ESTIMATES ON PATH FUNCTIONALS OVER WASSERSTEIN SPACES

S. BIANCHINI AND A. BRANCOLINI

Abstract. In this paper we consider the class a functionals (introduced in [BBS]) $\mathcal{G}_{r,p}(\gamma)$ defined on Lipschitz curves γ valued in the p-Wasserstein space. The problem considered is the following: given a measure μ, give conditions in order to assure the existence a curve γ such that $\gamma(0) = \mu$, $\gamma(1) = \delta_{x_0}$, and $\mathcal{G}_{r,p}(\gamma) < +\infty$.

To this end, new estimates on $\mathcal{G}_{r,p}(\mu)$ are given and a notion of dimension of a measure (called *path dimension*) is introduced: the path dimension specifies the values of the parameters (r, p) for which the answer to the previous reachability problem is positive. Finally, we compare the path dimension with other known dimensions.

Contents

1. Introduction 1
1.1. Notation 4
1.2. Settings 5
2. Reachability results 6
2.1. Lower semicontinuity 7
2.2. Extremality of a discrete plan 8
2.3. Estimate $\mathcal{G}_{r,p}(\gamma_{\mathrm{opt}})$. 8
2.4. Reachability of measures 14
2.5. Regularity 16
2.6. Path functional dimension 18
3. Other notions of dimension 22
3.1. Hausdorff measure and dimension 22
3.2. Minkowski dimension 24
3.3. Renyi dimension or q-dimension 26
3.4. Resolution dimension 29
3.5. Irrigation dimension 31
4. Upper and lower bounds for path dimension 32
4.1. Special distributed measures 35
References 38

1. Introduction

Optimal transportation problems aroused great interest in the last years, both on the point of view of the classical Monge-Kantorovich approach and its many applications and the so-called *branched transportation*.

Date: Thursday 5^{th} March, 2009.
2000 *Mathematics Subject Classification.* Primary: 49K27; Secondary: 93B03, 28A78.
Key words and phrases. Optimal transportation problems, Path Functionals.

The problem of branched transportation was first stated in [MMS] and [X1] with the purpose to give a simple mathematical modelling of branched structures which arise in nature, i.e. cardiovascular systems and lungs, or in artificial systems, i.e. roads. In [MMS] the authors consider a functional, defined on the set of all possible trajectories of the particles, with a subadditive term which assigns less cost to motions where the particles move together. In [X1] the author introduced a functional defined on weighted directed graphs (here the motion of a single particle is not considered) with a similar subadditive term which penalizes the transport in spread masses. It has been shown that these model turn out to be equivalent, meaning that the underlined optimal structure is the same. From a purely mathematical point of view the main questions are the existence of a minimum (which is the easy part) and the regularity of the optimal structure (see [X2] or [BCM2] for the equivalent formulation of [BCM1]).

Another approach to the branched transportation problem is the one proposed in [BBS]. Here, the moving particles are represented by a curve γ valued in the set of probability measures equipped by the Wasserstein distance. An initial measure μ_0 and a final one μ_1 are given. Then, they consider the functional $\mathcal{G}_{r,p}(\gamma)$, defined on Lipschitz curves $\gamma : [0,1] \to (\mathcal{W}_p(\overline{\Omega}), W_p)$, $(\mathcal{W}_p, W_p)$ being the Wasserstein space of order $p \geq 1$ (we will consider the general case $p > 0$), such that $\gamma(0) = \mu$ and $\gamma(1) = \nu$, given by

$$\mathcal{G}_{r,p}(\gamma) = \int_0^1 G_r(\gamma(t))|\gamma'|_p(t)dt, \tag{1.1}$$

where (given a parameter $r \in [0,1[$)

$$G_r(\mu) = \begin{cases} \sum_i a_i^r & \text{if } \mu = \sum_i a_i \delta_{x_i}, \\ +\infty & \text{otherwise.} \end{cases} \tag{1.2}$$

Since the r-th power is sub-addictive and G_r is finite only on discrete measures the resulting minimal curves are expected to give the branched structure.

The main result of [BBS] is that, given $\mu, \nu \in \mathcal{W}_p(\overline{\Omega})$, the functional (1.1) admits a minimum, provided there exists a curve of finite cost, i.e. a curve γ such that $\mathcal{G}_{r,p}(\gamma) < +\infty$. Actually, in the paper the existence result for the functional (1.1) follows from a result of the same kind for a general type of functionals. In the same paper it is also proved that if μ, ν are discrete measures, then there exists such a curve; the same is true for every couple of measures, provided $r > 1 - 1/N$, where N is the linear dimension of the space, while if $r \leq 1 - 1/N$ a Dirac mass cannot be connected to an $\mathcal{L}^N$-absolutely continuous measure keeping the cost finite.

The question of the reachability of a measure from a Dirac mass arises then naturally in this context and this paper is mainly devoted to provide an exhaustive answer to this problem as far as the functional (1.1) is concerned. Actually, the same question is of interest in the Maddalena-Morel-Solimini model, which has been answered in [DS2]. Let us mention a different approach on path functionals introduced in [B].

The main results of this paper are the followings.

In Section 2, we study equivalent formulations of the property that there exists a path γ connecting $\mu \neq \nu$. It turns out that the fact that a measure μ can be connected to another measure $\nu \neq \mu$ with a path γ with finite cost $\mathcal{G}_{r,p}(\gamma)$ is independent on ν. The next definition is then natural (Definition 2.12).

Definition. A probability measure μ is *reachable w.r.t.* r, p (or (r,p)*-reachable*) if there exists $\gamma \in \mathrm{Lip}([0,\varepsilon], \mathcal{W}_p(\overline{\Omega}))$ such that $\gamma(0) = \mu$ and $|\gamma'|(t) = 1$ for a.e. $t \in [0,\varepsilon]$ and

$$\mathcal{G}_{r,p}(\gamma) = \int_0^\varepsilon G_r(\gamma(t))\mathrm{d}t < +\infty.$$

The main theorem of this section is the following (Theorem 2.13 of Section 2.4).

