Reverse Order Law for the Core Inverse in Rings

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Abstract: In this paper, necessary and sufficient conditions of the one-sided reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, the two-sided reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ and $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$ for the core inverse are given in rings with involution. In addition, the mixed-type reverse order laws, such as $(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$, $a^{\oplus} = b(ab)^{\#}$ and $(ab)^{\#} = b^{\oplus}a^{\oplus}$, are also considered.

Keywords: core inverse; group inverse; reverse order law

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1 Introduction

Let $\mathbb{C}_{n\times n}$ denote the set of all $n \times n$ complex matrices. It is well known that $(AB)^{-1} = B^{-1}A^{-1}$, where $A, B \in \mathbb{C}_{n\times n}$ are invertible. The previous equality is called the reverse order law for the ordinary inverse. In general, the equality doesn't hold when the ordinary inverse is replaced by the generalized inverse. In 1966, Greville [13] first gave a necessary and sufficient condition of the reverse order law $(AB)^{\dagger} = B^{\dagger}A^{\dagger}$ for the Moore-Penrose inverse. Since then, many authors studied the reverse order law for various classes of generalized inverses in the setting of complex matrices, operators, and elements of rings with involution. For example, Deng [10] investigated some necessary and sufficient conditions of the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ for the group inverse of linear bounded operators on Hilbert spaces. In [23], Mosić and Djordjević extended the results of [10] to the ring case, giving some new conditions and providing simpler and more transparent proofs to already existing conditions. Recently, Mary [19] provided equivalent conditions for the two-sided reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ and $(ba)^{\#} = a^{\#}b^{\#}$ for the group inverse in semigroups and rings. In [20, 22, 24], Mosić

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et al. considered the mixed-type reverse order laws in rings, such as $(ab)^\dagger = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\# = b^\dagger(a^\dagger abb^\dagger)^\dagger a^\dagger$, $(ab)^\# = (a^\dagger ab)^\dagger a^\dagger$, and $(ab)^\# = b^\dagger a^\dagger$. More results on the reverse order law for the generalized inverse can be found in [3, 7–9, 11, 16–18, 21, 25, 29].

The following problem on the reverse order law for the core inverse was proposed by Baksalary and Trenkler [2]:

$$\text{If } A^{\oplus}, B^{\oplus}, \text{ and } (AB)^{\oplus} \text{ exist, does it follow that } (AB)^{\oplus} = B^{\oplus}A^{\oplus}?$$

This problem attracted researchers' attention. Later, Cohen, Herman and Jayaraman [6] gave several counterexamples for the problem. In [30], Wang and Liu obtained equivalent conditions of the reverse order $(AB)^{\oplus} = B^{\oplus}A^{\oplus}$ by the ranks of matrices. Next, we will continue to consider the reverse law for the core inverse in rings with involution.

The article is motivated by the papers [19, 22, 30]. We present some equivalent conditions for the one-sided reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, the two-sided reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ and $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$ for the core inverse in rings with involution. We also study the mixed-type reverse order laws, such as $(ab)^\# = b^{\oplus}(abb^{\oplus})^{\oplus}$, $a^{\oplus} = b(ab)^\#$ and $(ab)^\# = b^{\oplus}a^{\oplus}$.

2 Preliminaries

Throughout this paper, R denotes a unital $*$-ring, that is, a ring with unity 1 and an involution $a \mapsto a^*$ satisfying $(a^*)^* = a$, $(a+b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$.

For the readers' convenience, we first recall the definitions of some generalized inverses. An element $a \in R$ is said to be Moore-Penrose invertible with respect to the involution $*$ if the following equations

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (4)\ (xa)^* = xa$$

have a common solution [26]. Such solution is unique if it exists, and is denoted by $a^\dagger$.

The Drazin inverse [12] of $a \in R$ is the element $x \in R$ which satisfies

$$(1^k)\ a^k = a^{k+1}x, \quad (2)\ xax = x, \quad (5)\ ax = xa, \quad \text{for some} \quad k \geq 1.$$

The element x above is unique if it exists and is denoted by a^D. The smallest such k is called the index of a, and denoted by ind(a). In particular, when ind(a)=1, the Drazin inverse a^D is called the group inverse of a and it is denoted by $a^\#$.

Baksalary and Trenkler [1] introduced the core inverse for complex matrices. Let $A \in \mathbb{C}_{n\times n}$. A matrix $A^{\oplus} \in \mathbb{C}_{n\times n}$ is called core inverse of A if

$$\text{(i) } AA^{\oplus} = P_A \quad \text{and} \quad \text{(ii) } \mathcal{R}(A^{\oplus}) \subseteq \mathcal{R}(A),$$

where P_A is the orthogonal projector onto $\mathcal{R}(A)$, and $\mathcal{R}(A)$ is the column space of A.

Later, the notion of the core inverse for complex matrices was extended to the ring case by Rakić, Dinčić and Djordjević [28]. The core inverse [28] of $a \in R$ is the element $x \in R$ which satisfies

$$(1)\ axa = a, \quad (2)\ xax = x, \quad (3)\ (ax)^* = ax, \quad (6)\ xa^2 = a \quad (7)\ ax^2 = x.$$

The element x above is unique if it exists and is denoted by $a^{\textcircled{\#}}$.

Let $\delta = \{1, 2, 3, 4, 5, 6, 7\}$. If $x \in R$ satisfies the equations in (i) for all $i \in \delta$, then x is called a δ-inverse of a. The set of all δ-inverses of a is denoted by $a\{\delta\}$. For example, $a\{1, 2, 5\} = \{a^{\#}\}$. The element $a \in R$ is regular if $a\{1\} \neq \varnothing$. $R^{\#}$, $R^{\{1,3\}}$, $R^{\dagger}$, $R^{\textcircled{\#}}$ stand for the set of all group, {1,3}, Moore-Penrose, core invertible elements of R, respectively. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R.

Given $a \in R$, the following notations will be used:

$$a^0 = \{x \in R : ax = 0\} \text{ and } {}^0a = \{x \in R : xa = 0\}.$$

An element $a \in R$ is said to be EP if $a \in R^{\#} \cap R^{\dagger}$ and $a^{\#} = a^{\dagger}$. An element $a \in R$ satisfying $a^* = a$ is called Hermitian.

Now, we present several known characterizations for some generalized inverses, which play an important role in the sequel.

Lemma 2.1. [31, Theorem 3.1] *Let $a, x \in R$. Then $a \in R^{\textcircled{\#}}$ with $x = a^{\textcircled{\#}}$ if and only if*

$$(ax)^* = ax, \quad xa^2 = a \quad \text{and} \quad ax^2 = x.$$

Lemma 2.2. [31, Theorem 2.6] *Let $a \in R$. Then $a \in R^{\textcircled{\#}}$ if and only if $a \in R^{\#} \cap R^{\{1,3\}}$. In this case, $a^{\textcircled{\#}} = a^{\#}aa^{(1,3)}$, where $a^{(1,3)} \in a\{1, 3\}$.*

Lemma 2.3. [15, Theorem 1] *Let $a \in R$. Then $a \in R^{\#}$ if and only if $a = a^2x = ya^2$ for some $x, y \in R$. In this case, $a^{\#} = yax = y^2a = ax^2$.*

Lemma 2.4. [14, p. 201] *Let $a, x \in R$. Then x is a $\{1, 3\}$-inverse of a if and only if $a = x^*a^*a$.*

The following lemmas will also be useful.

