_k sends a neighbourhood of P by a rotation of angle π onto a neighbourhood of Q, is the identity outside U and*

$$\sup_k \|F_k - Id\|_\infty < \varepsilon, \qquad \lim_{k \to \infty} \|F_k - Id\|_{1,p} = 0.$$

In addition, the functions F_k are $\mathbb{C}^\infty$ diffeomorphisms.

Proof. If $P = Q$ we choose $F_k = Id$. Assume otherwise. The properties that we want to prove are invariant relative to the choice of axis. Therefore, we can assume that the origin is the middle point of the line segment $\overline{PQ}$ and that there exists $c > 0$ such that $P = (-c, 0, \ldots, 0)$ and $Q = (c, 0, \ldots, 0)$. Consider $c < a < \varepsilon/2$, $0 < b \leq a$ such that the ellipsoid $\Sigma_{a,b}$ is contained in U. Choose $\mu > 0$ such that $(1-\mu)a > c$ and therefore $P, Q \in \Sigma_{a,b,\mu}$. We fix a and μ and so we drop them from the indexes, denoting $F_b = F_{a,b,\mu}$. Of course we have $\|F_b - Id\|_\infty \leq 2a < \varepsilon$. We will prove that $\|F_b - Id\|_{1,p}$ converges to 0 when b tends to 0. Notice that $\|F_b - Id\|_p = \|F_b - Id\|_{L^p(\Sigma_{a,b})} \leq \varepsilon\left(\lambda(\Sigma_{a,b})\right)^{1/p} = \varepsilon(w_n\, a\, b^{n-1})^{1/p}$. For the partial derivative we have, by Lemma 3.3,

$$\begin{aligned}
\left\|\frac{\partial(F_b - Id)_i}{\partial x_j}\right\|_{L^p(I^n)} &\leq \left\|\frac{\partial\,(F_b)_i}{\partial x_j}\right\|_{L^p(\Sigma_{a,b})} + \|1\|_{L^p(\Sigma_{a,b})} \\
&\leq \tfrac{a}{b}\tfrac{6\pi}{\mu}\lambda(\Sigma_{a,b})^{1/p} + \lambda(\Sigma_{a,b})^{1/p} \\
&\leq \tfrac{6\pi\, a}{\mu b}\cdot\left(w_n a b^{n-1}\right)^{1/p} + (w_n a b^{n-1})^{1/p}.
\end{aligned}$$

Then

$$\|F_b - Id\|_{1,p} \leq \varepsilon(w_n\, a\, b^{n-1})^{1/p} + n^2\left(\tfrac{6\pi\, a}{\mu b}\cdot\left(w_n a b^{n-1}\right)^{1/p} + (w_n a b^{n-1})^{1/p}\right)$$

and the conclusion follows since $\frac{n-1}{p} > 1$. □

One can prove that, if $p \geq n-1$ then the construction in previous proposition does not work. Of course we only need to consider $p = n-1$. In this case if (for example)

$$U = \left\{x \in \mathbb{R}^n : 1 - \frac{\mu}{2} \leq \sqrt{\left(\frac{x_1}{a}\right)^2 + \sum_{i=2}^{n}\left(\frac{x_i}{b}\right)^2} \leq 1 - \frac{\mu}{4},\ x_1, \ldots, x_n \geq 0\right\}$$

then there exists C, independent of a, b, μ such that $\left\|\frac{\partial(F - Id)_1}{\partial x_n}\right\|_{L^p(U)} \geq C\,\frac{a^n}{\mu^{n-2}}$. The key ingredients when we evaluate the integral are the monotonicity of the derivative of h_μ in $]1-\frac{\mu}{2}, 1[$ and the fact that $h_\mu(1-\frac{\mu}{4})$ and $h_\mu(1-\frac{\mu}{2})$ do not depend on μ.

Next result is an upgrade of Proposition 3.5 towards the proof of Theorem 3.1.

