UNIVERSITY OF COPENHAGEN

Classification of reductive real spherical pairs II
the semisimple case

Knop, F.; Krötz, B.; Pecher, T.; Schlichtkrull, H.

Published in:
Transformation Groups

DOI:
10.1007/s00031-019-09515-w

Publication date:
2019

Document version
Peer reviewed version

Document license:
Unspecified

Citation for published version (APA):
Knop, F., Krötz, B., Pecher, T., & Schlichtkrull, H. (2019). Classification of reductive real spherical pairs II: the semisimple case. *Transformation Groups*, *24*(2), 467-510. https://doi.org/10.1007/s00031-019-09515-w

CLASSIFICATION OF REDUCTIVE REAL SPHERICAL PAIRS II. THE SEMISIMPLE CASE

FRIEDRICH KNOP

Department Mathematik, Emmy-Noether-Zentrum
FAU Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany

BERNHARD KRÖTZ

Universität Paderborn, Institut für Mathematik
Warburger Straße 100, D-33098 Paderborn, Deutschland

TOBIAS PECHER

Universität Paderborn, Institut für Mathematik
Warburger Straße 100, D-33098 Paderborn, Deutschland

HENRIK SCHLICHTKRULL

University of Copenhagen, Department of Mathematics
Universitetsparken 5, DK-2100 Copenhagen Ø, Denmark

ABSTRACT. If $\mathfrak{g}$ is a real reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ is a subalgebra, then the pair $(\mathfrak{g}, \mathfrak{h})$ is called real spherical provided that $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ for some choice of a minimal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$. This paper concludes the classification of real spherical pairs $(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{h}$ is a reductive real algebraic subalgebra. More precisely, we classify all such pairs which are strictly indecomposable, and we discuss (in Section 6) how to construct from these all real spherical pairs. A preceding paper treated the case where $\mathfrak{g}$ is simple. The present work builds on that case and on the classification by Brion and Mikityuk for the complex spherical case.

E-mail addresses: `friedrich.knop@fau.de`, `bkroetz@gmx.de`, `tpecher@math.upb.de`, `schlicht@math.ku.dk`.

Date: September 6, 2018.

2000 *Mathematics Subject Classification.* 14M17, 20G20, 22E15, 22F30, 53C30.

The second author was supported by ERC Advanced Investigators Grant HARG 268105.

0. Introduction

One of the most remarkable accomplishments of early Lie theory was the discovery by W. Killing that the simple complex Lie algebras are classified (up to isomorphism) by the irreducible root systems. The classification was completed and extended to real Lie algebras by E. Cartan, who determined all real forms of the algebras from Killing's list and later used that to obtain a classification (up to local isomorphism) of all Riemannian symmetric spaces. In turn, Cartan's list of symmetric spaces was extended to pseudo-Riemannian symmetric spaces by M. Berger [2]. These classifications have played a profound role in the development of the theory of semi-simple Lie groups and their symmetric spaces, for example by providing important examples for explicit calculations.

More recently the notion of a symmetric space has been generalized. Given a complex reductive Lie group G and a closed complex subgroup H the homogeneous space $Z := G/H$ is said to be *spherical* if there exists a Borel subgroup $B \subset G$ such that the orbit $Bz \subset Z$ is open for some $z \in Z$. In particular, this is the case when G/H is symmetric. The condition of being spherical is local, and it can be stated in terms of the corresponding Lie algebras as $\mathfrak{g} = \mathfrak{b} + \mathfrak{h}$, a vector space sum, for some Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$. The pair $(\mathfrak{g}, \mathfrak{h})$ is called *spherical* when this condition is fulfilled, and it is called *reductive* if $\mathfrak{h}$ is reductive in $\mathfrak{g}$. By extending Cartan's list of symmetric pairs the spherical reductive pairs of the simple complex Lie algebras were classified (up to isomorphism) by M. Krämer [17]. Subsequently such a classification, but without the assumption of $\mathfrak{g}$ being simple, was obtained by M. Brion [4] and I.V. Mikityuk [21].

A notion of spherical homogeneous spaces exists also for real Lie groups. If G is a real reductive Lie group and H a closed subgroup, the space G/H is called real spherical if there exists a minimal parabolic subgroup $P \subset G$ such that the orbit $Pz \subset Z$ is open for some $z \in Z$. Again the notion is local and translates into $\mathfrak{g} = \mathfrak{p} + \mathfrak{h}$ for some minimal parabolic subalgebra $\mathfrak{p}$, in which case $(\mathfrak{g}, \mathfrak{h})$ is called a *real spherical pair*. It is clear that the pair $(\mathfrak{g}, \mathfrak{h})$ is real spherical if the complexified pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$ is spherical, since (up to conjugation) the Borel subalgebra $\mathfrak{b}$ is contained in the complexification $\mathfrak{p}_{\mathbb{C}}$ of $\mathfrak{p}$. In this case the real pair is said to be *absolutely spherical*. In particular, this is always the case if G/H is a symmetric space. However, there do exist real spherical pairs which are not absolutely spherical.

In recent years a vivid research activity has taken place for real spherical spaces, motivated in part by [25]. See for example [5], [7], [9], [10], [11], [12], [14], [15], [18], [19], [20], [6]. This paper was driven by the desire to determine the scope of this new area, and by the need to find relevant examples by which one can support investigations through explicit computations.

In the preceding paper [8], from now on referred to as part I, we obtained a classification of the real spherical reductive pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple, thus providing a real analogue of the list of Krämer. The aim of this second part is then to obtain a real analogue of the subsequent classification by Brion and Mikityuk. More precisely, all strictly indecomposable (a notion which will be explained below) real spherical reductive pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ semi-simple will be classified up to isomorphism.

To be specific we consider a real reductive Lie algebra $\mathfrak{g}$, that is, $\mathfrak{g}$ is a real Lie algebra such that $\mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus [\mathfrak{g}, \mathfrak{g}]$ is a direct Lie algebra sum with $\mathfrak{z}(\mathfrak{g})$ the center of $\mathfrak{g}$, and with semi-simple derived subalgebra $[\mathfrak{g}, \mathfrak{g}]$. A subalgebra $\mathfrak{h} \subset \mathfrak{g}$ is called

- *reductive (in $\mathfrak{g}$)* if $\mathrm{ad}_{\mathfrak{g}}|_{\mathfrak{h}}$ is completely reducible,
- *compact (in $\mathfrak{g}$)* if $e^{\mathrm{ad}_{\mathfrak{g}} \mathfrak{h}} \subset \mathrm{Aut}(\mathfrak{g})$ is compact,

- *elementary (in $\mathfrak{g}$)* if $\mathfrak{h}$ is reductive in $\mathfrak{g}$ and $[\mathfrak{h}, \mathfrak{h}]$ is compact,
- *symmetric (in $\mathfrak{g}$)* if $\mathfrak{h}$ is the fixed point set of an involutive automorphism of $\mathfrak{g}$.

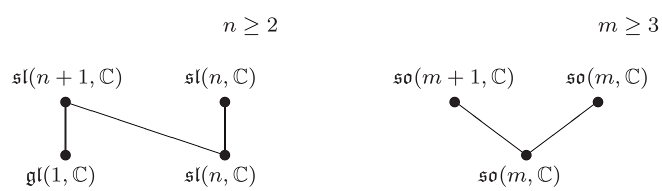
From now on we let $\mathfrak{h} \subset \mathfrak{g}$ be a reductive subalgebra. It decomposes into $\mathfrak{h} = \mathfrak{h}_{\rm n} \oplus \mathfrak{h}_{\rm el}$ where $\mathfrak{h}_{\rm el}$ is its largest elementary ideal and $\mathfrak{h}_{\rm n}$ is the sum of its non-elementary simple ideals. In addition we require that $\mathfrak{h}$ is real algebraic, that is $\mathfrak{h}_\mathbb{C} := \mathfrak{h} \otimes_\mathbb{R} \mathbb{C}$ is an algebraic subalgebra of $\mathfrak{g}_\mathbb{C}$. Recall that this is always the case when $\mathfrak{h}$ is semi-simple.

The classification of real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ readily reduces to the case where $\mathfrak{g}$ is semi-simple with all simple factors being non-compact (see Lemma 3.2). We assume this from now on and let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ be the decomposition into simple factors. For a subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we let $\mathfrak{h}_i$ be the projection of $\mathfrak{h}$ to $\mathfrak{g}_i$. It is easy to see that if $(\mathfrak{g}, \mathfrak{h})$ is real spherical then all $(\mathfrak{g}_i, \mathfrak{h}_i)$ are real spherical (see Lemma 3.1). In this sense the real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple, which were classified in part I, serve as building blocks for the general classification.

The pair $(\mathfrak{g}, \mathfrak{h})$ is called *decomposable* if $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{g}_i \neq \{0\}$ and $\mathfrak{h} = (\mathfrak{h} \cap \mathfrak{g}_1) \oplus (\mathfrak{h} \cap \mathfrak{g}_2)$, and *indecomposable* otherwise. Clearly it suffices to classify indecomposable pairs. The step from simple to semi-simple is rather straightforward in the classification of symmetric pairs, as the only indecomposable symmetric pairs with $\mathfrak{g}$ not simple are the so-called group cases $(\mathfrak{g}, \mathfrak{h}) \simeq (\mathfrak{h} \oplus \mathfrak{h}, \operatorname{diag} \mathfrak{h})$. For spherical pairs the situation is far more complicated. This is seen already in the complex case where Brion and Mikityuk found a multitude of indecomposable complex spherical pairs in addition to the group case.

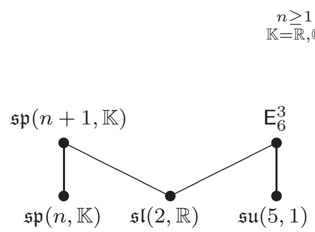
It would then be desirable to give a classification of all indecomposable reductive real spherical pairs, but in order to obtain an efficient classification an additional requirement is necessary. We call $(\mathfrak{g}, \mathfrak{h})$ *strictly indecomposable* provided that $(\mathfrak{g}, \mathfrak{h}_{\rm n})$ is indecomposable, and we shall only completely classify the strictly indecomposable real spherical pairs. Such a stronger assumption appears also in the work of Brion and Mikityuk, as explained in [21, Sect. 5] (see also [26, p. 45-46]). In Section 6 we describe the reduction of the indecomposable case to the strictly indecomposable case.

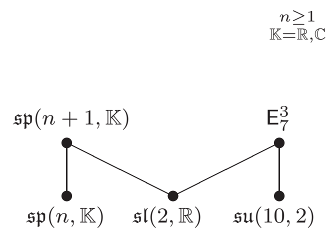
With the requirement of strict indecomposability it is a particular feature of the Brion-Mikityuk classification that the number of simple factors of $\mathfrak{g}$ is at most 3. Among the most prominent examples of Brion and Mikityuk are the Gross-Prasad spaces:



Here one finds $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ in the top row, $\mathfrak{h}$ in the lower row, and the various lines indicate the factors of $\mathfrak{g}$ into which the given factors $\mathfrak{h}'$ of $\mathfrak{h}$ embed (diagonally if more than one line branches from $\mathfrak{h}'$). It is noteworthy that the second case specialized to $m = 3$ yields the well-known triple case $(\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}), \operatorname{diag} \mathfrak{sl}(2, \mathbb{C}))$.

We now describe our classification in more detail. As in the complex case it turns out that the number of factors is at most 3. The cases with two factors are listed in Theorem 4.1 and the remaining ones in Theorem 5.2. In both theorems we use diagrams similar to those above to describe $\mathfrak{g}$ and $\mathfrak{h}$, and in total there are approximately 50 such diagrams. Given the classification of Brion-Mikityuk the absolutely spherical pairs are fairly easy to determine, and they are marked by *a.s.* in our tables. The majority of the diagrams contain cases which are not absolutely spherical. We highlight here our two most exotic cases:





.

A real spherical pair $(\mathfrak{g},\mathfrak{h})$ such that $(\mathfrak{g}\oplus\mathfrak{h}, \operatorname{diag}\mathfrak{h})$ is real spherical, as for example the second Gross-Prasad pair above, is called *strongly spherical*; see Theorem 4.1 1b) which lists those with both $\mathfrak{g}$ and $\mathfrak{h}$ simple. Strongly spherical pairs deserve special attention in view of their relevance for branching laws from $\mathfrak{g}$ to $\mathfrak{h}$ (see [14], [6]). Prior to this work strongly spherical pairs $(\mathfrak{g},\mathfrak{h})$ where $\mathfrak{h}\subset\mathfrak{g}$ is symmetric were classified in [14]. The cases found in [14] can easily be extracted from our tables together with all additional cases for non-symmetric $\mathfrak{h}$, see Table 10 at the end of the paper.

Let us now explain our approach. Given a minimal parabolic subalgebra $\mathfrak{p}$ such that $\mathfrak{g}=\mathfrak{h}+\mathfrak{p}$ we recall from [11] the infinitesimal version of the local structure theorem: There exists a unique parabolic subalgebra $\mathfrak{q}\supset\mathfrak{p}$ of $\mathfrak{g}$ with Levi decomposition $\mathfrak{q}=\mathfrak{l}\ltimes\mathfrak{u}$ such that

- $\mathfrak{q}\cap\mathfrak{h}=\mathfrak{l}\cap\mathfrak{h}$,
- $\mathfrak{l}\cap\mathfrak{h}$ contains all non-compact simple ideals of $\mathfrak{l}$.

The subalgebra $\mathfrak{s}(\mathfrak{g},\mathfrak{h}):=\mathfrak{l}\cap\mathfrak{h}$ of $\mathfrak{h}$ is reductive and is an invariant of the real spherical pair $(\mathfrak{g},\mathfrak{h})$. It is called the *structural algebra* (cf. [13] where $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ is called principal subalgebra). Suppose now that $\mathfrak{g}=\mathfrak{g}_1\oplus\mathfrak{g}_2$ and that $\mathfrak{h}\subset\mathfrak{g}$ is a reductive subalgebra with projection $\mathfrak{h}_i$ to $\mathfrak{g}_i$ for $i=1,2$. A necessary condition for $(\mathfrak{g},\mathfrak{h})$ to be real spherical is that both $(\mathfrak{g}_1,\mathfrak{h}_1)$ and $(\mathfrak{g}_2,\mathfrak{h}_2)$ are real spherical. Let us assume this and set $\mathfrak{h}':=\mathfrak{h}_1\oplus\mathfrak{h}_2$. Then $\mathfrak{h}'$ is a real spherical subalgebra of $\mathfrak{g}$ which contains $\mathfrak{h}$. Our main tool for the classification is the fact (see Lemma 3.1) that in this case $\mathfrak{h}\subset\mathfrak{g}$ is real spherical if and only if there exist minimal parabolic subalgebras $\mathfrak{p}_i\subset\mathfrak{s}(\mathfrak{g}_i,\mathfrak{h}_i)$ such that

$$\mathfrak{h}'=\mathfrak{h}+\mathfrak{p}_1+\mathfrak{p}_2.$$

In particular it is a necessary condition for $(\mathfrak{g},\mathfrak{h})$ to be real spherical that

$$\mathfrak{h}'=\mathfrak{h}+\mathfrak{s}(\mathfrak{g}_1,\mathfrak{h}_1)+\mathfrak{s}(\mathfrak{g}_2,\mathfrak{h}_2)\,.$$

This condition turns out to be rather restrictive in view of the factorization results of Onishchik [22], which were already used in Part I, and which are recalled in Proposition 2.8. In the first step we therefore determine all structural Lie algebras $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ for $\mathfrak{g}$ simple. Some part was already done in Part I and in Appendix A we provide a complete list in Tables 3 - 7. It is then a matter of efficient bookkeeping to obtain the classification for $k=2$ factors, which is recorded in Theorem 4.1. This theorem is divided into three parts: $\mathfrak{h}$ simple, $\mathfrak{h}$ semi-simple but not simple, and $\mathfrak{h}$ reductive but not semi-simple. In all cases we also determine the subalgebra $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{h}$. Having obtained the classification for $k=2$ together with this information, it is then a rather quick task to derive both the classification for $k = 3$ and the exclusion of $k \geq 4$ (see Theorem 5.2 and Proposition 5.3).

In Appendix B at the end of the paper we prove a general result on the geometric structure of restricted root spaces for symmetric spaces, which generalizes a theorem of Kostant for Riemannian symmetric spaces, [16] Thm. 2.1.7. The result is applied in Section 4 to a particular symmetric space of E_7.

Acknowledgment. It is a pleasure to thank a meticulous referee for some valuable suggestions which resulted in an improved exposition. In particular, Prop. 6.2 was suggested by him. In addition we thank Jan Frahm for pointing out some inaccuracies in an earlier version of the paper.

1. Notation for classical and exceptional Lie groups

Fix a real reductive Lie algebra $\mathfrak{g}$ and let $G_\mathbb{C}$ be a linear complex algebraic group with Lie algebra $\mathfrak{g}_\mathbb{C} = \mathfrak{g} \otimes_\mathbb{R} \mathbb{C}$. If $\mathfrak{g}_\mathbb{C}$ is classical, then we shall denote by $G_\mathbb{C}$ the corresponding classical group, i.e. $G_\mathbb{C} = \mathrm{SL}(n, \mathbb{C}), \mathrm{SO}(n, \mathbb{C}), \mathrm{Sp}(n, \mathbb{C})$. To avoid confusion let us stress that we use the notation $\mathrm{Sp}(n, \mathbb{R})$, $\mathrm{Sp}(n, \mathbb{C})$ to indicate that the underlying classical vector space is $\mathbb{R}^{2n}$, $\mathbb{C}^{2n}$. Further $\mathrm{Sp}(n)$ denotes the compact real form of $\mathrm{Sp}(n, \mathbb{C})$ and likewise the underlying vector space for $\mathrm{Sp}(p, q)$ is $\mathbb{C}^{2p+2q}$. Finally we use $\mathrm{SO}_0(p, q)$ for the identity component of $\mathrm{O}(p, q)$, the indefinite orthogonal group on $\mathbb{R}^{p+q}$.

For the exceptional Lie algebras we use the notation of Berger, [2, p. 117] , and write $\mathsf{E}_6^\mathbb{C}, \mathsf{E}_7^\mathbb{C}$ etc. for the complex simple Lie algebras of type E_6, E_7 etc., and $\mathsf{E}_6, \mathsf{E}_7$ etc. for the corresponding compact real forms. For the non-compact real forms we write

$$\begin{array}{rcl} \mathsf{E}_6^1, \mathsf{E}_6^2, \mathsf{E}_6^3, \mathsf{E}_6^4 & \text{for} & \mathrm{E\,I}, \mathrm{E\,II}, \mathrm{E\,III}, \mathrm{E\,IV} \\ \mathsf{E}_7^1, \mathsf{E}_7^2, \mathsf{E}_7^3 & \text{for} & \mathrm{E\,V}, \mathrm{E\,VI}, \mathrm{E\,VII} \\ \mathsf{E}_8^1, \mathsf{E}_8^2 & \text{for} & \mathrm{E\,VIII}, \mathrm{E\,IX} \\ \mathsf{F}_4^1, \mathsf{F}_4^2 & \text{for} & \mathrm{F\,I}, \mathrm{F\,II} \end{array}$$

and finally G_2^1 for the unique non-compact real form of $\mathsf{G}_2^\mathbb{C}$.

By slight abuse of notation we also denote the simply connected Lie groups with these Lie algebras by the same symbols.

2. Real spherical pairs

2.1. **Preliminaries.** Let $\mathfrak{g}$ be a real reductive Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ an algebraic subalgebra. Recall from the introduction that the pair $(\mathfrak{g}, \mathfrak{h})$ is called *real spherical* provided there exists a minimal parabolic subalgebra $\mathfrak{p} \subset \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} + \mathfrak{p}\,.$$

For later reference we record the obvious necessary condition

$$\dim \mathfrak{g} \leq \dim \mathfrak{h} + \dim \mathfrak{p}. \tag{2.1}$$

A pair $(\mathfrak{g}, \mathfrak{h})$ of a complex Lie algebra and a complex subalgebra is called *complex spherical* or just *spherical* if it is real spherical when regarded as a pair of real Lie algebras. Note that in this case the minimal parabolic subalgebras of $\mathfrak{g}$ are precisely the Borel subalgebras. We recall also that a real reductive pair $(\mathfrak{g}, \mathfrak{h})$ for which $(\mathfrak{g}_\mathbb{C}, \mathfrak{h}_\mathbb{C})$ is spherical is said to be *absolutely spherical*, and from [12, Lemma 2.1] we record:

Lemma 2.1. *All absolutely spherical pairs are real spherical.*

We denote by G the connected Lie subgroup of $G_\mathbb{C}$ with Lie algebra $\mathfrak{g}$, and for an algebraic subalgebra $\mathfrak{h} \subset \mathfrak{g}$ we denote by $H \subset G$ the corresponding connected Lie subgroup. Let $P \subset G$ be a minimal parabolic subgroup. A real variety Z on which G acts is called real spherical if there is an open P-orbit on Z. In particular, this applies to the homogeneous space $Z := G/H$. If $L \subset G$ we write $L_H := L \cap H$ for the intersection with H.

2.2. **Local structure theorem.** Let $Z = G/H$ be a real spherical space and choose a minimal parabolic subgroup $P \subset G$ such that PH is open in G. The local structure theorem, [11, Th. 2.3 and Th. 2.8], asserts that there is a unique parabolic subgroup $Q \supset P$ which admits a Levi decomposition $Q = L \ltimes U$ for which:

$$PH = QH\,, \tag{2.2}$$

$$Q_H = L_H\,, \tag{2.3}$$

$$L_\mathrm{n} \subset L_H\,, \tag{2.4}$$

where $L_\mathrm{n} \triangleleft L$ is the connected normal subgroup with Lie algebra $\mathfrak{l}_\mathrm{n}$, the sum of all non-compact simple ideals of $\mathfrak{l}$.

The parabolic subgroup Q depends on the homogeneous space Z, and on the choice of minimal parabolic subgroup P. We refer to it as being *adapted* to Z and P or, for short, just adapted to Z when P is clear from the context. We emphasize that it is Q, but not the Levi part L satisfying (2.2)-(2.4), which is uniquely determined by Z and P. However, it follows from (2.3) that L_H is unique in the same sense as Q. By (2.4), L_H is a reductive subgroup of L and hence also of G.

On the Lie algebra level we let $\mathfrak{s}(\mathfrak{g},\mathfrak{h}) := \mathfrak{l}_\mathfrak{h} = \mathfrak{l} \cap \mathfrak{h}$, and we call the subalgebra $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ of $\mathfrak{h}$ the *structural algebra* associated with $(\mathfrak{g},\mathfrak{h})$ and P. The dependence of $\mathfrak{s}(\mathfrak{g},\mathfrak{h}) \subset \mathfrak{h}$ on P is explained in the following remark, see also [7, Lemma 13.5].

Remark 2.2. Suppose that $P'H$ is open for some minimal parabolic $P' \subset G$. Then $P' = P_g := g^{-1}Pg$ for some $g \in G$ such that PgH is open. Since $P_\mathbb{C}H_\mathbb{C}$ is the unique open $P_\mathbb{C} \times H_\mathbb{C}$-double coset in $G_\mathbb{C}$, it follows that the number of open $P \times H$-double cosets is finite and hence represented by a finite set $W \subset G$. Moreover, the local structure theorem implies that W can be chosen such that elements $w \in W$ are of the form $w = t_\mathbb{C}h_\mathbb{C}$ with $t_\mathbb{C} \in Z(L_\mathbb{C})$ (the center of $L_\mathbb{C}$) and $h_\mathbb{C} \in H_\mathbb{C}$. In particular $\mathrm{Ad}(t_\mathbb{C})\mathfrak{s} = \mathfrak{s}$. We may assume that $g = wh$ for some $w \in W$ and $h \in H$ and then the structural algebra with respect to P_g is $\mathfrak{s}_g = \mathrm{Ad}(g)^{-1}\mathfrak{s}$. In particular, we see that all possible structural algebras are $\mathrm{Ad}(H)$-conjugate to some $\mathfrak{s}_w$, $w \in W$. Finally note that $\mathfrak{s}_w = \mathrm{Ad}(h_\mathbb{C})^{-1}\mathfrak{s} \subset \mathfrak{h}$ is a "real point" of $\mathrm{Ad}(H_\mathbb{C})\mathfrak{s}$.

The structural algebra $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ is the Lie algebra of the group L_H, for which we use the notation $\mathrm{S}(G,H)$ in case we need to specify the pair (G,H) to which it is associated.

We recall from Part I, Lemma 2.8 the following result.

Lemma 2.3. *Let G/H be real spherical and let $Q \supset P$ be adapted to it. Then P_H is a minimal parabolic subgroup of L_H.*

2.3. **Iwasawa decomposition.** The local structure theorem enables us to choose an Iwasawa decomposition for G which corresponds well with the structure of G/H. We denote by N the unipotent radical of the minimal parabolic subgroup P, and remark that any unipotent subgroup of P is contained in N. The fact that P is minimal is equivalent to the property that P/N is elementary.

With $Q = LU$ as above we choose a Cartan involution of L and inflate it to a Cartan involution of G. Corresponding to that we can obtain an Iwasawa decomposition $G = KAN$ of G, such that $A \subset L$ and such that N is exactly the unipotent radical of P. With $M = Z_K(A)$, the centralizer of A in K, we have the decomposition $P = (MA) \ltimes N$ with the elementary Levi part MA. This decomposition and its Lie algebra version $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ are usually referred to as Langlands decompositions.

Finally we recall that $\mathfrak{g}$ is called *split* if $\mathfrak{m}$, as defined above, is zero, and *quasi-split* if $\mathfrak{m}$ is abelian, or equivalently, if $\mathfrak{p}_\mathbb{C}$ is a Borel subalgebra of $\mathfrak{g}_\mathbb{C}$.

2.4. **Towers of spherical subgroups.** By a tower of subgroups of a group we mean just a nested sequence of two subgroups. Let

$$H \subset H' \subset G \tag{2.5}$$

be a tower of reductive subgroups in G for which the upper quotient $Z' = G/H'$ is real spherical. We let $P \subset G$ be a minimal parabolic subgroup such that $PH' \subset G$ is open, and let Q' be the parabolic subgroup adapted to Z' and P. According to Lemma 2.3, $P_{H'}$ is a minimal parabolic subgroup of $S' := \mathrm{S}(G, H') = Q'_{H'}$.