Theorem. *Let $\mu \in \mathcal{W}_p(\overline{\Omega})$. The following conditions are equivalent:*

(1) *μ is reachable;*
(2) *there exists $\varepsilon > 0$ (equivalently for all $\varepsilon > 0$)*

$$\int_0^\varepsilon \min\{G_r(\nu) : W_p(\nu,\mu) \leq t\}\mathrm{d}t < +\infty;$$

(3) *there exists $\gamma \in \mathrm{Lip}([0,1], \mathcal{W}_p(\overline{\Omega}))$ which satisfies $\mathrm{Var}(\gamma) > 0, \gamma(0) = \mu, \gamma(1) = \delta_{x_0}$ and*

$$\int_0^1 G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t < +\infty.$$

The proof is based on careful estimates of the functional $\mathcal{G}_{r,p}$ on paths connecting finitely atomic measure (Proposition 2.8 of Section 2.3). This result allows us to prove that in the case $r < 1 - \frac{1}{N}$ the set of reachable measures is of first category in the Wasserstein space (Proposition 2.15 of Section 2.4). Finally in Section 2.6 we introduce the *path dimension*,

$$\dim_{path,p}(\mu) = \frac{\min\{1,p\}}{1 - r^*},$$

where

$$r^* = \inf\{0 \leq r < 1 : \mu \text{ is reachable w.r.t. } p, r\}.$$

The quantity $\dim_{path,p}(\mu)$ will be compared in the next sections to other known measure-theoretic dimensions.

In Section 3 we recall some classical and more recent notions of dimensions for sets and measures:

- Hausdorff dimension $\dim_{\mathcal{H}}(\mu)$ in Section 3.1,
- Minkowski dimension $\dim_M(\mu)$ in Section 3.2,
- Renyi dimension or q-dimension $\dim_q(\mu)$ in Section 3.3,
- resolution dimension $\dim_{\mathcal{W}_p}(\mu)$ in Section 3.4,
- irrigation dimension $\dim_{irr}(\mu)$ in Section 3.5.

In the same sections we compare the various dimensions on the same measure μ, with the idea that in different cases one dimension can be easier estimated that the others and its value gives bounds to the others. The results here are certainly not new: we however think that this collections of definitions and properties can be useful and simplify the reading of the paper.

In Section 4 we consider the comparison of the path dimension of a measure μ with the classical dimensions studied in the previous sections. Among the many comparisons, the most useful are

$$\dim_{\mathcal{H}}(\mu) \leq \dim_{path,p}(\mu) \leq \max\{\dim_M(\mu), 1\},$$

(which holds also for the irrigation dimension), and

$$\dim_{irr}(\mu) \leq \frac{p}{p-1}\dim_{path,p}(\mu),$$

providing an answer to the questions left open in [BBS].

In Section 4.1 we study the case where uniform bounds on the local dimension (Definition 4.5) hold. In this case it turns out that all the above notions of dimension are equivalent. Finally a simple example (Example 4.8) concludes the section and the paper.

1.1. **Notation.** A list of notations used in this paper is given.

$\mathbb{N}, \mathbb{N}_0$	natural numbers, natural numbers with 0
$\mathbb{Q}$, $\mathbb{R}$	rational numbers, real numbers
$:=$	the left side is defined by the right side
$\sharp A$	the cardinality of the set A
(X, d)	Polish space, for the purpose of of this paper locally compact
$B_\varepsilon(x)$	the open ball of radius ε centered in x
Ω	open subset of a Polish space
$\bar{A}$	the closure of $A \subseteq X$, X topological
A^c	the complementary of a set (in a specified ambient set)
Q_R	cube of size R in $\mathbb{R}^N$
$P_{i_1 \dots i_I}$	projection of $x \in \Pi_i X_i$ into its $(i_1, \dots, i_I)$ coordinates
$\operatorname{diam} A$	diameter of $A \subseteq X$
$d(x, A)$	distance of the point x from the set A
$D_{\mathrm{Haus}}(A, B)$	Hausdorff distance of A, B closed (1.7)
$\lvert\gamma'\rvert(t)$	metric derivative of $\gamma(t)$ (1.9)
$\mathcal{B}$	the Borel σ-algebra on X topological
$\mathcal{M}(X)$	measures on a measurable space $(X, \mathcal{B})$
$\mathcal{P}(X)$	probability measures on a measurable space $(X, \mathcal{B})$
$\mathcal{L}$	Lebesgue measure on $\mathbb{R}$
$\mu\lvert_A$	the measure $\mu\lvert_A(A') = \mu(A' \cap A)$
$\Pi(\mu, \nu)$	probability measures on $X \times X$ with marginals μ, ν (1.3)
$\operatorname{spt} \mu$	the support of the measure μ
$\Pi_{\mathrm{opt}}(\mu, \nu)$	probability measures in $\Pi(\mu, \nu)$ minimizing $\int c\pi$
$\Pi_{\mathrm{ext}}(\mu, \nu)$	extremal points of the convex set $\Pi(\mu, \nu)$
Γ	a subset of $\bar{\Omega} \times \bar{\Omega}$ where a measure π is concentrated
$W_p(\mu, \nu)$	the p-Wasserstein distance of μ, ν (1.4)
D_n	set of measures with n atoms
$W_p(\mu, D_n)$	distance of μ from D_n (3.21)
$\mathcal{W}_p(\overline{\Omega})$	Wasserstein space of exponent $p \in [0, +\infty]$ (1.5)
$G_r(\mu)$	l.s.c. functional on $\mathcal{P}(\Omega)$ defined in (1.2)
$\mathcal{G}_{r,p}(\mu, \nu)$	path functional defined in (1.1)
$g_{r,p,\mu}(t)$	functional defined in (2.11)
$\mathcal{H}^\alpha_\delta$	outer Hausdorff measure (3.1)
$\mathcal{H}^\alpha$	Hausdorff measure (3.2)
$\dim_{\mathcal{H}}(A)$	Hausdorff dimension of the set A (3.3)
$\dim_{\mathcal{H}}(\mu)$	Hausdorff dimension of the measure μ (3.4)
$N(A, \varepsilon)$	minimal number of balls or radius ε needed to cover A (3.7)
$\dim_M(B)$	Minkowski dimension of a set B (3.8)
$P(A, \varepsilon)$	packing number of A (3.10)
$Q(A, m)$	number of dyadic cubes of size 2^{-m} intersecting A, Definition 3.6
$\dim_M(\mu)$	Minkowski dimension of a measure μ, Definition 3.8