Proposition 3.6. *Let $n \geq 3$, $\varepsilon > 0$, P, Q, R be distinct interior points of I^n with $|P-Q|, |P-R|, |Q-R| < \varepsilon$. Consider $\delta > 0$ such that $U = V_\delta(\overline{PR} \cup \overline{RQ})$ is compactly included in the interior of I^n and diam$(U) < \varepsilon$. Let $p < n-1$.*

The function $H_k = F_{k,R,Q} \circ F_{k,P,R}$, where $F_{k,R,Q}, F_{k,P,R}$ are given by Proposition 3.5 satisfies $H_k(P) = Q$, sends a neighbourhood of P by translation onto a neighbourhood of Q, is equal to the identity in $\mathbb{R}^n \setminus U$, and

$$\sup_k \|H_k - Id\|_\infty < \varepsilon, \qquad \lim_{k\to\infty} \|H_k - Id\|_{1,p} = 0.$$

Proof. Let $c = \frac{|P-R|}{2}$, $c' = \frac{|Q-R|}{2}$ and a, a' be such that $c < a < c + \delta$, $c' < a' < c' + \delta$, $a, a' < \varepsilon/2$. Consider $T, T' : \mathbb{R}^n \to \mathbb{R}^n$ rotations such that $T(-c, 0, \ldots, 0) = P$, $T(c, 0, \ldots, 0) = R$, $T'(-c', 0, \ldots, 0) = R$, $T'(c', 0, \ldots, 0) = Q$.

Consider $0 < \mu, \mu' < 1$ such that $(1-\mu)a > c$ and $(1-\mu')a' > c'$ and, for $0 < b, b' < \delta$, the ellipsoids $E_{a,b} = T(\Sigma_{a,b})$ and $E_{a',b'} = T'(\Sigma_{a',b'})$, which are contained in U.

Then, since a, a', μ, μ', T, T' are fixed we will drop them from the indexes. If $F_b = F_{a,b,\mu,T}$ and $G_{b'} = F_{a',b',\mu',T'}$ and $H_{b,b'} = G_{b'} \circ F_b$, we have $\|H_{b,b'} - Id\|_\infty < \mathrm{diam}(U) < \varepsilon$ and

$$\|H_{b,b'} - Id\|_{L^p(I^n)} \leq \|2\varepsilon\|_{L^p(E_{a,b} \cup E_{a',b'})} = 2\varepsilon \lambda(E_{a,b} \cup E_{a',b'})^{1/p}.$$

Hence, using Corollary 3.4 and noticing that $H_{b,b'} = G_{b'}$ outside $E_{a,b}$, we obtain

$$\begin{aligned}
\left\| \frac{\partial (H_{b,b'} - Id)_i}{\partial x_j} \right\|_{L^p(I^n)} &\leq \left\| \frac{\partial (H_{b,b'})_i}{\partial x_j} \right\|_{L^p(E_{a,b} \cup E_{a',b'})} + \|1\|_{L^p(E_{a,b} \cup E_{a',b'})} \\
&\leq \left\| \frac{\partial (H_{b,b'})_i}{\partial x_j} \right\|_{L^p(E_{a,b})} + \left\| \frac{\partial (G_{b'})_i}{\partial x_j} \right\|_{L^p(E_{a',b'} \setminus E_{a,b})} + \|1\|_{L^p(E_{a,b} \cup E_{a',b'})} \\
&\leq C_1 \frac{a}{b\mu} \frac{a'}{b'\mu'} \lambda(E_{a,b})^{1/p} + C_0 \frac{a'}{b'\mu'} \lambda(E_{a',b'})^{1/p} + \lambda(E_{a,b} \cup E_{a',b'})^{1/p} \\
&\leq \frac{C_1 a a'}{b b' \mu \mu'} \left(w_n \, a \, b^{n-1} \right)^{1/p} + \frac{C_0 a'}{b'\mu'} \left(w_n \, a' \, b'^{n-1} \right)^{1/p} + \lambda(E_{a,b} \cup E_{a',b'})^{1/p}.
\end{aligned}$$

Choosing s such that $0 < s < \frac{n-1}{p} - 1$ and $b' = b^s$ then the conclusion follows letting b converge to 0.