We recall from Part I the following necessary and sufficient condition that also the deeper quotient $Z = G/H$ of (2.5) is real spherical.

Proposition 2.4. *G/H is real spherical if and only if H'/H is real spherical for the action of S', i.e. $P_{H'}$ has an open orbit on H'/H. If this is the case then $H' = S'H$, and*

$$H'/H \simeq S'/S'_H\,. \tag{2.6}$$

Proof. See Prop. 2.9 and Cor. 2.10 in Part I. □

We now assume that G/H is real spherical, and we want to relate the structural algebras of the various quotients in (2.5) to each other. For that we assume that P has been chosen such that PH is open in G. This implies the previous assumption that PH' is open, and hence we can maintain the notation related to Q' from above.

Proposition 2.5. *Let $Q \supset P$ be adapted to G/H and let $S := \mathrm{S}(G, H) = Q_H$. Then the following assertions hold:*

1. *$Q \subset Q'$ and $S \subset S'$.*
2. *$Q_{S'} = Q_{H'} \supset P_{H'}$, and hence $Q_{H'}$ is a parabolic subgroup of S'*
3. *$P_{H'}S'_H$ is open in S'.*
4. *$Q_{H'}$ is adapted to the real spherical space S'/S'_H and $P_{H'} \subset S'$.*
5. *The associated structural algebras satisfy*

$$\mathfrak{s}\big[\mathfrak{s}(\mathfrak{g}, \mathfrak{h}'), \mathfrak{s}(\mathfrak{g}, \mathfrak{h}') \cap \mathfrak{h}\big] = \mathfrak{s}(\mathfrak{g}, \mathfrak{h})\,. \tag{2.7}$$

Proof. (1) It is sufficient to establish $Q \subset Q'$ as this implies $Q_H \subset Q'_{H'}$. We recall the characterization of $Q_\mathbb{C}$ from [9, Lemma 3.7]: $Q_\mathbb{C} = \{g \in G_\mathbb{C} \mid gP_\mathbb{C}H_\mathbb{C} = P_\mathbb{C}H_\mathbb{C}\}$. Likewise $Q'_\mathbb{C} = \{g \in G_\mathbb{C} \mid gP_\mathbb{C}H'_\mathbb{C} = P_\mathbb{C}H'_\mathbb{C}\}$. In particular $Q_\mathbb{C}H_\mathbb{C} = P_\mathbb{C}H_\mathbb{C}$, and by multiplying this with $H'_\mathbb{C}$ from the right we obtain $Q_\mathbb{C}H'_\mathbb{C} = P_\mathbb{C}H'_\mathbb{C}$. Thus $Q_\mathbb{C} \subset Q'_\mathbb{C}$ by the aforementioned characterization of Q'. Intersecting the obtained inclusion with G finally yields $Q \subset Q'$.

(2) The asserted equality $Q_{H'} = Q_{S'}$ amounts to showing $Q_{H'} \subset S'$. Since $S' = Q'_{H'}$ by definition, this follows from $Q \subset Q'$ which we established in (1).

(3) The fact that PH is open in G implies that $P_{H'}H$ is open in H'. It then follows from the isomorphism in (2.6) that $P_{H'}S'_H$ is open in S'.

(4) Let $\mathcal{L} \subset Q_{H'}$ be any choice of Levi subgroup with $\mathcal{L} \supset S$. Since $\mathcal{L} \subset Q = LU$ we have $\mathcal{L}_H = Q_H = S$ and $\mathcal{L}_{\mathrm{n}} = L_{\mathrm{n}} \subset S$. This then yields a Levi decomposition $Q_{H'} = \mathcal{L} \ltimes U_{H'}$ for which we need to verify (2.2) - (2.4) for S'/S'_H. Now $PH = QH$ yields $P_{H'}H = Q_{H'}H$ which is (2.2). Further $Q_{H'} \cap H = S = Q_H = \mathcal{L} \cap H$ which is (2.3). Finally, we have $\mathcal{L}_H = S \supset \mathcal{L}_{\mathrm{n}} = L_{\mathrm{n}}$, that is (2.4).

(5) This follows immediately from (4) since $Q_{H'} \cap H = Q_H$. □

Remark 2.6. It is shown above that $Q \subset Q'$ for the parabolic subgroups of G adapted to Z and Z'. By a careful analysis of the proof of the local structure theorem in [11] one can show in addition that Levi decompositions $Q = LU$ and $Q' = L'U'$ can be chosen such that $L \subset L'$ and such that all requirements in the local structure theorem are satisfied. However, we do not need this in the current paper.

Another observation, which likewise will not be used here, is that Proposition 2.5 and the additional property just mentioned, are valid without the assumption that H and H' are reductive.

Corollary 2.7. *Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h}$ a reductive subalgebra. Assume $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ is a decomposition in ideals, and consider $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}_2$ by the inclusion map $x \mapsto (x, x_2)$, where x_2 denotes the $\mathfrak{h}_2$-component of x. Suppose that $(\mathfrak{g}, \mathfrak{h})$ is real spherical, and let $\mathfrak{s}_2$ denote the projection of $\mathfrak{s}(\mathfrak{g}, \mathfrak{h})$ to $\mathfrak{h}_2$. Then the following statements are equivalent:*

(1) $(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ *is real spherical.*
(2) $(\mathfrak{h}_2 \oplus \mathfrak{s}_2, \mathfrak{s}_2)$ *is real spherical.*

Moreover if these conditions hold, then

$$\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h}) \simeq (\mathfrak{s}(\mathfrak{g}, \mathfrak{h}) \cap \mathfrak{h}_1) \oplus \mathfrak{s}(\mathfrak{h}_2 \oplus \mathfrak{s}_2, \mathfrak{s}_2)\,. \tag{2.8}$$

Here the second summand on the right-hand side of (2.8) embeds into $\mathfrak{h} \subset \mathfrak{g} \oplus \mathfrak{h}_2$ by the restriction of the diagonal embedding $\mathfrak{s}_2 \to \mathfrak{s}(\mathfrak{g}, \mathfrak{h}) \oplus \mathfrak{s}_2$.

Proof. We apply Propositions 2.4- 2.5 to the tower

$$H \subset H' := H \times H_2 \subset G \times H_2$$

where the first inclusion map is given by $x \mapsto (x, x_2)$ as above. Then $H'/H \simeq H_2$ and $\mathrm{S}(G \times H_2, H') = S \times H_2$ where $S = \mathrm{S}(G, H)$. The action of S on H'/H factors through the projection $H \to H_2$, and hence identifies with the action of its projection S_2 on H_2. Thus H'/H is real spherical for $S \times H_2$ if and only if $[H_2 \times S_2]/S_2$ is real spherical, and the equivalence of (1) and (2) follows. Moreover (2.8) follows from (2.7). □

2.5. **Factorizations.** In the situation of Proposition 2.4(2) we have $\mathfrak{h}' = \mathfrak{s}(\mathfrak{g}, \mathfrak{h}') + \mathfrak{h}$. In general, if $\mathfrak{g}$ is a reductive Lie algebra, then a *factorization* of $\mathfrak{g}$ is a decomposition $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ of the vector space $\mathfrak{g}$ as the sum of two reductive subalgebras $\mathfrak{g}_1$, $\mathfrak{g}_2$. Factorizations of simple Lie algebras were classified by Onishchik ([22]). That classification was extensively used already in Part I. For convenience we repeat it here.

It is easily seen that for a real Lie algebra $\mathfrak{g}$ and any two subalgebras $\mathfrak{g}_1, \mathfrak{g}_2$ we have the factorization $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ if and only if $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{1,\mathbb{C}} + \mathfrak{g}_{2,\mathbb{C}}$ holds for the complexifications. Hence it suffices to deal with the complex case.

Proposition 2.8. *Let $\mathfrak{g}$ be a complex simple Lie algebra and let $\mathfrak{g}_1$, $\mathfrak{g}_2$ be proper reductive subalgebras. Then $\mathfrak{g} = \mathfrak{g}_1 + \mathfrak{g}_2$ if and only if the algebras occur in the following Table 1 (up to isomorphism and interchange of $\mathfrak{g}_1$ and $\mathfrak{g}_2$).*

$\mathfrak{g}$	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_1 \cap \mathfrak{g}_2$		
$\mathfrak{sl}(2n, \mathbb{C})$	$\mathfrak{sl}(2n-1, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{sp}(n, \mathbb{C})$	$\mathfrak{sp}(n-1, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{z} \subset \mathbb{C}$	$n \geq 2$
$\mathfrak{so}(2n, \mathbb{C})$	$\mathfrak{so}(2n-1, \mathbb{C})$	$\mathfrak{sl}(n, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{sl}(n-1, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{z} \subset \mathbb{C}$	$n \geq 4$
$\mathfrak{so}(4n, \mathbb{C})$	$\mathfrak{so}(4n-1, \mathbb{C})$	$\mathfrak{sp}(n, \mathbb{C}) + \mathfrak{f}$	$\mathfrak{sp}(n-1, \mathbb{C}) + \mathfrak{f}$	$\mathfrak{f} \subset \mathfrak{sp}(1, \mathbb{C})$	$n \geq 2$
$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(5, \mathbb{C}) + \mathfrak{z}$	$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(2, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{z} \subset \mathbb{C}$	
$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{so}(6, \mathbb{C})$	$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(3, \mathbb{C})$		
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(5, \mathbb{C}) + \mathfrak{f}$	$\mathfrak{spin}(7, \mathbb{C})$	$\mathfrak{sl}(2, \mathbb{C}) + \mathfrak{f}$	$\mathfrak{f} \subset \mathfrak{sp}(1, \mathbb{C})$	
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(6, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{spin}(7, \mathbb{C})$	$\mathfrak{sl}(3, \mathbb{C}) + \mathfrak{z}$	$\mathfrak{z} \subset \mathbb{C}$	
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{so}(7, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$	$\mathsf{G}_2^{\mathbb{C}}$		
$\mathfrak{so}(8, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})_+$	$\mathfrak{spin}(7, \mathbb{C})_-$	$\mathsf{G}_2^{\mathbb{C}}$		
$\mathfrak{so}(16, \mathbb{C})$	$\mathfrak{so}(15, \mathbb{C})$	$\mathfrak{spin}(9, \mathbb{C})$	$\mathfrak{spin}(7, \mathbb{C})$		

Table 1

In Table 1 the subscripts $\pm$ to $\mathfrak{spin}(7, \mathbb{C})$ indicate representatives from its two conjugacy classes in $\mathfrak{so}(8, \mathbb{C})$.

3. Generalities and basic reductions

3.1. **Two lemmas.** We state two simple lemmas in which $\mathfrak{g}_1, \mathfrak{g}_2$ are reductive Lie algebras, $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, and $\mathfrak{h} \subset \mathfrak{g}$ is a reductive subalgebra with projections $\mathfrak{h}_1$, $\mathfrak{h}_2$ to $\mathfrak{g}_1, \mathfrak{g}_2$. Moreover, $Z_i = G_i/H_i$.

Lemma 3.1. *The pair $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if the following two conditions are both satisfied*

(1) $(\mathfrak{g}_1, \mathfrak{h}_1)$ *and* $(\mathfrak{g}_2, \mathfrak{h}_2)$ *are real spherical, and*

(2) $(H_1 \times H_2)/H$ *is a real spherical variety for the action of* $\mathrm{S}(G_1, H_1) \times \mathrm{S}(G_2, H_2)$.

In this case the action in (2) is transitive.

Proof. We consider the tower $H \subset H' := H_1 \times H_2$ in $G = G_1 \times G_2$. If G/H is real spherical, then so is G/H', and hence (1) holds. On the other hand, if we assume (1) and apply Proposition 2.4, we see that G/H is real spherical if and only if (2) holds. The final statement comes from (2.6). □

Note that as a consequence of the final statement in the lemma, condition (2) can be replaced by the following Lie algebraic version. There exists for $i = 1, 2$ some minimal parabolic subalgebra $\mathfrak{p}_i$ of $\mathfrak{s}(\mathfrak{g}_i, \mathfrak{h}_i)$ such that

$$\mathfrak{h}_1 \oplus \mathfrak{h}_2 = (\mathfrak{p}_1 \oplus \mathfrak{p}_2) + \mathfrak{h}\,.$$

Recall from the introduction that a reductive Lie algebra $\mathfrak{g}$ is called *elementary* if $[\mathfrak{g}, \mathfrak{g}]$ is compact.

Lemma 3.2. *The following assertions hold:*

(1) *If* $\mathfrak{g}_2$ *is elementary, then* $(\mathfrak{g}, \mathfrak{h})$ *is real spherical if and only if* $(\mathfrak{g}_1, \mathfrak{h}_1)$ *is real spherical.*
(2) *If* $(\mathfrak{g}, \mathfrak{h})$ *is real spherical, then so is* $(\mathfrak{g}_1 \oplus \mathfrak{h}_2, \mathfrak{h})$.

Proof. For the proof of (1) we only need to observe that when the ideal $\mathfrak{g}_2$ is elementary then it is contained in $\mathfrak{p}$.

For (2) we consider the following two towers

$$\mathfrak{h} \subset \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{g} \tag{3.1}$$

$$\mathfrak{h} \subset \mathfrak{h}_1 \oplus \mathfrak{h}_2 \subset \mathfrak{g}_1 \oplus \mathfrak{h}_2. \tag{3.2}$$

Lemma 3.1 applied to (3.1) gives that $(H_1 \times H_2)/H$ is real spherical for $\mathrm{S}(G_1, H_1) \times \mathrm{S}(G_2, H_2)$. Hence it is real spherical also for $\mathrm{S}(G_1, H_1) \times H_2$, and then the opposite implication of Lemma 3.1 applied to (3.2) implies the sphericality of $(\mathfrak{g}_1 \oplus \mathfrak{h}_2, \mathfrak{h})$. □

3.2. **Notions of indecomposability and equivalence.** We say that the pair $(\mathfrak{g}, \mathfrak{h})$ is *decomposable* if there exists a non-trivial decomposition $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ in ideals of $\mathfrak{g}$ such that $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$ with $\mathfrak{h}_i = \mathfrak{h} \cap \mathfrak{g}_i$. It is clear that in this case $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if both components $(\mathfrak{g}_i, \mathfrak{h}_i)$ are real spherical. If $(\mathfrak{g}, \mathfrak{h})$ is not decomposable, then we call it *indecomposable*. We conclude that we only need to classify the indecomposable real spherical pairs.

Furthermore, it follows immediately from Lemma 3.2(1) that it suffices to classify those real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ for which $\mathfrak{g}$ is semi-simple and without compact factors.

However, as mentioned in the introduction we need to exclude a certain circumstance in order to prevent that the classification becomes unwieldy. Recall that by $\mathfrak{h}_{\mathrm{n}}$ we denote the largest non-elementary ideal of $\mathfrak{h}$. We say that $(\mathfrak{g}, \mathfrak{h})$ is *strictly indecomposable* provided $(\mathfrak{g}, \mathfrak{h}_{\mathrm{n}})$ is indecomposable.

By definition we consider two pairs $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ *equivalent* if there exists an isomorphism of $\mathfrak{g}_1$ onto $\mathfrak{g}_2$ which carries $\mathfrak{h}_1$ onto $\mathfrak{h}_2$.

Outline. The classification that will be given in Sections 4 and 5 consists of *all strictly indecomposable real spherical pairs* $(\mathfrak{g}, \mathfrak{h})$*, up to equivalence, for which* $\mathfrak{g}$ *is semi-simple and without compact factors and* $\mathfrak{h} \subset \mathfrak{g}$ *is algebraic reductive.* In Section 6 we discuss the indecomposable real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ as above which are not strictly indecomposable.

3.3. **A basic condition.** Before we commence with the classification we need one more observation. We assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ is semi-simple with each $\mathfrak{g}_i$ simple and non-compact. On the group level we let $G = G_1 \times \ldots \times G_k$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $H \subset G$ be the connected subgroup corresponding to $\mathfrak{h}$.

We write $p_i : \mathfrak{g} \to \mathfrak{g}_i$ for the various projections and set $\mathfrak{h}_i := p_i(\mathfrak{h})$. We also use p_i for the projection on the group level $G \to G_i$ and set $H_i = p_i(H)$. It follows from Lemma 3.1(1) that if $(\mathfrak{g}, \mathfrak{h})$ is real spherical then $(\mathfrak{g}_i, \mathfrak{h}_i)$ is real spherical for all $1 \leq i \leq k$.

For the purpose of proving our classification we can thus assume from the outset that each pair $(\mathfrak{g}_i, \mathfrak{h}_i)$ is real spherical.

4. The case of two factors

We assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ consists of two simple non-compact factors and that $\mathfrak{h} \subset \mathfrak{g}$ is an algebraic reductive subalgebra such that $(\mathfrak{g}, \mathfrak{h})$ is a strictly indecomposable real spherical pair. A complete classification of such pairs will be given in Theorem 4.1.

4.1. **Preliminaries.** Let

$$H_1' = \{x_1 \in G_1 \mid (x_1, \mathbf{1}) \in H\}, \quad H_2' = \{x_2 \in G_2 \mid (\mathbf{1}, x_2) \in H\}.$$

These are easily seen to be normal subgroups of H_1 and H_2, respectively. It follows that $H_1' \times H_2'$ is a normal subgroup of $H_1 \times H_2$, hence also of H. Let

$$H^0 := H/[H_1' \times H_2']$$

with projection map $p^0 : H \to H^0$. Note that H^0 is not elementary as otherwise $(\mathfrak{g}, \mathfrak{h}_n)$ would be decomposable.

For $x \in H$ we have $p_1(x) \in H_1' \Leftrightarrow p_2(x) \in H_2'$, and hence p_1 and p_2 induce diffeomorphisms $H^0 \to H_1/H_1'$ and $H^0 \to H_2/H_2'$. By means of the first we regard H^0 as a homogeneous space for H_1 (with trivial H_1'-action). Likewise for the other, except that it will be more convenient to equip H^0 with the right action of H_2 associated to $H^0 \to H_2' \backslash H_2$.

It is now easily seen that by

$$(p_1(x), p_2(y)) \mapsto p^0(xy^{-1}), \quad x, y \in H$$

we obtain a well-defined map $H_1 \times H_2 \to H^0$. This map is equivariant and induces an isomorphism of homogeneous spaces

$$[H_1 \times H_2]/H \xrightarrow{\sim} H^0. \tag{4.1}$$

Let $\mathfrak{h}_i'' \triangleleft \mathfrak{h}_i$ be a complementary ideal to $\mathfrak{h}_i'$. Observe that $\mathfrak{h}_i'' \simeq \mathfrak{h}^0$ and

$$\mathfrak{h} \simeq \mathfrak{h}_1' \oplus \mathfrak{h}^0 \oplus \mathfrak{h}_2' \tag{4.2}$$

Let $\mathrm{S}_i' := \mathrm{S}(G, H_i) \cap H_i'$ and let $S_i'' \subset H_i/H_i' \simeq H^0$ be the projection of $\mathrm{S}(G, H_i)$ for $i = 1, 2$. This leads to the exact sequence

$$\mathbf{1} \longrightarrow S_i' \longrightarrow \mathrm{S}(G, H_i) \longrightarrow S_i'' \longrightarrow \mathbf{1},$$

which splits on the Lie algebra level and yields a local isomorphism (that is, Lie algebras are isomorphic)

$$\mathrm{S}(G, H_i) \simeq S_i' \times S_i''. \tag{4.3}$$

We recall from Lemma 3.1(2) that $Z = G/H$ is real spherical if and only if $[H_1 \times H_2]/H$ is real spherical for $\mathrm{S}(G, H_1) \times \mathrm{S}(G, H_2)$. Under the identifications (4.1), (4.3) this means that H^0 is real spherical as $S_1'' \times S_2''$-variety. Moreover, we record that in this case

$$H^0 = [S_1'' \times S_2'']/[S_1'' \cap S_2''] \tag{4.4}$$

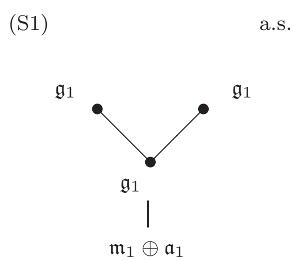
and by (2.7) there is a local isomorphism

$$\mathrm{S}(G, H) \simeq S_1' \times S_2' \times \mathrm{S}(S_1'' \times S_2'', S_1'' \cap S_2''). \tag{4.5}$$

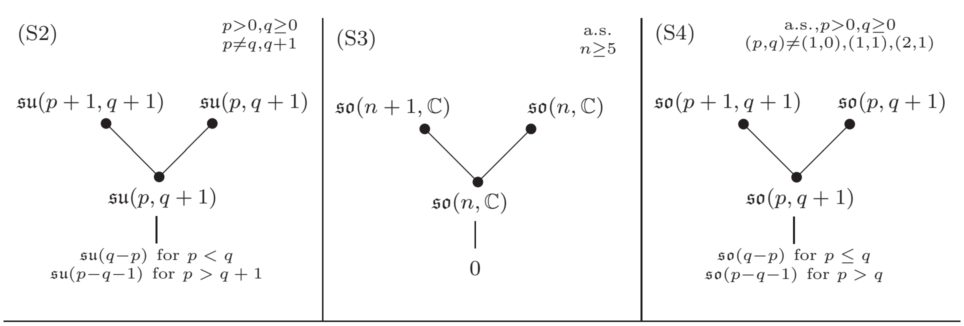
4.2. **The diagrams.** In the following diagrams we describe a real spherical pair $(\mathfrak{g}, \mathfrak{h})$ together with its structural algebra $\mathfrak{s}(\mathfrak{g}, \mathfrak{h})$ by a three-layered graph. The vertices in the top row consist of the simple factors $\mathfrak{g}_i$ of $\mathfrak{g}$ and those in the second row of the simple (or central) factors $\mathfrak{h}^{(j)}$ of $\mathfrak{h}$. There is an edge between $\mathfrak{g}_i$ and $\mathfrak{h}^{(j)}$ if $p_i(\mathfrak{h}^{(j)})$ is non-zero. Likewise in the third row we list the factors of $\mathfrak{s}(\mathfrak{g}, \mathfrak{h})$ and the edges between the middle and last row correspond to non-zero embeddings into the various of factors of $\mathfrak{h}$. It might happen in the third rows that certain indices become negative, e.g. $\mathfrak{su}(p-2, q)$ for $p = 0, 1$, and then our convention is that the corresponding factor of $\mathfrak{s}(\mathfrak{g}, \mathfrak{h})$ is $\{0\}$. Cases which are absolutely spherical are marked with a.s. in the upper right corner of the corresponding box.