$\underline{a}$	a finite probability vector
$H_q(\underline{a})$	q-entropy for a probability vector (3.16), (3.17)
$\dim_q(\mu)$	Renyi dimension or q-dimension of the measure μ (3.18), (3.19)
$\dim_{path,p}(\mu)$	path dimension or reachability dimension of the measure μ, Definition 2.19
$\dim_{\mathcal{W}_p}(\mu)$	resolution dimension of μ (3.22)
χ	set of fibers, Definition 3.20
$\mathbf{P}$	family of fibers χ, Definition 3.20
$\mathbf{P}_S$	family of fibers χ starting from $S \in \mathbb{R}^N$, Definition 3.20
σ_χ	absorption time, Definition 3.22
$A_t(\chi)$	absorbed points at time t (3.26)
$M_t(\chi)$	moving points at time t, $M_t(\chi) = A_t(\chi)^c$ (3.27)
$I_\alpha(\chi)$	irrigation cost (3.28)
$\dim_{irr}(\mu)$	irrigation dimension of μ (3.29)
$\dim_{loc}(x,\mu)$	local dimension of μ at x (4.3)

1.2. **Settings.** Let (X, d) be a Polish space, and $\mathcal{P}(X)$ be the Borel probability measures on X. Given $\mu, \nu \in \mathcal{P}(X)$, let $\Pi(\mu, \nu)$ be the set of transport plans

$$\Pi(\mu,\nu) = \Big\{\pi \in \mathcal{P}(X \times X) : (P_1)_\sharp \pi = \mu, (P_2)_\sharp \pi = \nu\Big\}, \tag{1.3}$$

Let $\Pi_{\text{opt}}(\mu, \nu)$ denote the set of optimal transport plans with respect to the functional

$$\pi \mapsto \int_{X\times X} c(x,y)\pi(dxdy).$$

Recall that both $\Pi(\mu, \nu), \Pi_{\text{opt}}(\mu, \nu)$ are convex sets. Then, let $\Pi_{\text{ext}}(\mu, \nu)$ denote the subset of extreme points of $\Pi(\mu, \nu)$.

On $\mathcal{P}(X)$ consider the Wasserstein metric W_p, $p \in [0, +\infty]$,

$$W_p(\mu,\nu) = \inf\bigg\{\bigg(\int d(x,y)^p \pi(dxdy)\bigg)^{\min\{1,1/p\}}, \pi \in \Pi(\mu,\nu)\bigg\}, \tag{1.4}$$

and define the Wasserstein space

$$\mathcal{W}_p(X) = \big\{\mu : W_p(\mu, \delta_x) < +\infty\big\}. \tag{1.5}$$

It is easy to see that $\mathcal{W}_p(X)$ does not depend on $x \in X$.

Particular cases are $p = 0$ and $p = \infty$: for $p = 0$ set

$$W_0(\mu,\nu) = \inf_{\Pi(\mu,\nu)} \pi\{x \neq y\} = \frac{1}{2}\|\mu - \nu\|, \tag{1.6}$$

while for $p = +\infty$ define

$$W_\infty(\mu,\nu) = \inf\bigg\{D_{\text{Haus}}\big(\{x = y\}, \operatorname{spt}\pi\big), \pi \in \Pi(\mu,\nu)\bigg\}, \tag{1.7}$$

where D_{Haus} is the Hausdorff distance between closed sets:

$$D_{\text{Haus}}(A,B) = \max\Big\{\sup_{y\in B}\inf_{x\in A} d(x,y), \sup_{x\in A}\inf_{y\in B} d(x,y)\Big\}. \tag{1.8}$$

For any Lipschitz curve $\gamma : [0,1] \mapsto \mathcal{P}(X)$, let $|\gamma'|_p(t)$ (or just $|\gamma'|(t)$ when no confusion occurs) be the metric derivative w.r.t. W_p,

$$|\gamma'|_p(t) = \liminf_{s\to 0^+} \frac{W_p(\gamma(t+s),\gamma(t))}{s}. \tag{1.9}$$

By the assumption of Lipschitz continuity of γ, it follows that the limit exists $\mathcal{L}^1$-a.e. t, and the length of γ is

$$L_p(\gamma) = \int_0^1 |\gamma'|_p(t)dt. \tag{1.10}$$

See, for example, Chapter 3 of [AT].

2. Reachability results

The following lemma is an easy, but useful, tool we will use in the analysis of the problem.

Lemma 2.1 (Reparametrization of paths functionals)**.** *Every path $\gamma : [0,1] \to \mathcal{W}_p(\overline{\Omega})$ can be extended to a Lipschitz curve $\tilde{\gamma} : [0,+\infty] \to \mathcal{W}_p(\overline{\Omega})$ such that $\mathrm{Var}(\tilde{\gamma}) = \mathrm{Var}(\gamma)$ and*

$$\int_0^1 G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^{+\infty} G_r(\tilde{\gamma}(t))|\tilde{\gamma}'|(t)\mathrm{d}t.$$

Vice versa, every Lipschitz curve $\gamma : [0,+\infty] \to \mathcal{W}_p(\overline{\Omega})$ such that $\mathrm{Var}(\gamma) < +\infty$ can be re-parametrized as a path $\tilde{\gamma} : [0, \mathrm{Var}(\gamma)] \to \mathcal{W}_p(\overline{\Omega})$ with $|\tilde{\gamma}'|(t) = 1$ such that

$$\int_0^{+\infty} G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^{\mathrm{Var}(\gamma)} G_r(\tilde{\gamma}(t))|\tilde{\gamma}'|(t)\mathrm{d}t.$$

Proof. First of all note that any path functional $\tilde{\gamma} : [0,1] \to \mathcal{W}_p(\overline{\Omega})$ can be extended on $[0,+\infty[$ setting $\tilde{\gamma}(t) = \gamma(1)$ when $t > 1$ without changing the value of the integral:

$$\int_0^1 G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^{+\infty} G_r(\tilde{\gamma}(t))|\tilde{\gamma}'|(t)\mathrm{d}t.$$