Since, by Proposition 3.5, F_b restricted to a neighbourhood of P and $G_{b'}$ restricted to a neighbourhood of R are rotations of angle π on the plane defined by P, R and Q, then $H_{b,b'} = G_{b'} \circ F_b$ is locally a translation around P. $\square$

We can now prove Theorem 3.1.

PROOF. (of Theorem 3.1) Let $\mathcal{P} = \{P_i, Q_i : i = 1, \dots, N\}$ and $J = \{i : P_i \neq Q_i\}$. Let $\{R_i\}_{i \in J}$ be a set of distinct points of the interior of I^n not in $\mathcal{P}$ such that $\overline{P_i R_i} \cap \overline{P_j R_j} = \overline{R_i Q_i} \cap \overline{R_j Q_j} = \emptyset$, if $i \neq j$, $|P_i - R_i|, |R_i - Q_i| < \varepsilon/2$ and

$$\overline{P_i R_i} \cap \overline{R_j Q_j} = \begin{cases} R_i & \text{if } i = j \\ P_i & \text{if } P_i = Q_j \\ \emptyset & \text{otherwise}\,. \end{cases}$$

This can be done by a step by step argument. To avoid overweight of notation, we suppose that $J = I$. First we choose R_1 in a small ball around the midpoint $\frac{P_1+Q_1}{2}$ and such that $(\overline{P_1 R_1} \cup \overline{R_1 Q_1}) \cap \mathcal{P} = \{P_1, Q_1\}$. Then, for $i = 2, \dots, N$, we choose R_i in a small ball around the midpoint $\frac{P_i+Q_i}{2}$ and such that $(\overline{P_i R_i} \cup \overline{R_i Q_i}) \cap (\mathcal{P} \cup \bigcup_{j=1}^{i-1} \overline{P_j R_j} \cup \overline{R_j Q_j}) = \{P_i, Q_i\}$.

Let $\delta > 0$ be such that, if L_1, L_2 are two line segments, each one of the form $\overline{P_i R_i}$ or $\overline{R_j Q_j}$, then $V_\delta(L_1) \cap V_\delta(L_2) = \emptyset$ if and only if $L_1 \cap L_2 = \emptyset$ (see Figure 1).

For each $i \in I$, consider ellipsoids E_i and $\tilde{E}_i$ such that $\overline{P_i R_i} \subseteq E_i \subseteq V_\delta(\overline{P_i R_i})$ and $\overline{R_i Q_i} \subseteq \tilde{E}_i \subseteq V_\delta(\overline{R_i Q_i})$.

Let $G_{k,i} = F_{k,P_i,R_i}$ and $H_{k,i} = F_{k,R_i,Q_i}$ be given by Proposition 3.5, if $i \in I$, and $G_i = H_i = Id$, otherwise.

Define $G_k = G_{k,1} \circ \cdots \circ G_{k,N}$, $H_k = H_{k,1} \circ \cdots \circ H_{k,N}$ and $F_k = H_k \circ G_k$. We have that,

$$F_k - Id = \sum_{i,j} \chi_{E_i \cap G_{k,i}^{-1}(\tilde{E}_j)} \left(H_{k,j} \circ G_{k,i} - Id\right) + \sum_i \chi_{E_i \setminus \left(\cup_j G_{k,i}^{-1}(\tilde{E}_j)\right)} \left(G_{k,i} - Id\right) + \sum_j \chi_{\tilde{E}_j \setminus (\cup_i E_i)} \left(H_{k,j} - Id\right)\,.$$

Notice that, when applying the function $F_k - Id$ to a point, at most one of the terms is non-zero. Then, by the choice of the points R_i, we have that

$$\|F_k - Id\|_\infty = \max_{i,j}\{\|H_{k,j} \circ G_{k,i} - Id\|_\infty, \|G_{k,i} - Id\|_\infty, \|H_{k,j} - Id\|_\infty\} < \varepsilon\,.$$