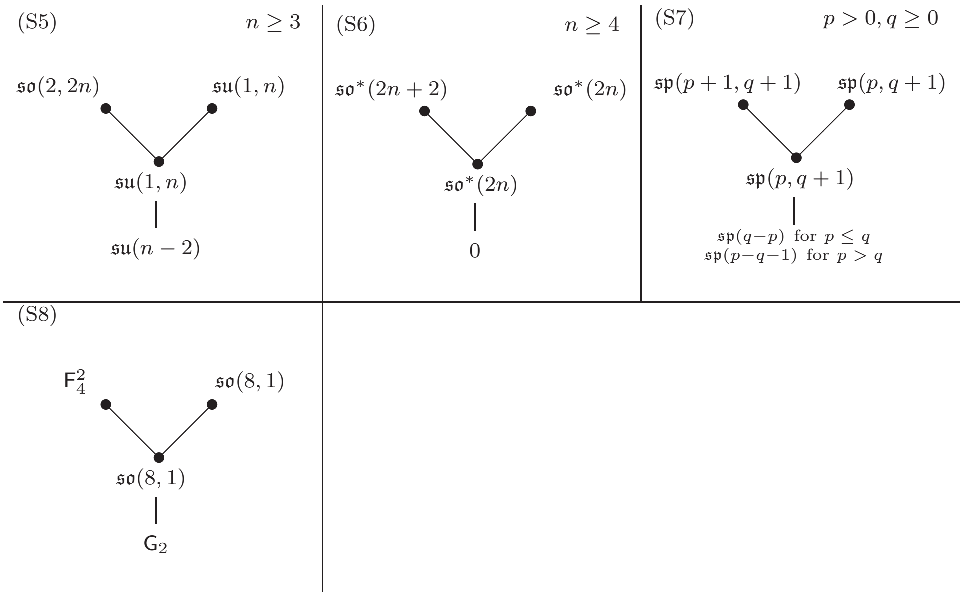
Theorem 4.1. *Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ be the decomposition of $\mathfrak{g}$ into simple non-compact subalgebras, and let $\mathfrak{h}$ be an algebraic reductive subalgebra such that $(\mathfrak{g}, \mathfrak{h})$ is strictly indecomposable. Then $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if it is equivalent to an element of the following list.*

1) *If $\mathfrak{h}$ is simple, then $(\mathfrak{g}, \mathfrak{h})$ is either*
   1a) *of group type, i.e. $\mathfrak{g}_1 = \mathfrak{g}_2$ and*



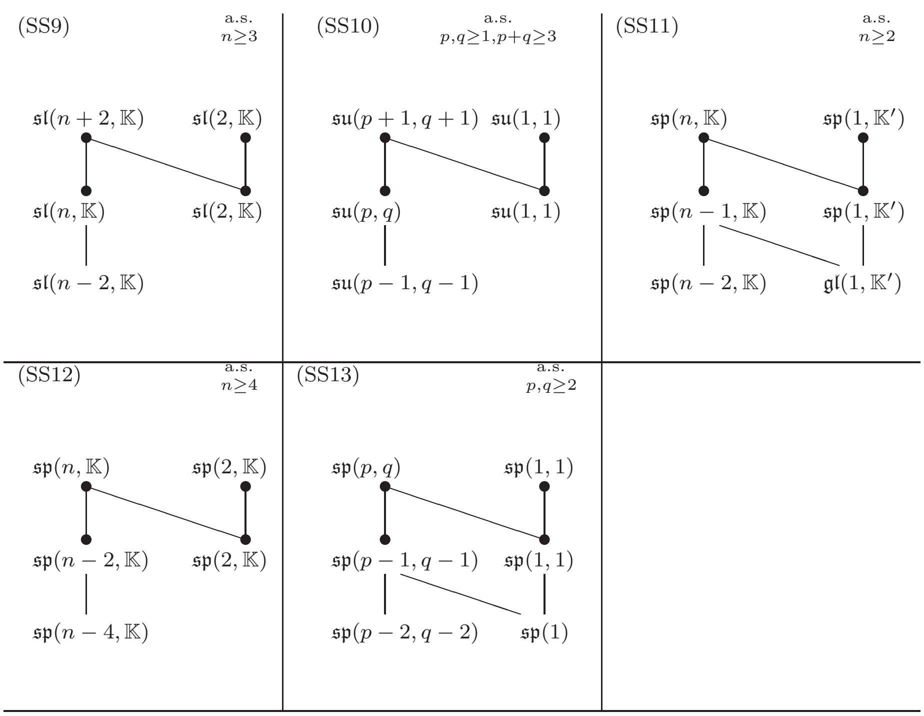
   with $\mathfrak{g}_1$ non-compact simple and with $\mathfrak{m}_1$ and $\mathfrak{a}_1$ defined as in Subsection 2.3.
   1b) *or exactly one $\mathfrak{h}_i$ equals $\mathfrak{g}_i$. In this case, $(\mathfrak{g}, \mathfrak{h})$ is equivalent to one of the following pairs:*

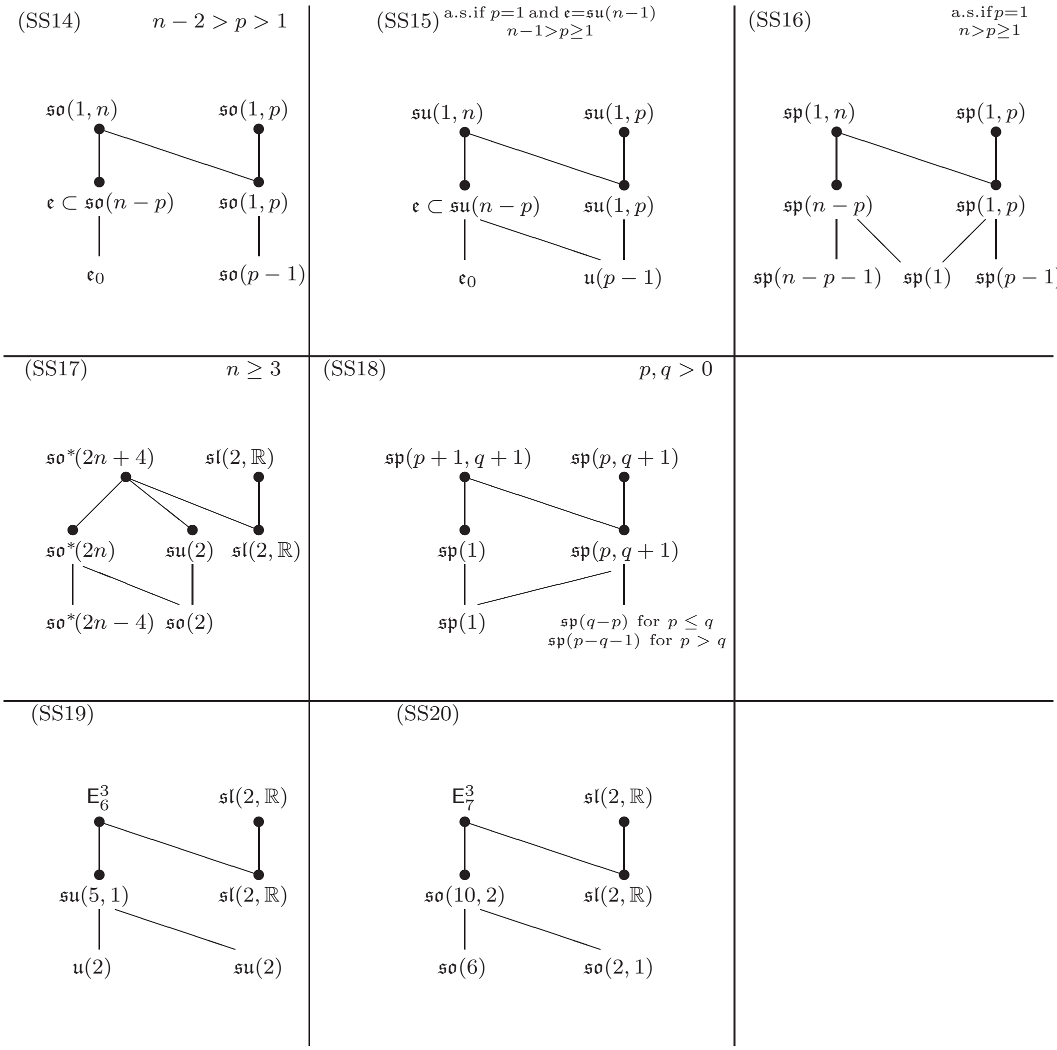




2) *If* $\mathfrak{h}$ *is semi-simple but not simple, then either*

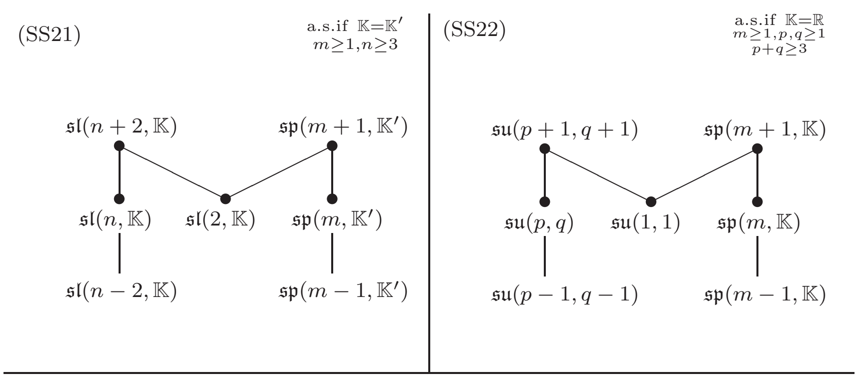
2a) $\mathfrak{h}_2 = \mathfrak{g}_2$ *and* $(\mathfrak{g}, \mathfrak{h})$ *is equivalent to one of the following pairs, where* $\mathbb{K}, \mathbb{K}' = \mathbb{R}$ *or* $\mathbb{C}$ *and* $\mathbb{K}' \subset \mathbb{K}$

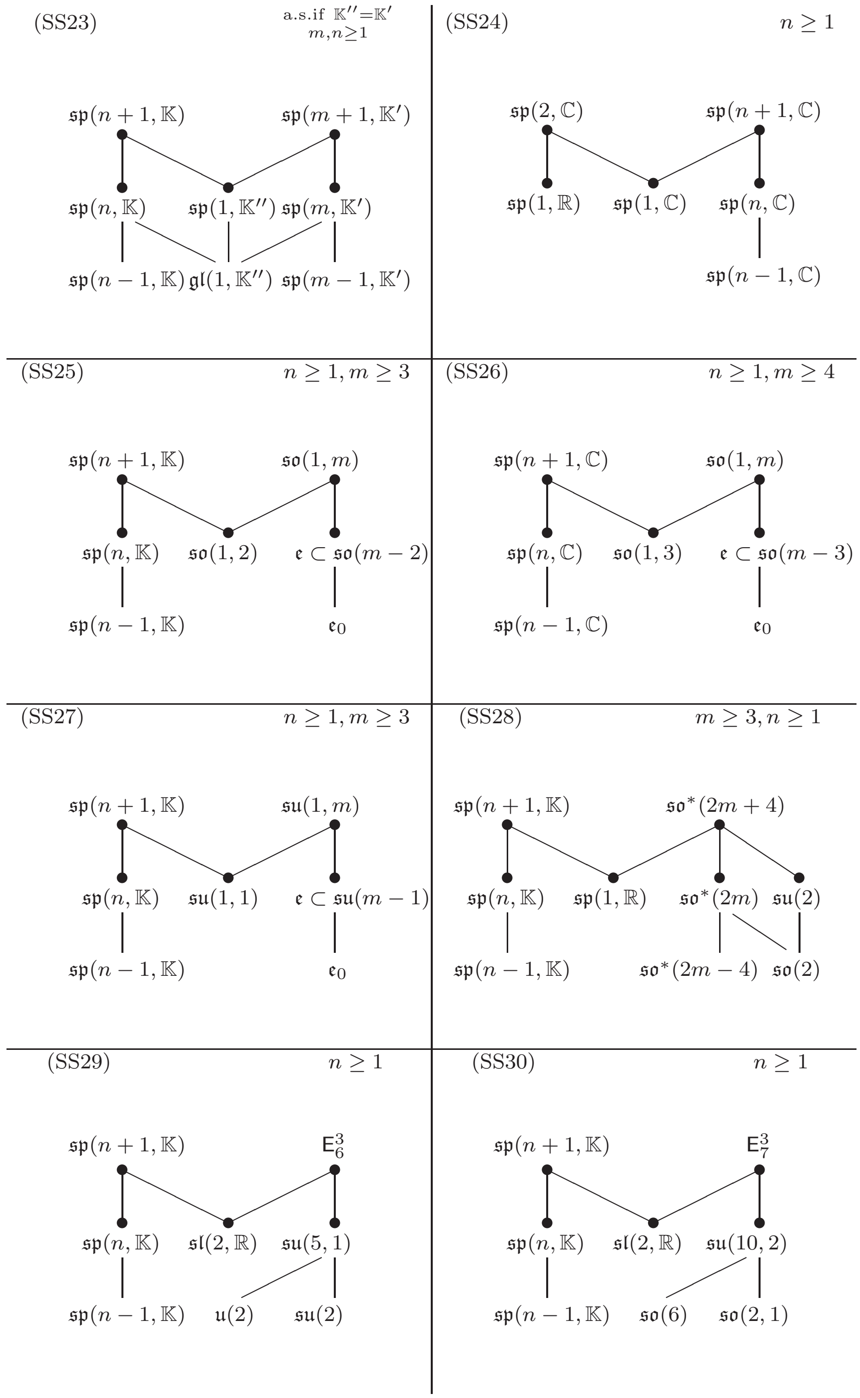




For the various possibilities of $\mathfrak{e}$ *and* $\mathfrak{e}_0$ *in* (SS14) *and* (SS15) *see Table 8.*

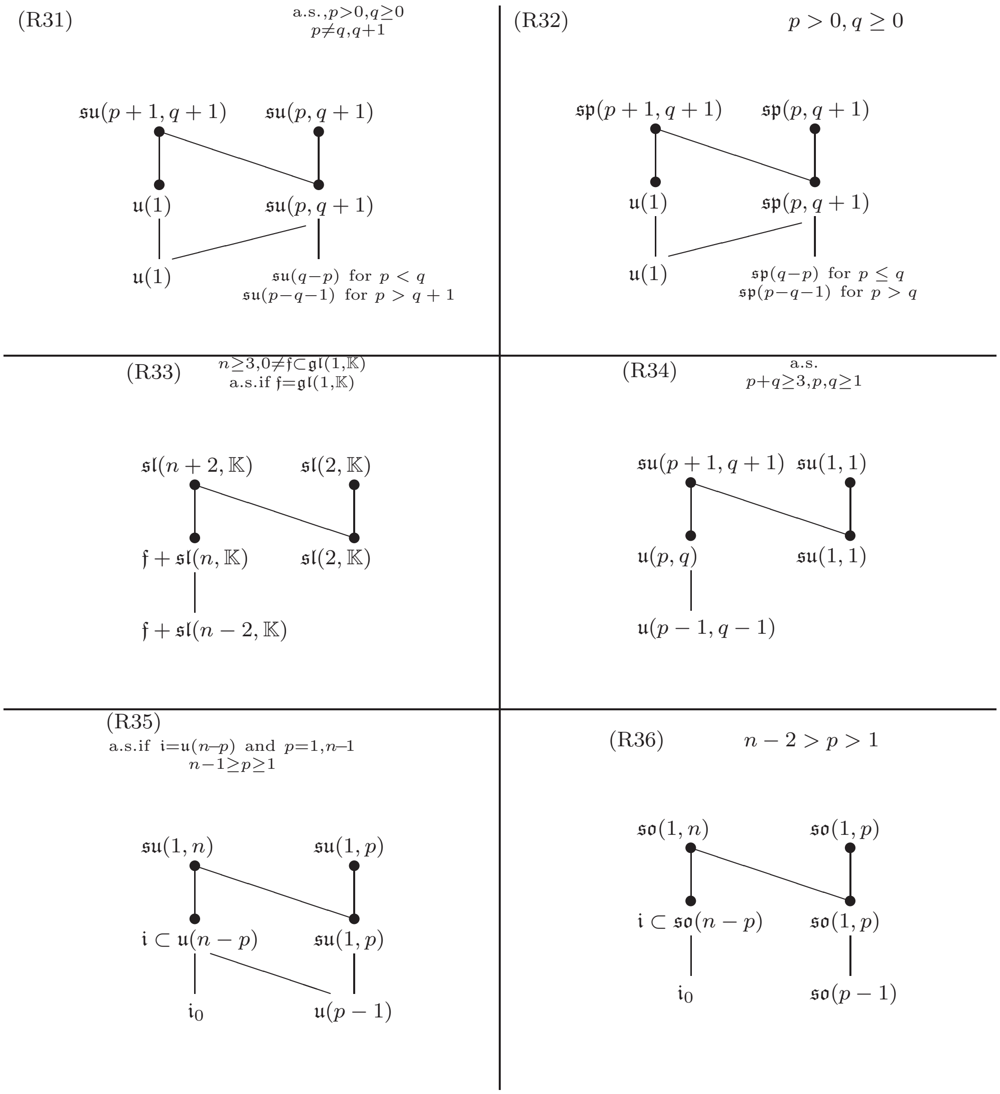
2b) *or* $\mathfrak{h}_i \neq \mathfrak{g}_i$ *for* $i = 1, 2$ *and* $(\mathfrak{g}, \mathfrak{h})$ *is equivalent to one of the pairs where* $\mathbb{K}, \mathbb{K}', \mathbb{K}'' = \mathbb{R}, \mathbb{C}$ *and* $\mathbb{K}'' \subset \mathbb{K} \subset \mathbb{K}'$

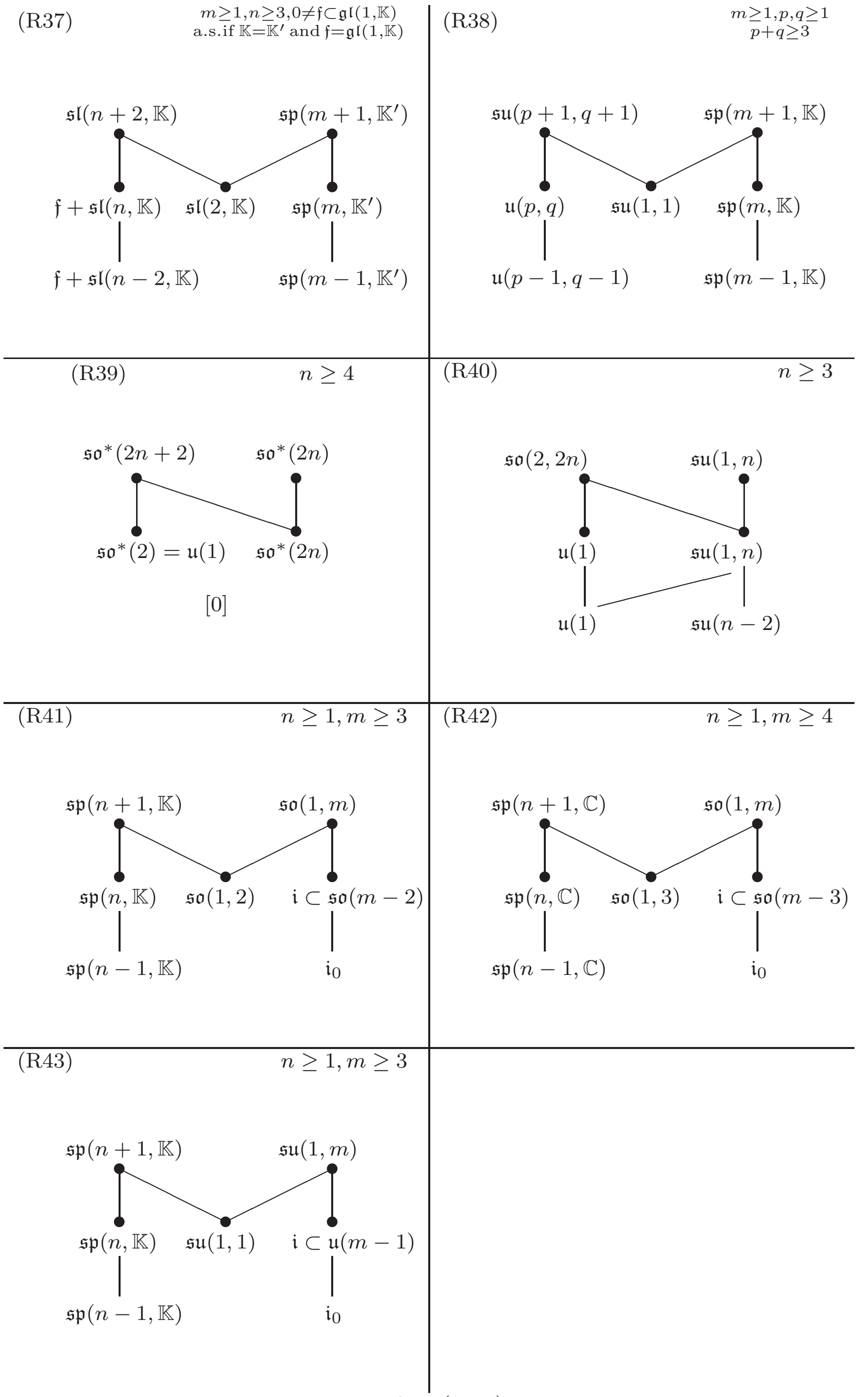




3) *If $\mathfrak{h}=[\mathfrak{h},\mathfrak{h}]+\mathfrak{z}$ is not semi-simple, then there are two classes:*

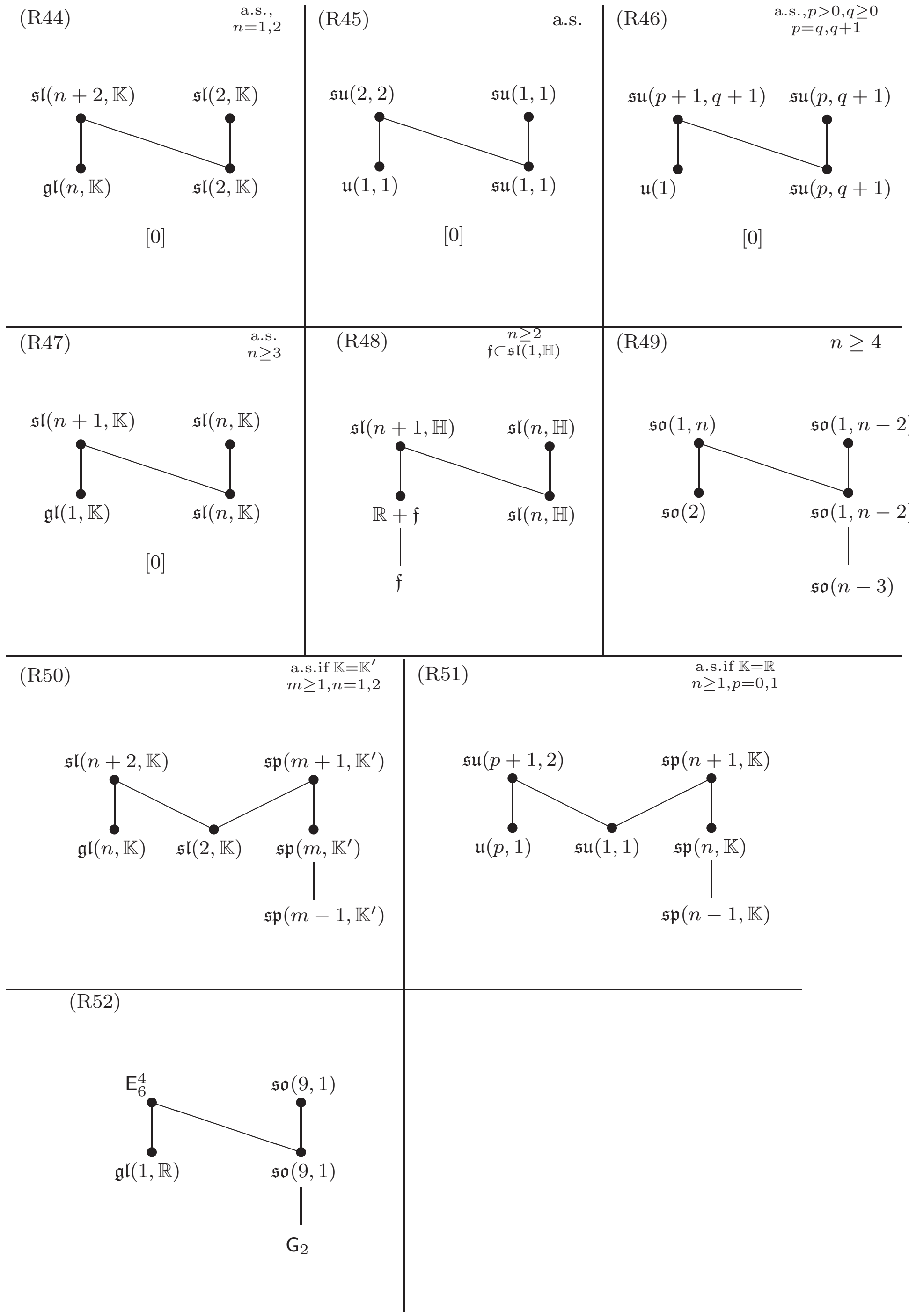
3a) $(\mathfrak{g},[\mathfrak{h},\mathfrak{h}])$ *is already real spherical, hence belongs to* 1) *or* 2). *Then* $(\mathfrak{g},\mathfrak{h})$ *appears in the following list, where* $\mathbb{K}\subset\mathbb{K}'$*:*


(R31)
a.s., $p>0, q\geq 0$
$p\neq q, q+1$
$\mathfrak{su}(p+1,q+1)$
$\mathfrak{su}(p,q+1)$
$\mathfrak{u}(1)$
$\mathfrak{su}(p,q+1)$
$\mathfrak{u}(1)$
$\mathfrak{su}(q-p)$ for $p<q$
$\mathfrak{su}(p-q-1)$ for $p>q+1$
(R32)
$p>0, q\geq 0$
$\mathfrak{sp}(p+1,q+1)$
$\mathfrak{sp}(p,q+1)$
$\mathfrak{u}(1)$
$\mathfrak{sp}(p,q+1)$
$\mathfrak{u}(1)$
$\mathfrak{sp}(q-p)$ for $p\leq q$
$\mathfrak{sp}(p-q-1)$ for $p>q$
(R33)
$n\geq 3, 0\neq \mathfrak{f}\subset \mathfrak{gl}(1,\mathbb{K})$
a.s.if $\mathfrak{f}=\mathfrak{gl}(1,\mathbb{K})$
$\mathfrak{sl}(n+2,\mathbb{K})$
$\mathfrak{sl}(2,\mathbb{K})$
$\mathfrak{f}+\mathfrak{sl}(n,\mathbb{K})$
$\mathfrak{sl}(2,\mathbb{K})$
$\mathfrak{f}+\mathfrak{sl}(n-2,\mathbb{K})$
(R34)
a.s.
$p+q\geq 3, p,q\geq 1$
$\mathfrak{su}(p+1,q+1)$
$\mathfrak{su}(1,1)$
$\mathfrak{u}(p,q)$
$\mathfrak{su}(1,1)$
$\mathfrak{u}(p-1,q-1)$
(R35)
a.s.if $\mathfrak{i}=\mathfrak{u}(n-p)$ and $p=1,n-1$
$n-1\geq p\geq 1$
$\mathfrak{su}(1,n)$
$\mathfrak{su}(1,p)$
$\mathfrak{i}\subset\mathfrak{u}(n-p)$
$\mathfrak{su}(1,p)$
$\mathfrak{i}_0$
$\mathfrak{u}(p-1)$
(R36)
$n-2>p>1$
$\mathfrak{so}(1,n)$
$\mathfrak{so}(1,p)$
$\mathfrak{i}\subset\mathfrak{so}(n-p)$
$\mathfrak{so}(1,p)$
$\mathfrak{i}_0$
$\mathfrak{so}(p-1)$



For the various possibilities for $(\mathfrak{i},\mathfrak{i}_0)$ *see Table 9.*

3b) $(\mathfrak{g},[\mathfrak{h},\mathfrak{h}])$ *is not real spherical. Then* $(\mathfrak{g},\mathfrak{h})$ *appears in the following list where* $\mathbb{K}\subset\mathbb{K}'$*:*



This theorem will be proven in various subsequent lemmas.

4.3. **The case where $\mathfrak{h}$ is simple.** If $\mathfrak{h}$ is simple, we have $\mathfrak{h}=\mathfrak{h}^0$ with respect to (4.2). We can assume that $(\mathfrak{g},\mathfrak{h})$ is not of group type. The building blocks for this situation will be when $\mathfrak{h}$ is isomorphic to one factor, say $\mathfrak{h}\simeq\mathfrak{g}_2$. For the treatment of this case we use simplified notation: Let $\mathfrak{g}$ be a reductive Lie algebra and $\mathfrak{h}\subset\mathfrak{g}$ a reductive subalgebra. Recall that we say that $(\mathfrak{g},\mathfrak{h})$ is *strongly real spherical* if the pair $(\mathfrak{g}\oplus\mathfrak{h},\mathrm{diag}(\mathfrak{h}))$ is real spherical. In the first step we classify all strongly real spherical pairs with $\mathfrak{g}$ and $\mathfrak{h}$ both non-compact simple.