Lemma 2.5. [4, Corollary 3.4] *Let $a, x \in R$ with $xa = ax$ and $xa^* = a^*x$. If $a \in R^{\textcircled{\#}}$, then $xa^{\textcircled{\#}} = a^{\textcircled{\#}}x$.*

Lemma 2.6. [5, Cline's formula] *Let $a, b \in R$. If ab is Drazin invertible, then ba is Drazin invertible and $(ba)^D = b((ab)^D)^2a$.*

Lemma 2.7. [28, Theorem 3.1] *Let $a \in R$. Then the following are equivalent:*
(i) *a is EP;*
(ii) *$a \in R^{\textcircled{\#}}$ and $a^{\#} = a^{\textcircled{\#}}$.*
(iii) *$a \in R^{\textcircled{\#}}$ and $aa^{\textcircled{\#}} = a^{\textcircled{\#}}a$.*
(iv) *$a \in R^{\#} \cap R^{\dagger}$ and $a^{\dagger} = a^{\textcircled{\#}}$*

Lemma 2.8. [28, Lemma 2.5, Lemma 2.6] *Let $a, b \in R$.*
(i) *If $aR \subseteq bR$, then ${}^{\circ}b \subseteq {}^{\circ}a$.*
(ii) *If b is regular and ${}^{\circ}b \subseteq {}^{\circ}a$, then $aR \subseteq bR$.*
(iii) *If $Ra \subseteq Rb$, then $b^{\circ} \subseteq a^{\circ}$.*
(iv) *If b is regular and $b^{\circ} \subseteq a^{\circ}$, then $Ra \subseteq Rb$.*

3 One-sided and Two-sided Reverse Order Laws

In this section, we will give some equivalent conditions of the one-sided $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, the two-sided reverse order laws $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ and $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$ for the core inverse in rings.

First, in order to prove our main results, we give new existence criterion for the core inverse as follows:

Theorem 3.1. *Let $a, x \in R$. Then the following are equivalent:*
(i) *$a \in R^{\oplus}$ and $x = a^{\oplus}$;*
(ii) *$axa = a$, $xR = aR$ and $Rx \subseteq Ra^*$;*
(iii) *$axa = a$, ${}^0x = {}^0a$ and $(a^*)^0 \subseteq x^0$;*
(iv) *$xax = x$, $xR = aR$ and $Rx = Ra^*$;*
(v) *$xax = x$, $xR = aR$ and $Ra^* \subseteq Rx$;*
(vi) *$xax = x$, ${}^0x = {}^0a$ and $x^0 \subseteq (a^*)^0$;*
(vii) *$a \in R^{\#}$, $axa = a$, $(ax)^* = ax$, and $xR \subseteq aR$;*
(viii) *$a \in R^{\#}$, $xax = x$, $(ax)^* = ax$, and $aR \subseteq xR$.*

Proof. (i) $\Rightarrow$ (ii) and (iv) $\Rightarrow$ (v) are trivial.

(ii) $\Rightarrow$ (iii) and (v) $\Rightarrow$ (vi) follow directly from Lemma 2.8.

(iii) $\Rightarrow$ (iv) Note that a^* is regular. Since $(a^*)^0 \subseteq x^0$, using Lemma 2.8, we have $Rx \subseteq Ra^*$, which implies that $x = t_1a^*$ for some $t_1 \in R$. Thus, we get

$$x = t_1a^* = t_1(axa)^* = t_1a^*x^*a^* = xx^*a^* = x(ax)^*.$$

Multiplying the previous equality by a from the left side, we obtain $ax = ax(ax)^*$, which gives that $ax = (ax)^*$. Therefore, we get $x = x(ax)^* = xax$.

From $a = axa = (ax)^*a = x^*a^*a$, it follows that $a^* = a^*ax$. So, $Ra^* \subseteq Rx$. Therefore, $Ra^* = Rx$. According to the condition ${}^0x = {}^0a$, we deduce that $xR = aR$ by Lemma 2.8.

(vi) $\Rightarrow$ (i) Clearly, we have $Ra^* \subseteq Rx$, which yields $a = x^*t_2$ for some $t_2 \in R$. So, we have

$$a = x^*t_2 = (xax)^*t_2 = x^*a^*x^*t_2 = x^*a^*a = (ax)^*a.$$

Hence, we get $ax = (ax)^*ax$, which immediately yields $ax = (ax)^*$. Then, $a = (ax)^*a = axa$.

On one hand, we can see that $x = xax = x(ax)^* = xx^*a^*$, which implies $Rx \subseteq Ra^*$, leading to $Rx = Ra^*$, since $Ra^* \subseteq Rx$. On the other hand, it is easy to get $xR = aR$. Finally, by the definition of the core inverse, we claim that $a \in R^{\oplus}$ and $x = a^{\oplus}$.

(i) $\Rightarrow$ (vii) and (i) $\Rightarrow$ (viii) can be obtained by Lemma 2.2 and the definition of core inverse.

(vii) $\Rightarrow$ (i) Note that $x \in a\{1, 3\}$ and $x = at_3$ for some $t_3 \in R$. From Lemma 2.2, it follows that $a \in R^{\oplus}$ and $a^{\oplus} = a^{\#}ax = a^{\#}aat_3 = at_3 = x$.

(viii) $\Rightarrow$ (i) This is analogous to the proof of (ii) $\Rightarrow$ (i). Indeed, there exists $r_4 \in R$ such that $a = xt_4$. Then, we have

$$axa = axaaa^{\#} = a(xax)t_4a^{\#} = a(xt_4)a^{\#} = a^2a^{\#} = a,$$

which gives that $x \in a\{1,3\}$, since $(ax)^* = ax$. Hence, $a \in R^{\oplus}$ with

$$a^{\oplus} = a^{\#}ax = aa^{\#}x = xt_4a^{\#}x = xa(xt_4)a^{\#}x = xa^2a^{\#}x = xax = x. \quad \square$$

Now, we state a sufficient condition for the reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ to hold, which extends [4, Theorem 3.5].

Theorem 3.2. *Let $a, b \in R^{\oplus}$, $ab \in R^{\#}$. If $abb^{\oplus} = bb^{\oplus}a$ and $baa^{\oplus} = aa^{\oplus}b$, then $ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.*

Proof. According to Theorem 3.1 (i) and (ii), it suffices to prove that $abb^{\oplus}a^{\oplus}ab = ab$, $b^{\oplus}a^{\oplus}R = abR$, and $Rb^{\oplus}a^{\oplus} \subseteq R(ab)^*$.

The condition $abb^{\oplus} = bb^{\oplus}a$ implies $a^*bb^{\oplus} = bb^{\oplus}a^*$. Then, from Lemma 2.5, it follows that $a^{\oplus}bb^{\oplus} = bb^{\oplus}a^{\oplus}$. Similarly, we obtain $b^{\oplus}aa^{\oplus} = aa^{\oplus}b^{\oplus}$. Hence, we get $abb^{\oplus}a^{\oplus}ab = aa^{\oplus}abb^{\oplus}b = ab$.

On one hand, we have

$$\begin{aligned} b^{\oplus}a^{\oplus} &= (b^{\oplus}aa^{\oplus})a^{\oplus} = aa^{\oplus}b^{\oplus}a^{\oplus} = a(a^{\oplus}bb^{\oplus})b^{\oplus}a^{\oplus} \\ &= abb^{\oplus}a^{\oplus}b^{\oplus}a^{\oplus} = ab(b^{\oplus}a^{\oplus})^2, \end{aligned}$$

which gives $b^{\oplus}a^{\oplus}R \subseteq abR$. On the other hand, combining $b^{\oplus}a^{\oplus} = ab(b^{\oplus}a^{\oplus})^2$ and $abb^{\oplus}a^{\oplus}ab = ab$, we get

$$\begin{aligned} ab &= (ab)^{\#}ab(b^{\oplus}a^{\oplus})(ab)^2 = (ab)^{\#}abab(b^{\oplus}a^{\oplus})^2(ab)^2 \\ &= (ab(b^{\oplus}a^{\oplus})^2)(ab)^2 = b^{\oplus}a^{\oplus}(ab)^2, \end{aligned}$$

which yields $abR \subseteq b^{\oplus}a^{\oplus}R$. Thus, we have $b^{\oplus}a^{\oplus}R = abR$.

Finally, by the following equalities

$$\begin{aligned} b^{\oplus}a^{\oplus} &= b^{\oplus}(bb^{\oplus}a^{\oplus})aa^{\oplus} = b^{\oplus}a^{\oplus}bb^{\oplus}aa^{\oplus} = b^{\oplus}a^{\oplus}(aa^{\oplus}bb^{\oplus})^* \\ &= b^{\oplus}a^{\oplus}(abb^{\oplus}a^{\oplus})^* = b^{\oplus}a^{\oplus}(b^{\oplus}a^{\oplus})^*(ab)^*, \end{aligned}$$

we have $Rb^{\oplus}a^{\oplus} \subseteq R(ab)^*$.