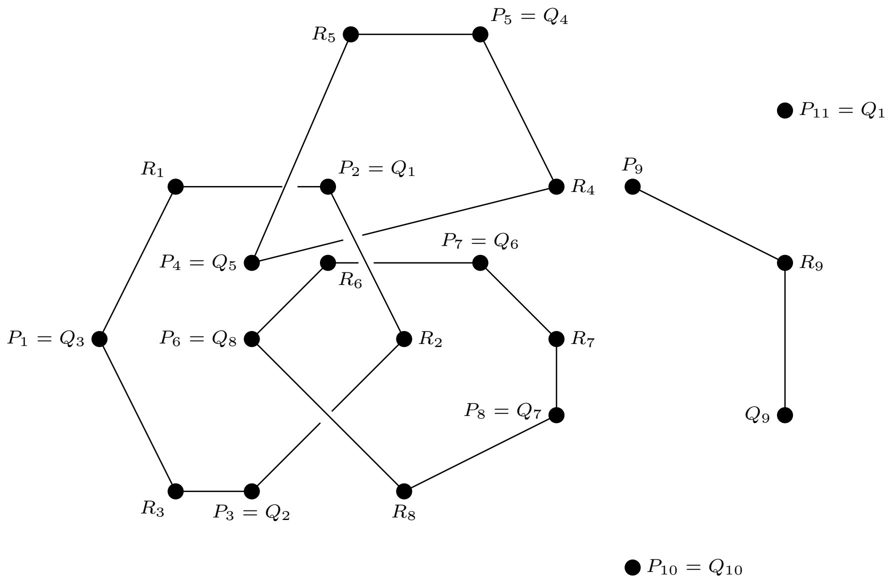

Figure 1: An illustrative example of the setting of Theorem 3.1.

Furthermore,

$$\begin{aligned}\|F_k - Id\|_{1,p} \leq & \sum_{i,j} \|H_{k,j} \circ G_{k,i} - Id\|_{1,p} \\ & + \sum_{i} \|G_{k,i} - Id\|_{1,p} + \sum_{j} \|H_{k,j} - Id\|_{1,p}.\end{aligned}$$

Hence, the conclusion follows from the previous proposition. □

4. Volume preserving Sobolev weak Lusin theorem

In this section we prove a volume preserving weak Lusin theorem for the Sobolev class $\mathbb{W}^{1,p}_{\lambda}(X)$, for $p < n-1$ and show that if $p > n$ the result is not true.

Theorem A (Volume preserving Sobolev weak Lusin theorem). *Let X be a closed connected n-dimensional manifold, $n \geq 3$. Let $\varepsilon > 0$ and $g \in \mathcal{G}(X)$ with $\|g - Id\|_\infty < \varepsilon$.*

Let $1 \leq p < n-1$. Then given any weak topology neighbourhood $\mathcal{W}$ of g, there exists $f \in \mathbb{W}^{1,p}_\lambda(X)$ such that $f \in \mathcal{W}$ and $\|f - Id\|_{\infty;1,p} < \varepsilon$.

In addition, $f \in \mathbb{W}^{1,\infty}_\lambda(X)$.

Since any smooth closed connected n-manifold can be obtained from I^n by making boundary identifications, it follows, using a Moser's result [8], that the proof of Theorem A reduces to the proof on the unit cube. For this, we will follow the strategy of the proof of [3, Theorem 6.2], adapting it to the Sobolev setting. As a key step, we will prove that a dyadic permutation of I^n can be approximated by a Sobolev-$(1,p)$ volume preserving homeomorphism of I^n, in the sense described in Theorem 4.1.

We recall that a *dyadic permutation of I^n of order m* is a bijection $\mathscr{P}\colon I^n \to I^n$ which permutes by simple translation the dyadic open cubes that are products of intervals of the form $]k/2^m, (k+1)/2^m[$.

Theorem 4.1. *Let $\varepsilon > 0$ and $\mathscr{P}$ be a dyadic permutation of the cube I^n, $n \geq 3$, with $\|\mathscr{P} - Id\|_\infty < \varepsilon$.*

Let $1 \leq p < n-1$. Then given any $\gamma > 0$, there is $f \in \mathbb{W}^{1,p}_\lambda(I^n)$, with $\|f - Id\|_{\infty;1,p} < \varepsilon$, and equal to the identity on a neighbourhood of the boundary, satisfying

$$\lambda\{x\colon \mathscr{P}(x) \neq f(x)\} < \gamma\,.$$

In addition, $f \in \mathbb{W}^{1,\infty}_\lambda(I^n)$.