We let $Q = LU$ be the parabolic subgroup adapted to G/H, as in the local structure theorem, and put $L_H = L \cap H$. An immediate consequence of Corollary 2.7 then is:

Corollary 4.2. *Let $(\mathfrak{g}, \mathfrak{h})$ be a real spherical pair with structural algebra $\mathfrak{s} = \mathfrak{s}(\mathfrak{g}, \mathfrak{h})$, where $\mathfrak{g}$ is reductive and $\mathfrak{h}$ reductive in $\mathfrak{g}$. Then the following conditions are equivalent:*

(1) *$(\mathfrak{g}, \mathfrak{h})$ is strongly real spherical.*
(2) *$(\mathfrak{h}, \mathfrak{s})$ is strongly real spherical.*

Moreover if these conditions hold, then

$$\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{h}, \mathfrak{h}) = \mathfrak{s}(\mathfrak{h} \oplus \mathfrak{s}, \mathfrak{s})\,. \tag{4.6}$$

Proof. Let $\mathfrak{h}_1 = \{0\}$ and $\mathfrak{h}_2 = \mathfrak{h}$ in Corollary 2.7. □

Note that in particular, if (G, H) is strongly real spherical, we have that L_H is a spherical subgroup of H.

Lemma 4.3. *Let $\mathfrak{g}$ be a simple Lie algebra and $\mathfrak{h} \subsetneq \mathfrak{g}$ a simple non-compact real spherical subalgebra. Let $\mathfrak{s} = \mathfrak{s}(\mathfrak{g}, \mathfrak{h})$ denote the associated structural algebra.*

(a) *The pair $(\mathfrak{h}, \mathfrak{s})$ is real spherical if and only if $(\mathfrak{g}, \mathfrak{h})$ belongs (up to equivalence) to the following list of triples $(\mathfrak{g}, \mathfrak{h}, \mathfrak{s})$:*
   (1) $(\mathfrak{sl}(n+2, \mathbb{K}), \mathfrak{sl}(n+1, \mathbb{K}), \mathfrak{sl}(n, \mathbb{K}))$ *for* $\mathbb{K} = \mathbb{R}, \mathbb{C}, \mathbb{H}$ *and* $n \geq 1$
   (2) $(\mathfrak{su}(p+1, q+1), \mathfrak{su}(p, q+1), \mathfrak{su}(p, q))$ *for* $p + q \geq 1$
   (3) $(\mathfrak{so}(n+1, \mathbb{C}), \mathfrak{so}(n, \mathbb{C}), \mathfrak{so}(n-1, \mathbb{C}))$ *for* $n \geq 5$
   (4) $(\mathfrak{so}(p+1, q+1), \mathfrak{so}(p, q+1), \mathfrak{so}(p, q))$ *for* $p + q \geq 2$ *and* $(p, q) \neq (1, 1), (2, 1)$
   (5) $(\mathfrak{so}^*(2n+4), \mathfrak{so}^*(2n+2), \mathfrak{so}^*(2n))$ *for* $n \geq 3$
   (6) $(\mathfrak{so}(2, 2n), \mathfrak{su}(1, n), \mathfrak{su}(n-1) \oplus \mathfrak{su}(1, 1))$ *for* $n \geq 3$
   (7) $(\mathfrak{so}(7, \mathbb{C}), \mathsf{G}_2^{\mathbb{C}}, \mathfrak{sl}(3, \mathbb{C}))$
   (8) $(\mathfrak{so}(3, 4), \mathsf{G}_2^1, \mathfrak{sl}(3, \mathbb{R}))$
   (9) $(\mathfrak{sp}(p+1, q+1), \mathfrak{sp}(p, q+1), \mathfrak{sp}(p, q))$ *for* $p > 0$
   (10) $(\mathsf{F}_4^{\mathbb{C}}, \mathfrak{so}(9, \mathbb{C}), \mathfrak{spin}(7, \mathbb{C}))$
   (11) $(\mathsf{F}_4^1, \mathfrak{so}(4, 5), \mathfrak{spin}(3, 4))$
   (12) $(\mathsf{F}_4^2, \mathfrak{so}(1, 8), \mathfrak{spin}(7))$
   (13) $(\mathsf{G}_2^{\mathbb{C}}, \mathfrak{sl}(3, \mathbb{C}), \mathfrak{sl}(2, \mathbb{C}))$
   (14) $(\mathsf{G}_2^1, \mathfrak{sl}(3, \mathbb{R}), \mathfrak{sl}(2, \mathbb{R}))$
   (15) $(\mathsf{G}_2^1, \mathfrak{su}(2, 1), \mathfrak{sl}(2, \mathbb{R}))$

(b) *The strongly real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ are:*
   (2) *for* $p \notin \{q, q+1\}$, (3), (4), (5), (6), (9) *and* (12).

Moreover, the structural algebras $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ *are given by*

$(\mathfrak{g},\mathfrak{h})$	$\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$
(2)	$\begin{cases}\mathfrak{su}(q-p) & \text{for } p<q\\ \mathfrak{su}(p-q-1) & \text{for } p>q+1\end{cases}$
(3)	$\{0\}$
(4)	$\begin{cases}\mathfrak{so}(q-p) & \text{for } p\leq q\\ \mathfrak{so}(p-q-1) & \text{for } p>q\end{cases}$
(5)	$\{0\}$
(6)	$\mathfrak{su}(n-2)$
(9)	$\begin{cases}\mathfrak{sp}(q-p) & \text{for } p\leq q\\ \mathfrak{sp}(p-q-1) & \text{for } p>q\end{cases}$
(12)	G_2

Proof. Part (a) can be easily obtained from Tables 3 - 7. We proceed with (b) and let $(\mathfrak{g},\mathfrak{h})$ be a pair such that $(\mathfrak{g},\mathfrak{h},\mathfrak{s})$ satisfies (a).

In the cases where $(\mathfrak{g},\mathfrak{h})$ is complex, we can refer to [4], [21]. Hence, it follows that (3) is strongly real spherical, while (7), (10) and (13) are not. Further (11) is a split real form of (10) and hence is not strongly real spherical. However, (4) is a real form of (3), hence it is also strongly real spherical by Lemma 2.1.

There is an alternative to verify that (3) is strongly spherical, which has the advantage that it also yields the structural algebra $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$. Indeed by Corollary 4.2, $(\mathfrak{so}(n+1,\mathbb{C}),\mathfrak{so}(n,\mathbb{C}))$ is strongly real spherical if and only if $(\mathfrak{so}(n,\mathbb{C}),\mathfrak{so}(n-1,\mathbb{C}))$ is strongly real spherical, and the structural algebras for these two pairs are isomorphic. Hence iteration brings us down to the case $(\mathfrak{g},\mathfrak{h})=(\mathfrak{so}(4,\mathbb{C}),\mathfrak{so}(3,\mathbb{C}))$. Because of the isomorphism $\mathfrak{so}(4,\mathbb{C})=\mathfrak{so}(3,\mathbb{C})\oplus\mathfrak{so}(3,\mathbb{C})$ this pair is known to be strongly real spherical with trivial structural algebra.

In a similar way we can treat (4). If $p=q$, then both $\mathfrak{g}$ and $\mathfrak{h}$ are quasisplit, and $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\{0\}$ by what we have just shown. Suppose that $p<q$. Iteration brings us down to $(\mathfrak{so}(1,q+1-p),\mathfrak{so}(q-p+1))$ which is strongly real spherical by Lemma 3.2(1). Moreover, $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathfrak{so}(q-p)$. If $p>q$, then iteration leaves us with $(\mathfrak{so}(p-q,1),\mathfrak{so}(p-q))$ which has structural algebra $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathfrak{so}(p-q-1)$.

We proceed analogously in (2) and (9). For $p\leq q$ the sequence for $(\mathfrak{g},\mathfrak{h})=(\mathfrak{su}(p+1,q+1),\mathfrak{su}(p,q+1))$ ends up at $(\mathfrak{su}(1,q-p+1),\mathfrak{su}(q-p+1))$ which is strongly real spherical if and only if $p<q$ by Lemma 3.2(1). Moreover, if $p<q$ one has $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathfrak{su}(q-p)$. Similarly, for $p>q$, the sequence terminates at $(\mathfrak{su}(1,p-q),\mathfrak{su}(p-q))$ which is strongly real spherical if and only if $p>q+1$ and then has $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})=\mathfrak{su}(p-q-1)$. The sequence for (9) terminates at $(\mathfrak{g},\mathfrak{h})=(\mathfrak{sp}(1,q-p+1),\mathfrak{sp}(q-p+1))$ when $p\leq q$, and at $(\mathfrak{g},\mathfrak{h})=(\mathfrak{sp}(1,p-q),\mathfrak{sp}(p-q))$ when $p>q$. Both are strongly real spherical by Lemma 3.2(1). Moreover we obtain for $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h},\mathfrak{h})$ that it is equal to $\mathfrak{sp}(q-p)$ for $p\leq q$ and $\mathfrak{sp}(p-q-1)$ if $p>q$.

The case (1) can also be determined by iteration. Here we end with $(\mathfrak{g},\mathfrak{h})=(\mathfrak{sl}(2,\mathbb{K}),\mathfrak{sl}(1,\mathbb{K}))$ which cannot be strongly real spherical as it is not even real spherical (see Part I). Hence (1) is not real spherical.

We turn to (6). It follows from Corollary 2.7 that $(\mathfrak{so}(2,2n),\mathfrak{su}(1,n))$ is strongly real spherical if and only if $(\mathfrak{su}(1,n),\mathfrak{su}(1,1)\oplus\mathfrak{su}(n-1))$ is strongly real spherical. Applying Corollary 2.7 again we obtain that this is the case if and only if $(\mathfrak{su}(1,1)\oplus\mathfrak{su}(n-1),\mathrm{diag}(\mathfrak{u}(1))\oplus\mathfrak{su}(n-$

2)) is strongly spherical. Since $\mathfrak{su}(n-1)$ and $\mathfrak{u}(1)$ are both compact the latter is equivalent to $(\mathfrak{su}(1,1), \mathfrak{u}(1))$ being real spherical, which is true. Further we obtain from Corollary 2.7 that

$$\begin{aligned}\mathfrak{s}(\mathfrak{so}(2,2n)\oplus\mathfrak{su}(1,n),\mathfrak{su}(1,n)) &= \mathfrak{s}(\mathfrak{su}(1,1)\oplus\mathfrak{su}(n-1)\oplus\mathfrak{u}(n-2),\mathfrak{u}(n-2)) \\ &= \mathfrak{su}(n-2)\oplus\mathfrak{s}(\mathfrak{su}(1,1)\oplus\mathfrak{u}(1),\mathfrak{u}(1)) = \mathfrak{su}(n-2)\,.\end{aligned}$$

The case (5) is again determined by induction. The sequence terminates at $(\mathfrak{so}^*(8), \mathfrak{so}^*(6)) \simeq (\mathfrak{so}(2,6), \mathfrak{su}(1,3))$ which is strongly real spherical with structural algebra $\{0\}$ by case (6).

It follows from Corollary 4.2 and the dimension bound (2.1), that the cases (8), (14) and (15) are not strongly real spherical.

Finally (12) is strongly real spherical if and only if $(\mathfrak{so}(8,1), \mathfrak{spin}(7))$ is strongly real spherical, and by Lemma 3.2(1) if and only if that pair is real spherical. This is actually the case, by the classification in Part I. Moreover, since $\mathrm{Spin}(7) \cap \mathrm{SO}(7) = \mathsf{G}_2$ (see [27, p. 170]) it follows that $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h}, \mathfrak{h}) = \mathsf{G}_2$. ☐

Let us now discuss the general case with $\mathfrak{h}$ simple.

Proposition 4.4. *For $k = 2$, a strictly indecomposable real spherical pair $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{h}$ simple arises from a strongly real spherical pair, i.e. $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}'\oplus\mathfrak{h}', \mathfrak{h}')$ with $(\mathfrak{g}', \mathfrak{h}')$ strongly spherical. The cases are listed in Theorem 4.1 (1).*

Proof. The strongly spherical pairs were determined in Lemma 4.3(b) and they correspond exactly to diagrams (S2) – (S8). Thus it only remains to exclude further cases.

For this we can assume that $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h})$ with $\mathfrak{g}_1, \mathfrak{g}_2 \neq \mathfrak{h}$. Let $S_i = \mathrm{S}(G, H_i)$ for $i = 1, 2$. Then $H = H^0$ is real spherical for the transitive action of $S_1 \times S_2$ (see Lemma 3.1). Further it follows from Lemma 3.2 (2) that both $[G_1 \times H]/H$ and $[G_2 \times H]/H$ need to be real spherical, i.e. both $(\mathfrak{g}_i, \mathfrak{h})$ need to be one of the strongly real spherical pairs of Lemma 4.3. Using the lemma we can easily check that there are only the following possibilities, and we claim that none of these lead to a real spherical pair $(\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h})$.

$\mathfrak{g}_1 = \mathfrak{so}(n+1,\mathbb{C})$	$\mathfrak{g}_2 = \mathfrak{g}_1$	$\mathfrak{h} = \mathfrak{so}(n,\mathbb{C})$
$\mathfrak{g}_1 = \mathfrak{so}(p+1,q)$	$\mathfrak{g}_2 = \mathfrak{g}_1$ or $\mathfrak{so}(p,q+1)$	$\mathfrak{h} = \mathfrak{so}(p,q)$
$\mathfrak{g}_1 = \mathfrak{su}(p+1,q)$	$\mathfrak{g}_2 = \mathfrak{g}_1$ or $\mathfrak{su}(p,q+1)$	$\mathfrak{h} = \mathfrak{su}(p,q)$
$\mathfrak{g}_1 = \mathfrak{sp}(p+1,q)$	$\mathfrak{g}_2 = \mathfrak{g}_1$ or $\mathfrak{sp}(p,q+1)$	$\mathfrak{h} = \mathfrak{sp}(p,q)$
$\mathfrak{g}_1 = \mathsf{F}_4^2$	$\mathfrak{g}_2 = \mathfrak{g}_1$ or $\mathfrak{so}(2,8)$ or $\mathfrak{so}(1,9)$	$\mathfrak{h} = \mathfrak{so}(1,8)$

Consider first $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{so}(n+1,\mathbb{C})$ and $\mathfrak{h} = \mathfrak{so}(n,\mathbb{C})$ with $n \geq 5$. Note that $S_1 = S_2 = \mathrm{SO}(n-1,\mathbb{C})$ up to isomorphism, and hence $H = \mathrm{SO}(n,\mathbb{C})$ is not homogeneous for $S_1 \times S_2$ by the factorization result of Onishchik (see Proposition 2.8), except when $n = 8$. However, a simple dimension count with (2.1) excludes that $\mathrm{SO}(8,\mathbb{C})$ is real spherical for $\mathrm{SO}(7,\mathbb{C}) \times \mathrm{SO}(7,\mathbb{C})$.

The second case is more subtle. If this leads to a real spherical pair, then $\mathfrak{h} = \mathfrak{so}(p,q) = \mathfrak{s}_1 + \mathfrak{s}_2$ with $\mathfrak{s}_1, \mathfrak{s}_2 = \mathfrak{so}(p,q-1)$ or $\mathfrak{so}(p-1,q)$, up to isomorphism. According to Onishchik's list, this is only possible if $p + q = 8$, so that $\mathfrak{h}_\mathbb{C} = \mathfrak{so}(8,\mathbb{C}) = \mathfrak{so}(7,\mathbb{C}) + \mathfrak{spin}(7,\mathbb{C})$. In addition, it follows from Part I Lemma 2.4 that $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are both non-compact. Since moreover H is real spherical for the action of $S_1 \times S_2$ we conclude by dimension count that $\mathfrak{s}_1$ or $\mathfrak{s}_2$ must be isomorphic to $\mathfrak{so}(1,6)$. This forces that $\mathfrak{h}$ is either isomorphic to $\mathfrak{so}(2,6)$ or

$\mathfrak{so}(1,7)$, and hence in particular its real rank is at most 2. The embedding of $\mathfrak{spin}(7,\mathbb{C})$ in $\mathfrak{so}(8,\mathbb{C})$ is given by the 8-dimensional spin representation. This representation is known to yield a real representation of $\mathfrak{spin}(p,q)$ where $p+q=7$ if and only if $p-q$ is congruent to 0, 1 or 7 modulo 8. This then happens only for $\mathfrak{spin}(7)$ and $\mathfrak{spin}(3,4)$. The first possibility is excluded since $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are supposed to be non-compact, and the other one is excluded as $\mathfrak{spin}(3,4)$ has real rank 3 and hence does not embed into $\mathfrak{h}$.

In the remaining three cases the result of Onishchik immediately excludes the relevant factorizations of $\mathfrak{h}$. □

4.4. **The case where $\mathfrak{h}$ is semi-simple and not simple.** We begin with the case where the projection $\mathfrak{h}\to\mathfrak{h}_1$ is an isomorphism and where $\mathfrak{g}_2=\mathfrak{h}_2$. The general case will be built on this. The next lemma classifies these cases.

Lemma 4.5. *For $k=2$, the list of all spherical pairs $(\mathfrak{g},\mathfrak{h})$ with $\mathfrak{h}$ semi-simple but not simple, $\mathfrak{h}\simeq\mathfrak{h}_1$ and $\mathfrak{g}_2=\mathfrak{h}_2$ is given by part* (2a) *of Theorem 4.1. Moreover, the structural algebras are as indicated in the diagrams.*

Proof. Let us first translate the problem and slightly change notation. Let $\mathfrak{g}$ be non-compact simple and $\mathfrak{h}\subset\mathfrak{g}$ be a semi-simple real spherical subalgebra such that $\mathfrak{h}=\mathfrak{h}_1\oplus\mathfrak{h}_2$ with $\mathfrak{h}_2$ non-compact simple. Let $\mathfrak{s}=\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ be the structural algebra and $\mathfrak{s}_2$ the projection of $\mathfrak{s}$ to $\mathfrak{h}_2$. The pairs which are called for in the lemma arise as $(\mathfrak{g}\oplus\mathfrak{h}_2,\mathfrak{h})$, and according to Corollary 2.7 the pair $(\mathfrak{g}\oplus\mathfrak{h}_2,\mathfrak{h})$ is real spherical if and only if $(\mathfrak{h}_2\oplus\mathfrak{s}_2,\mathfrak{s}_2)$ is real spherical. In particular, a necessary condition is that $(\mathfrak{h}_2,\mathfrak{s}_2)$ is real spherical.

We begin with the classical cases. From Tables 3 - 7 we obtain the following Table 2, of real spherical pairs $(\mathfrak{g},\mathfrak{h})$ as just mentioned, for which $(\mathfrak{h}_2,\mathfrak{s}_2)$ is real spherical.

	$(\mathfrak{g},\mathfrak{h})$	$(\mathfrak{h}_2,\mathfrak{s}_2)$
(1)	$(\mathfrak{sl}(n,\mathbb{C}),\mathfrak{sl}(n-2,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C}))$	$(\mathfrak{sl}(2,\mathbb{C}),\mathbb{C})$ for $n\geq 5$
(2)	$(\mathfrak{sl}(n,\mathbb{R}),\mathfrak{sl}(n-2,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R}))$	$(\mathfrak{sl}(2,\mathbb{R}),\mathbb{R})$ for $n\geq 5$
(3)	$(\mathfrak{su}(p,q),\mathfrak{su}(p-1,q-1)+\mathfrak{su}(1,1))$	$(\mathfrak{su}(1,1),i\mathbb{R})$ for $(p,q)\neq(2,2),(1,2)$
(4)	$(\mathfrak{su}(1,q),\mathfrak{c}+\mathfrak{su}(1,q_2))$	$(\mathfrak{su}(1,q_2),\mathfrak{u}(q_2))$ for $q_2\geq 1$ and $q-q_2\geq 2$
(5)	$(\mathfrak{sl}(n,\mathbb{H}),\mathfrak{sl}(1,\mathbb{H})+\mathfrak{sl}(n-1,\mathbb{H}))$	$(\mathfrak{sl}(n-1,\mathbb{H}),\mathfrak{sl}(n-2,\mathbb{H}))$
(6)	$(\mathfrak{so}(1,q),\mathfrak{c}+\mathfrak{so}(1,q_2))$	$(\mathfrak{so}(1,q_2),\mathfrak{so}(q_2))$ for $q_2>1$ and $q-q_2\geq 3$
(7)	$(\mathfrak{so}^*(2n),\mathfrak{so}^*(2n-4)+\mathfrak{su}(2)+\mathfrak{sl}(2,\mathbb{R}))$	$(\mathfrak{sl}(2,\mathbb{R}),\mathfrak{so}(2))$
(8)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{C}))$	$(\mathfrak{sp}(1,\mathbb{C}),\mathfrak{sp}(1,\mathbb{C}))$
(9)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{C}))$	$(\mathfrak{sp}(n-1,\mathbb{C}),\mathfrak{sp}(n-2,\mathbb{C})+\mathfrak{sp}(1,\mathbb{C}))$
(10)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{R}))$	$(\mathfrak{sp}(1,\mathbb{R}),\mathfrak{sp}(1,\mathbb{R}))$
(11)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{R}))$	$(\mathfrak{sp}(n-1,\mathbb{C}),\mathfrak{sp}(n-2,\mathbb{C})+\mathfrak{sp}(1,\mathbb{R}))$
(12)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1))$	$(\mathfrak{sp}(n-1,\mathbb{C}),\mathfrak{sp}(n-2,\mathbb{C})+\mathfrak{sp}(1))$
(13)	$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{sp}(n-1,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R}))$	$(\mathfrak{sp}(1,\mathbb{R}),\mathfrak{sp}(1,\mathbb{R}))$
(14)	$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{sp}(n-1,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R}))$	$(\mathfrak{sp}(n-1,\mathbb{R}),\mathfrak{sp}(n-2,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R}))$
(15)	$(\mathfrak{sp}(n,\mathbb{C}),\mathfrak{sp}(n-2,\mathbb{C})+\mathfrak{sp}(2,\mathbb{C}))$	$(\mathfrak{sp}(2,\mathbb{C}),\mathfrak{sp}(1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{C}))$
(16)	$(\mathfrak{sp}(n,\mathbb{R}),\mathfrak{sp}(n-2,\mathbb{R})+\mathfrak{sp}(2,\mathbb{R}))$	$(\mathfrak{sp}(2,\mathbb{R}),\mathfrak{sp}(1,\mathbb{R})+\mathfrak{sp}(1,\mathbb{R}))$
(17)	$(\mathfrak{sp}(p,q),\mathfrak{sp}(p-1,q-1)+\mathfrak{sp}(1,1))$	$(\mathfrak{sp}(1,1),\mathfrak{sp}(1)+\mathfrak{sp}(1))$
(18)	$(\mathfrak{sp}(p,q),\mathfrak{sp}(1)+\mathfrak{sp}(p-1,q))$	$(\mathfrak{sp}(p-1,q),\mathfrak{sp}(p-1,q-1)+\mathfrak{sp}(1))$
(19)	$(\mathfrak{sp}(1,q),\mathfrak{sp}(q-q_2)+\mathfrak{sp}(1,q_2))$	$(\mathfrak{sp}(1,q_2),\mathfrak{sp}(1)+\mathfrak{sp}(q_2-1))$

Table 2

For the possibilities for $\mathfrak{c}$ in (4) and (6), see Table 8.

We claim that the following cases from above give rise to a strongly spherical pair $(\mathfrak{h}_2, \mathfrak{s}_2)$, and hence $(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ is real spherical. In each case we refer to the diagram in Theorem 4.1(2a) which displays $(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$.

(1)–(SS9)	(4)–(SS15)	(8)–(SS11)	(15)–(SS12)	(18)–(SS18)
(2)–(SS9)	(6)–(SS14)	(10)–(SS11)	(16)–(SS12)	(19)–(SS16)
(3)–(SS10)	(7)–(SS17)	(13)–(SS11)	(17)–(SS13)	

For (1) and (2) this is well known. We also note that all cases where $\mathfrak{s}_2$ is compact are strongly spherical, since they reduce to $(\mathfrak{h}_2, \mathfrak{s}_2)$ by Lemma 3.2(1). These are (3), (4), (6), (7), (17) and (19). As for (5), we can exclude it as $(\mathfrak{h}_2, \mathfrak{s}_2)$ does not show up in Lemma 4.3. In (8), (10), (13) strong sphericality is obvious. Both (15) and its real form (16) produce a spherical pair $(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ since the complex form of this pair already belongs to the list of Brion-Mikityuk. Further, (18) is an $\mathfrak{sp}(1)$-extension of (S7), hence strongly spherical. In all these cases it is then easy to verify $\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ in the corresponding diagram by means of (2.8).

Among the classical cases we are left with (9), (11), (12), and (14). Note that (9) and (14) are special cases of (8), (15) and (13), (16) when $n = 2, 3$. Apart from these the remaining cases can be excluded with the dimension bound (2.1).

We turn to the exceptional cases. We recall from [4], [21] that here there are no complex strongly spherical pairs so that we can disregard all cases where $\mathfrak{g}$ is quasisplit (see Subsection 2.3 for the definition of quasisplit). Inspecting Tables 4 - 7 we are left to check the following two cases for $(\mathfrak{g}, \mathfrak{h}, \mathfrak{h}_2)$:

(20) $(\mathsf{E}_6^3, \mathfrak{su}(5,1) + \mathfrak{sl}(2,\mathbb{R}), \mathfrak{sl}(2,\mathbb{R}))$ with $\mathfrak{s}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{u}(2) + \mathfrak{u}(2)$.

(21) $(\mathsf{E}_7^3, \mathfrak{so}(10,2) + \mathfrak{sl}(2,\mathbb{R}), \mathfrak{sl}(2,\mathbb{R}))$ with $\mathfrak{s}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{so}(6) \oplus \mathfrak{so}(2) \oplus \mathfrak{so}(2,1)$.

In these two cases we need to determine $\mathfrak{s}_2$. Moreover, the particular case is spherical if and only if $\mathfrak{s}_2 \neq 0$. We will now show that this happens in both cases, and by that obtain the diagrams (SS19) and (SS20).