Vice versa, given $\gamma : [0,+\infty] \to \mathcal{W}_p(\overline{\Omega})$ such that

$$\int_0^{+\infty} G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t < +\infty,$$

since $G_r(\gamma(t)) \geq 1$, we have:

$$\mathrm{Var}(\gamma) = \int_0^{+\infty} |\gamma'|(t)\mathrm{d}t < +\infty.$$

Setting

$$l(t) := \mathrm{Var}_0^t(\gamma) = \int_0^t |\gamma'|(\tau)\mathrm{d}\tau,$$

we get a Lipschitz function $l : [0,+\infty] \to [0, \mathrm{Var}(\gamma)]$ with $\mathrm{Lip}(l) \leq \mathrm{Lip}(\gamma)$. Let $\lambda : [0, \mathrm{Var}(\gamma)] \to [0,+\infty]$ be the pseudo-inverse function of l:

$$\lambda(s) := \inf\{t : l(t) = s\}.$$

If $\lambda(\mathrm{Var}(\gamma)) < +\infty$, then it can be seen as in [AT] that $\tilde{\gamma} : [0, \mathrm{Var}(\gamma)] \to \mathcal{W}_p(\overline{\Omega})$ defined by

$$\tilde{\gamma}(s) := \gamma(\lambda(s))$$

satisfies $|\tilde\gamma'|(t)=1$ and

$$\int_0^{+\infty} G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^{\mathrm{Var}(\gamma)} G_r(\tilde\gamma(t))|\tilde\gamma'|(t)\mathrm{d}t.$$

Otherwise, if $\lambda(\mathrm{Var}(\gamma)) = +\infty$ we get a Lipschitz curve $\tilde\gamma : [0,\mathrm{Var}(\gamma)[\to \mathcal{W}_p(\overline{\Omega})$ with can be extended (as a uniformly continuous function to a complete metric space) to the closed interval $[0,\mathrm{Var}(\gamma)]$. □

2.1. **Lower semicontinuity.**

Theorem 2.2. *Suppose that $\mu_n \to \mu, \nu_n \to \nu$ w.r.t. the Wasserstein distance of order p. Then*

$$\mathcal{G}_{r,p}(\mu,\nu) \le \liminf_{n\to+\infty} \mathcal{G}_{r,p}(\mu_n,\nu_n).$$

Proof. Consider $\gamma_n : [0,1] \to \mathcal{W}_p(\overline{\Omega})$ such that $\gamma_n(0)=\mu_n, \gamma_n(1)=\nu_n$ and

$$\mathcal{G}_{r,p}(\mu_n,\nu_n) = \int_0^1 G_r(\gamma_n(t))|\gamma_n'|(t)\mathrm{d}t.$$

If we parametrize γ_n by arc length and map linearly the interval $[0,\mathrm{Var}(\gamma_n)]$ on $[0,1]$, we can suppose that $|\gamma_n'|$ is a constant (depending on n, actually $|\gamma_n'| = \mathrm{Lip}(\gamma_n) = \mathrm{Var}(\gamma_n)$ since $\mathrm{Lip}(\gamma) = \sup_{t\in[0,1]} |\gamma'|(t)$).

If $\liminf_n \mathcal{G}_{r,p}(\mu_n,\nu_n) = +\infty$ the inequality is trivial. So, we suppose that

$$\liminf_n \mathcal{G}_{r,p}(\mu_n,\nu_n) < +\infty$$

and extract a subsequence without relabelling such that

$$\liminf_n \mathcal{G}_{r,p}(\mu_n,\nu_n) = \lim_n \mathcal{G}_{r,p}(\mu_n,\nu_n).$$

The sequence $\{\gamma_n\}_{n\in\mathbb{N}}$ is equi-bounded and equi-lipschitzean since

$$\mathrm{Var}(\gamma_n) = \int_0^1 |\gamma_n'|(t)\mathrm{d}t \le \mathcal{G}_{r,p}(\mu_n,\nu_n) \le C < +\infty,$$

that is $\gamma_n([0,1]) \subseteq B_C(\mu_n)$ in the p-Wasserstein Space and $W_p(\mu_n,\mu)$ is bounded. Up to a subsequence we can suppose by Ascoli-Arzelà theorem that the sequence is uniformly convergent: $\gamma_n \to \gamma$ in $\mathcal{W}_p(\Omega)$ and $\mu_n = \gamma_n(0) \to \gamma(0) = \mu, \nu_n = \gamma_n(1) \to \gamma(1) = \nu$ in $\mathcal{W}_p(\Omega)$. Up to a subsequence we can suppose that $\liminf_n \mathrm{Lip}(\gamma_n) = \lim_n \mathrm{Lip}(\gamma_n) = \lim_n |\gamma_n'|(t)$. We have that

$$|\gamma'|(t) \le \lim_{n\to+\infty} \mathrm{Lip}(\gamma_n),$$

so, by lower semicontinuity of G_r,

$$G_r(\gamma(t)) \le \liminf_{n\to+\infty} G_r(\gamma_n(t)),$$

and finally

$$G_r(\gamma(t))|\gamma'|(t) \le \liminf_{n\to+\infty} G_r(\gamma_n(t))|\gamma_n'|(t).$$

The statement of the theorem then follows integrating on $[0,1]$ and applying the Fatou Lemma. □

2.2. **Extremality of a discrete plan.** We recall the following theorem (Theorem 3 of [HW]). We define a set $\Gamma \subset \bar{\Omega} \times \bar{\Omega}$ *acyclic* if for any finite sets of points $(x_i, y_i) \in \Gamma$, $i = 1, \ldots, I$,

$$\Big\{(x_{i+1}, y_i), i = 1, \ldots, I, x_{I+1} = x_1\Big\} \not\subseteq \Gamma.$$

Theorem 2.3. *If $\pi \in \Pi_{ext}(\mu, \nu)$, then there exists a σ-compact acyclic subset $\Gamma \subset \bar{\Omega} \times \bar{\Omega}$ such that $\pi(\Gamma) = 1$.*

We will say that π is *acyclic* if it satisfies the second part of the theorem, i.e. there exists an acyclic set $\Gamma \subset \bar{\Omega} \times \bar{\Omega}$ such that $\pi(\Gamma) = 1$. When at least one of the marginals of π is purely atomic, it is easy to prove that this condition is also equivalent to extremality (see for example Theorem 1 of [LE]).