The proof is completed. $\square$

Proposition 3.3. [4, Theorem 3.5] *Let $a, b \in R^{\oplus}$ with $ab = ba$ and $ab^* = b^*a$. Then $ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.*

Proof. From $ab = ba$ and $ab^* = b^*a$, we have $ab^{\oplus} = b^{\oplus}a$ and $ba^{\oplus} = a^{\oplus}b$ by Lemma 2.5, which imply $abb^{\oplus} = bb^{\oplus}a$ and $baa^{\oplus} = aa^{\oplus}b$. In addition, we know that $ab \in R^{\#}$, since $a, b \in R^{\#}$ with $ab = ba$. Then, $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ by Theorem 3.2. $\square$

Remark 3.4. *We have seen that the condition of Proposition 3.3 can imply the condition of Theorem 3.2. But, in general, the condition of Theorem 3.2 does not imply the condition of Proposition 3.3. Indeed, take non-commutative invertible elements $a, b \in R$. Obviously, a and b satisfy the condition of Theorem 3.2. However, $ab \neq ba$.*

Next, we give an equivalent condition which ensures the reverse order law $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ holds.

Theorem 3.5. *Let $a, b \in R^{\oplus}$. Then the following are equivalent:*
(i) *$ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;*
(ii) *$ab \in R^{\#}$, $ab^{\oplus}R \subseteq b^{\oplus}a^{\oplus}R$, and $b^*a^{\#}(1-(abb^{\oplus}a^{\oplus})^*)a = 0$.*

Proof. (i) $\Rightarrow$ (ii) Obviously, $ab \in R^{\#}$. Note that

$$ab^{\oplus} = ab(b^{\oplus})^2 = b^{\oplus}a^{\oplus}(ab)^2(b^{\oplus})^2,$$

which implies $ab^{\oplus}R \subseteq b^{\oplus}a^{\oplus}R$. By Lemma 2.2, we have $a^{\oplus}a = a^{\#}aa^{(1,3)}a = a^{\#}a$, together with $b^* = b^*bb^{\oplus}$, we obtain

$$\begin{aligned} b^*aa^{\#} &= b^*bb^{\oplus}a^{\oplus}a^2a^{\#} = b^*bb^{\oplus}(a^{\oplus}a) = b^*b(ab)^{\oplus}a = b^*b(ab)^{\oplus}ab(ab)^{\oplus}a \\ &= b^*bb^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}a = (b^*bb^{\oplus})(a^{\oplus}a)bb^{\oplus}a^{\oplus}a = b^*a^{\#}abb^{\oplus}a^{\oplus}a \\ &= b^*a^{\#}(abb^{\oplus}a^{\oplus})^*a. \end{aligned}$$

Therefore, $b^*a^{\#}(1-(abb^{\oplus}a^{\oplus})^*)a = 0$.

(ii) $\Rightarrow$ (i) Observe that $a^{\oplus} = a^{\oplus}aa^{(1,3)}$ and $a^{\#}aa^{\oplus} = a^{\oplus}$ by Lemma 2.2. Then, applying the condition $b^*a^{\#}a = b^*a^{\#}(abb^{\oplus}a^{\oplus})^*a$, we obtain

$$\begin{aligned} b^*a^{\oplus} &= (b^*a^{\#}a)a^{\oplus} = b^*a^{\#}(abb^{\oplus}a^{\oplus})^*aa^{\oplus} = b^*a^{\#}(a^{\oplus})^*bb^{\oplus}a^*aa^{\oplus} \\ &= b^*a^{\#}(a^{\oplus})^*bb^{\oplus}a^* = b^*a^{\#}(a^{\oplus}aa^{(1,3)})^*bb^{\oplus}a^* \\ &= b^*a^{\#}aa^{(1,3)}(a^{\oplus})^*bb^{\oplus}a^* \\ &= b^*a^{\oplus}(a^{\oplus})^*bb^{\oplus}a^*, \end{aligned}$$

which yields $(a^{\oplus})^*b = abb^{\oplus}a^{\oplus}(a^{\oplus})^*b$. By the previous equality, we get

$$\begin{aligned} (a^{\oplus})^*(b^{\oplus})^* &= (a^{\oplus})^*(b^{\oplus}bb^{\oplus})^* = ((a^{\oplus})^*b)b^{\oplus}(b^{\oplus})^* = abb^{\oplus}a^{\oplus}(a^{\oplus})^*bb^{\oplus}(b^{\oplus})^* \\ &= abb^{\oplus}a^{\oplus}(a^{\oplus})^*(b^{\oplus})^* = abb^{\oplus}a^{\oplus}(b^{\oplus}a^{\oplus})^*, \end{aligned}$$

which gives $b^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}(abb^{\oplus}a^{\oplus})^*$. Then, $abb^{\oplus}a^{\oplus} = abb^{\oplus}a^{\oplus}(abb^{\oplus}a^{\oplus})^*$. So, we have $abb^{\oplus}a^{\oplus} = (abb^{\oplus}a^{\oplus})^*$. This immediately yields $b^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus}$.

The condition $ab^{\oplus}R \subseteq b^{\oplus}a^{\oplus}R$ ensures that $ab^{\oplus} = b^{\oplus}a^{\oplus}r$ for some $r \in R$. Hence, we get $ab = (ab^{\oplus})b^2 = b^{\oplus}a^{\oplus}rb^2$, which gives $abR \subseteq b^{\oplus}a^{\oplus}R$. From Theorem 3.1, it follows that $ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. □

In order to simplify the proof of the following theorems, we present a useful lemma.

Lemma 3.6. *Let $a, b, ab \in R^{\oplus}$ with $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. Then*
(i) *$ab = bb^{\oplus}ab = b^{\oplus}bab$;*
(ii) *$abR \subseteq baR$;*
(iii) *$ab(ab)^{\oplus}bb^{\oplus} = bb^{\oplus}ab(ab)^{\oplus} = ab(ab)^{\oplus}$;*
(iv) *$bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{3,6\}$.*

Proof. (i) Suppose that $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. Then, we have

$$ab = b^{\oplus}a^{\oplus}(ab)^2 = bb^{\oplus}(b^{\oplus}a^{\oplus}(ab)^2) = bb^{\oplus}ab,$$

and

$$ab = b^{\oplus}a^{\oplus}(ab)^2 = b^{\oplus}b(b^{\oplus}a^{\oplus}(ab)^2) = b^{\oplus}bab.$$

(ii) From (i), it follows that

$$ab = bb^{\oplus}ab = bb^{\oplus}a^{\oplus}a^2b = b(ab)^{\oplus}a^2b = bab((ab)^{\oplus})^2a^2b.$$

Hence, $abR \subseteq baR$.

(iii) Clearly, we have

$$\begin{aligned} ab(ab)^{\oplus}bb^{\oplus} &= (ab(ab)^{\oplus})^*(bb^{\oplus})^* = (bb^{\oplus}ab(ab)^{\oplus})^* = (bb^{\oplus}abb^{\oplus}a^{\oplus})^* \\ &= (abb^{\oplus}a^{\oplus})^* = ab(ab)^{\oplus}, \end{aligned}$$

which implies that $ab(ab)^{\oplus}bb^{\oplus} = (ab(ab)^{\oplus}bb^{\oplus})^* = bb^{\oplus}ab(ab)^{\oplus}$.

(iv) Since $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, we have $abb^{\oplus}bb^{\oplus}a^{\oplus} = abb^{\oplus}a^{\oplus} = ab(ab)^{\oplus}$, which gives $bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{3\}$. Also, we have

$$\begin{aligned} bb^{\oplus}a^{\oplus}(abb^{\oplus})^2 &= b(b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus})a^2bb^{\oplus} = bb^{\oplus}(a^{\oplus}a^2)bb^{\oplus} = (bb^{\oplus}ab)b^{\oplus} \\ &= abb^{\oplus}. \end{aligned}$$

Thus, $bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{3, 6\}$. □

Deng [10] and Mosić [23] studied the reverse order law $(ab)^{\#} = b^{\#}a^{\#}$ for the group inverse under the condition $ba = a^2$. Motivated by this, we will consider the one-sided reverse law for the core inverse under the same condition, using Theorem 3.5.