We begin with (20) and $\mathfrak{g} = \mathsf{E}_6^3$. Note that the fact that $\mathfrak{s} = \mathfrak{l}_\mathfrak{h}$ is compact implies that $\mathfrak{l}$ is elementary, hence the Levi part of a minimal parabolic subalgebra of $\mathfrak{g}$. Observe that $\mathfrak{g}$ is Hermitian with maximal compact subalgebra $\mathfrak{k} = \mathfrak{so}(10) \oplus \mathfrak{so}(2)$. We recall that $\mathfrak{z}(\mathfrak{k}) = \mathfrak{so}(2) \subset \mathfrak{m} = \mathfrak{u}(4) \subset \mathfrak{l}$ with $\mathfrak{m}$ defined in Subsection 2.3. Hence $\mathfrak{z}(\mathfrak{k}) \subset \mathfrak{s}$ as $\mathfrak{s} = \mathfrak{u}(2) \oplus \mathfrak{u}(2)$ contains the center of $\mathfrak{m}$.

We claim that $\mathfrak{z}(\mathfrak{k}) \not\subset \mathfrak{h}_1 = \mathfrak{su}(5,1)$ which implies $\mathfrak{s}_2 \neq 0$. Indeed, if $\mathfrak{z}(\mathfrak{k}) \subset \mathfrak{h}_1$, then $\mathfrak{z}(\mathfrak{k})$ would act trivially on $\mathfrak{h}_2 = \mathfrak{sl}(2,\mathbb{R})$, and since $\mathfrak{h}_2 \cap \mathfrak{k}^\perp \neq \{0\}$ that would be a contradiction to the fact that $\mathfrak{z}(\mathfrak{k})$ acts without non-trivial fixed points on $\mathfrak{k}^\perp$. We have thus shown that $(\mathfrak{h}_2, \mathfrak{s}_2) = (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2,\mathbb{R}))$ and thus $\mathfrak{s}(\mathfrak{h}_2 \oplus \mathfrak{s}_2, \mathfrak{s}_2) = \{0\}$. Moreover, $\mathfrak{s} \cap \mathfrak{h}_1 = \mathfrak{su}(2) \oplus \mathfrak{su}(2) \oplus \mathfrak{u}(1)$ and thus it follows from (2.8) that $\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ is stated in (SS19).

We move on with the exceptional case $\mathfrak{g} = \mathsf{E}_7^3$ with $\mathfrak{h} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 = \mathfrak{so}(10,2) \oplus \mathfrak{sl}(2,\mathbb{R})$. Notice that the procedure mentioned in Appendix A to determine the structural algebra $\mathfrak{s}$ also gives $\mathfrak{l}$. In this particular case we obtain $\mathfrak{l} = \mathfrak{l}_1 \oplus \mathfrak{l}_2 \oplus \mathfrak{a}_Z := \mathfrak{so}(8) \oplus \mathfrak{so}(2,1) \oplus \mathbb{R}^2$ with $\mathfrak{a}_Z = \mathbb{R}^2$ the center of $\mathfrak{l}$. Further $\mathfrak{s} = \mathfrak{so}(6) \oplus \mathfrak{so}(2,1) \oplus \mathfrak{so}(2)$ with $\mathfrak{so}(2,1) \subset \mathfrak{h}_1$ by Part I, Remark 8.2 (a). In order to show that $\mathfrak{s}_2 \neq 0$ it is sufficient to show that $\mathfrak{so}(2) \not\subset \mathfrak{h}_1$. Assume the contrary and consider the $\mathfrak{h}$-module $V := \mathfrak{g}_\mathbb{C}/\mathfrak{h}_\mathbb{C}$. Let V_{32} be the 32-dimensional spin representation of $\mathfrak{h}_{1,\mathbb{C}}$, then it is easy to see that $V = V_{32} \otimes \mathbb{C}^2$ as $\mathfrak{h}_\mathbb{C}$-module. Hence if

$\mathfrak{so}(2) \subset \mathfrak{h}_1$, we deduce that V, considered as $\mathfrak{s}_\mathbb{C}$-module, decomposes with even multiplicities. On the other hand $V \simeq (\mathfrak{l}/\mathfrak{s} \oplus \mathfrak{u})_\mathbb{C}$ as $\mathfrak{s}_\mathbb{C}$-module.

In the next step we investigate the $\mathfrak{s}$-module $\mathfrak{u}$. We recall from [23] that the root system $\Sigma(\mathfrak{g}, \mathfrak{a}_Z)$ is of type BC_2,

$$\Sigma(\mathfrak{g}, \mathfrak{a}_Z) = \{\pm\epsilon_1, \pm 2\epsilon_1, \pm\epsilon_2, \pm 2\epsilon_2, \pm\epsilon_1 \pm \epsilon_2\}.$$

We decompose $\mathfrak{u} = \mathfrak{u}_0 \oplus \mathfrak{u}_1$ where $\mathfrak{u}_0$ corresponds to the short roots and $\mathfrak{u}_1$ to the non-short roots. All root spaces are $\mathfrak{l}$-modules.

We first claim that $\mathfrak{u}_1$ decomposes with even multiplicities as an $\mathfrak{s}$-module. According to [23], the long root spaces are real one-dimensional, hence trivial as $[\mathfrak{l}, \mathfrak{l}]$-modules and thus also as $\mathfrak{s}$-modules. We now assert that root spaces corresponding to roots from $\{\pm\epsilon_1 \pm \epsilon_2\}$ are isomorphic as $\mathfrak{s}$-modules. Indeed, we have just seen that $\mathfrak{g}^{\pm 2\epsilon_i} \simeq \mathbb{R}$ are trivial $\mathfrak{s}$-modules. Now take for example the root space $\mathfrak{g}^{\epsilon_1 - \epsilon_2}$ and $0 \neq X_i \in \mathfrak{g}^{2\epsilon_i}$. Then, finite dimensional $\mathfrak{sl}(2)$-representation theory yields that bracketing with X_2 results in an $\mathfrak{s}$-equivariant isomorphism $\mathfrak{g}^{\epsilon_1 - \epsilon_2} \to \mathfrak{g}^{\epsilon_1 + \epsilon_2}$. Iterating then implies that all $\mathfrak{g}^\alpha$ with $\alpha \in \{\pm\epsilon_1 \pm \epsilon_2\}$ are isomorphic as $\mathfrak{s}$-modules (even as $[\mathfrak{l}, \mathfrak{l}]$-modules). Our claim follows.

We are left with $\mathfrak{u}_0$ which consists of short roots $\mathfrak{g}^{\pm\epsilon_i}$. First note that $\mathfrak{g}^{\epsilon_i}$ is isomorphic to $\mathfrak{g}^{-\epsilon_i}$ as $\mathfrak{l}_\mathfrak{h}$-module via the bracketing argument from above. According to [23] the $\mathfrak{g}^{\epsilon_i}$ are real 8-dimensional. Let us fix one, say $W := \mathfrak{g}^{\epsilon_1}$. Since $W_\mathbb{C}$ is a prehomogeneous $\mathfrak{l}_\mathbb{C}$-module by Proposition B.1 it follows that there are three possibilities for W (triality):

$$\mathbb{R}^8 \qquad \mathfrak{spin}(8)_+ \qquad \mathfrak{spin}(8)_-$$

where $\mathbb{R}^8$ indicates the standard representation. Observe that W is uniquely determined by its branching to $\mathfrak{so}(6) \simeq \mathfrak{su}(4)$ which is respectively

$$\mathbb{R}^6 \oplus \mathbb{R}^2 \qquad \mathbb{C}^4 \qquad (\mathbb{C}^4)^* . \tag{4.7}$$

To sum up: if $\mathfrak{g}^{\epsilon_1}$ and $\mathfrak{g}^{\epsilon_2}$ are isomorphic, then $\mathfrak{u}_0$ decomposes with even multiplicities as an $\mathfrak{l}_\mathfrak{h}$-module. Otherwise, the $\mathfrak{l}_\mathfrak{h}$-module $\mathfrak{u}_0$ is a sum of two which are listed in (4.7).

Recall that V was assumed to decompose with even multiplicities. We have just shown that $\mathfrak{u}_1$ decomposes with even multiplicities and determined the possible branchings for $\mathfrak{u}_0$. Now, $\mathfrak{so}(8,\mathbb{C})/\mathfrak{so}(6,\mathbb{C}) \oplus \mathfrak{so}(2,\mathbb{C}) = \mathbb{C}^6 \oplus (\mathbb{C}^6)^*$ and we observe that $\mathbb{C}^6$ and $(\mathbb{C}^6)^*$ are inequivalent as $\mathfrak{so}(6,\mathbb{C}) \otimes \mathfrak{so}(2,\mathbb{C})$-modules. Together with our branching results for $\mathfrak{u}_0$ and $\mathfrak{u}_1$ we thus obtain a contradiction.

It follows that $(\mathfrak{h}_2, \mathfrak{s}_2) = (\mathfrak{sl}(2,\mathbb{R}), \mathfrak{so}(2,\mathbb{R}))$ and hence $\mathfrak{s}(\mathfrak{h}_2 \oplus \mathfrak{s}_2, \mathfrak{s}_2) = \{0\}$. In addition $\mathfrak{s} \cap \mathfrak{h}_1 = \mathfrak{so}(6) \oplus \mathfrak{so}(2,1)$ by Remark 8.2 in Part I, and we obtain $\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{h}_2, \mathfrak{h})$ via (2.8). □

Proof of Theorem 4.1 (2b): Let us now consider the case of a semi-simple and not simple real spherical subalgebra $\mathfrak{h} \subset \mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathfrak{h}_i \neq \mathfrak{g}_i$ for $i = 1, 2$. We use the notation of (4.2). Since $\mathfrak{h}^0 \neq 0$ both $\mathfrak{h}_1$ and $\mathfrak{h}_2$ cannot be simple. Hence it is no loss of generality to assume that $\mathfrak{h}_1 = \mathfrak{h}_1' \oplus \mathfrak{h}^0$ is semi-simple and not simple. First observe that $(\mathfrak{g}_1 \oplus \mathfrak{h}_2, \mathfrak{h})$ is real spherical by Lemma 3.2 (2). Then, by Lemma 3.1 (1) we deduce that $(\mathfrak{g}_1 \oplus \mathfrak{h}^0, \mathfrak{h}_1)$ is real spherical. Let $\mathfrak{h}^{0,+} \triangleleft \mathfrak{h}^0$ be a simple non-compact factor. It follows that $(\mathfrak{g}_1 \oplus \mathfrak{h}^{0,+}, \mathfrak{h}_1)$ must show up in the list of Theorem 4.1 (2a). From this list we now deduce that $\mathfrak{h}_1$ has only two simple factors, except for $\mathfrak{g}_1 = \mathfrak{so}^*(2n+4)$ or $(\mathfrak{g}_1, \mathfrak{h}_1) = (\mathfrak{so}(1,n), \mathfrak{so}(4) + \mathfrak{so}(1, n-4))$.

Arguing similarly we also obtain that $(\mathfrak{h}^{0,+} \oplus \mathfrak{g}_2, \mathfrak{h}_2)$ is real spherical and thus must show up in the lists of Theorem 4.1 (1b), (2a).

Recall from Section 4.1 that $(\mathfrak{g},\mathfrak{h})$ is real spherical if and only if H^0 is spherical as an $S_1''\times S_2''$-variety, and (see (4.3)) that then $\mathfrak{h}^0=\mathfrak{s}_1''+\mathfrak{s}_2''$, as a necessary condition for real sphericality.

Let us assume first that $\mathfrak{h}^0=\mathfrak{h}^{0,+}$. Inspecting the list in Theorem 4.1 (2a) we see that $\mathfrak{h}^0$ is either symplectic or of rank one. We begin with the symplectic case and recall that symplectic Lie algebras do not admit non-trivial factorizations. Hence if $(\mathfrak{g},\mathfrak{h})$ is real spherical, then this forces that $\mathfrak{s}_1''$ or $\mathfrak{s}_2''$ equals $\mathfrak{h}^0$. If $\mathfrak{s}_1''=\mathfrak{h}^0$, then this means that we are in the situations (8), (10) or (13) in Table 2. We can draw the same conclusion if $\mathfrak{s}_2''=\mathfrak{h}^0$. Thus in case $\mathfrak{h}^0=\mathfrak{h}^{0,+}$ is symplectic we may assume that $(\mathfrak{g}_1\oplus\mathfrak{h}^0,\mathfrak{h}_1)$ is one of the pairs $(\mathfrak{g}\oplus\mathfrak{h}_2,\mathfrak{h})$ in (8), (10) or (13). In particular $\mathfrak{h}^0=\mathfrak{s}_1''=\mathfrak{sp}(1,\mathbb{K})$. Being in this situation $(\mathfrak{g},\mathfrak{h})$ then is real spherical if and only if $\mathfrak{s}_2''$ contains $\mathfrak{gl}(1,\mathbb{K})$, $\mathfrak{so}(2,\mathbb{K})$ or $\mathfrak{sp}(1,\mathbb{R})$.

This situation only occurs for $(\mathfrak{h}^0\oplus\mathfrak{g}_2,\mathfrak{h}_2)$ being from the list of Theorem 4.1 (2a), to be precise for

$$(1),(2),(3),(4)\text{ for }q_2=1,(6)\text{ for }q_2=2,3,(7),(8),(10),(11)\text{ for }n=2,(13),(20),(21)$$

in Table 2. In all these cases the structural algebras $\mathfrak{s}$ can be read off via (4.5). In particular, we have $\mathfrak{s}=\mathfrak{s}_1'\oplus\mathfrak{s}_2'$ except when $(\mathfrak{h}^0\oplus\mathfrak{g}_2,\mathfrak{h}_2)$ is also of type (8), (10) or (13). In the latter case we have $\mathfrak{s}=\mathfrak{s}_1'\oplus\mathfrak{gl}(1,\mathbb{K})\oplus\mathfrak{s}_2'$, where $\mathbb{K}=\mathbb{R},\mathbb{C}$ is determined by $\mathfrak{h}^0=\mathfrak{sp}(1,\mathbb{K})$, and where $\mathfrak{gl}(1,\mathbb{K})$ is embedded tridiagonally into $\mathfrak{h}=\mathfrak{h}_1'\oplus\mathfrak{h}^0\oplus\mathfrak{h}_2'$.

We move on where $\mathfrak{h}^0=\mathfrak{h}^{0,+}$ is of rank one and not symplectic, and hence equals $\mathfrak{so}(1,p)$ or $\mathfrak{su}(1,p)$. For that we first recall the following fact from Part I, Lemma 2.4: If $\mathfrak{r}=\mathfrak{r}_1+\mathfrak{r}_2$ is a non-trivial factorization of a simple algebra $\mathfrak{r}$ with one factor compact, then $\mathfrak{r}$ is compact.

We begin with the case where $\mathfrak{h}^0=\mathfrak{so}(1,p)$ for $p\geq 4$. Then $\mathfrak{s}_1''$ is compact and hence we must have $\mathfrak{s}_2''=\mathfrak{h}^0$ which is not possible. Likewise we can exclude the case where $\mathfrak{h}^0=\mathfrak{su}(1,p)$ for $p>1$.

To summarize, the cases where $\mathfrak{h}^0=\mathfrak{h}^{0,+}$ give rise to (SS21) - (SS30). It remains to show that $\mathfrak{h}^0=\mathfrak{h}^{0,+}$. Suppose the contrary and let $\mathfrak{h}^0=\mathfrak{h}^{0,+}\oplus\mathfrak{h}^{0,-}$ with $\mathfrak{h}^{0,-}\neq\{0\}$. Let $\mathfrak{h}_i^-$ be the projection of $\mathfrak{h}^{0,-}$ to $\mathfrak{h}_i$. We consider $\widetilde{\mathfrak{h}}=\mathfrak{h}_1^-\oplus\mathfrak{h}$ and observe that $\widetilde{\mathfrak{h}}^0=\mathfrak{h}^{0,+}$. Hence the pair $(\mathfrak{g},\widetilde{\mathfrak{h}})$ must be one of the cases (SS21) - (SS30). Inspecting this list we quickly see that this is impossible. Indeed, let us begin with the case where $\widetilde{\mathfrak{h}}$ has three factors. Then $\widetilde{\mathfrak{h}}_1'\simeq\widetilde{\mathfrak{h}}_2'$ since $\widetilde{\mathfrak{h}}\simeq\mathfrak{h}_1^-\oplus\mathfrak{h}^{0,+}\oplus\mathfrak{h}_1^-$. Hence $(\mathfrak{g},\widetilde{\mathfrak{h}})$ needs to be (SS23) for $n=m$ and $\mathbb{K}=\mathbb{K}'$. A simple dimension count then shows that (2.1) is violated and thus $(\mathfrak{g},\mathfrak{h})$ is not real spherical. Likewise the case where $\widetilde{\mathfrak{h}}$ has four factors is ruled out right away. □

4.5. The case where $\mathfrak{h}$ is not semi-simple.

Proof of Theorem 4.1 (3a): We only have to determine the cases in Theorem 4.1(1),(2) where one can enlarge $\mathfrak{h}_i$ inside $\mathfrak{g}_i$ for $i=1$ or 2 to have a non-trivial center. Going through the cases one by one we see that this is possible for (S2), (S5), (S6), (S7), (SS9), (SS10), (SS14), (SS15), (SS21), (SS22), (SS25), (SS26), and (SS27). This then results in the asserted list. □

For the proof of (3b) of the theorem we consider as earlier first the case where $\mathfrak{g}_2=\mathfrak{h}_2$. We assume that $\mathfrak{h}$ is reductive, but not semi-simple.

Lemma 4.6. *Suppose that $\mathfrak{g}$ is non-compact simple and $\mathfrak{h}=\mathfrak{h}'\oplus\mathfrak{h}''$ is a reductive subalgebra of $\mathfrak{g}$ with $\mathfrak{h}'$ not semi-simple and $\mathfrak{h}''$ non-compact simple. Assume $(\mathfrak{g},\mathfrak{h})$ is real spherical, and* *let $\mathfrak{s}''$ be the projection of $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{h}''$. Then $(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h})$ is real spherical with $(\mathfrak{g}\oplus\mathfrak{h}'',[\mathfrak{h},\mathfrak{h}])$ not real spherical if and only if $(\mathfrak{g},\mathfrak{h},\mathfrak{h}'',\mathfrak{s}'')$ is one of the following:*

(1) $(\mathfrak{sl}(n+2,\mathbb{K}),\mathfrak{s}(\mathfrak{gl}(n,\mathbb{K})\oplus\mathfrak{gl}(2,\mathbb{K})),\mathfrak{sl}(2,\mathbb{K}),\mathfrak{gl}(1,\mathbb{K}))$ *for* $\mathbb{K}=\mathbb{R},\mathbb{C}$ *and* $n=1,2$,
(2) $(\mathfrak{su}(2,2),\mathfrak{s}(\mathfrak{u}(1,1)\oplus\mathfrak{u}(1,1)),\mathfrak{su}(1,1),\mathfrak{u}(1))$,
(3) $(\mathfrak{su}(p+1,q+1),\mathfrak{u}(p,q+1),\mathfrak{su}(p,q+1),\mathfrak{u}(p,q))$ *for* $p=q,q+1$,
(4) $(\mathfrak{sl}(n,\mathbb{K}),\mathfrak{gl}(n-1,\mathbb{K}),\mathfrak{sl}(n-1,\mathbb{K}),\mathfrak{gl}(n-2,\mathbb{K}))$ *for* $\mathbb{K}=\mathbb{R},\mathbb{C}$ *and* $n\geq 3$,
(5) $(\mathfrak{sl}(n+1,\mathbb{H}),\mathfrak{gl}(n,\mathbb{H})+\mathfrak{f},\mathfrak{sl}(n,\mathbb{H}),\mathfrak{gl}(n-1,\mathbb{H})+\mathfrak{f})$ *for* $\mathfrak{f}\subset\mathfrak{sl}(1,\mathbb{H})$,
(6) $(\mathfrak{so}(1,n),\mathfrak{so}(2)+\mathfrak{so}(1,n-2),\mathfrak{so}(1,n-2),\mathfrak{so}(n-2))$,
(7) $(\mathsf{E}_6^4,\mathfrak{gl}(1,\mathbb{R})+\mathfrak{so}(9,1),\mathfrak{so}(9,1),\mathfrak{so}(1,1)+\mathfrak{spin}(7))$.

Moreover, the structural algebras $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h})$ are given by

$(\mathfrak{g},\mathfrak{h})$	$\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h})$
(1)	0
(2)	0
(3)	0
(4)	0
(5)	$\mathfrak{f}$
(6)	$\mathfrak{so}(n-3)$
(7)	G_2

Proof. First note that $(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h})$ is real spherical if and only if $(\mathfrak{h}'',\mathfrak{l}'')$ is strongly real spherical by Corollary 2.7. Under the additional assumption that $(\mathfrak{g}\oplus\mathfrak{h}'',[\mathfrak{h},\mathfrak{h}])$ is not real spherical the asserted list is extracted from Tables 3 - 7.

The table for $\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h})$ follows inductively via (2.8). This is straightforward for (1) - (6). For (7) we obtain with Corollary 2.7 applied iteratively:

$$\begin{aligned}\mathfrak{s}(\mathfrak{g}\oplus\mathfrak{h}'',\mathfrak{h}) &\simeq \mathfrak{s}(\mathfrak{so}(9,1)\oplus[\mathfrak{so}(1,1)\oplus\mathfrak{spin}(7)],\mathfrak{so}(1,1)\oplus\mathfrak{spin}(7))\\ &\simeq \mathfrak{s}([\mathfrak{so}(1,1)\oplus\mathfrak{spin}(7)]\oplus\mathsf{G}_2,\mathsf{G}_2)\\ &\simeq \mathsf{G}_2\,.\end{aligned}$$ □

Proof of Theorem 4.1(3b): Let us assume first that $(\mathfrak{g}_1\oplus\mathfrak{g}_2,\mathfrak{h})$ is of the type where $\mathfrak{g}_2=\mathfrak{h}_2$. Then we can use Lemma 4.6, from which we obtain all diagrams of (3b) except (R50) and (R51).

Having classified all cases with $\mathfrak{h}$ reductive and $\mathfrak{g}_2=\mathfrak{h}_2$ we can now complete the proof. Suppose that $(\mathfrak{g}_1\oplus\mathfrak{g}_2,\mathfrak{h})$ is real spherical with $\mathfrak{h}$ reductive and not semi-simple, that $(\mathfrak{g}_1\oplus\mathfrak{g}_2,[\mathfrak{h},\mathfrak{h}])$ is not real spherical, and that $\mathfrak{h}_i\neq\mathfrak{g}_i$ for $i=1,2$. It is no loss of generality to assume that $\mathfrak{h}_1$ is not semi-simple. We let $\mathfrak{h}^0\triangleleft\mathfrak{h}$ be a simple non-compact ideal with non-zero projections to $\mathfrak{h}_1$ and $\mathfrak{h}_2$. Then $\mathfrak{h}_i\simeq\widetilde{\mathfrak{h}}_i:=\mathfrak{h}_i'\oplus\mathfrak{h}^0$ for $i=1,2$. The fact that $(\mathfrak{g},\mathfrak{h})$ is real spherical implies that $(\mathfrak{g}_1\oplus\mathfrak{h}^0,\widetilde{\mathfrak{h}}_1)$ and $(\mathfrak{h}^0\oplus\mathfrak{g}_2,\widetilde{\mathfrak{h}}_2)$ are both real spherical. Note that $(\mathfrak{g}_1\oplus\mathfrak{h}^0,\widetilde{\mathfrak{h}}_1)$ is part of what was already just classified in Theorem 4.1(3a,b). Likewise all cases for $(\mathfrak{h}^0\oplus\mathfrak{g}_2,\widetilde{\mathfrak{h}}_2)$ with $\widetilde{\mathfrak{h}}_2$ possibly semi-simple were already classified. Moreover if we let $\mathfrak{l}_i^0$ be the projection of $\mathfrak{s}(\mathfrak{g}_i,\mathfrak{h}_i)$ to $\mathfrak{h}^0\subset\mathfrak{h}_i$, then $\mathfrak{h}^0$ has to be real spherical for the action of $\mathfrak{l}_1^0\oplus\mathfrak{l}_2^0$.

Using the already obtained tables it is now an elementary procedure to determine all the remaining cases. As an example we do the computation for the case where $(\mathfrak{g}_1\oplus\mathfrak{h}^0,\widetilde{\mathfrak{h}}_1)$ is (R44). In this case $(\mathfrak{g}_1\oplus\mathfrak{h}^0,\widetilde{\mathfrak{h}}_1)=(\mathfrak{sl}(n+2,\mathbb{K})\oplus\mathfrak{sl}(2,\mathbb{K}),\mathfrak{gl}(2,\mathbb{K}))$ with $n=1,2$ and

$\mathfrak{l}_1^0 = \mathfrak{gl}(1, \mathbb{K})$. A quick inspection of the already obtained lists implies that $(\mathfrak{g}_2 \oplus \mathfrak{h}^0, \widetilde{\mathfrak{h}}_2)$ is of one of the forms:

- $(\mathfrak{g}_1 \oplus \mathfrak{h}^0, \widetilde{\mathfrak{h}}_1)$,
- $(\mathfrak{h}^0 \oplus \mathfrak{h}^0, \mathfrak{h}^0)$,
- (SS11) with the reversed roles of $\mathbb{K}$ and $\mathbb{K}'$, and $\mathfrak{l}_2^0 = \mathfrak{sl}(2, \mathbb{K})$.

The first two of these give nothing new, and the last case gives (R50).

The remaining cases (R31)-(R36), (R45)-(R49) and (R52) are treated similarly. The two cases where $(\mathfrak{g}_1 \oplus \mathfrak{h}^0, \widetilde{\mathfrak{h}}_1)$ is (R45) and (R46) contribute the values $p = 1$ and $p = 0$ respectively, of (R51). The other cases do not contribute anything new. ☐

This completes the proof of Theorem 4.1.

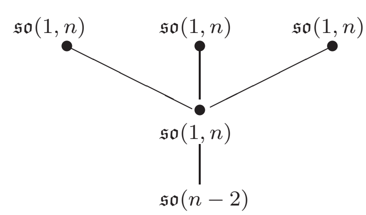
5. More than two factors

Having classified all cases with two factors and their corresponding structural algebras, it is now a manageable task to complete the classification. The key is Proposition 5.3 which limits the investigation to the case $k = 3$. We start our examination with the real spherical triple spaces.