Using Theorem 2.3 and the elementary fact that the extremal measures in $\Pi_{\rm opt}(\mu, \nu)$ are given by $\Pi_{\rm opt}(\mu, \nu) \cap \Pi_{\rm ext}(\mu, \nu)$, we conclude the following corollary.

Corollary 2.4. *Let μ, ν be discrete measures given by*

$$\mu = \sum_{i=1}^m a_i \delta_{x_i}, \qquad \nu = \sum_{j=1}^n b_j \delta_{y_j}.$$

Let $\pi \in \Pi_{opt}(\mu, \nu)$ be extremal. Then, given two sets of different indexes $\{i_1, i_2, \ldots, i_k\}$ and $\{j_1, j_2, \ldots, j_k\}$ (so, with $k \leq \min\{\sharp \operatorname{spt} \mu, \sharp \operatorname{spt} \nu\}$) we must have

$$\prod_{l=1}^k (\pi_{i_l j_l} \pi_{i_l j_{l+1}}) = 0.$$

Since the function $G_r(\mu)$ is concave, the choice of an extremal optimal plan will play an important role in the next section.

2.3. **Estimate $\mathcal{G}_{r,p}(\gamma_{\mathbf{opt}})$.**

Proposition 2.5. *Let μ, ν be discrete measures given by*

$$\mu = \sum_{i=1}^m a_i \delta_{x_i}, \qquad \nu = \sum_{j=1}^n b_j \delta_{y_j}.$$

Let $\pi \in \Pi_{ext}(\mu, \nu)$. Then,

$$\sum_{i,j} \pi_{ij}^r \leq \sum_i a_i^r + \sum_j b_j^r. \tag{2.1}$$

The constraints to be satisfied by a transport plan is given by $m + n$ equations which are not linearly independent (summing the first m and subtracting the last n equations gives zero). We recall that the marginal equations are

$$\sum_{j=1}^n \pi_{ij} - a_i = 0, \tag{2.2}$$

for $i = 1, 2, \ldots, m$, and

$$\sum_{i=1}^m \pi_{ij} - b_j = 0, \tag{2.3}$$

for $j = 1, 2, \ldots, n$. We also recall the following lemma from [DT2, DT1].

Lemma 2.6. *$m+n-1$ of the equations above are linearly independent.*

Proof. Drop the one with $j=n$, and consider the remaining $m+n-1$. Suppose the λ_i and μ_j (with $i=1,2,\dots,m$, $j=1,2,\dots,n-1$) are coefficient of a null linear combination. Since the variables π_{in} (with $i=1,2,\dots m$) appear only once, we must have that $\lambda_i=0$ for all i. Then also $\mu_j=0$ for all $j=1,2,\dots,n-1$. □

We recall that in Linear Programming, the *feasible set* is the set

$$\big\{\pi\in\mathbb{R}^{m\times n}:\pi_{ij}\geq 0 \text{ and (2.2), (2.3) holds}\big\}.$$

Lemma 2.7. *At most $m+n-1$ of the π_{ij} are non-zero if π is extremal.*

Proof. By the equation from the marginal conditions we can determine $m+n-1$ of the π_{ij}. Since the minimum is reached also on the extremal points of the feasible set, a minimum which is also an extremal must satisfy equality in exactly $mn-(m+n-1)$ of the non-negativity constraints $\pi_{ij}\geq 0$. □

Proof of Proposition 2.5. The proof follows immediately if we can prove that there exists an injective function

$$h:\{\pi_{ij}>0\}\to\{a_i,b_j\}$$

such that $\pi_{ij}\leq h(\pi_{ij})$. We divide the proof into several steps.

Step 1. Suppose that $m\leq n$. Since the sum of the entries of the i-th row is $a_i>0$ and of the j-th column is $b_j>0$, in every row or column there must be at least a non-zero entry. In order to satisfy this condition at least n non-zero entries are needed (a diagonal of the matrix $(\pi_{ij})_{ij}$ plus the remaining entries of the last row). It is not possible that on each column one finds at least two non-zero entries, otherwise the non-zero entries would be at least $2n>m+n-1$. Let j_0 be the index of that column and let then $\pi_{i_0j_0}$ the only non-zero entry in that column. We thus define $h(\pi_{i_0j_0})=b_{j_0}$.

Step 2. We proceed by (finite) induction. Assume that at the $k-1$ step we have defined h on the $k-1$ points $\pi_{i_\ell j_\ell}$, $\ell=1,\dots,k$, such that the marginals of the reduced transference plan $\pi_{k-1}=\pi-\sum_\ell\pi_{i_\ell j_\ell}$ satisfies

$$(\mu_{k-1})_i=\big((P_1)_\sharp\pi_{k-1}\big)_i=\sum_{j\not\ni j_\ell}\pi_{ij}\leq\mu,\quad(\nu_{k-1})_j=\big((P_2)_\sharp\pi_{k-1}\big)_j=\sum_{i\not\ni i_\ell}\pi_{ij}\leq\nu,$$

and the cardinality m_{k-1} of the support of μ_{k-1} plus the cardinality n_{k-1} of the support of ν_{k-1} is bounded by $n+m-1-k$.

Step 3. If $m_{k-1}\leq n_{k-1}$, then the procedure of Point (1) yields an entry $\pi_{i_kj_k}$ such that $\pi_{i_kj_k}=b'_{j_k}\leq b_{j_k}$, and moreover by removing this entry it follows that $\mu_k=(P_1)_\sharp\pi\leq\mu$, $\nu_k=(P_2)_\sharp\pi_k\leq\nu$ and the cardinality m_k of the support of μ_k plus the cardinality n_k of the support of ν_k is bounded by $n+m-k-2$. Define the $h(i_k,j_k)=b_{j_k}$.

Step 4. If $m_{k-1}\leq n_{k-1}$, then repeat the procedure of Point (1) to find an entry $\pi_{i_kj_k}$ such that it is the unique non-zero entry on the i_k-row: hence $\pi_{i_kj_k}=a'_{j_k}\leq a_{j_k}$, and moreover by removing this entry it follows that $\mu_k=(P_1)_\sharp\pi\leq\mu$, $\nu_k=(P_2)_\sharp\pi_k\leq\nu$ and the cardinality m_k of the support of μ_k plus the cardinality n_k of the support of ν_k is bounded by $n+m-k-2$. Define in this case $h(i_k,j_k)=a_{i_k}$.