Proof. Without loss of generality we can suppose that $\mathscr{P}$ is a permutation of dyadic cubes σ_i, $i = 1, \ldots, N$, with diameter less than $(\varepsilon - \|\mathscr{P} - Id\|_\infty)/3$. For $0 < \beta < 1$, we denote by σ_i^β the cubes concentric to σ_i with parallel faces and such that $\lambda(\sigma_i^\beta) = \beta\,\lambda(\sigma_i)$. We denote by P_i the center of the cube σ_i and let $Q_i = \mathscr{P}(P_i)$, $i = 1, \ldots, N$. By hypothesis, $|P_i - Q_i| < \varepsilon$ for every i. Since $\|\mathscr{P} - Id\|_\infty < \frac{1}{3}\varepsilon + \frac{2}{3}\|\mathscr{P} - Id\|_\infty$, applying Theorem 3.1 to the sets $\{P_i\}_{i=1}^N$ and $\{Q_i\}_{i=1}^N$, we obtain a volume preserving C^∞ diffeomorphism of I^n equal to the identity on a neighbourhood of the boundary of I^n and

$$\|F - Id\|_{\infty;1,p} < \tfrac{1}{3}\varepsilon + \tfrac{2}{3}\|\mathscr{P} - Id\|_\infty. \tag{4}$$

Furthermore, since F sends a neighbourhood of P_i by translation onto a neighbourhood of Q_i, there exists $0 < \alpha < 1$ such that $F = \mathscr{P}$ on σ_i^α, for every i. Consequently, $\lambda\{x : \mathscr{P}(x) \neq F(x)\} \leq 1 - \alpha$. If $1 - \alpha < \gamma$, the Theorem is proved taking $f = F$. Otherwise, it is clearly enough to obtain a map f which coincides with $\mathscr{P}$ on the cubes σ_i^β, for some β such that $\beta > 1-\gamma$. Fix $\beta > 1-\gamma$. To obtain f we first define a map $T\colon I^n \to I^n$ (not volume preserving) which leaves each cube σ_i invariant in the following way: T sends σ_i^α radially onto σ_i^β, with constant Jacobian matrix $JT = \sqrt[n]{\frac{\beta}{\alpha}}.Id$, and sends $\sigma_i \setminus \sigma_i^\alpha$ radially onto $\sigma_i \setminus \sigma_i^\beta$, with constant Jacobian matrix $JT = \sqrt[n]{\frac{1-\beta}{1-\alpha}}.Id$.

Finally, we define the map $f := TFT^{-1}$. This map satisfies the following:

(i) $\lambda\{x\colon \mathscr{P}(x) \neq f(x)\} < \gamma$, by construction.

(ii) $Jf(x) = JF(T^{-1}(x))$, except in the boundary of σ_i, σ_i^α and σ_i^β. For this, just notice that $x \in \cup_i \operatorname{int}(\sigma_i^\beta)$ if and only if $f(T^{-1})(x)) \in \cup_i \operatorname{int}(\sigma_i^\alpha)$.

(iii) f is volume preserving. Since, by (ii), $Jf(x) = JF(T^{-1}(x))$, except in the boundary of σ_i, σ_i^α and σ_i^β, we have that $\det Jf(x) = 1$.

We will now obtain the control of $\|f - Id\|_{\infty;1,p}$. By Theorem 3.1, we have that $\|f - Id\|_\infty \leq \|F - Id\|_\infty + 2\|T - Id\|_\infty < \varepsilon$.