Consider $G_1 \times \cdots \times G_k$. For $i, j, \ldots \in \{1, 2, \ldots k\}$ we write $H_{ij\ldots}$ for the projection of H to $G_{ij\ldots} = G_i \times G_j \times \cdots$ and we set $Z_{ij\ldots} = G_{ij\ldots}/H_{ij\ldots}$.

Lemma 5.1. *Let* $\mathfrak{g}$ *be a simple non-compact Lie algebra. Then* $(\mathfrak{g} \oplus \mathfrak{g} \oplus \mathfrak{g}, \operatorname{diag}(\mathfrak{g}))$ *is real spherical if and only if* $\mathfrak{g} = \mathfrak{so}(1, n)$ *for* $n \geq 2$*. The corresponding diagram reads*

(BM53) $n \geq 2$



Proof. Recall the Langlands decomposition $\mathfrak{p} = \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ of the be a minimal parabolic of $\mathfrak{g}$ (see Subsection 2.3), and note that Z_{12} is real spherical with structural algebra $\mathfrak{m} + \mathfrak{a}$ according to diagram (S1). We consider the tower $H' = H_{12} \times H_3 = \operatorname{diag}_2(G) \times G \supset H = \operatorname{diag}_3(G)$. Then according to Proposition 2.4, Z is real spherical if and only if $MA \times P$ has an open orbit on G and in this case the structural algebra is $\mathfrak{s}(\mathfrak{g} \oplus \mathfrak{m} \oplus \mathfrak{a}, \mathfrak{m} \oplus \mathfrak{a})$. This means that MA has an open orbit on N. Since MA preserves restricted root spaces, this implies that $\dim A$ equals the number of positive roots for the restricted root system of $\mathfrak{g}$ with respect to $\mathfrak{a}$. It follows that the restricted root system is of type A_1, i.e. $\mathfrak{g} \simeq \mathfrak{so}(1, n)$ for some $n \geq 2$. Conversely, in these cases MA has an open orbit on N. ☐

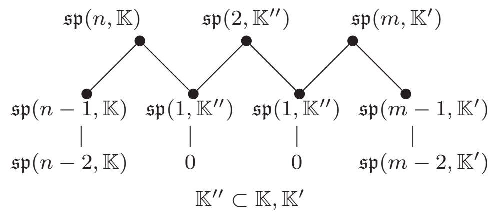
Suppose that $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ are reductive pairs such that $\mathfrak{h}_1 = \mathfrak{h}_1' \oplus \mathfrak{h}^0$ and $\mathfrak{h}_2 = \mathfrak{h}_2' \oplus \mathfrak{h}^0$ share a common ideal $\mathfrak{h}^0$ (up to isomorphism). Then we refer to the reductive pair

$$(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{g}_1 \oplus \mathfrak{g}_2, \mathfrak{h}_1' \oplus \mathfrak{h}_2' \oplus \operatorname{diag}(\mathfrak{h}^0)). \tag{5.1}$$

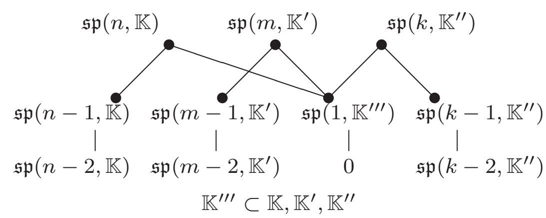
as the *glueing* of $(\mathfrak{g}_1, \mathfrak{h}_1)$ and $(\mathfrak{g}_2, \mathfrak{h}_2)$ *along* $\mathfrak{h}^0$.

Theorem 5.2. *The strictly indecomposable real spherical spaces with $k = 3$ are the triple cases from Lemma 5.1 and the following:*

(BM54) $m, n \geq 1$



(BM55) $m, n, k \geq 1$



Here $\mathbb{K},\mathbb{K}',\mathbb{K}'',\mathbb{K}''' = \mathbb{R},\mathbb{C}$ subject to the following additional conditions:

(1) *In* (BM54) *one has $\mathbb{K}' = \mathbb{K}''$ if $m = 1$, resp. $\mathbb{K} = \mathbb{K}''$ if $n = 1$.*

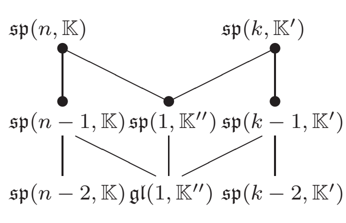
(2) *In* (BM55) *one has $\mathbb{K}''' = \mathbb{K}$ if $n = 1$, $\mathbb{K}''' = \mathbb{K}'$ if $m = 1$ and $\mathbb{K}''' = \mathbb{K}''$ if $k = 1$.*

and the absolutely spherical cases are those with $\mathbb{K} = \mathbb{K}' = \mathbb{K}'' = \mathbb{K}'''$.

Proof. Let $(\mathfrak{g},\mathfrak{h})$ be a strictly indecomposable spherical pair with $k = 3$. Without loss of generality we may assume that Z_{12} and Z_{23} are strictly indecomposable. Hence both show up in the list of Theorem 4.1.

We begin with the case where $\mathfrak{g}_3 = \mathfrak{h}_3$, and write $\mathfrak{h}_{12} = \mathfrak{h}'_{12} \oplus \mathfrak{h}^0$ with $\mathfrak{h}^0 \simeq \mathfrak{h}_3$. Let $\mathfrak{s}^0_{12}$ be the projection of $\mathfrak{s}(\mathfrak{g}_{12},\mathfrak{h}_{12})$ to $\mathfrak{h}^0$. Then a necessary and sufficient condition for Z to be real spherical is that $(\mathfrak{h}^0,\mathfrak{s}^0_{12})$ is strongly real spherical. By inspecting the lists in Theorem 4.1 we see that this can only happen if either $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{h}^0$ (case (S1)) or $\mathfrak{h}^0 = \mathfrak{sp}(1,\mathbb{K})$ with $\mathfrak{s}^0_{12} = \mathfrak{gl}(1,\mathbb{K})$ (cases (SS11), (SS23)). The first case was treated in Lemma 5.1, and leads to $\mathfrak{g}_1 = \mathfrak{g}_2 = \mathfrak{g}_3 = \mathfrak{so}(n,1)$, $n \geq 2$. The other cases can be combined in the following diagram

(5.2)



with $n, k \geq 1$ and $\mathbb{K},\mathbb{K}',\mathbb{K}'' = \mathbb{R},\mathbb{C}$ with $\mathbb{K}'' \subset \mathbb{K},\mathbb{K}'$. If $n = 1$, resp. $k = 1$, we require in addition that $\mathbb{K}'' = \mathbb{K}$, resp. $\mathbb{K}'' = \mathbb{K}'$. Note that with $n = k = 1$ also to the first two cases of Lemma 5.1 are included.

There are two possible choices of $\mathfrak{h}^0$ in $\mathfrak{h}_{12}$. It can be the middle factor $\mathfrak{sp}(1,\mathbb{K}'')$, or it can be one of the others when n or k is 2, say $k=2$ and $\mathfrak{h}^0=\mathfrak{sp}(1,\mathbb{K}')$. The first choice leads to the cases $m=1$ of (BM55). For the second choice we need $(\mathfrak{sp}(1,\mathbb{K}'),\mathfrak{gl}(1,\mathbb{K}''))$ to be strongly spherical, and hence $\mathbb{K}'=\mathbb{K}''$. This then leads to the cases $m=1$ of (BM54).

Let us now consider the case where $\mathfrak{h}_3\subsetneq\mathfrak{g}_3$. Since Z is strictly indecomposable, there exists a non-compact simple common factor $\mathfrak{h}^{0,+}$ of $\mathfrak{h}_{12}$ and $\mathfrak{h}_3$. According to Lemma 3.2(2) and Lemma 3.1, the pair $(\mathfrak{g}_{12}\oplus\mathfrak{h}^{0,+},\mathfrak{h}_{12})$ is then real spherical, hence one of the pairs we just determined. In particular Z_{12} is either a group case of $\mathfrak{so}(n,1)$, or it is of type (5.2). Moreover $\mathfrak{h}^{0,+}$ is either $\mathfrak{so}(n,1)$ for some n, or it is $\mathfrak{sp}(1,\mathbb{R})$, $\mathfrak{sp}(1,\mathbb{C})$.

By switching the roles of Z_{12} and Z_{23} we see that the same limitations apply to Z_{23}. In particular, if Z_{12} is a group case with $\mathfrak{g}_1=\mathfrak{g}_2=\mathfrak{so}(n,1)$ for $n\geq 4$, then Z_{23} has to be of the same type with $\mathfrak{g}_3=\mathfrak{g}_2=\mathfrak{so}(n,1)$, and we are in the case of Lemma 5.1. Excluding that case we infer that Z_{12} is of type (5.2) and that for some $m>1$

$$(\mathfrak{g}_3,\mathfrak{h}_3)=(\mathfrak{sp}(m,\mathbb{K}'''),\mathfrak{sp}(1,\mathbb{K}'''')\oplus\mathfrak{sp}(m-1,\mathbb{K}''')) \tag{5.3}$$

with $\mathbb{K}''''\subset\mathbb{K}'''$. Moreover $\mathfrak{s}(\mathfrak{g}_3,\mathfrak{h}_3)=\mathfrak{sp}(1,\mathbb{K}'''')\oplus\mathfrak{sp}(m-2,\mathbb{K}''')$.

If $\mathfrak{h}^0=\mathfrak{h}^{0,+}$ is simple the glueing between Z_3 and Z_{12} takes place along the factor $\mathfrak{sp}(1,\mathbb{K}'''')$ of $\mathfrak{h}_3$, and as before there are two choices in $\mathfrak{h}_{12}$. These two choices then lead to the remaining cases of (BM55) and (BM54). Finally, the possibility for $n=k=m=2$ that Z_3 is glued to Z_{12} along both factors of $\mathfrak{h}_3$ is easily excluded by dimension count. □

Proposition 5.3. (Exclusion of four and more factors) *Let Z be a strictly indecomposable real spherical space attached to $G=G_1\times\ldots\times G_k$. Then $k\leq 3$.*

Proof. It is sufficient to consider the case of $k=4$. We argue by contradiction and assume that there exists a strictly indecomposable real spherical space Z of this length.

We observe that after reordering we may assume that Z_{123} and Z_{234} are strictly indecomposable (this follows for example from the simple fact that every connected graph with 4 vertices contains a path of length 3).

It follows in particular that both Z_{123} and Z_{234} has to be one of the spaces listed in Theorem 5.2. We claim that it cannot be the triple space. Indeed, if Z_{123} were the triple space with $\mathfrak{g}_i=\mathfrak{so}(1,n)$ for $i=1,2,3$, then Z_{123} and Z_4 would be glued together along $\mathfrak{h}^0\simeq\mathfrak{so}(1,n)$ and $\mathfrak{h}^0$ would be spherical for the action of $\mathfrak{h}^0\oplus[\mathfrak{l}\cap\mathfrak{h}]_{123}$. Now $[\mathfrak{l}\cap\mathfrak{h}]_{123}=\mathfrak{so}(n-2)$ is compact and that would mean that $\mathfrak{so}(n-2)$ is a spherical subalgebra of $\mathfrak{h}^0=\mathfrak{so}(1,n)$ which is not the case by the classification of Part I (note that the spherical subalgebra $\mathfrak{su}(4)$ of $\mathfrak{so}(1,8)$ is isomorphic to $\mathfrak{so}(6)$, but the subalgebras are not conjugate within $\mathfrak{so}(1,8)$).

Hence Z_{123} has to be one of the cases (BM54) or (BM55). By symmetry this holds for Z_{234} as well. In particular we have $(\mathfrak{g}_4,\mathfrak{h}_4)=(\mathfrak{sp}(s,\widetilde{\mathbb{K}}),\mathfrak{sp}(1,\widetilde{\mathbb{K}}')\oplus\mathfrak{sp}(s-1,\widetilde{\mathbb{K}}))$ with $\widetilde{\mathbb{K}},\widetilde{\mathbb{K}}'=\mathbb{R},\mathbb{C}$, $\widetilde{\mathbb{K}}'\subset\widetilde{\mathbb{K}}$ and $s\geq 1$. Then Z_{123} and Z_4 are glued together along $\mathfrak{h}^0=\mathfrak{sp}(1,\widetilde{\mathbb{K}}')$ or $\mathfrak{h}^0=\mathfrak{sp}(1,\mathbb{K})$ if $s=2$. In order for the resulting space $Z=Z_{1234}$ to be real spherical this would require that the projection of $[\mathfrak{l}\cap\mathfrak{h}]_{123}$ to $\mathfrak{h}^0$ is real spherical. But this projection is 0 by Theorem 5.2 and we obtain a contradiction. □

6. Indecomposability versus strict indecomposability

Let $\mathfrak{g}$ be a real semisimple Lie algebra and $\mathfrak{h}$ an algebraic reductive subalgebra. The goal of this section is to describe how one can determine whether an indecomposable, but not strictly indecomposable, pair $(\mathfrak{g},\mathfrak{h})$ is real spherical.

Lemma 6.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be an indecomposable real spherical pair which not strictly indecomposable. Then there is a splitting $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ into ideals such that with $\mathfrak{h}_i$ the projection of $\mathfrak{h}$ to $\mathfrak{g}_i$ one has:*

(1) $\mathfrak{h}_{\rm n} = (\mathfrak{h}_1)_{\rm n} \oplus \ldots \oplus (\mathfrak{h}_k)_{\rm n}$.
(2) $(\mathfrak{g}_i, \mathfrak{h}_i)$ *is a strictly indecomposable real spherical pair for all* $1 \leq i \leq k$.

In particular, if $\mathfrak{g}$ is semisimple without compact factors, then each $(\mathfrak{g}_i, \mathfrak{h}_i)$ appears either in Theorem 4.1 or Theorem 5.2.

Proof. By definition, if $(\mathfrak{g}, \mathfrak{h})$ is not strictly indecomposbale, there exists a splitting of $\mathfrak{g}$ and $\mathfrak{h}_{\rm n}$ as indicated. Moreover, as G/H projects onto G/H_i, it follows that each $(\mathfrak{g}_i, \mathfrak{h}_i)$ is real spherical (see also Lemma 3.1). We assume that k is maximal and then each $(\mathfrak{g}_i, \mathfrak{h}_i)$ is strictly indecomposable. □

For our objective to describe all indecomposable real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ which are not strictly indecomposable, we may thus assume that $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ with $\mathfrak{h}_{\rm n} = (\mathfrak{h}_1)_{\rm n} \oplus \ldots \oplus (\mathfrak{h}_k)_{\rm n}$ such that each $(\mathfrak{g}_i, \mathfrak{h}_i)$ be a strictly indecomposable real spherical pair.

Let $\mathfrak{h}' := \mathfrak{h}_1 \oplus \ldots \oplus \mathfrak{h}_k$ and note that $\mathfrak{s}(\mathfrak{g}, \mathfrak{h}') = \mathfrak{s}(\mathfrak{g}_1, \mathfrak{h}_1) \oplus \ldots \oplus \mathfrak{s}(\mathfrak{g}_k, \mathfrak{h}_k)$. We denote by $\mathfrak{c}_i$ the projection of $\mathfrak{s}(\mathfrak{g}_i, \mathfrak{h}_i)$ to $(\mathfrak{h}_i)_{\rm el}$. Since $(\mathfrak{h}_{\rm n})' = \mathfrak{h}_{\rm n}$ we thus obtain from Proposition 2.4 applied to the tower $\mathfrak{g} \supset \mathfrak{h}' \supset \mathfrak{h}$ the following criterion:

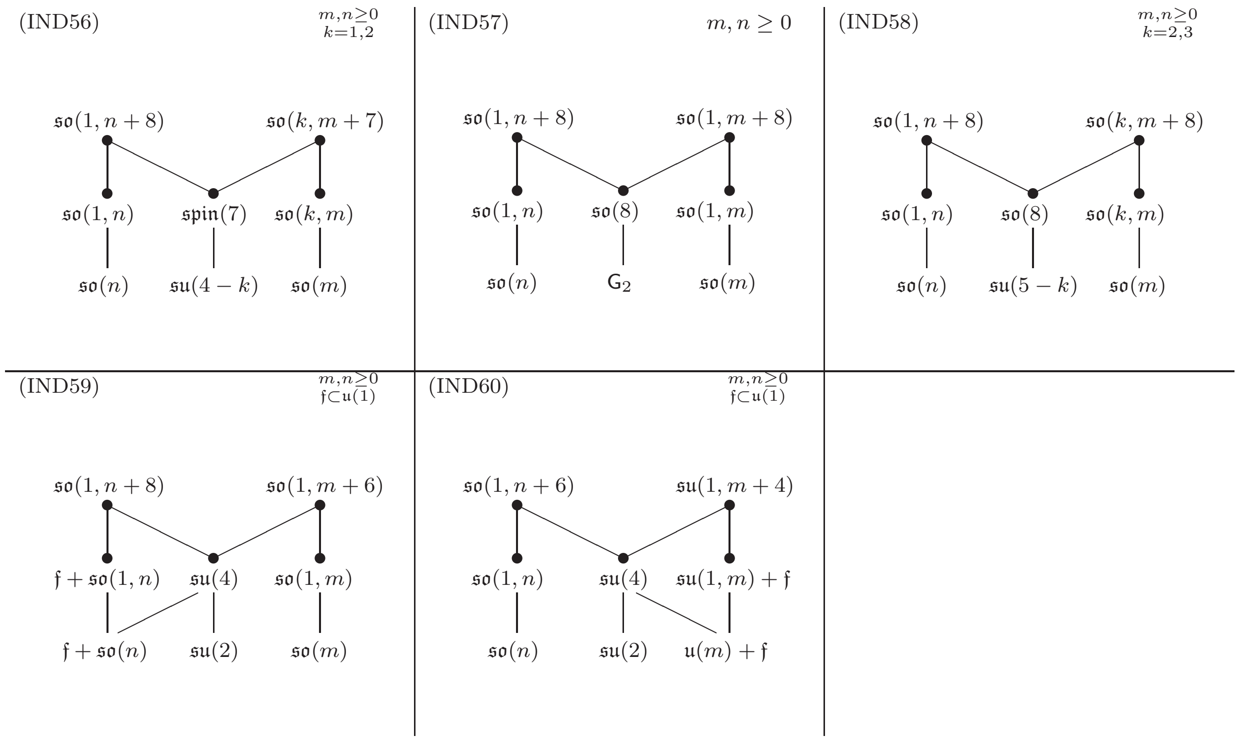
Proposition 6.2. (Criterion for sphericality) *In the setup described above, $(\mathfrak{g}, \mathfrak{h})$ is real spherical if and only if*

$$(\mathfrak{h}_1)_{\rm el} \oplus \ldots \oplus (\mathfrak{h}_k)_{\rm el} = \mathfrak{h}_{\rm el} + \mathfrak{c}_1 + \ldots + \mathfrak{c}_k$$

In the sequel we will use this criterion to analyse the situation further. We begin with the building blocks of $k = 2$ and each $\mathfrak{g}_i$ non-compact simple.

Proposition 6.3. (Indecomposable pairs with two factors) *Suppose that $\mathfrak{g}_1, \mathfrak{g}_2$ are both non-compact simple and set $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$. Let $\mathfrak{h} \subset \mathfrak{g}$ be a reductive subalgebra such that $(\mathfrak{g}, \mathfrak{h})$ is an indecomposable real spherical pair which is not strictly indecomposable. Set $\mathfrak{h}^0 := \mathfrak{h}/[(\mathfrak{g}_1 \cap \mathfrak{h}) \oplus (\mathfrak{g}_2 \cap \mathfrak{h})]$.*

Then either $\mathfrak{h}^0 \simeq \mathbb{R}^l \oplus (i\mathbb{R})^m \oplus \mathfrak{sp}(1)^n$ for some $l, m, n \leq 1$, or $\mathfrak{h}^0$ is simple and $(\mathfrak{g}, \mathfrak{h})$ is equivalent to one of the following pairs:



In diagram (IND56) *the left diagonal embedding of* $\mathfrak{spin}(7)\simeq\mathfrak{so}(7)$ *is the spin embedding of* $\mathfrak{spin}(7)$ *into* $\mathfrak{so}(8)$ *whereas the right diagonal embedding is the standard embedding of* $\mathfrak{so}(7)$*. In diagrams* (IND57)-(IND58) *the left diagonal embedding of* $\mathfrak{so}(8)$ *into* $\mathfrak{so}(1,n+8)$ *is the triality automorphism followed by the standard embedding of* $\mathfrak{so}(8)$*, and the right diagonal embedding is just the standard embedding of* $\mathfrak{so}(8)$*.*

In diagrams (IND59) *and* (IND60) *we employ the standard isomorphism* $\mathfrak{su}(4)\simeq\mathfrak{so}(6)$*. Moreover, the lower diagonal lines refer to morphisms which are only non-trivial on* $\mathfrak{f}$*.*

Proof. The proof is a matter of bookkeeping followed by a simple application of Onishchik's list of factorizations (see Proposition 2.8). From Tables 3–7 one collects all pairs $(\mathfrak{g},\mathfrak{h})$ for which $\mathfrak{h}$ contains a compact simple ideal $\mathfrak{h}^0$ that admits a non-trivial factorization according to Onishchik (that is, $\mathfrak{h}^0=\mathfrak{su}(2n)$ $(n\geq 2)$, $\mathfrak{so}(2n)$ $(n\geq 3)$, or $\mathfrak{so}(7)$). For every such pair the projection $\mathfrak{c}^0$ of $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{h}^0$ is determined from the tables. Then for two such pairs $(\mathfrak{g},\mathfrak{h}_1)$ and $(\mathfrak{g},\mathfrak{h}_2)$ where $\mathfrak{h}_1$ and $\mathfrak{h}_2$ have a common simple compact factor, say $\mathfrak{h}^0$, one checks with Onishchik's list whether $\mathfrak{h}^0=\mathfrak{c}_1^0+\mathfrak{c}_2^0$. This happens precisely in the cases listed in the proposition. □

Corollary 6.4. *Let* $\mathfrak{g}$ *be semi-simple without compact factors, and let* $(\mathfrak{g},\mathfrak{h})$ *be an indecomposable real spherical reductive pair. Suppose that* $(\mathfrak{g},\mathfrak{h})$ *is not strictly indecomposable, such that* $\mathfrak{h}^0:=\mathfrak{h}/[(\mathfrak{g}_1\cap\mathfrak{h})\oplus\ldots\oplus(\mathfrak{g}_k\cap\mathfrak{h})]$ *is not of the the type* $\mathbb{R}^l\oplus(i\mathbb{R})^m\oplus\mathfrak{sp}(1)^n$ *for any* $l,m,n\geq 0$*. Then* $(\mathfrak{g},\mathfrak{h})$ *is one of the pairs* (IND56)-(IND60) *listed in Proposition 6.3.*

Proof. By assumption, we first note that that there exists a pair of simple ideals of $\mathfrak{g}$, say $\mathfrak{g}_1$ and $\mathfrak{g}_2$, such that the projection of $(\mathfrak{g},\mathfrak{h})$ to $\mathfrak{g}_1\oplus\mathfrak{g}_2$ belongs to (IND56)-(IND60). Moreover $\mathfrak{s}(\mathfrak{g},\mathfrak{h})$ is elementary in all cases (IND56)-(IND60), and the list of Onishchik (see Proposition 2.8) readily excludes further possibilities. □

6.1. **Examples with arbitrarily many factors.** In case $\mathfrak{c}_i = (\mathfrak{h}_i)_{\mathrm{el}} = \mathbb{R}^l \oplus (i\mathbb{R})^m \oplus \mathfrak{sp}(1)^n$ one can construct a lot of examples of real spherical pairs with arbitrarily many factors, i.e. the bound of Proposition 5.3 of $k \leq 3$ is not valid for general indecomposable pairs.

This was first observed by Mikityuk [21, page 545]) who listed all complex spherical pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple such that $\mathfrak{c} = \mathfrak{gl}(1, \mathbb{C})$ with $\mathfrak{h} = \mathfrak{c} \oplus [\mathfrak{h}, \mathfrak{h}]$. This is for example the case for $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{so}(2n, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ for $n \geq 5$ odd. In case each $(\mathfrak{g}_i, \mathfrak{h}_i)$ is complex spherical with $\mathfrak{c}_i = \mathfrak{gl}(1, \mathbb{C})$ one obtains via

$$(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, ([\mathfrak{h}_1, \mathfrak{h}_1] \oplus \ldots \oplus [\mathfrak{h}_k, \mathfrak{h}_k]) + \operatorname{diag} \mathfrak{gl}(1, \mathbb{C}))$$

a complex spherical pair with arbitrarily many factors.

More interesting are perhaps the cases of strictly indecomposable real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ for which $\mathfrak{h}$ contains $\mathfrak{sp}(1)$ as a factor such that $\mathfrak{h} = \mathfrak{sp}(1) \oplus \widetilde{\mathfrak{h}}$ as a direct sum of Lie algebras.

By inspecting our tables we see that this happens if and only if $(\mathfrak{g}, \mathfrak{h})$ belongs to the family of pairs

$$(\mathfrak{so}(n, 1), \mathfrak{so}(n-4q, 1) \oplus \mathfrak{sp}(q) \oplus \mathfrak{sp}(1)) \tag{6.1}$$

(see Table 7),

$$(\mathfrak{sp}(p, q+1), \mathfrak{sp}(p, q) \oplus \mathfrak{sp}(1)) \tag{6.2}$$

(see Table 3) or the family (see (SS18))

$$(\mathfrak{sp}(p+1, q) \oplus \mathfrak{sp}(p, q), \mathfrak{sp}(p, q) \oplus \mathfrak{sp}(1))\,. \tag{6.3}$$

Suppose that each $(\mathfrak{g}_i, \mathfrak{h}_i)$ is of the type (6.1) - (6.3) and decompose $\mathfrak{h}_i = \mathfrak{sp}(1) \oplus \widetilde{\mathfrak{h}}_i$. Then we obtain via

$$(\mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k, (\widetilde{\mathfrak{h}}_1 \oplus \ldots \oplus \widetilde{\mathfrak{h}}_k) + \operatorname{diag} \mathfrak{sp}(1))$$

real spherical pairs with arbitrarily many factors.