Step 5. The proof of the existence of h now follows by finite induction, since the measures are finitely atomic. □

A simple approximation argument implies that (2.1) holds also for purely atomic measure μ, ν.

Proposition 2.8. *Let μ and ν be discrete probability measures with finite support. Then the following estimates hold:*

- *if $0 \leq p \leq 1$, then*

$$\mathcal{G}_{r,p}(\gamma_{opt}) \leq \frac{1}{1+r} W_p(\mu,\nu)(G_r(\mu)+G_r(\nu)); \tag{2.4}$$

- *if $p > 1$, then*

$$\mathcal{G}_{r,p}(\gamma_{opt}) \leq W_p(\mu,\nu)(G_r(\mu)+G_r(\nu)); \tag{2.5}$$

- *if $p = 1$ and $r = 0$, then*

$$\mathcal{G}_{r,p}(\gamma_{opt}) \leq W_1(\mu,\nu)\max\{G_r(\mu),G_r(\nu)\}. \tag{2.6}$$

Proof. We split the proof in three parts.

Proof of inequality (2.4). Consider the curve given by:

$$\gamma(t) = (1-t)\mu + t\nu. \tag{2.7}$$

For this curve

$$G_r(\gamma(t)) = \sum_{i=1}^{m}(1-t)^r a_i^r + \sum_{j=1}^{n} t^r b_j^r = (1-t)^r G_r(\mu) + t^r G_r(\nu).$$

We have now to evaluate the metric derivative. Consider an optimal transport plan π between μ and ν. In the time interval $[t,t']$ a portion of mass $\pi_{ij}(t'-t)$ disappears in x_i and appears in y_j. The cost of the transportation is then at most

$$\sum_{i,j} \pi_{ij}(t'-t)|x_i-y_j|^p. \tag{2.8}$$

Passing to the limit as $t' \to t$, the metric derivative is then

$$\sum_{i,j} \pi_{ij}|x_i-y_j|^p = W_p(\mu,\nu).$$

The curve is actually a geodesic since the optimal transport plan between $(1-t)\mu+t\nu$ and $(1-t')\mu+t'\nu$ has only to move the masses $(t'-t)a_i$ on the point x_i on the masses $(t'-t)b_j$ on the points y_j. The optimal transport plan between these two measures is $(t'-t)\pi$ and the Wasserstein distance is then $(t'-t)W_p(\mu,\nu)$. Integrating in the interval $[0,1]$, we then have

$$\begin{aligned}\mathcal{G}_{p,r}(\gamma_{\mathrm{opt}}) &\leq \int_0^1 \big((1-t)^r G_r(\mu) + t^r G_r(\nu)\big) W_p(\mu,\nu)dt \\ &\leq \frac{1}{1+r}(G_r(\mu)+G_r(\nu))W_p(\mu,\nu).\end{aligned}$$

Proof of inequality (2.5). First, to clarify computations, we consider the case $\nu = \delta_{y_1}$. Let the curve $\gamma : [0,1] \to \mathcal{W}_p(\Omega)$ be given by:

$$\gamma(t) := \sum_{i=1}^{m} a_i \delta_{x_i + t(y_1 - x_i)}.$$

We have $|\gamma'|(t) = W_p(\mu,\delta_{y_1})$, and $G_r(\gamma) = G_r(\mu)$. So, we have

$$\mathcal{G}_{r,p}(\gamma_{\mathrm{opt}}) \leq W_p(\mu,\delta_{y_1})G_r(\mu) \leq W_p(\mu,\delta_{y_1})\big(G_r(\mu)+1\big).$$

In the general case, we consider an acyclic optimal transference plan $\pi = \sum_{ij} \pi_{ij}\delta_{(x_i,y_j)}$ and the path

$$t \mapsto \gamma(t) = \sum_{ij} \pi_{ij}\delta_{(1-t)x_i+ty_j}.$$

The functional $\int_0^1 G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t$ can be computed to be $\sum_{ij}\pi_{ij}^r W_p(\mu,\nu)$. Thanks to Proposition 2.5 we have

$$\sum_{ij}\pi_{ij}^r \le G_r(\mu)+G_r(\nu).$$

The proof of the inequality is complete.

Proof of inequality (2.6). In this case we are evaluating

$$\mathcal{G}_{0,1}(\gamma) = \int_0^1 \sharp\,\mathrm{spt}\,\gamma(t)|\gamma'|(t)dt.$$

The main point is that if $p = 1$, then both paths of the kind

$$t \mapsto \delta_{x_1+t(y_1-x_1)}, \tag{2.9}$$

and

$$t \mapsto (1-t)\delta_{x_1} + t\delta_{y_1}, \tag{2.10}$$

are Lipschitz curves. Roughly speaking, we use paths of the first kind (2.9) to move the mass in a point y_j where there is still no mass and the paths of type (2.10) to move the mass in a point y_j where there is already some mass without incrementing the cardinality of the support of the measure $\gamma(t)$.

Let π be an extremal transference plan, and assume that $\mu_{k-1} \le \mu$, $\nu_{k-1} \le \nu$ have been determined in such a way $G_0(\mu_{k-1}) + G_0(\nu_{k-1}) \le \max\{G_0(\mu), G_0(\nu)\}$.

If there is a row i_{k-1} with a unique non-zero element $\pi_{i_{k-1}j_{k-1}}$, then using then the path of the form (2.9),

$$\gamma_{k-1}(t) = \sum_{i\neq i_{k-1}} (\mu_{k-1})_i\delta_{x_i} + \sum_j (\nu_{k-1})_j\delta_{y_j} + (\mu_{k-1})_{i_{k-1}}\delta_{(1-t)x_{i_{k-1}}+ty_{j_{k-1}}},$$

we obtain that $|\gamma'_{k-1}|(t) = \pi_{i_{k-1}j_{k-1}}$ and $G_0(\gamma(t)) = G_0(\mu_{k-1}) + G_0(\nu_{k-1})$.