Theorem 3.7. *Let $a, b \in R^{\oplus}$ with $ba = a^2$. Then,*

(i) *$ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.*

(ii) *$abb^{\oplus} \in R^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$.*

Proof. (i) First, we prove $ab \in R^{\#}$. Note that $a \in R^{\#}$ and $ba = a^2$. Then, we have

$$ab = aa^2a^2(a^{\#})^4b = ababa(a^{\#})^4b = (ab)^2(a^{\#})^3b$$

and

$$ab = a^2a^{\#}b = baa^{\#}b = b(a^{\#})^3aa^2b = b(a^{\#})^3abab = b(a^{\#})^3(ab)^2,$$

which imply $ab \in R^{\#}$ by Lemma 2.3.

Next, we see that

$$a = a^2a^{\#} = baa^{\#} = b^{\oplus}b^2aa^{\#} = b^{\oplus}b(ba)a^{\#} = b^{\oplus}ba^2a^{\#} = b^{\oplus}ba = b^{\oplus}a^2.$$

Hence, $ab^{\oplus} = b^{\oplus}a^2b^{\oplus} = b^{\oplus}a^{\oplus}a^3b^{\oplus}$, which yields $ab^{\oplus}R \subseteq b^{\oplus}a^{\oplus}R$. Note that $a = b^{\oplus}ba$, then we deduce that $a^{\oplus} = a(a^{\oplus})^2 = b^{\oplus}ba(a^{\oplus})^2 = b^{\oplus}ba^{\oplus}$. In addition, the assumption $ba = a^2$ ensures that $aa^{\oplus} = a^2(a^{\oplus})^2 = ba(a^{\oplus})^2 = ba^{\oplus}$. Therefore,

$$abb^{\oplus}a^{\oplus} = abb^{\oplus}b^{\oplus}ba^{\oplus} = ab^{\oplus}ba^{\oplus} = a(b^{\oplus}a)a^{\oplus} = a(b^{\oplus}a^2)a^{\#}a^{\oplus} = aaa^{\#}a^{\oplus} = aa^{\oplus},$$

which implies $(abb^{\oplus}a^{\oplus})^* = abb^{\oplus}a^{\oplus}$. So, we conclude that

$$b^*a^{\#}(1-(abb^{\oplus}a^{\oplus})^*)a = b^*a^{\#}a - b^*a^{\#}(abb^{\oplus}a^{\oplus})a = b^*a^{\#}a - b^*a^{\#}aa^{\oplus}a = 0.$$

By Theorem 3.5, we claim that $ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$.

(ii) By (i), Lemma 3.6(iv), and Lemma 2.1, we only need to prove $bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{7\}$.

In the proof of (i), we obtain $a^{\oplus} = b^{\oplus}ba^{\oplus}$ and $ba^{\oplus} = aa^{\oplus}$ which imply that $bb^{\oplus}a^{\oplus} = bb^{\oplus}b^{\oplus}(ba^{\oplus}) = b^{\oplus}aa^{\oplus}$. Hence,

$$bb^{\oplus}a^{\oplus}(abb^{\oplus})^2 = b^{\oplus}(aa^{\oplus}a)(bb^{\oplus}ab)b^{\oplus} = b^{\oplus}aabb^{\oplus} = (b^{\oplus}bab)b^{\oplus} = abb^{\oplus}. \quad \square$$

Using Lemma 3.6 and Theorem 3.1, we deduce the following result.

Theorem 3.8. *Let $a, b, ab \in R^{\oplus}$. Then the following are equivalent:*

(i) $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;

(ii) $b(ab)^{\oplus} = bb^{\oplus}a^{\oplus}$ *and* $abb^{\oplus} = b^{\oplus}babb^{\oplus}$.

Proof. (i) $\Rightarrow$ (ii) can be obtained by Lemma 3.6(i).

(ii) $\Rightarrow$ (i) Since $b(ab)^{\oplus} = bb^{\oplus}a^{\oplus}$, we have

$$a(bb^{\oplus}a^{\oplus})ab = ab(ab)^{\oplus}ab = ab \text{ and } abb^{\oplus}a^{\oplus} = ab(ab)^{\oplus},$$

which implies $b^{\oplus}a^{\oplus} \in ab\{1,3\}$. In addition, observe that

$$\begin{aligned} b^{\oplus}a^{\oplus} &= b^{\oplus}(bb^{\oplus}a^{\oplus}) = b^{\oplus}b(ab)^{\oplus} = b^{\oplus}bab((ab)^{\oplus})^2 \\ &= (b^{\oplus}babb^{\oplus})b((ab)^{\oplus})^2 = abb^{\oplus}b((ab)^{\oplus})^2, \end{aligned}$$

which yields $b^{\oplus}a^{\oplus}R \subseteq abR$. Hence, by Theorem 3.1 (i) and (vii), we obtain $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. $\square$

In the following theorem, we investigate the one-sided reverse order law for the core inverse under the condition $Rb^*a \subseteq Rab^*$.

Theorem 3.9. *Let $a, b \in R^{\oplus}$ with $Rb^*a \subseteq Rab^*$. Then the following are equivalent:*

(i) $ab \in R^{\oplus}$ *and* $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;

(ii) $ab \in R^{\#}$, $abR \subseteq baR$, $b^{\oplus}aR \subseteq abR$, *and* $aa^{\oplus}bb^{\oplus} = bb^{\oplus}aa^{\oplus}$;

(iii) $ab, abb^{\oplus} \in R^{\oplus}$ *with* $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$ *and* $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$.

Proof. (i) $\Rightarrow$ (ii) Obviously, $ab \in R^{\#}$. From $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ and Lemma 3.6, it follows that $abR \subseteq baR$. The equality $b^{\oplus}a = b^{\oplus}a^{\oplus}a^2 = ab((ab)^{\oplus})^2a^2$ implies $b^{\oplus}aR \subseteq abR$.

Next, our aim is to prove $aa^{\oplus}bb^{\oplus} = bb^{\oplus}aa^{\oplus}$. The assumption $Rb^*a \subseteq Rab^*$ guarantees $b^*a = w_1ab^*$ for some $w_1 \in R$. Then, we have

$$b^*a = w_1a(bb^{\oplus}b)^* = w_1ab^*bb^{\oplus} = b^*abb^{\oplus}.$$

Note that $ab = bb^{\oplus}ab$ by Lemma 3.6(i), we obtain

$$\begin{aligned} abb^{\oplus}a^{\oplus} &= bb^{\oplus}abb^{\oplus}a^{\oplus} = (bb^{\oplus})^*abb^{\oplus}a^{\oplus} = (b^{\oplus})^*(b^*abb^{\oplus})a^{\oplus} \\ &= (b^{\oplus})^*b^*aa^{\oplus} = bb^{\oplus}aa^{\oplus}. \end{aligned}$$

Therefore, the following equalities hold:

$$abb^{\oplus}a^{\oplus} = (abb^{\oplus}a^{\oplus})^* = (bb^{\oplus}aa^{\oplus})^* = aa^{\oplus}bb^{\oplus}.$$

So, we have $aa^{\oplus}bb^{\oplus} = bb^{\oplus}aa^{\oplus}$.

(ii) $\Rightarrow$ (i) Since $abR \subseteq baR$, there exists $w_2 \in R$ such that $ab = baw_2$. Therefore, we obtain $ab = bb^{\oplus}(baw_2) = bb^{\oplus}ab$. Then, the equality $abb^{\oplus}a^{\oplus} = bb^{\oplus}aa^{\oplus}$ can be obtained in a similar way as in the proof of (i) $\Rightarrow$ (ii). Hence, we deduce that

$$\begin{aligned} (b^{\oplus}a^{\oplus})^*(ab)^*ab &= (abb^{\oplus}a^{\oplus})^*ab = (bb^{\oplus}aa^{\oplus})^*ab = aa^{\oplus}bb^{\oplus}ab \\ &= bb^{\oplus}aa^{\oplus}ab = bb^{\oplus}ab = ab, \end{aligned}$$

which implies $b^{\oplus}a^{\oplus} \in ab\{1,3\}$ by Lemma 2.4.