Appendix A. Tables for the structural algebra $\mathfrak{l} \cap \mathfrak{h}$

In this appendix we list the structural algebra $\mathfrak{s}(\mathfrak{g}, \mathfrak{h}) = \mathfrak{l} \cap \mathfrak{h}$ for all real spherical pairs $(\mathfrak{g}, \mathfrak{h})$ with $\mathfrak{g}$ simple and $\mathfrak{h}$ reductive. The recipe how this can be obtained is explained in Part I, Sect. 8. We start with the symmetric cases (Berger's list [2]), move on to the non-symmetric absolutely spherical cases obtained from Krämer's list [17] (see Part I for the determination of all real forms in Krämer's list), and finally catalogue the remaining cases obtained in Part I.

In the tables below the following conventions hold: $n = p + q = k + l$ with $p \leq q$ and $k \leq l$. Further $p = p_1 + p_2$, $q = q_1 + q_2$ and $k = p_1 + q_1$, $l = p_2 + q_2$. Notice that both $p \leq q$ and $k \leq l$ force $q_2 \geq p_1$. If $\mathfrak{h} \subset \mathfrak{gl}(n, \mathbb{C})$, then we use the notation $\mathfrak{s}[\mathfrak{h}] := \mathfrak{h} \cap \mathfrak{sl}(n, \mathbb{C})$.

Tables 3 - 6 are separated by horizontal lines with each segment corresponding to the real forms of a complex spherical space listed on top of the segment.

In most of the cases the embedding of $\mathfrak{l} \cap \mathfrak{h}$ into $\mathfrak{h}$ is unique up to conjugation. The cases which cannot be decided just by the form of $\mathfrak{h}$ and $\mathfrak{l} \cap \mathfrak{h}$, and which are of relevance for this paper, are discussed in Part I, Remark 8.2.

A.1. **The classical symmetric Lie algebras.** [See also [1].]

	$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$	
(1)	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{so}(n,\mathbb{C})$	0	
(2)	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{so}(p,q)$	0	
(3)	$\mathfrak{su}(p,q)$	$\begin{cases}\mathfrak{so}(p,q)\\ \mathfrak{so}^*(2p),\ p=q\end{cases}$	$\mathfrak{so}(q-p)$	
(4)	$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{so}^*(2n)$	$(i\mathbb{R})^n$	
(5)	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(1,\mathbb{C})^n$	
(6)	$\mathfrak{sl}(2n,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})$	$\mathfrak{sp}(1,\mathbb{R})^n$	
(7)	$\mathfrak{su}(2p,2q)$	$\begin{cases}\mathfrak{sp}(p,q)\\ \mathfrak{sp}(2p,\mathbb{R}),\ p=q\end{cases}$	$\mathfrak{sl}(2,\mathbb{C})^p+\mathfrak{sp}(q-p,0)$	
(8)	$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(1,0)^n$	
(9)	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{s}[\mathfrak{gl}(p,\mathbb{C})+\mathfrak{gl}(q,\mathbb{C})]$	$\mathfrak{s}[\mathfrak{gl}(q-p,\mathbb{C})+\mathbb{C}^p]$	
(10)	$\mathfrak{sl}(n,\mathbb{R})$	$\begin{cases}\mathfrak{sl}(p,\mathbb{R})+\mathfrak{sl}(q,\mathbb{R})+\mathbb{R}\\ \mathfrak{sl}(p,\mathbb{C})+i\mathbb{R},\ p=q\end{cases}$	$\mathfrak{s}[\mathfrak{gl}(q-p,\mathbb{R})+\mathbb{R}^p]$	
(11a)	$\mathfrak{su}(p,q)$	$\mathfrak{s}[\mathfrak{u}(p_1,q_1)+\mathfrak{u}(p_2,q_2)]$	$\begin{cases}\mathfrak{s}[\mathfrak{u}(p_2-q_1,q_2-p_1)+(i\mathbb{R})^{p_1+q_1}]\\ \mathfrak{s}[\mathfrak{u}(q_1-p_2)+\mathfrak{u}(q_2-p_1)+(i\mathbb{R})^{p_1+p_2}]\end{cases}$	$p_2\geq q_1$ $p_2\leq q_1$
(11b)	$\mathfrak{su}(p,p)$	$\mathfrak{sl}(p,\mathbb{C})+\mathbb{R}$	$(i\mathbb{R})^{p-1}$	
(12a)	$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(p,\mathbb{H})+\mathfrak{sl}(q,\mathbb{H})+\mathbb{R}$	$\mathfrak{s}[\mathfrak{gl}(q-p,\mathbb{H})+\mathfrak{gl}(1,\mathbb{H})^p]$	
(12b)	$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{C})+i\mathbb{R}$	$\begin{cases}\mathfrak{s}[\mathfrak{gl}(1,\mathbb{H})^{\frac{n}{2}}]\\ \mathfrak{s}[\mathfrak{gl}(1,\mathbb{H})^{[\frac{n}{2}]}]+\mathfrak{u}(1)\end{cases}$	n even n odd
(13)	$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	$\begin{cases}\mathfrak{sl}(2,\mathbb{C})^{\frac{n}{2}}\\ \mathfrak{sl}(2,\mathbb{C})^{\frac{n-1}{2}}+\mathbb{C}\end{cases}$	n even n odd
(14a)	$\mathfrak{so}(p,q)$	$\mathfrak{u}(\frac{p}{2},\frac{q}{2})$	$\mathfrak{u}(\frac{q-p}{2})+\mathfrak{sl}(2,\mathbb{R})^{\frac{p}{2}}$	
(14b)	$\mathfrak{so}(p,p)$	$\mathfrak{gl}(p,\mathbb{R})$	$\begin{cases}\mathfrak{sl}(2,\mathbb{R})^{\frac{p}{2}}\\ \mathfrak{sl}(2,\mathbb{R})^{\frac{p-1}{2}}+\mathbb{R}\end{cases}$	p even p odd
(15a)	$\mathfrak{so}^*(2n)$	$\mathfrak{u}(p,q)$	$\begin{cases}\mathfrak{sl}(1,\mathbb{H})^{\frac{n}{2}}\\ \mathfrak{sl}(1,\mathbb{H})^{\frac{n}{2}-1}+\mathfrak{su}(1,1)+i\mathbb{R}\\ \mathfrak{sl}(1,\mathbb{H})^{\frac{n-1}{2}}+i\mathbb{R}\end{cases}$	n even, p even n even, p odd n odd
(15b)	$\mathfrak{so}^*(2n)$	$\mathfrak{gl}(\frac{n}{2},\mathbb{H})$	$\mathfrak{sl}(1,\mathbb{H})^{\frac{n}{2}}$	n even
(16)	$\mathfrak{so}(n,\mathbb{C})$	$\mathfrak{so}(p,\mathbb{C})+\mathfrak{so}(q,\mathbb{C})$	$\mathfrak{so}(q-p,\mathbb{C})$	
(17a)	$\mathfrak{so}(p,q)$	$\mathfrak{so}(p_1,q_1)+\mathfrak{so}(p_2,q_2)$	$\begin{cases}\mathfrak{so}(p_2-q_1,q_2-p_1)\\ \mathfrak{so}(q_1-p_2)+\mathfrak{so}(q_2-p_1)\end{cases}$	$p_2\geq q_1$ $p_2\leq q_1$
(17b)	$\mathfrak{so}(p,p)$	$\mathfrak{so}(p,\mathbb{C})$	0	
(18a)	$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2p)+\mathfrak{so}^*(2q)$	$\mathfrak{so}^*(2q-2p)+(i\mathbb{R})^p$	
(18b)	$\mathfrak{so}^*(2n)$	$\mathfrak{so}(n,\mathbb{C})$	$(i\mathbb{R})^{[\frac{n}{2}]}$	
(19)	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{gl}(n,\mathbb{C})$	0	
(20)	$\mathfrak{sp}(n,\mathbb{R})$	$\begin{cases}\mathfrak{u}(p,q)\\ \mathfrak{gl}(n,\mathbb{R})\end{cases}$	0	
(21)	$\mathfrak{sp}(p,q)$	$\begin{cases}\mathfrak{u}(p,q)\\ \mathfrak{gl}(p,\mathbb{H}),\ p=q\end{cases}$	$\mathfrak{u}(q-p)+(i\mathbb{R})^p$	
(1)(22)	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{sp}(p,\mathbb{C})+\mathfrak{sp}(q,\mathbb{C})$	$\mathfrak{sp}(q-p,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C})^p$	
(23)	$\mathfrak{sp}(n,\mathbb{R})$	$\begin{cases}\mathfrak{sp}(p,\mathbb{R})+\mathfrak{sp}(q,\mathbb{R})\\ \mathfrak{sp}(p,\mathbb{C}),\ p=q\end{cases}$	$\mathfrak{sp}(q-p,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})^p$	
(24a)	$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p_1,q_1)+\mathfrak{sp}(p_2,q_2)$	$\begin{cases}\mathfrak{sp}(p_2-q_1,q_2-p_1)+\mathfrak{sp}(1)^{p_1+q_1}\\ \mathfrak{sp}(q_1-p_2)+\mathfrak{sp}(q_2-p_1)+\mathfrak{sp}(1)^{p_1+p_2}\end{cases}$	$p_2\geq q_1$ $p_2\leq q_1$
(24b)	$\mathfrak{sp}(p,p)$	$\mathfrak{sp}(p,\mathbb{C})$	$\mathfrak{sp}(1)^p$	

Table 3

A.2. **The exceptional symmetric Lie algebras.**

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$
$\mathsf{E}_6^{\mathbb{C}}$	$\mathfrak{sp}(4,\mathbb{C})$	0
E_6^1	$\mathfrak{sp}(4,\mathbb{R})$	0
	$\mathfrak{sp}(4)$	
	$\mathfrak{sp}(2,2)$	
E_6^2	$\mathfrak{sp}(4,\mathbb{R})$	0
	$\mathfrak{sp}(1,3)$	
E_6^3	$\mathfrak{sp}(2,2)$	$\mathfrak{so}(4)$
E_6^4	$\mathfrak{sp}(1,3)$	$\mathfrak{so}(4)+\mathfrak{so}(4)$
$\mathsf{E}_6^{\mathbb{C}}$	$\mathfrak{sl}(6,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C})$	$\mathbb{C}^2\subseteq\mathfrak{sl}(6,\mathbb{C})$
E_6^1	$\mathfrak{sl}(6,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$	$\mathbb{R}^2$
	$\mathfrak{sl}(3,\mathbb{H})+\mathfrak{su}(2)$	
E_6^2	$\mathfrak{su}(6)+\mathfrak{su}(2)$	$(i\mathbb{R})^2$
	$\mathfrak{su}(2,4)+\mathfrak{su}(2)$	
	$\mathfrak{su}(3,3)+\mathfrak{sl}(2,\mathbb{R})$	
E_6^3	$\mathfrak{su}(2,4)+\mathfrak{su}(2)$	$\mathfrak{u}(2)+\mathfrak{u}(2)$
	$\mathfrak{su}(1,5)+\mathfrak{sl}(2,\mathbb{R})$	
E_6^4	$\mathfrak{sl}(3,\mathbb{H})+\mathfrak{su}(2)$	$\mathfrak{so}(5)+\mathfrak{so}(3)+\mathbb{R}$
$\mathsf{E}_6^{\mathbb{C}}$	$\mathfrak{so}(10,\mathbb{C})+\mathbb{C}$	$\mathfrak{sl}(4,\mathbb{C})+\mathbb{C}$
E_6^1	$\mathfrak{so}(5,5)+\mathbb{R}$	$\mathfrak{sl}(4,\mathbb{R})+\mathbb{R}$
E_6^2	$\mathfrak{so}(4,6)+i\mathbb{R}$	$\mathfrak{u}(2,2)$
	$\mathfrak{so}^*(10)+i\mathbb{R}$	
E_6^3	$\mathfrak{so}(10)+i\mathbb{R}$	$\mathfrak{u}(4)$
	$\mathfrak{so}(2,8)+i\mathbb{R}$	
	$\mathfrak{so}^*(10)+i\mathbb{R}$	
E_6^4	$\mathfrak{so}(1,9)+\mathbb{R}$	$\mathfrak{spin}(7)+\mathbb{R}$
$\mathsf{E}_6^{\mathbb{C}}$	$\mathsf{F}_4^{\mathbb{C}}$	$\mathfrak{so}(8,\mathbb{C})$
E_6^1	F_4^1	$\mathfrak{so}(4,4)$
E_6^2	F_4^1	$\mathfrak{so}(3,5)$
E_6^3	F_4^2	$\mathfrak{so}(1,7)$
E_6^4	F_4^2	$\mathfrak{so}(8)$
	F_4	
$\mathsf{E}_7^{\mathbb{C}}$	$\mathfrak{sl}(8,\mathbb{C})$	0
E_7^1	$\mathfrak{sl}(8,\mathbb{R})$	0
	$\mathfrak{su}(8)$	
	$\mathfrak{su}(4,4)$	
	$\mathfrak{sl}(4,\mathbb{H})$	
E_7^2	$\mathfrak{su}(2,6)$	$\mathfrak{so}(2)^3$
	$\mathfrak{su}(4,4)$	
E_7^3	$\mathfrak{su}(2,6)$	$\mathfrak{so}(4)+\mathfrak{so}(4)$
	$\mathfrak{sl}(4,\mathbb{H})$	
$\mathsf{E}_7^{\mathbb{C}}$	$\mathfrak{so}(12,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C})^3\subseteq\mathfrak{so}(12,\mathbb{C})$
E_7^1	$\mathfrak{so}(6,6)+\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})^3$
	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	
E_7^2	$\mathfrak{so}(12)+\mathfrak{su}(2)$	$\mathfrak{su}(2)^3$
	$\mathfrak{so}(4,8)+\mathfrak{su}(2)$	
	$\mathfrak{so}^*(12)+\mathfrak{sl}(2,\mathbb{R})$	
E_7^3	$\mathfrak{so}(2,10)+\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}(6)+\mathfrak{so}(2)+\mathfrak{sl}(2,\mathbb{R})$
	$\mathfrak{so}^*(12)+\mathfrak{su}(2)$	
$\mathsf{E}_7^{\mathbb{C}}$	$\mathsf{E}_6^{\mathbb{C}}+\mathbb{C}$	$\mathfrak{so}(8,\mathbb{C})$
E_7^1	$\mathsf{E}_6^1+\mathbb{R}$	$\mathfrak{so}(4,4)$
	$\mathsf{E}_6^2+i\mathbb{R}$	
E_7^2	$\mathsf{E}_6^2+i\mathbb{R}$	$\mathfrak{so}(2,6)+\mathfrak{so}(2)$
	$\mathsf{E}_6^3+i\mathbb{R}$	
E_7^3	$\mathsf{E}_6^3+i\mathbb{R}$	$\mathfrak{so}(8)$
	$\mathsf{E}_6^4+\mathbb{R}$	
	$\mathsf{E}_6+i\mathbb{R}$	
$\mathsf{E}_8^{\mathbb{C}}$	$\mathfrak{so}(16,\mathbb{C})$	0
E_8^1	$\mathfrak{so}(8,8)$	0
	$\mathfrak{so}(16)$	
	$\mathfrak{so}^*(16)$	
E_8^2	$\mathfrak{so}(4,12)$	$\mathfrak{so}(4)+\mathfrak{so}(4)$
	$\mathfrak{so}^*(16)$	
$\mathsf{E}_8^{\mathbb{C}}$	$\mathsf{E}_7^{\mathbb{C}}+\mathfrak{sl}(2,\mathbb{C})$	$\mathfrak{so}(8,\mathbb{C})$
E_8^1	$\mathsf{E}_7^1+\mathfrak{sl}(2,\mathbb{R})$	$\mathfrak{so}(4,4)$
	$\mathsf{E}_7^2+\mathfrak{su}(2)$	
E_8^2	$\mathsf{E}_7^2+\mathfrak{su}(2)$	$\mathfrak{so}(8)$
	$\mathsf{E}_7^3+\mathfrak{sl}(2,\mathbb{R})$	
	$\mathsf{E}_7+\mathfrak{su}(2)$	
$\mathsf{F}_4^{\mathbb{C}}$	$\mathfrak{sp}(3,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C})$	0
F_4^1	$\mathfrak{sp}(3,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$	0
	$\mathfrak{sp}(3)+\mathfrak{su}(2)$	
	$\mathfrak{sp}(1,2)+\mathfrak{su}(2)$	
F_4^2	$\mathfrak{sp}(1,2)+\mathfrak{su}(2)$	$\mathfrak{so}(4)+\mathfrak{so}(3)$
$\mathsf{F}_4^{\mathbb{C}}$	$\mathfrak{so}(9,\mathbb{C})$	$\mathfrak{spin}(7,\mathbb{C})$
F_4^1	$\mathfrak{so}(4,5)$	$\mathfrak{spin}(3,4)$
F_4^2	$\mathfrak{so}(9)$	$\mathfrak{spin}(7)$
	$\mathfrak{so}(1,8)$	
$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(2,\mathbb{C})+\mathfrak{sl}(2,\mathbb{C})$	0
G_2^1	$\mathfrak{sl}(2,\mathbb{R})+\mathfrak{sl}(2,\mathbb{R})$	0
	$\mathfrak{su}(2)+\mathfrak{su}(2)$	

Table 4

A.3. **Two general cases of symmetric Lie algebras.**

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$	
$\mathfrak{g}_{\mathbb{C}}$	$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{c}_{\mathbb{R}}$	$\mathfrak{g}_{\mathbb{C}}$ complex simple, $\mathfrak{g}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{C}}$ real form, $\mathfrak{c}_{\mathbb{R}}\subset\mathfrak{g}_{\mathbb{R}}$ fund. Cartan.
$\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{g}$	$\mathfrak{g}$ simple

A.4. **The non-symmetric absolutely spherical cases.**

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$	
$\mathfrak{sl}(2n+1,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})+\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{C}$
$\mathfrak{sl}(2n+1,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})+\mathfrak{f}$	$\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{R}$
$\mathfrak{su}(n,n+1)$	$\mathfrak{sp}(n,\mathbb{R})+i\mathfrak{f}$	$i\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{R}$
$\mathfrak{su}(2p,2q+1)$	$\mathfrak{sp}(p,q)+i\mathfrak{f}$	$\mathfrak{sp}(q-p)+i\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{R}$
$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{sl}(p,\mathbb{C})+\mathfrak{sl}(q,\mathbb{C})+\mathfrak{f}$	$\mathfrak{s}[\mathfrak{sl}(q-p,\mathbb{C})+\mathbb{C}^p]+\mathfrak{f}$	$p<q, \mathfrak{f}\subset\mathbb{C}, \dim_{\mathbb{R}}\mathfrak{f}\leq 1$
$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(p,\mathbb{R})+\mathfrak{sl}(q,\mathbb{R})$	$\mathfrak{s}[\mathfrak{sl}(q-p,\mathbb{R})+\mathbb{R}^p]$	$p<q$
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p_1,q_1)+\mathfrak{su}(p_2,q_2)$	$\begin{cases}\mathfrak{s}[\mathfrak{su}(p_2-q_1,q_2-p_1)+(i\mathbb{R})^{p_1+q_1}], & p_2\geq q_1\\ \mathfrak{s}[\mathfrak{s}[\mathfrak{u}(q_1-p_2)+\mathfrak{u}(q_2-p_1)]+(i\mathbb{R})^{p_1+p_2}], & p_2\leq q_1\end{cases}$	$p_1+q_1<p_2+q_2$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(p,\mathbb{H})+\mathfrak{sl}(q,\mathbb{H})$	$\mathfrak{s}[\mathfrak{sl}(q-p,\mathbb{H})+\mathfrak{gl}(1,\mathbb{H})^p]$	
$\mathfrak{so}(2n+1,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})+\mathbb{C}$	0	
$\mathfrak{so}(n,n+1)$	$\mathfrak{sl}(n,\mathbb{R})+\mathbb{R}$	0	
$\mathfrak{so}(2p,2q+1)$	$\mathfrak{su}(p,q)+i\mathbb{R}$	$\mathfrak{u}(q-p)$	
$\mathfrak{so}(2n,\mathbb{C})$	$\mathfrak{sl}(n,\mathbb{C})+\mathfrak{f}$	$\mathfrak{sl}(2,\mathbb{C})^{\frac{n-1}{2}}+\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{C}, \dim_{\mathbb{R}}\mathfrak{f}\leq 1, n$ odd
$\mathfrak{so}(n,n)$	$\mathfrak{sl}(n,\mathbb{R})$	$\mathfrak{sl}(2,\mathbb{R})^{\frac{n-1}{2}}$	n odd
$\mathfrak{so}(2p,2q)$	$\mathfrak{su}(p,q)$	$\mathfrak{su}(q-p)+\mathfrak{sl}(2,\mathbb{R})^p$	$p+q$ odd
$\mathfrak{so}^*(2n)$	$\mathfrak{su}(p,q)$	$\mathfrak{sl}(1,\mathbb{H})^{\frac{n-1}{2}}$	$n=p+q$ odd
$\mathfrak{sp}(n+1,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})+\mathbb{C}$	$\mathfrak{sp}(n-1,\mathbb{C})+\mathbb{C}$	
$\mathfrak{sp}(n+1,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})+\mathfrak{sp}(1)$	$\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1)$	
$\mathfrak{sp}(n+1,\mathbb{C})$	$\mathfrak{sp}(n,\mathbb{C})+\mathfrak{sp}(1,\mathbb{R})$	$\mathfrak{sp}(n-1,\mathbb{C})+\mathfrak{sp}(1,\mathbb{R})$	
$\mathfrak{sp}(n+1,\mathbb{R})$	$\mathfrak{sp}(n,\mathbb{R})+\mathfrak{f}$	$\mathfrak{sp}(n-1,\mathbb{R})+\mathfrak{f}$	$\mathfrak{f}\in\{\mathbb{R},i\mathbb{R}\}$
$\mathfrak{sp}(p,q)$	$\begin{cases}\mathfrak{sp}(p-1,q)+i\mathbb{R}\\ \mathfrak{sp}(p,q-1)+i\mathbb{R}\end{cases}$	$\mathfrak{sp}(p-1,q-1)+i\mathbb{R}$	

Table 5

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$	
$\mathfrak{so}(7,\mathbb{C})$	$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(3,\mathbb{C})$	
$\mathfrak{so}(3,4)$	G_2^1	$\mathfrak{sl}(3,\mathbb{R})$	
$\mathfrak{so}(8,\mathbb{C})$	$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(2,\mathbb{C})$	
$\mathfrak{so}(4,4)$	G_2^1	$\mathfrak{sl}(2,\mathbb{R})$	
$\mathfrak{so}(3,5)$	G_2^1	$\mathfrak{sl}(2,\mathbb{R})$	
$\mathfrak{so}(1,7)$	G_2	$\mathfrak{su}(3)$	
$\mathfrak{so}(9,\mathbb{C})$	$\mathfrak{spin}(7,\mathbb{C})$	$\mathfrak{sl}(3,\mathbb{C})$	
$\mathfrak{so}(4,5)$	$\mathfrak{spin}(3,4)$	$\mathfrak{sl}(3,\mathbb{R})$	
$\mathfrak{so}(1,8)$	$\mathfrak{spin}(7)$	G_2	
$\mathfrak{so}(10)$	$\mathfrak{spin}(7,\mathbb{C})+\mathbb{C}$	$\mathfrak{sl}(2,\mathbb{C})$	
$\mathfrak{so}(5,5)$	$\mathfrak{spin}(3,4)+\mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R})$	
$\mathfrak{so}(4,6)$	$\mathfrak{spin}(3,4)+i\mathbb{R}$	$\mathfrak{sl}(2,\mathbb{R})$	
$\mathfrak{so}(2,8)$	$\mathfrak{spin}(7)+i\mathbb{R}$	$\mathfrak{su}(3)$	
$\mathfrak{so}(1,9)$	$\mathfrak{spin}(7)+\mathbb{R}$	G_2	
$\mathfrak{so}^*(10)$	$\begin{cases}\mathfrak{spin}(1,6)+i\mathbb{R}\\ \mathfrak{spin}(2,5)+i\mathbb{R}\end{cases}$	$\mathfrak{sl}(1,\mathbb{H})+i\mathbb{R}$	
$\mathsf{G}_2^{\mathbb{C}}$	$\mathfrak{sl}(3,\mathbb{C})$	$\mathfrak{sl}(2,\mathbb{C})$	
G_2^1	$\begin{cases}\mathfrak{sl}(3,\mathbb{R})\\ \mathfrak{su}(1,2)\end{cases}$	$\mathfrak{sl}(2,\mathbb{R})$	
$\mathsf{E}_6^{\mathbb{C}}$	$\mathfrak{so}(10,\mathbb{C})+\mathfrak{f}$	$\mathfrak{sl}(4,\mathbb{C})+\mathfrak{f}$	$\mathfrak{f}\subset\mathbb{C}, \dim_{\mathbb{R}}\mathfrak{f}\leq 1$
E_6^1	$\mathfrak{so}(5,5)$	$\mathfrak{sl}(4,\mathbb{R})$	
E_6^2	$\begin{cases}\mathfrak{so}(4,6)\\ \mathfrak{so}^*(10)\end{cases}$	$\mathfrak{su}(2,2)$	
E_6^3	$\begin{cases}\mathfrak{so}(10)\\ \mathfrak{so}(2,8)\\ \mathfrak{so}^*(10)\end{cases}$	$\mathfrak{su}(4)$	
E_6^4	$\mathfrak{so}(1,9)$	$\mathfrak{spin}(7)$	