If instead there exist only columns with a single non-zero element $\pi_{i_{k-1}j_{k-1}}$, two cases can happen:

(1) if there is an element $0 < \nu_{k-1}(y_{j_{k-1}}) < \nu(y_{j_{k-1}})$, then we take $\pi_{i_{k-1}j_{k-1}} \neq 0$ and we can use the path

$$\begin{aligned}\gamma_{k-1}(t) = &\sum_{i\neq i_{k-1}} (\mu_{k-1})_i\delta_{x_i} + \sum_{j\neq j_{k-1}} (\nu_{k-1})_j\delta_{y_j}\\ &+ \big((\mu_{k-1})_{i_{k-1}} - t\pi_{i_{k-1}j_{k-1}}\big)\delta_{x_{i_{k-1}}}\\ &+ \big((\nu_{k-1})_{i_{k-1}} + t\pi_{i_{k-1}j_{k-1}}\big)\delta_{y_{j_{k-1}}},\end{aligned}$$

for which $|\gamma'_{k-1}|(t) = \pi_{i_{k-1}j_{k-1}}$ and

$$G_0(\gamma_{k-1}(t)) = G_0(\mu_{k-1}) + G_0(\nu_{k-1}) \le \max\{G_0(\mu), G_0(\nu)\};$$

(2) if $\nu_{k-1}(y_{j_{k-1}}) = 0$, then by using the path

$$\begin{aligned}\gamma_{k-1}(t) = &\sum_{i \neq i_{k-1}} (\mu_{k-1})_i \delta_{x_i} + \sum_j (\nu_{k-1})_j \delta_{y_j} \\ &+ \big((\mu_{k-1})_{i_{k-1}} - \pi_{i_{k-1} j_{k-1}} \big) \delta_{x_{i_{k-1}}} \\ &+ \pi_{i_{k-1} j_{k-1}} \delta_{(1-t) x_{i_{k-1}} + t y_{j_{k-1}}},\end{aligned}$$

we obtain $|\gamma'|(t) = \pi_{i_{k-1} j_{k-1}}$

$$G_0(\gamma_{k-1}(t)) = G_0(\mu_{k-1}) + G_0(\nu_{k-1}) + 1.$$

In this last case $\pi_{i_{k-1} j_{k-1}} = \nu_{j_{k-1}}$.

Define μ_k, ν_k as the final measures obtained by $\gamma_{k-1}(1)$.

It remains to show that in the last case

$$\begin{aligned} G(\gamma_{k-1}(t)) &= G_0(\mu_k) + G_0(\nu_k) \\ &= G_0(\mu_{k-1}) + G_0(\nu_{k-1}) + 1 \leq \max\{G_0(\mu), G_0(\nu)\}.\end{aligned}$$

Assume that in each row there are at least two elements $\pi_{ij} > 0$, and that for some index m_0 up to re-parametrization

$$0 < (\nu_{k-1})_j \leq \nu_j \ j = 1, \dots, m_0, \quad (\nu_{k-1})_j = 0 \ j = m_0 + 1, \dots, m.$$

The above conditions imply that $m - m_0 > G_0(\mu_{k-1})$, otherwise there is certainly a row i_{k-1} with a unique non-zero element.

Hence we have that

$$G_0(\mu_{k-1}) + G_0(\nu_{k-1}) < m - m_0 + m_0 \leq m = G_0(\nu) \leq \max\{G_0(\mu), G_0(\nu)\}.$$

By finite induction we conclude that $|\gamma_k'|(t) = \pi_{i_k j_k}$,

$$G_r(\gamma_k(t)) \leq \max\{G_0(\mu), G_0(\nu)\},$$

so that by piecing together the γ_k one constructs a path satisfying the last inequality. □

The estimates of Proposition 2.8 will be extended to all measures in Corollary 2.10. We begin with a lemma.

Lemma 2.9. *Let Ω be a subset of $\mathbb{R}^N$. Let $\mu \in \mathcal{W}_p(\overline{\Omega})$ such that $G_r(\mu) < +\infty$ (so, a discrete measure with possibly infinite support). Then, given $\varepsilon > 0$ there exists a measure $\tilde{\mu} \in \mathcal{W}_p(\overline{\Omega})$ with finite support such that*

$$G_r(\tilde{\mu}) \leq G_r(\mu),$$

and

$$W_p(\mu, \tilde{\mu}) < \varepsilon.$$

Proof. Let

$$\mu = \sum_{i=0}^{+\infty} a_h \delta_{x_h}.$$

Define (the parameters H, b are to be chosen, actually b may be chosen arbitrarily)

$$\tilde{\mu} = \sum_{i=0}^{H} a_h \delta_{x_h} + \left(\sum_{i=H+1}^{+\infty} a_h \right) \delta_b.$$

By subadditivity we easily have that

$$G_r(\tilde{\mu}) \leq G_r(\mu).$$

The estimate on the p-Wasserstein distance follows. The transport plan that fixes the masses in x_h for $h = 0, 1, \ldots, H$ and moves those in x_h for $h \geq H + 1$ in b gives the upper estimate:

$$W_p(\mu, \tilde{\mu}) \leq \left(\sum_{h=H+1}^{+\infty} a_h |x_h - b|^p \right)^{\min\{1,1/p\}}.$$

Since the momentum of order p of μ is finite, we can find H such that

$$\left(\sum_{h=H+1}^{+\infty} a_h |x_h - b|^p \right)^{\min\{1,1/p\}} < \varepsilon.$$

This concludes the proof. Note that there is no boundedness assumption on Ω thanks to the finiteness of the momentum of order p of μ. □

Corollary 2.10. *The estimates of Proposition 2.8 are true for all measures.*

Proof. Let μ, ν be generic measures in $W_p(\overline{\Omega})$. The only non-trivial case is if both $G_r(\mu) < +\infty$ and $G_r(\nu) < +\infty$. This means that both μ and ν are discrete measures and $\sharp(\operatorname{spt} \mu) = +\infty, \sharp(\operatorname{spt} \nu) = +\infty$.