From $b^{\oplus}aR \subseteq abR$, we know that $b^{\oplus}a = abw_3$ for some $w_3 \in R$. Then, we obtain $b^{\oplus}a^{\oplus} = b^{\oplus}a(a^{\oplus})^2 = abw_3(a^{\oplus})^2$, which gives that $b^{\oplus}a^{\oplus}R \subseteq abR$. Hence, $ab \in R^{\oplus}$ with $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ by Theorem 3.1.

(i) $\Rightarrow$ (iii) According to the condition (i) and Lemma 3.6 (iv), we get $bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{3,6\}$. In addition, from the proof of (i) $\Rightarrow$ (ii), we see that $abb^{\oplus}a^{\oplus} = bb^{\oplus}aa^{\oplus} = aa^{\oplus}bb^{\oplus}$. Then, we obtain

$$\begin{aligned} abb^{\oplus}(bb^{\oplus}a^{\oplus})^2 &= (abb^{\oplus}a^{\oplus})bb^{\oplus}a^{\oplus} = aa^{\oplus}bb^{\oplus}bb^{\oplus}a^{\oplus} = aa^{\oplus}bb^{\oplus}a^{\oplus} \\ &= bb^{\oplus}aa^{\oplus}a^{\oplus} = bb^{\oplus}a^{\oplus}. \end{aligned}$$

By Lemma 2.1, it follows that $abb^{\oplus} \in R^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$. Then, it is easy to see that $(ab)^{\oplus} = b^{\oplus}(bb^{\oplus}a^{\oplus}) = b^{\oplus}(abb^{\oplus})^{\oplus}$.

(iii) $\Rightarrow$ (i) Suppose that $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$. Then, we deduce

$$(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus} = b^{\oplus}bb^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}. \quad \square$$

Remark 3.10. *(1) Note that Corollary 3.3 can also be obtained by Theorem 3.9. Indeed, we have* $ab^{\oplus} = b^{\oplus}a$, $ba^{\oplus} = a^{\oplus}b$, *and* $a^{\oplus}b^{\oplus} = b^{\oplus}a^{\oplus}$ *by Lemma 2.5. Then,* $aa^{\oplus}bb^{\oplus} = bb^{\oplus}aa^{\oplus}$ *and* $b^{\oplus}a = ab^{\oplus} = ab(b^{\oplus})^2$, *which gives* $b^{\oplus}aR \subseteq abR$. *Hence, we have* $ab \in R^{\oplus}$ *and* $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ *by Theorem 3.9.*

(2) In general, any item of (i)−(iii) in Theorem 3.9 does not imply $Rb^*a \subseteq Rab^*$. For example, let $\mathbb{Z}_2$ be the ring of integers modulo 2, take $R = M_2(\mathbb{Z}_2)$ with the transpose of matrices as involution. Setting $a = b = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 0 \end{smallmatrix}\right] \in R$. Then $a^2 = a$, which implies that $a \in R^{\#}$ with $a^{\#} = a$. Note that $a = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right] a^*a \in Ra^*a$, then $a \in R^{\{1,3\}}$. Hence, $a \in R^{\oplus}$ with $a^{\oplus} = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]$. Then, it is easy to check that such a and b satisfy any item of (i)−(iii) in Theorem 3.9. However, $ab^* = 0$ and $b^*a \neq 0$, so $Rb^*a \not\subseteq Rab^*$.

(3) If we replace $Rb^*a \subseteq Rab^*$ with $ba = a^2$ in Theorem 3.9 then the items (i),(ii), and (iii) hold. Indeed, by Theorem 3.7 and Lemma 3.6, we only need to show that $b^{\oplus}aR \subseteq abR$ and $aa^{\oplus}bb^{\oplus} = bb^{\oplus}aa^{\oplus}$. Since $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$ by Theorem 3.7, then $b^{\oplus}aR = (b^{\oplus}a^{\oplus})a^2R = (ab)^{\oplus}a^2R = ab((ab)^{\oplus})^2a^2R \subseteq abR$. In addition, from the proof of Theorem 3.7, we see that $a^{\oplus} = b^{\oplus}ba^{\oplus}$ and $aa^{\oplus} = ba^{\oplus}$, which gives $b(b^{\oplus}aa^{\oplus}) = ba^{\oplus} = aa^{\oplus}$ is Hermitian. Thus, we get $bb^{\oplus}aa^{\oplus} = (bb^{\oplus}aa^{\oplus})^* = (aa^{\oplus})^*(bb^{\oplus})^* = aa^{\oplus}bb^{\oplus}$.

Next, we continue to consider the one-sided reverse law for the core inverse under certain conditions.

Theorem 3.11. *Let $a, b \in R^{\oplus}$ with $Rb \subseteq Rab$ (or $aR \subseteq abR$). If a is EP, then the following are equivalent:*

(i) *$ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;*

(ii) *$(a^{\oplus})^*b \in R^{\oplus}$ and $((a^{\oplus})^*b)^{\oplus} = b^{\oplus}a^*$.*

Proof. (i) ⇒ (ii) Since $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, the we have that

$$(a^{\oplus})^*bb^{\oplus}a^* = (a^{\oplus})^*(bb^{\oplus})^*a^* = (abb^{\oplus}a^{\oplus})^* = abb^{\oplus}a^{\oplus}$$

is Hermitian, i.e., $b^{\oplus}a^* \in (a^{\oplus})^*b\{3\}$.

Note that a is EP, then $a^{\oplus}a = aa^{\oplus}$ is Hermitian by Lemma 2.7. So, $a^* = (a(a^{\oplus}a))^* = a^{\oplus}aa^*$. Thus, we have

$$\begin{aligned} (a^{\oplus})^*b(b^{\oplus}a^*)^2 &= ((a^{\oplus})^*(bb^{\oplus})^*a^*)b^{\oplus}a^* = (abb^{\oplus}a^{\oplus})b^{\oplus}a^* \\ &= (abb^{\oplus}a^{\oplus}b^{\oplus}a^{\oplus})aa^* = b^{\oplus}a^{\oplus}aa^* = b^{\oplus}a^*. \end{aligned}$$

If $Rb \subseteq Rab$, then $b = xab$ for some $x \in R$. Therefore, we can deduce that

$$\begin{aligned} b^{\oplus}a^*((a^{\oplus})^*b)^2 &= b^{\oplus}a^*(a^{\oplus})^*b(a^{\oplus})^*b = b^{\oplus}(a^{\oplus}a)^*b(a^{\oplus})^*b \\ &= b^{\oplus}a^{\oplus}ab(a^{\oplus})^*(bb^{\oplus})^*b = b^{\oplus}a^{\oplus}ab(bb^{\oplus}a^{\oplus})^*b \\ &= b^{\oplus}a^{\oplus}ab(xabb^{\oplus}a^{\oplus})^*b = (b^{\oplus}a^{\oplus}abab)b^{\oplus}a^{\oplus}x^*b \\ &= abb^{\oplus}a^{\oplus}x^*b = (abb^{\oplus}a^{\oplus})^*x^*b \\ &= (xabb^{\oplus}a^{\oplus})^*b = (bb^{\oplus}a^{\oplus})^*b \\ &= (a^{\oplus})^*(bb^{\oplus})^*b = (a^{\oplus})^*b. \end{aligned}$$

If $aR \subseteq abR$, then $a = aby$ for some $y \in R$. Thus, we get

$$\begin{aligned} b^{\oplus}a^*((a^{\oplus})^*b)^2 &= b^{\oplus}a^{\oplus}ab(a^{\oplus})^*b = b^{\oplus}a^{\oplus}ab(a^{\oplus}aa^{\oplus})^*b \\ &= b^{\oplus}a^{\oplus}abaa^{\oplus}(a^{\oplus})^*b = (b^{\oplus}a^{\oplus}abab)ya^{\oplus}(a^{\oplus})^*b \\ &= abya^{\oplus}(a^{\oplus})^*b = aa^{\oplus}(a^{\oplus})^*b = (a^{\oplus})^*b. \end{aligned}$$

Finally, from Lemma 2.1, it follows that $(a^{\oplus})^*b \in R^{\oplus}$ and $((a^{\oplus})^*b)^{\oplus} = b^{\oplus}a^*$.