Table 6

A.5. **Non-absolutely spherical simple pairs** $(\mathfrak{g},\mathfrak{h})$**.** The table below treats the cases classified in Part I.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{s}(\mathfrak{g},\mathfrak{h})$	
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n-1,\mathbb{H})+\mathfrak{f},\ \mathfrak{f}\subseteq\mathbb{C}$	$\mathfrak{sl}(n-2,\mathbb{H})+\mathfrak{f}$	$n\geq 3$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n,\mathbb{C})$	$\mathfrak{gl}(1,\mathbb{H})^{[\frac{n}{2}]}$	n odd
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p_1,q_1)+\mathfrak{su}(p_2,q_2)$	$\begin{cases}\mathfrak{s}[\mathfrak{su}(p_2-q_1,q_2-p_1)+(i\mathbb{R})^{p_1+q_1}],\ p_2\geq q_1\\ \mathfrak{s}[\mathfrak{s}[\mathfrak{u}(q_1-p_2)+\mathfrak{u}(q_2-p_1)]+(i\mathbb{R})^{p_1+p_2}],\ p_2\leq q_1\end{cases}$	$(p_1,q_1)\neq(q_2,p_2)$
$\mathfrak{su}(1,n)$	$\mathfrak{su}(1,n-2q)+\mathfrak{sp}(q)+\mathfrak{f},\ \mathfrak{f}\subset\mathfrak{u}(1)$	$\mathfrak{su}(n-2q)+\mathfrak{sp}(q-1)+i\mathbb{R}+\mathfrak{f}$	$1\leq q\leq\frac{n}{2}$
$\mathfrak{sp}(p,q)$	$\mathfrak{su}(p,q)$	$\mathfrak{s}[\mathfrak{u}(q-p)+i\mathbb{R}^q]$	
$\mathfrak{sp}(p,q)$	$\begin{cases}\mathfrak{sp}(p-1,q)\\ \mathfrak{sp}(p,q-1)\end{cases}$	$\mathfrak{sp}(p-1,q-1)$	
$\mathfrak{so}(2p,2q)$	$\mathfrak{su}(p,q)$	$\mathfrak{su}(q-p)+\mathfrak{sl}(2,\mathbb{R})^p$	$p<q$
$\mathfrak{so}(2p+1,2q)$	$\mathfrak{su}(p,q)$	$\mathfrak{su}(q-p)$	$p\neq q-1,q$
$\mathfrak{so}(3,n)$	$\mathfrak{so}(3,n-8)+\mathfrak{spin}(7)$	$\mathfrak{so}(n-8)+\mathfrak{so}(3),\ \mathfrak{so}(3)\subset\mathfrak{spin}(7)$	$n\geq 8$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,n-8)+\mathfrak{spin}(7)$	$\mathfrak{so}(n-8)+\mathfrak{su}(3)$	$n\geq 8$
$\mathfrak{so}(2,n)$	$\mathfrak{so}(2,n-7)+\mathsf{G}_2$	$\mathfrak{so}(n-7)+\mathfrak{su}(2)$	$n\geq 7$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-8)+\mathfrak{spin}(7)$	$\mathfrak{so}(n-8)+\mathsf{G}_2$	$n\geq 8$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-7)+\mathsf{G}_2$	$\mathfrak{so}(n-7)+\mathfrak{su}(3)$	$n\geq 7$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-16)+\mathfrak{spin}(9)$	$\mathfrak{so}(n-16)+\mathfrak{spin}(7)$	$n\geq 16$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-2r)+\mathfrak{su}(r)+\mathfrak{f},\ \mathfrak{f}\subset\mathfrak{u}(1)$	$\mathfrak{so}(n-2r)+\mathfrak{su}(r-1)+\mathfrak{f}$	$2\leq r\leq\frac{n}{2}$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-4r)+\mathfrak{sp}(r)+\mathfrak{f},\ \mathfrak{f}\subset\mathfrak{sp}(1)$	$\mathfrak{so}(n-4r)+\mathfrak{sp}(r-1)+\mathfrak{f}$	$2\leq r\leq\frac{n}{4}$
$\mathfrak{so}(3,6)$	$\mathfrak{so}(2)+\mathsf{G}_2^1$	0	
$\mathfrak{so}(4,7)$	$\mathfrak{so}(3)+\mathfrak{spin}(3,4)$	0	
$\mathfrak{so}^*(2n)$	$\mathfrak{so}^*(2n-2)$	$\mathfrak{so}^*(2n-4)$	$n\geq 5$
$\mathfrak{so}^*(10)$	$\begin{cases}\mathfrak{spin}(1,6)\\ \mathfrak{spin}(2,5)\end{cases}$	$\mathfrak{sl}(1,\mathbb{H})$	
E_6^4	$\mathfrak{sl}(3,\mathbb{H})+\mathfrak{f},\ \mathfrak{f}\subset\mathfrak{u}(1)$	$\mathfrak{so}(5)+\mathfrak{f}$	
E_7^2	$\begin{cases}\mathsf{E}_6^3\\ \mathsf{E}_6^2\end{cases}$	$\mathfrak{so}(2,6)$	
F_4^2	$\mathfrak{sp}(1,2)+\mathfrak{f},\ \mathfrak{f}\subset\mathfrak{u}(1)$	$\mathfrak{so}(4)+\mathfrak{f}$	

Table 7

The following two tables related to rank one groups are used in Thm. 4.1.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{c}$	$\mathfrak{c}_0$	
$\mathfrak{su}(1,n)$	$\mathfrak{su}(1,p)+\mathfrak{su}(n-p)$	$\mathfrak{su}(n-p)$	$\mathfrak{su}(n-p-1)$	$1\leq p<n$
$\mathfrak{su}(1,n)$	$\mathfrak{su}(1,n-2p)+\mathfrak{sp}(p)$	$\mathfrak{sp}(p)$	$\mathfrak{sp}(p-1)$	$1\leq p<\frac{n}{2}$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,p)+\mathfrak{so}(n-p)$	$\mathfrak{so}(n-p)$	$\mathfrak{so}(n-p-1)$	$2\leq p\leq n-3$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-8)+\mathfrak{spin}(7)$	$\mathfrak{spin}(7)$	G_2	$n\geq 10$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-7)+\mathsf{G}_2$	G_2	$\mathfrak{su}(3)$	$n\geq 9$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-16)+\mathfrak{spin}(9)$	$\mathfrak{spin}(9)$	$\mathfrak{spin}(7)$	$n\geq 18$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-2r)+\mathfrak{su}(r)$	$\mathfrak{su}(r)$	$\mathfrak{su}(r-1)$	$n-2r,r\geq 2$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-4r)+\mathfrak{sp}(r)$	$\mathfrak{sp}(r)$	$\mathfrak{sp}(r-1)$	$n-4r,r\geq 2$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-4r)+\mathfrak{sp}(r)+\mathfrak{sp}(1)$	$\mathfrak{sp}(r)$	$\mathfrak{sp}(r-1)+\mathfrak{sp}(1)$	$n-4r,r\geq 2$

Table 8

The restriction on the indices in Table 8 is such that $\mathfrak{h}\neq\mathfrak{g}$ is semi-simple and non-compact.

$\mathfrak{g}$	$\mathfrak{h}$	$\mathfrak{i}$	$\mathfrak{i}_0$	
$\mathfrak{su}(1,n)$	$\mathfrak{s}[\mathfrak{u}(1,p)+\mathfrak{u}(n-p)]$	$\mathfrak{u}(n-p)$	$\mathfrak{u}(n-p-1)$	$1\leq p<n$
$\mathfrak{su}(1,n)$	$\mathfrak{su}(1,n-2p)+\mathfrak{sp}(p)+\mathfrak{u}(1)$	$\mathfrak{sp}(p)+\mathfrak{u}(1)$	$\mathfrak{sp}(p-1)+\mathfrak{u}(1)$	$1\leq p<\frac{n}{2}$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-2)+\mathfrak{so}(2)$	$\mathfrak{so}(2)$	0	
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-2r)+\mathfrak{u}(r)$	$\mathfrak{u}(r)$	$\mathfrak{u}(r-1)$	$n-2r,r\geq 2$
$\mathfrak{so}(1,n)$	$\mathfrak{so}(1,n-4r)+\mathfrak{sp}(r)+\mathfrak{u}(1)$	$\mathfrak{sp}(r)+\mathfrak{u}(1)$	$\mathfrak{sp}(r-1)+\mathfrak{u}(1)$	$n-4r,r\geq 2$

Table 9

A.6. **Strongly spherical pairs.** We conclude with a classification of the strictly indecomposable strongly spherical pairs $(\mathfrak{g}, \mathfrak{h})$, with $\mathfrak{g}$ semi-simple and non-compact. This classification is easily extracted from our tables, in view of Lemma 3.2. In the following table we exclude symmetric pairs, as they were previously tabled in [14], and cases with $\mathfrak{h}$ compact.

$\mathfrak{g}$	$\mathfrak{h}$		
$\mathfrak{su}(p,q)$	$\mathfrak{su}(p-1,q)$		$p,q \geq 1;\ p-1, p \neq q$
$\mathfrak{su}(n,1)$	$\mathfrak{su}(n-q,1)+\mathfrak{su}(q)$		$n \geq 3; 1 \leq q \leq n$
$\mathfrak{su}(n,1)$	$\mathfrak{su}(n-2q,1)+\mathfrak{sp}(q)+\mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$	$1 \leq q \leq \frac{n}{2}$
$\mathfrak{sl}(n,\mathbb{H})$	$\mathfrak{sl}(n-1,\mathbb{H})+\mathbb{R}+\mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$	$n \geq 3$
$\mathfrak{sp}(p,q)$	$\mathfrak{sp}(p-1,q)+\mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$	$p,q \geq 1$
$\mathfrak{so}(2n,2)$	$\mathfrak{su}(n,1)$		$n \geq 2$
$\mathfrak{so}^*(2n+2)$	$\mathfrak{so}^*(2n)$		$n \geq 3$
$\mathfrak{so}(n,1)$	$\mathfrak{so}(n-2q,1)+\mathfrak{su}(q)+\mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{u}(1)$	$2 \leq q \leq \frac{n}{2}$
$\mathfrak{so}(n,1)$	$\mathfrak{so}(n-4q,1)+\mathfrak{sp}(q)+\mathfrak{f}$	$\mathfrak{f} \subseteq \mathfrak{sp}(1)$	$2 \leq q \leq \frac{n}{4}$
$\mathfrak{so}(n,1)$	$\mathfrak{so}(n-16,1)+\mathfrak{spin}(9)$		$n \geq 16$
$\mathfrak{so}(n,1)$	$\mathfrak{so}(n-7,1)+\mathsf{G}_2$		$n \geq 7$
$\mathfrak{so}(n,1)$	$\mathfrak{so}(n-8,1)+\mathfrak{spin}(7)$		$n \geq 8$

Table 10

Appendix B. The geometry of restricted root spaces in symmetric spaces

Throughout this appendix we let $\mathfrak{g}$ be a semi-simple real Lie algebra and $\mathfrak{h} \subset \mathfrak{g}$ a symmetric subalgebra. Let $\sigma : \mathfrak{g} \to \mathfrak{g}$ be the involution with fixed point algebra $\mathfrak{h}$ and θ a Cartan involution of $\mathfrak{g}$ which commutes with σ. We denote by $\mathfrak{g} = \mathfrak{k} + \mathfrak{s}$ the decomposition into θ-eigenspaces where $\mathfrak{k} \subset \mathfrak{g}$ corresponds to the $+1$-eigenspace and constitutes a maximal compact subalgebra of $\mathfrak{g}$. Notice that $\mathfrak{s} = \mathfrak{k}^\perp$ where the orthogonal complement is taken with respect to the Cartan-Killing form. Likewise we record the decomposition into σ-eigenspaces $\mathfrak{g} = \mathfrak{h} + \mathfrak{h}^\perp$. Let now $\mathfrak{a}_0 \subset \mathfrak{k}^\perp \cap \mathfrak{h}^\perp$ be a maximal abelian subspace which we inflate to a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{k}^\perp$. Let $\Sigma = \Sigma(\mathfrak{g}, \mathfrak{a}) \subset \mathfrak{a}^*$ be the corresponding restricted root system and recall a result of Rossmann [24] which asserts that $\Sigma_0 := \Sigma|_{\mathfrak{a}_0} \backslash \{0\}$ is a root system (as Σ, possibly reduced). We let $\Sigma_0^+ \subset \Sigma_0$ be a set of positive roots which we inflate to a set Σ^+ of positive roots for Σ. Set $\mathfrak{m} := \mathfrak{z}_\mathfrak{k}(\mathfrak{a})$, the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$, and let $\mathfrak{n} \subset \mathfrak{g}$ be the sum of the root spaces for the roots of Σ^+. Then $\mathfrak{p} := \mathfrak{m} + \mathfrak{a} + \mathfrak{n}$ defines a minimal parabolic subalgebra of $\mathfrak{g}$ with $\mathfrak{g} = \mathfrak{h} + \mathfrak{p}$ and thus exhibiting $(\mathfrak{g}, \mathfrak{h})$ as a spherical pair. For any $\alpha \in \Sigma_0$ we denote by $\mathfrak{g}^\alpha$ the corresponding root space and define a unipotent subalgebra $\mathfrak{u} := \bigoplus_{\alpha \in \Sigma_0^+} \mathfrak{g}^\alpha$. Then with $\mathfrak{l} := \mathfrak{z}_\mathfrak{g}(\mathfrak{a}_0)$ we obtain via $\mathfrak{q} := \mathfrak{l} \ltimes \mathfrak{u}$ a parabolic subalgebra of $\mathfrak{g}$ which contains $\mathfrak{p}$. Moreover $\mathfrak{l} = \mathfrak{q} \cap \theta(\mathfrak{q}) = \mathfrak{q} \cap \sigma(\mathfrak{q})$ is σ-stable with $\mathfrak{l}_\mathrm{n} \subset \mathfrak{l} \cap \mathfrak{h}$. This shows that $\mathfrak{q}$ is the unique parabolic subalgebra containing $\mathfrak{p}$, which is adapted to $(\mathfrak{g}, \mathfrak{h})$. Furthermore $\mathfrak{a}_0 \simeq \mathfrak{a}/\mathfrak{a} \cap \mathfrak{h} = \mathfrak{a}_Z$.

Consider the involution $\tau := \sigma \circ \theta$ and note that $\mathfrak{a}_0$ is τ-fixed. Hence every root space $\mathfrak{g}^\alpha$ is τ-stable and every root space $\mathfrak{g}^\alpha$ decomposes into $\mathfrak{g}^{\alpha,+} \oplus \mathfrak{g}^{\alpha,-}$ according to the eigenspaces of τ. Moreover $\mathfrak{g}^+ := \mathfrak{g}^\tau$ is σ-stable and σ coincides with θ on $\mathfrak{g}^+$; in other words: $(\mathfrak{g}^+, \mathfrak{g}^+ \cap \mathfrak{h})$ is a Riemannian subsymmetric pair of $(\mathfrak{g}, \mathfrak{h})$. Notice that $\mathfrak{l}^+ := \mathfrak{l} \cap \mathfrak{g}^+ = \mathfrak{a}_0 + \mathfrak{m}^+$ with $\mathfrak{m}^+$ the centralizer of $\mathfrak{a}_0$ in $\mathfrak{k} \cap \mathfrak{h}$.

Let $G = \mathrm{Int}(\mathfrak{g})$ be the adjoint group of $\mathfrak{g}$, and let $M^+ \subset L \subset G$ denote the connected subgroups corresponding to the subalgebras $\mathfrak{m}^+ \subset \mathfrak{l} \subset \mathfrak{g}$. Then each root space $\mathfrak{g}^\alpha$, $\alpha \in \Sigma_0 =$

$\Sigma(\mathfrak{g}, \mathfrak{a}_0)$, is naturally a module for L, and also for the extension $L_\tau := L \rtimes \{\mathbf{1}, \tau\}$, and the subspaces $\mathfrak{g}^{\alpha,\pm}$ are τ- and M^+-invariant.

This appendix is concerned with the basic L-geometry of the root spaces $\mathfrak{g}^\alpha$, summarized in the next proposition:

Proposition B.1. *Let $(\mathfrak{g}, \mathfrak{h})$ be a symmetric pair and $\mathfrak{a}_0 \subset \mathfrak{h}^\perp \cap \mathfrak{k}^\perp$ be a maximal abelian subspace. Then with notation as above the following holds for every root space $\mathfrak{g}^\alpha$ for $\alpha \in \Sigma_0$:*

1. *$\mathfrak{g}^\alpha$ decomposes into finitely many L-orbits. In particular, there exists an open L-orbit, and hence $\mathfrak{g}^\alpha_\mathbb{C}$ is a prehomogeneous vector space.*
2. *If $\dim \mathfrak{g}^{\alpha,\epsilon} > 1$ then M^+ acts transitively on the unit sphere of $\mathfrak{g}^{\alpha,\epsilon}$ ($\epsilon \in \{\pm\}$).*
3. *$\mathfrak{g}^\alpha$ is irreducible as a real representation of L_τ.*

Remark B.2. (a) In case $\sigma = \theta$ we have $\mathfrak{g}^+ = \mathfrak{g}$, and M^+ is the identity component of the centralizer of $\mathfrak{a}$ in K. In this case the transitivity in (2) is a well-known result of Kostant [16, Thm. 2.1.7] of which (1) and (3) are also immediate consequences.

(b) In general $\mathfrak{g}^\alpha$ is not irreducible as a real L-module. This is for example easy to see in the case of $(\mathfrak{g}, \mathfrak{h}) = (\mathfrak{h} \oplus \mathfrak{h}, \mathrm{diag}(\mathfrak{h}))$.

Proof. Fix $\alpha \in \Sigma_0^+$ and consider the graded subalgebra

$$\mathfrak{g}(\alpha) := \mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{l} \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha} \tag{B.1}$$

This subalgebra is σ and θ-stable and the same holds for the semi-simple subalgebra $\mathfrak{g}[\alpha]$ generated by $\mathfrak{g}^\alpha$ and $\mathfrak{g}^{-\alpha}$. With $\mathfrak{l}[\alpha] = \mathfrak{l} \cap \mathfrak{g}[\alpha]$ we then obtain

$$\mathfrak{g}[\alpha] = \mathfrak{g}^{-2\alpha} \oplus \mathfrak{g}^{-\alpha} \oplus \mathfrak{l}[\alpha] \oplus \mathfrak{g}^\alpha \oplus \mathfrak{g}^{2\alpha}$$

and $\mathfrak{a}_0[\alpha] = \mathfrak{a}_0 \cap \mathfrak{l}[\alpha]$ is one-dimensional. It is clear that it suffices to show the assertions of the proposition for $\mathfrak{g}[\alpha]$, and from now on we assume that $\mathfrak{g} = \mathfrak{g}(\alpha) = \mathfrak{g}[\alpha]$.

To prove (1) we observe that (B.1) provides a $\mathbb{Z}$-grading of $\mathfrak{g}$. Then a result of Vinberg (cf. [28, Prop. 2]) implies that the action $L_\mathbb{C}$ on $(\mathfrak{g}^\alpha)_\mathbb{C}$ has only finitely many orbits. Let $\mathcal{O} \subset (\mathfrak{g}^\alpha)_\mathbb{C}$ be one of them and set $\mathcal{O}_\mathbb{R} := \mathcal{O} \cap \mathfrak{g}^\alpha$. Then $\mathcal{O}_\mathbb{R}$ is a finite union of L-orbits by a result of Whitney [29] on connected components of real points of smooth complex varieties.

For (2) we first note that M^+ is compact and preserves the Cartan-Killing form, hence it acts on the unit sphere of $\mathfrak{g}^{\alpha,\epsilon}$ and the orbits are closed. It then suffices to show that there is an open orbit. Since A_0 acts by scalar multiplication, it is equivalent to show that $L^+ := M^+ A_0$ has an open orbit on $\mathfrak{g}^{\alpha,\epsilon}$. This follows by applying (1) to the σ-invariant Lie subalgebra of $\mathfrak{g}$,

$$\mathfrak{g}^\epsilon := \mathfrak{g}^{-2\alpha,+} + \mathfrak{g}^{-\alpha,\epsilon} + \mathfrak{l}^+ + \mathfrak{g}^{\alpha,\epsilon} + \mathfrak{g}^{2\alpha,+}\,.$$

We move on to (3). Let us first observe that if $\mathfrak{a}_0 = \mathfrak{a}$, that is, if $\mathfrak{a}_0$ is maximal abelian in $\mathfrak{s}$, then by applying (2) for the involution $\sigma = \theta$ (as in Remark B.2(a)) it follows that $\mathfrak{g}^\alpha$ is irreducible already as an L-module. We assume from now on then that $\mathfrak{a}_0 \subsetneq \mathfrak{a}$, and let $Y \in \mathfrak{a} \cap \mathfrak{h}$ be non-zero. Since $\mathfrak{g} = \mathfrak{g}[\alpha]$ it follows that $[Y, \mathfrak{g}^\alpha] \neq \{0\}$. Note that $\tau(Y) = -Y$ and hence $[Y, \mathfrak{g}^{\alpha,\pm}] \subset \mathfrak{g}^{\alpha,\mp}$. Since $\mathrm{ad}(Y)$ acts semisimply, it follows that $[Y, \mathfrak{g}^{\alpha,\epsilon}] \neq \{0\}$ for each ϵ.

Let $U \subset \mathfrak{g}^\alpha$ be a non-zero L_τ-invariant subspace. Since U is τ-invariant then $U = U^+ \oplus U^-$ where $U^\pm = U \cap \mathfrak{g}^{\alpha,\pm}$, and each $U^\pm$ is M^+-invariant. It follows from (2) that U^ϵ equals $\{0\}$ or $\mathfrak{g}^{\alpha,\epsilon}$ for each ϵ. It now follows from the $\mathrm{ad}(Y)$-invariance that $U = \mathfrak{g}^\alpha$, and we are done. □

References

[1] K. Baba, *Local orbit types of the isotropy representations for semisimple pseudo-Riemannian symmetric spaces*, Diff. Geom. Appl. **38** (2015), 124–150.

[2] M. Berger, *Les espaces symétriques noncompacts*, Ann. Sci. École Norm. Sup. (3) **74** (1957), 85–177.

[3] P. Bravi and G. Pezzini, *The spherical systems of the wonderful reductive subgroups*, J. Lie Theory 25 (2015), 105–123.

[4] M. Brion, *Classification des espaces homogènes sphériques*, Compositio Math. **63**(2) (1987), 189–208.

[5] P. Delorme, B. Krötz and S. Souaifi, *The constant term of tempered functions on a real spherical space*, arXiv:1702.04678

[6] J. Frahm (= Möllers), *Symmetry breaking operators for strongly spherical reductive pairs and the Gross-Prasad conjecture for complex orthogonal groups*, arXiv:1705.06109

[7] F. Knop and B. Krötz, *Reductive group actions*, arXiv: 1604.01005

[8] F. Knop, B. Krötz, T. Pecher and H. Schlichtkrull, *Classification of reductive real spherical pairs I. The simple case*, Transformation Groups, doi 10.1007/s00031-017-9470-5

[9] F. Knop, B. Krötz, E. Sayag and H. Schlichtkrull, *Simple compactifications and polar decomposition of homogeneous real spherical spaces*, Selecta Math. N.S. **2** (2015), 1071–1097.

[10] ———, *Volume growth, temperedness and integrability of matrix coefficients on a real spherical space*, J. Funct. Anal. **271** (2016), 12–36.

[11] F. Knop, B. Krötz and H. Schlichtkrull, *The local structure theorem for real spherical spaces*, Compositio Math. **151** (2015), 2145–2159.

[12] ———, *The tempered spectrum of a real spherical space*, Acta Math. **218** (2017), 201–295.

[13] F. Knop and B. Van Steirteghem, *Classification of smooth affine spherical varieties*, Transformation Groups **11** (2006), 495–516.

[14] T. Kobayashi and T. Matsuki, *Classification of finite-multiplicity symmetric pairs*, Transformation Groups **19 (2)** (2014), 457–493

[15] T. Kobayashi and T. Oshima, *Finite multiplicity theorems for induction and restriction*, Adv. Math. **248** (2013), 921–944.

[16] B. Kostant, *On the existence and irreducibility of certain series of representations.* Lie groups and their representations (Proc. Summer School, Bolyai Jnos Math. Soc., Budapest, 1971), pp. 231329. Halsted, New York, 1975.

[17] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio math. **38 (2)** (1979), 129–153.

[18] B. Krötz, J.J. Kuit, E. Opdam and H. Schlichtkrull, *The infinitesimal characters of discrete series for real spherical spaces*, arXiv:1711.08635

[19] B. Krötz and H. Schlichtkrull, *Finite orbit decomposition of real flag manifolds*, J. EMS **18** (2016), 1391–1403.

[20] ———, *Multiplicity bounds and the subrepresentation theorem for real spherical spaces*, Trans. Amer. Math. Soc. **368** (2016), 2749–2762.

[21] I.V. Mikityuk, *On the integrability of Hamiltonian systems with homogeneous configuration spaces*, Math. USSR Sbornik, **57(2)** (1987), 527–546.

[22] A.L. Onishchik, *Inclusion relations among transitive compact transformation groups*, Trudy Moskov. Mat. Obshch. **11** (1962), 199–242; Amer. Math. Soc. Transl. (2) **50** (1966), 5–58.

[23] T. Oshima and J. Sekiguchi, *The restricted root system of a semisimple symmetric pair*, Group representations and systems of differential equations (Tokyo, 1982), 433–497, Adv. Stud. Pure Math., **4**, North-Holland, Amsterdam, 1984.

[24] W. Rossmann, *The structure of semisimple symmetric spaces*, Canad. J. Math. **31** (1979), 157–180.

[25] Y. Sakellaridis and A. Venkatesh, *Periods and harmonic analysis on spherical varieties*, Astérisque 396 (2017).

[26] D. Timashev, Homogeneous Spaces and Equivariant Embeddings, Enc. of Math. Sciences **138**, Springer Verlag 2011.

[27] V.S. Varadarajan, *Spin(7)-subgroups of SO(8) and Spin(8)*, Expo. Math. **19** (2001), 163–177.

[28] E.B. Vinberg, *The Weyl group of a graded Lie algebra*, Math. USSR Izv. **10** (1976), 463–495.

[29] H. Whitney, *Elementary Structure of Real Algebraic Varieties*, Annals of Math. **66**, (1957), 545–556.