Furthermore, using (ii), the change of variables defined by T^{-1} and the fact that $F = \mathscr{P}$ in σ_i^α, we obtain that

$$\begin{aligned}
\sum_{j,k} \left\| \frac{\partial (f - Id)_k}{\partial x_j} \right\|_p^p &= \sum_{j,k} \int_{I^n} \left| \frac{\partial (F - Id)_k}{\partial x_j}(T^{-1}(x)) \right|^p \, dx \\
&= \sum_{j,k} \int_{I^n} \left| \frac{\partial (F - Id)_k}{\partial x_j}(x) \right|^p \det JT(x)\, dx \\
&= \sum_{j,k} \int_{I^n \setminus \cup \sigma_i^\alpha} \left| \frac{\partial (F - Id)_k}{\partial x_j}(x) \right|^p \det JT(x)\, dx \\
&= \frac{1-\beta}{1-\alpha} \sum_{j,k} \int_{I^n \setminus \cup \sigma_i^\alpha} \left| \frac{\partial (F - Id)_k}{\partial x_j}(x) \right|^p \, dx \\
&\leq \frac{1-\beta}{\gamma} \sum_{j,k} \left\| \frac{\partial (F - Id)_i}{\partial x_j} \right\|_p^p .
\end{aligned}$$

Hence, taking β large enough, the conclusion follows. □

We can now complete the proof of Theorem A.

Proof. (of Theorem A) Let $\varepsilon > 0$ and $g \in \mathcal{G}(I^n)$ with $\|g - Id\|_\infty < \varepsilon$. Let $\delta, \gamma > 0$. We will obtain $f \in \mathbb{W}^{1,p}_\lambda(I^n)$, with $\|f - Id\|_{\infty;1,p} < \varepsilon$ and equal to the identity on the boundary of I^n, satisfying $\lambda\{x: |g(x) - f(x)| \geq \delta\} < \gamma$.

The automorphism $g \in \mathcal{G}(I^n)$ can be weakly approximated by a dyadic permutation R with $\rho(g, R)$ small. This follows from the denseness of dyadic permutations in $\mathcal{G}(I^n)$, in the weak topology (see [2, §2] or [3, Lemma 6.4]).

Furthermore, the dyadic permutation R can be weakly approximated by another dyadic permutation $\mathscr{P}$ such that $\lambda\{x : |g(x) - \mathscr{P}(x)| \geq \delta\} < \gamma$ and $\|\mathscr{P} - Id\|_\infty < \varepsilon$. The technique for this approximation is described in the proof of [3, Theorem 6.2, p. 46].

Set $\gamma_0 := \gamma - \lambda\{x : |g(x) - \mathscr{P}(x)| \geq \delta\}$. Applying Theorem 4.1 to the permutation $\mathscr{P}$ we obtain a map $f \in \mathbb{W}^{1,p}_\lambda(I^n)$, with $\|f - Id\|_{\infty;1,p} < \varepsilon$, and equal to the identity on a neighbourhood of the boundary, satisfying $\lambda\{x\colon \mathscr{P}(x) \neq f(x)\} < \gamma_0$. Hence, $\lambda\{x: |g(x) - f(x)| \geq \delta\} < \gamma$. In addition, $f \in \mathbb{W}^{1,\infty}_\lambda(I^n)$. □

Example 4.1 (Counterexample to Theorem A for $p > n$). *Consider $p > n$ and g any continuous element of $\mathcal{G}(I^n)$ different from the identity. Let $\varepsilon_0 > 0$ and $x_0 \in I^n$ be such that*

$$|g(x_0) - x_0| = \|g - Id\|_\infty = \varepsilon_0.$$

For $k > 0$ let $D_k = \left\{x \in I^n : |x - x_0| < \frac{1}{k},\ |g(x) - g(x_0)| < \frac{1}{k}\right\}$. Notice that $\delta_k = \lambda(D_k) > 0$, since D_k is a non-empty open set.

If the weak Lusin Theorem was valid for $\mathbb{W}^{1,p}_\lambda(I^n)$ then, for $k \in \mathbb{N}$ there would exist $f_k \in \mathbb{W}^{1,p}_\lambda(I^n)$ such that

$$\|f_k - Id\|_{\infty;1,p} < \varepsilon_0 + \tfrac{1}{k},\ \lambda(A_k) < \delta_k,\ \text{where } A_k = \{x \in I^n : |f_k(x) - g(x)| \geq \delta_k\}.$$