(ii) $\Rightarrow$ (i) By Lemma 2.7, we have $a^{\#} = a^{\oplus}$. Let $c = (a^{\oplus})^*$, then $c = (a^{\#})^* = (a^*)^{\#}$, which implies that $a^* = c^{\#}$. Note that c is EP. Thus, $a^* = c^{\oplus}$. According to the condition (ii), we obtain that $cb \in R^{\oplus}$ and $(cb)^{\oplus} = b^{\oplus}c^{\oplus}$.

If $Rb \subseteq Rab$, then $Rb \subseteq Rab = Raa^{\oplus}ab = Ra(a^{\oplus}a)^*b = Raa^*(a^{\oplus})^*b \subseteq R(a^{\oplus})^*b = Rcb$.

If $aR \subseteq abR$, then $a = aby$ for some $y \in R$. Since $((a^{\oplus})^*b)^{\oplus} = b^{\oplus}a^*$, then $b^{\oplus}a^*R = (a^{\oplus})^*bR$, which yields $(a^{\oplus})^*b = b^{\oplus}a^*z$ for some $z \in R$. Note that $a^* = a^{\oplus}aa^*$. Then, we obtain

$$\begin{aligned}(a^{\oplus})^* &= (aa^{\oplus}a^{\oplus})^* = (a^{\oplus})^*(a^{\oplus})^*a^* = (a^{\oplus})^*(a^{\oplus})^*a^{\oplus}aa^* \\ &= (a^{\oplus})^*(a^{\oplus})^*a^{\oplus}abya^* = (a^{\oplus})^*((a^{\oplus})^*(a^{\oplus}a)^*)bya^* \\ &= (a^{\oplus})^*((a^{\oplus})^*b)ya^* = (a^{\oplus})^*b^{\oplus}a^*zya^* \\ &= (a^{\oplus})^*b(b^{\oplus})^2a^*zya^*,\end{aligned}$$

which gives $(a^{\oplus})^*R \subseteq (a^{\oplus})^*bR$, i.e., $cR \subseteq cbR$.

Now, replacing a with c in the proof of (i) $\Rightarrow$ (ii), we can obtain that (i) holds. □

If we replace the assumption $Rb \subseteq Rab$ (or $aR \subseteq abR$) of Theorem 3.11 by $Rb \subseteq Rab$ and $aR \subseteq abR$, then we obtain the following result.

Theorem 3.12. *Let $a, b \in R^{\oplus}$ with $Rb \subseteq Rab$ and $aR \subseteq abR$. If a is EP, then the following are equivalent:*

(i) *$ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$;*

(ii) *$ab, abb^{\oplus} \in R^{\oplus}$ with $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$ and $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$.*

Proof. (i) $\Rightarrow$ (ii) In order to prove $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, it suffices to prove that $abb^{\oplus}(bb^{\oplus}a^{\oplus})^2 = bb^{\oplus}a^{\oplus}$ by Lemma 2.6 (iv).

Since $Rb \subseteq Rab$ and $aR \subseteq abR$, there exist t_1, t_2 such that $b = t_1ab$ and $a = abt_2$. Note that $ab = bb^{\oplus}ab$ and $aa^{\oplus} = a^{\oplus}a$. Thus, we get

$$\begin{aligned}abb^{\oplus}(bb^{\oplus}a^{\oplus})^2 &= abb^{\oplus}a^{\oplus}bb^{\oplus}a^{\oplus} = bb^{\oplus}abb^{\oplus}a^{\oplus}aa^{\oplus}bb^{\oplus}a^{\oplus}. \\ &= bb^{\oplus}(abb^{\oplus}a^{\oplus}ab)t_2a^{\oplus}bb^{\oplus}a^{\oplus} = t_1abb^{\oplus}(abt_2)a^{\oplus}bb^{\oplus}a^{\oplus} \\ &= t_1abb^{\oplus}aa^{\oplus}bb^{\oplus}a^{\oplus} = t_1(abb^{\oplus}a^{\oplus}ab)b^{\oplus}a^{\oplus} \\ &= (t_1ab)b^{\oplus}a^{\oplus} = bb^{\oplus}a^{\oplus}.\end{aligned}$$

(ii) $\Rightarrow$ (i) It is obvious. □

Now, necessary and sufficient conditions of the two-sided reverse laws for the core inverse are stated as follows.

Theorem 3.13. *Let $a, b \in R^{\oplus}$. Then the following are equivalent:*

(i) *$ab, ba \in R^{\oplus}$ with $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$;*

(ii) *$abb^{\oplus}, baa^{\oplus} \in R^{\oplus}$ with $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, $(baa^{\oplus})^{\oplus} = aa^{\oplus}b^{\oplus}$, $abR = (ab)^2R$, and $baR = (ba)^2R$.*

Proof. (i) $\Rightarrow$ (ii) Obviously, $abR = (ab)^2R$ and $baR = (ba)^2R$. By symmetry, we only need to prove $abb^{\oplus} \in R^{\oplus}$ with $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$. Note that $bb^{\oplus}a^{\oplus} \in abb^{\oplus}\{3,6\}$. Applying the hypothesis $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$ and Lemma 3.6(ii), we have $baR = abR$, i.e., $ba = abu$ for some $u \in R$. Then,

$$\begin{aligned} abb^{\oplus}(bb^{\oplus}a^{\oplus})^2 &= abb^{\oplus}a^{\oplus}bb^{\oplus}a^{\oplus} = ab(ab)^{\oplus}b(ab)^{\oplus} = ab(ab)^{\oplus}bab((ab)^{\oplus})^2 \\ &= (ab(ab)^{\oplus}ab)ub((ab)^{\oplus})^2 = abub((ab)^{\oplus})^2 = bab((ab)^{\oplus})^2 \\ &= b(ab)^{\oplus} = bb^{\oplus}a^{\oplus}. \end{aligned}$$

Using Lemma 2.1, we get $abb^{\oplus} \in R^{\oplus}$ with $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$.

(ii) $\Rightarrow$ (i) Since $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, we claim that

$$b^{\oplus}a^{\oplus}abb^{\oplus}a^{\oplus} = b^{\oplus}(bb^{\oplus}a^{\oplus}abb^{\oplus}bb^{\oplus}a^{\oplus}) = b^{\oplus}bb^{\oplus}a^{\oplus} = b^{\oplus}a^{\oplus}$$

and

$$abb^{\oplus}a^{\oplus} = abb^{\oplus}bb^{\oplus}a^{\oplus} = abb^{\oplus}(abb^{\oplus})^{\oplus},$$

which imply $b^{\oplus}a^{\oplus} \in ab\{2,3\}$. In addition, observe that

$$abb^{\oplus} = bb^{\oplus}a^{\oplus}(abb^{\oplus})^2 = b^{\oplus}b(bb^{\oplus}a^{\oplus}(abb^{\oplus})^2) = b^{\oplus}babb^{\oplus}.$$

Multiplying the previous equality by b from the right side, we get $ab = b^{\oplus}bab$. Symmetrically, $ba = a^{\oplus}aba$. Hence, we have $ab = b^{\oplus}(ba)b = b^{\oplus}a^{\oplus}(ab)^2$, which gives $abR \subseteq b^{\oplus}a^{\oplus}R$ and $Rab = R(ab)^2$. Note that $abR = (ab)^2R$, then $ab \in R^{\#}$ by Lemma 2.3. Finally, from Theorem 3.1, it follows that $ab \in R^{\oplus}$ with $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. The statement for ba can be obtained by symmetry. □

Recall that a ring R (with unit 1) is a Dedekind-finite ring if $ab = 1$ is sufficient for $ba = 1$. Let $p = q = 1 \in R$ in [27, Theorem 1]. Then we can see the fact: If $a \in R$ is regular, $a^- \in a\{1\}$, then $u = a^2a^- + 1 - aa^-$ is right (resp. left) invertible if and only if $aR = a^2R$ (resp. $Ra = Ra^2$). Form the previous fact, it immediately yields that $aR = a^2R$ if and only if $Ra = Ra^2$, where R is a Dedekind-finite ring, and a is regular in R. Thus, by Theorem 3.13, we get

Corollary 3.14. *Let R be a Dedekind-finite ring and $a, b \in R^{\oplus}$. Then the following are equivalent:*

(i) $ab, ba \in R^{\oplus}$ *with* $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$, $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$;

(ii) $abb^{\oplus}, baa^{\oplus} \in R^{\oplus}$ *with* $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, $(baa^{\oplus})^{\oplus} = aa^{\oplus}b^{\oplus}$.

Proof. According to the proof of Theorem 3.13, we only need to prove that ab and ba are regular. Suppose that $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, then we have

$$abb^{\oplus} = bb^{\oplus}a^{\oplus}(abb^{\oplus})^2 = bb^{\oplus}(bb^{\oplus}a^{\oplus}(abb^{\oplus})^2) = bb^{\oplus}abb^{\oplus},$$

which yields

$$ab = (abb^{\oplus})b = bb^{\oplus}abb^{\oplus}b = bb^{\oplus}ab = (bb^{\oplus}a^{\oplus})a^2b = abb^{\oplus}(bb^{\oplus}a^{\oplus})^2aab.$$

Therefore, ab is regular. Similarly, by the condition $(baa^{\oplus})^{\oplus} = aa^{\oplus}b^{\oplus}$, we can obtain that ba is regular. □

4 Mixed-type Reverse Order Laws

In this section, we will consider necessary and sufficient conditions for the mixed-type reverse laws: $(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$, $a^{\oplus} = b(ab)^{\#}$ and $(ab)^{\#} = b^{\oplus}a^{\oplus}$ to hold in rings.

Theorem 4.1. *Let $b \in R^{\oplus}$, $abb^{\oplus} \in R^{\oplus}$. Then the following are equivalent:*
(i) *$ab \in R^{\oplus}$ and $(ab)^{\oplus} = (ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$;*
(ii) *$b^{\oplus}(abb^{\oplus})^{\oplus} \in ab\{5\}$;*
(iii) *$b^{\oplus}bab = ab = bb^{\oplus}ab$ and $ba(abb^{\oplus})^{\oplus} = (abb^{\oplus})^{\oplus}ab$;*
(iv) *$ab \in R^{\#}$ and $(abb^{\oplus})^{\oplus} = b(ab)^{\#}$.*

Proof. (i) $\Rightarrow$ (ii) It is obvious.

(ii) $\Rightarrow$ (iii) Note that $abb^{\oplus}(abb^{\oplus})^{\oplus}ab = (abb^{\oplus}(abb^{\oplus})^{\oplus}abb^{\oplus})b = abb^{\oplus}b = ab$. Also, since $b^{\oplus}(abb^{\oplus})^{\oplus} \in ab\{5\}$, we have $ab = b^{\oplus}(abb^{\oplus})^{\oplus}(ab)^2$. From the previous equality, we get

$$ab = bb^{\oplus}(b^{\oplus}(abb^{\oplus})^{\oplus}(ab)^2) = bb^{\oplus}ab$$

and

$$ab = b^{\oplus}b(b^{\oplus}(abb^{\oplus})^{\oplus}(ab)^2) = b^{\oplus}bab.$$

Thus, we obtain $abb^{\oplus} = bb^{\oplus}abb^{\oplus}$. From the definition of the core inverse, it follows that $(abb^{\oplus})^{\oplus}R = abb^{\oplus}R$, i.e., $(abb^{\oplus})^{\oplus} = abb^{\oplus}u$ for some $u \in R$. Hence, we deduce

$$(abb^{\oplus})^{\oplus} = bb^{\oplus}abb^{\oplus}u = bb^{\oplus}(abb^{\oplus})^{\oplus}.$$

Thus, the following equations hold:

$$ba(abb^{\oplus})^{\oplus} = babb^{\oplus}(abb^{\oplus})^{\oplus} = bb^{\oplus}(abb^{\oplus})^{\oplus}ab = (abb^{\oplus})^{\oplus}ab.$$

(iii) $\Rightarrow$ (iv) Clearly, $b^{\oplus}(abb^{\oplus})^{\oplus} \in ab\{1, 2\}$.

Now, we prove $b^{\oplus}(abb^{\oplus})^{\oplus} \in ab\{5\}$. Since $ab = bb^{\oplus}ab$, from the proof of (ii) $\Rightarrow$ (iii), we can get $(abb^{\oplus})^{\oplus} = bb^{\oplus}(abb^{\oplus})^{\oplus}$. Then,

$$(ab)b^{\oplus}(abb^{\oplus})^{\oplus} = b^{\oplus}babb^{\oplus}(abb^{\oplus})^{\oplus} = b^{\oplus}ba(abb^{\oplus})^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}(ab),$$

which implies that $(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$. Thus, we get

$$(abb^{\oplus})^{\oplus} = abb^{\oplus}((abb^{\oplus})^{\oplus})^2 = bb^{\oplus}abb^{\oplus}((abb^{\oplus})^{\oplus})^2 = bb^{\oplus}(abb^{\oplus})^{\oplus} = b(ab)^{\#}.$$

(iv) $\Rightarrow$ (i) Since $(abb^{\oplus})^{\oplus} = b(ab)^{\#}$, we have

$$\begin{aligned} ab &= abb^{\oplus}b = (abb^{\oplus})^{\oplus}(abb^{\oplus})^2b = b(ab)^{\#}(abb^{\oplus})^2b \\ &= b^{\oplus}bb(ab)^{\#}(abb^{\oplus})^2b = b^{\oplus}b(abb^{\oplus})^{\oplus}(abb^{\oplus})^2b \\ &= b^{\oplus}babb^{\oplus}b = b^{\oplus}bab. \end{aligned}$$

Then, we get

$$(ab)^{\#} = ab((ab)^{\#})^2 = b^{\oplus}bab((ab)^{\#})^2 = b^{\oplus}b(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}.$$

Our next aim is to prove $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$. Obviously, $b^{\oplus}(abb^{\oplus})^{\oplus} \in ab\{3\}$. Note that

$$b^{\oplus}(abb^{\oplus})^{\oplus}(ab)^2 = (ab)^{\#}(ab)^2 = ab$$

and

$$ab(b^{\oplus}(abb^{\oplus})^{\oplus})^2 = ab((ab)^{\#})^2 = (ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus},$$

which imply $ab \in R^{\oplus}$ and $(ab)^{\oplus} = b^{\oplus}(abb^{\oplus})^{\oplus}$ by Lemma 2.1. □

Remark 4.2. *Any item of (i)−(iv) in Theorem 4.1 can imply that ab is EP. Indeed, this can be obtained by Lemma 2.7.*

Next, characterizations for the equality $a^{\oplus} = b(ab)^{\#}$ are presented. In particular, we will see that such equivalent conditions also ensure that ab is EP.

Theorem 4.3. *Let $a, b \in R^{\oplus}$, $ab \in R^{\#}$. Then the following are equivalent:*
(i) $a^{\oplus} = b(ab)^{\#}$;
(ii) $a^{\oplus}ab = baa^{\oplus}$ *and* $aR \subseteq abR$;
(iii) $ab \in R^{\oplus}$ *with* $(ab)^{\#} = (ab)^{\oplus} = b^{\oplus}a^{\oplus}$, *and* $aR \subseteq bR$;
(iv) $abb^{\oplus} \in R^{\oplus}$ *with* $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus}$, $(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$, *and* $aR \subseteq bR$;
(v) $abb^{\oplus} \in R^{\oplus}$ *with* $(abb^{\oplus})^{\oplus} = bb^{\oplus}a^{\oplus} = b(ab)^{\#}$, *and* $aR \subseteq bR$.

Proof. (i) ⇒ (ii) Suppose that $a^{\oplus} = b(ab)^{\#}$, then

$$a^{\oplus}ab = b(ab)^{\#}ab = ba(b(ab)^{\#}) = baa^{\oplus}.$$

Also, we have $a = aa^{\oplus}a = ab(ab)^{\#}a$, which implies $aR \subseteq abR$.

(ii) ⇒ (iii) There exists $r_1 \in R$ such that $a = abr_1$, since $aR \subseteq abR$. Then

$$a = a^{\oplus}a^2 = (a^{\oplus}ab)r_1a = baa^{\oplus}r_1a.$$

Let $r_2 = aa^{\oplus}r_1a$. Then, $a = br_2$, which gives $aR \subseteq bR$.