By construction $D_k \not\subseteq A_k$. Consider $x_1 \in D_k \setminus A_k$. Then

$$\|f_k - Id\|_\infty \geq |f_k(x_1) - x_1| \geq |g(x_1) - x_1| - |g(x_1) - f_k(x_1)| \geq \varepsilon_0 - \tfrac{2}{k} - \delta_k.$$

Since $\|f_k - Id\|_{\infty;1,p} = \|f_k - Id\|_\infty + \|D(f_k - Id)\|_p$, we obtain $\|D(f_k - Id)\|_p \leq \frac{3}{k} + \delta_k$. As $f_k = Id$ on the boundary of I^n we conclude that the sequence $(f_k)_{k\in\mathbb{N}}$ converges to Id in $W^{1,p}(I^n)$. Moreover, as $p > n$ then

$(f_k)_{k\in\mathbb{N}}$ *converges to* Id *in* $C^0(I^n)$ *which is absurd because* $\|f_k - Id\|_\infty \geq \varepsilon_0 - \frac{2}{k} - \delta_k$.

We observe that, this example also shows that Theorem A is not valid for $\mathbb{C}^{0,\alpha}_\lambda(I^n)$, *for all* $0 < \alpha \leq 1$. *Indeed, as before, we obtain* $f_k \in \mathbb{C}^{0,\alpha}_\lambda(I^n)$ *such that*

$$\|f_k - Id\|_\infty \geq \varepsilon_0 - \tfrac{2}{k} - \delta_k, \quad \sup_{x\neq y} \frac{|(f_k - Id)(x) - (f_k - Id)(y)|}{|x-y|^\alpha} < \tfrac{3}{k} + \delta_k.$$

Choosing y *such that* $f_k(y) = y$ *we have* $|(f_k - Id)(x)| < \left(\frac{3}{k} + \delta_k\right)(diam(I^n))^\alpha$, *which is a contradiction.*

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References

[1] Steve Alpern, *Approximation to and by measure preserving homeomorphisms*, J. London Math. Soc., 18(2) (1978), 305–315.

[2] Steve Alpern and Robert D. Edwards, *Lusin's theorem for measure preserving homeomorphisms*, Mathematika, 26(1) (1979), 33–43.

[3] Steve Alpern and Vidhu S. Prasad, *Typical Dynamics of Volume Preserving Homeomorphisms*, Cambridge Tracts in Mathematics, 2000.

[4] Edson de Faria, Peter Hazard, Charles Tresser, *Infinite entropy is generic in Hölder and Sobolev spaces*, C. R. Acad. Sci. Paris, Ser. I, 355(11) (2017), 1185–1189.

[5] Edson de Faria, Peter Hazard, Charles Tresser, *Genericity of Infinite Entropy for Maps with Low Regularity*, Preprint ArXiv 2017.

[6] Paul R. Halmos, *Lectures on ergodic theory*, The Mathematical Society of Japan, 1956. Publications of the Mathematical Society of Japan, no. 3.

[7] Tadeusz Iwaniec, Leonid V. Kovalev and Jani Onnine, *Diffeomorphic approximation of Sobolev homeomorphisms.* Arch. Rational Mech. Anal., 201 (2011), 1047–1067.

[8] Jürgen Moser, *On the volume elements of a manifold*, Trans. Amer. Math. Soc., 120 (1965), 286–294.

[9] John C. Oxtoby and Stanislaw M. Ulam, *Measure-preserving homeomorphisms and metrical transitivity*, Ann. of Math., 42(4) (1941), 874–920.

[10] John C. Oxtoby, *Approximation by measure-preserving homeomorphisms*, Recent Advances in Topological Dynamics, Proc. Conf. in Topological Dynamics, Yale University, New Haven, Conn., 1972; in honor of Gustav Arnold Hedlund, (Lecture Notes in Mathematics, 318) Springer, Berlin, 1973, 206–217.

[11] John C. Oxtoby, *Measure and category. A survey of the analogies between topological and measure spaces*, volume 2 of Graduate Texts in Mathematics, Springer-Verlag, New York, second edition, 1980.

[12] H. E. White Jr., *The approximation of one-one measurable transformations by measure preserving homeomorphisms*, Proc. Amer. Math. Soc., 44 (1974), 391–394.