Since

$$b^{\oplus}(a^{\oplus}ab) = b^{\oplus}baa^{\oplus} = b^{\oplus}bbr_2a^{\oplus} = br_2a^{\oplus} = aa^{\oplus}$$

and

$$abb^{\oplus}a^{\oplus} = abb^{\oplus}a(a^{\oplus})^2 = abb^{\oplus}br_2(a^{\oplus})^2 = abr_2(a^{\oplus})^2 = a^2(a^{\oplus})^2 = aa^{\oplus},$$

we have $b^{\oplus}a^{\oplus} \in ab\{3,5\}$.

Note that $(abb^{\oplus}a^{\oplus})ab = aa^{\oplus}ab = ab$ and $b^{\oplus}a^{\oplus}(abb^{\oplus}a^{\oplus}) = b^{\oplus}a^{\oplus}aa^{\oplus} = b^{\oplus}a^{\oplus}$, which gives $b^{\oplus}a^{\oplus} \in ab\{1,2\}$. Hence, $ab \in R^{\#}$ with $(ab)^{\#} = b^{\oplus}a^{\oplus}$. In addition, it is easy to see that $(ab)^{\oplus} = b^{\oplus}a^{\oplus}$. In fact,

$$b^{\oplus}a^{\oplus}(ab)^2 = (ab)^{\#}(ab)^2 = ab \text{ and } ab(b^{\oplus}a^{\oplus})^2 = ab((ab)^{\#})^2 = (ab)^{\#} = b^{\oplus}a^{\oplus}.$$

(iii) ⇒ (i) Obviously, $b(ab)^{\#} \in a\{2,3\}$. Observe that $a = br_3$ for some $r_3 \in R$. Then, we have

$$\begin{aligned} a &= a(a^{\oplus})^2a^2 = br_3(a^{\oplus})^2a^2 = bb^{\oplus}br_3(a^{\oplus})^2a^2 = bb^{\oplus}a(a^{\oplus})^2a^2 \\ &= bb^{\oplus}a^{\oplus}a^2 = b(ab)^{\#}a^2, \end{aligned}$$

which implies that $aR \subseteq b(ab)^{\#}R$. Therefore, $a \in R^{\oplus}$ and $a^{\oplus} = b(ab)^{\#}$ by Theorem 3.1.

(iii) ⇒ (iv) Since $a = br_4$ for some $r_4 \in R$, we get

$$bb^{\oplus}a^{\oplus} = bb^{\oplus}a(a^{\oplus})^2 = bb^{\oplus}br_4(a^{\oplus})^2 = br_4(a^{\oplus})^2 = a(a^{\oplus})^2 = a^{\oplus}.$$

Therefore, we only need to prove $abb^{\oplus} \in R^{\oplus}$ with $(abb^{\oplus})^{\oplus} = a^{\oplus}$. From $a(bb^{\oplus}a^{\oplus}) = aa^{\oplus}$, it follows that $a^{\oplus} \in abb^{\oplus}\{3\}$. Also, we have

$$abb^{\oplus}(a^{\oplus})^2 = (abb^{\oplus}a^{\oplus})a^{\oplus} = aa^{\oplus}a^{\oplus} = a^{\oplus}$$

and

$$a^{\oplus}(abb^{\oplus})^2 = a^{\oplus}(abb^{\oplus}a^{\oplus})a^2bb^{\oplus} = a^{\oplus}aa^{\oplus}a^2bb^{\oplus} = abb^{\oplus}.$$

Hence, we have $(abb^{\oplus})^{\oplus} = a^{\oplus}$ by Lemma 2.1. Then, it is easy to see that $(ab)^{\#} = b^{\oplus}(abb^{\oplus})^{\oplus}$.

(iv) ⇒ (v) It is obvious.

(v) ⇒ (i) From the proof of (iii) ⇒ (iv), we know $a^{\oplus} = bb^{\oplus}a^{\oplus}$, which gives $a^{\oplus} = b(ab)^{\#}$. □

The next theorem gives the result related to $a^{\oplus} = b(ab)^{\oplus}$.

Theorem 4.4. *Let $a \in R^{\#}$, $ab \in R^{\oplus}$. Then the following are equivalent:*

(i) *$a \in R^{\oplus}$ and $a^{\oplus} = b(ab)^{\oplus}$;*

(ii) *$aR \subseteq babR$.*

Proof. (i) ⇒ (ii) Suppose that $a^{\oplus} = b(ab)^{\oplus}$. Then, we have

$$a = a^{\oplus}a^2 = b(ab)^{\oplus}a^2 = bab((ab)^{\oplus})^2a^2,$$

which implies that $aR \subseteq babR$.

(ii) ⇒ (i) Since $aR \subseteq babR$, there exists $w \in R$ such that $a = babw$. Then, we obtain $a = b(ab)^{\oplus}(ab)^2w$, which gives $aR \subseteq b(ab)^{\oplus}R$. In addition, it is easy to verify $b(ab)^{\oplus} \in a\{2,3\}$. Hence, by Theorem 3.1, it follows that $a \in R^{\oplus}$ with $a^{\oplus} = b(ab)^{\oplus}$. □

Finally, we characterize the mixed-type reverse order laws $(ab)^{\#} = b^{\oplus}a^{\oplus}$ and $(ba)^{\#} = a^{\oplus}b^{\oplus}$ by the following necessary and sufficient conditions.

Theorem 4.5. *Let $a, b \in R^{\oplus}$. Then the following are equivalent:*

(i) *$ab, ba \in R^{\#}$ with $(ab)^{\#} = b^{\oplus}a^{\oplus}$, $(ba)^{\#} = a^{\oplus}b^{\oplus}$;*

(ii) *$a^{\oplus}ab = baa^{\oplus}$, $b^{\oplus}ba = abb^{\oplus}$, and $aa^{\oplus}b^{\oplus} = b^{\oplus}a^{\oplus}a$.*

Proof. (i) ⇒ (ii) Since $(ab)^{\#} = b^{\oplus}a^{\oplus}$, we have

$$ab = b^{\oplus}a^{\oplus}(ab)^2 = b^{\oplus}b(b^{\oplus}a^{\oplus}(ab)^2) = b^{\oplus}bab$$

and

$$ab = (ab)^2 b^{\oplus}a^{\oplus} = ((ab)^2 b^{\oplus}a^{\oplus})aa^{\oplus} = abaa^{\oplus}.$$

Similarly, from $(ba)^{\#} = a^{\oplus}b^{\oplus}$, it follows that $ba = a^{\oplus}aba = babb^{\oplus}$. Hence,

$$a^{\oplus}ab = (a^{\oplus}aba)a^{\oplus} = baa^{\oplus} \text{ and } b^{\oplus}ba = (b^{\oplus}bab)b^{\oplus} = abb^{\oplus}.$$

Since $ab, ba \in R^{\#}$, from Lemma 2.6, it follows that $a(ba)^{\#} = ab((ab)^{\#})^2 a = (ab)^{\#}a$, which immediately yields $aa^{\oplus}b^{\oplus} = b^{\oplus}a^{\oplus}a$.

(ii) ⇒ (i) By the hypotheses, we deduce

$$abb^{\oplus}(a^{\oplus}ab) = abb^{\oplus}baa^{\oplus} = a(baa^{\oplus}) = aa^{\oplus}ab = ab,$$

$$(b^{\oplus}a^{\oplus}a)bb^{\oplus}a^{\oplus} = aa^{\oplus}b^{\oplus}bb^{\oplus}a^{\oplus} = (aa^{\oplus}b^{\oplus})a^{\oplus} = b^{\oplus}a^{\oplus}aa^{\oplus} = b^{\oplus}a^{\oplus}$$

and

$$b^{\oplus}(a^{\oplus}ab) = (b^{\oplus}ba)a^{\oplus} = abb^{\oplus}a^{\oplus},$$

which imply $b^{\oplus}a^{\oplus} \in ab\{1,2,5\}$, i.e., $(ab)^{\#} = b^{\oplus}a^{\oplus}$. Similarly, we can obtain $(ba)^{\oplus} = a^{\oplus}b^{\oplus}$. □

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