Let μ_n and ν_n be approximating sequences as in Lemma 2.9. Then,

$$\mu_n \to \mu, \quad \nu_n \to \nu$$

w.r.t. W_p and

$$G_r(\mu_n) \to G_r(\mu), \quad G_r(\nu_n) \to G_r(\nu).$$

Then, for example for (2.5) one obtains using Theorem 2.2

$$\begin{aligned} \mathcal{G}_{r,p}(\mu, \nu) \leq & \liminf_{n \to +\infty} \mathcal{G}_{r,p}(\mu_n, \nu_n) \\ \leq & \liminf_{n \to +\infty} W_p(\mu_n, \nu_n)(G_r(\mu_n) + G_r(\nu_n)) \\ \leq & W_p(\mu, \nu)(G_r(\mu) + G_r(\nu)). \end{aligned}$$

This concludes the proof in that case. The other cases are completely similar. □

Corollary 2.11. *The following holds:*

1. *The estimate* (2.4) *holds for all metric spaces.*
2. *Let X be a metric space such that there exists a constant $C > 0$ such that for every couple (x, y) there exists a Lipschitz curve g such that $g(0) = x, g(1) = y$ and $\operatorname{Var}(g) \leq Cd(x, y)$. Then, the results of Corollary 2.10 are still true (except for a change in the constants in the r.h.s. of* (2.5), (2.6)*).*

Proof. The proof is the same, simply use the curve g to interpolate between points instead of straight lines in the second case. □

2.4. **Reachability of measures.**

Definition 2.12 (Reachable measure w.r.t. r, p)**.** A probability measure μ is *reachable w.r.t.* r, p (or (r, p)*-reachable*) if there exists $\gamma \in \mathrm{Lip}([0, \varepsilon], \mathcal{W}_p(\overline{\Omega}))$ such that $\gamma(0) = \mu$ and $|\gamma'|(t) = 1$ for a.e. $t \in [0, \varepsilon]$ and

$$\mathcal{G}_{r,p}(\gamma) = \int_0^\varepsilon G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^\varepsilon G_r(\gamma(t))\mathrm{d}t < +\infty.$$

Note that the existence of a curve $\gamma \in \mathrm{Lip}([0, \varepsilon], \mathcal{W}_p(\overline{\Omega}))$ such that $\gamma(0) = \mu$ and $|\gamma'|(t) = 1$ for a.e. $t \in [0, \varepsilon]$ is equivalent to the existence a curve $\gamma \in \mathrm{Lip}([0, \varepsilon'], \mathcal{W}_p(\overline{\Omega}))$ such that $\gamma(0) = \mu$ and $\mathrm{Var}(\gamma) > 0$.

Let

$$g_{r,p,\mu}(t) := \min\{G_r(\nu) : W_p(\nu, \mu) \le t\}. \tag{2.11}$$

Note that the infimum is actually a minimum since $\nu \mapsto G_r(\nu)$ is lower semicontinous w.r.t. the weak convergence of measures and the set $\{\nu : W_p(\nu, \mu) \le t\}$ is closed w.r.t. the topology induced by W_p (which is essentially the weak topology). Note moreover that the map $t \mapsto g_{r,p,\mu}(t)$ is monotone non-increasing.

The following is the main theorem of the paper.

Theorem 2.13. *Let $\mu \in \mathcal{W}_p(\overline{\Omega})$. The following conditions are equivalent:*

(1) *μ is reachable;*
(2) *there exists $\varepsilon > 0$ (equivalently for all $\varepsilon > 0$)*

$$\int_0^\varepsilon g_{r,p,\mu}(t)\mathrm{d}t < +\infty; \tag{2.12}$$

(3) *there exists $\gamma \in \mathrm{Lip}([0, 1], \mathcal{W}_p(\overline{\Omega}))$ which satisfies* $\mathrm{Var}(\gamma) > 0, \gamma(0) = \mu, \gamma(1) = \delta_{x_0}$ *and*

$$\int_0^1 G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t < +\infty.$$

Proof. We will show that (1. ⇒ 2.), (2. ⇒ 3.), (3. ⇒ 1.).

(1. ⇒ 2.) If μ is reachable there exists a curve γ as in Definition 2.12. Since

$$\int_0^\varepsilon g_{r,p,\mu}(t)\mathrm{d}t \le \int_0^\varepsilon G_r(\gamma(t))\mathrm{d}t < +\infty,$$

condition (2.12) is satisfied.

(2. ⇒ 3.) Suppose now that condition (2.12) is satisfied. Let μ_i a measure such that $G_r(\mu_i) = g_{r,p,\mu}(\varepsilon 2^{-i})$ and $W_p(\mu_i, \mu) \le \varepsilon 2^{-i}$. μ_i is a discrete measure with possibly infinite support. We now connect μ_i and μ_{i+1} with a curve $\gamma_i : [0, 1] \to \mathcal{W}_p(\overline{\Omega})$ as in Corollary 2.10 such that

$$\begin{aligned}\int_0^1 G_r(\gamma_i(t))|\gamma_i'|(t)\mathrm{d}t &\le (G_r(\mu_i) + G_r(\mu_{i+1}))W_p(\mu_i, \mu_{i+1}) \\ &\le 2G_r(\mu_i)(\varepsilon 2^{-i} + \varepsilon 2^{-(i+1)}) \\ &\le 2^{2-i}\varepsilon G_r(\mu_i) = 2^{2-i}\varepsilon g_{r,p,\mu}(\varepsilon 2^{-i}) \le 8\int_{\varepsilon 2^{-i-1}}^{\varepsilon 2^{-i}} g_{r,p,\mu}(t)dt.\end{aligned}$$

Gluing the paths γ_i together, we obtain a path γ (up to a change of variable to set the speed at a unitary value, see Lemma 2.1) with the desired properties.

(3. $\Rightarrow$ 1.) Directly from the definition. $\square$

Remark 2.14. If a measure μ is reached by some curve defined on an interval $[0,\varepsilon]$ with finite total variation, then the measure is reachable from a Dirac mass: just observe that $\gamma(\varepsilon)$ satisfies $G_r(\gamma(\varepsilon)) < +\infty$ and use Corollary 2.10.

We define the following equivalence relation: the measure μ is equivalent to ν if there exists a path γ such that $\gamma(0) = \mu, \gamma(1) = \nu$ and $\mathcal{G}_{r,p}(\gamma) < +\infty$. Definition 2.12 actually characterizes the measures in the equivalence class of a Dirac mass. All the other measures are "isolated".

Proposition 2.15. *If μ is not reachable from a Dirac mass, then its equivalence class consists of a single element (μ itself).*

Proof. Suppose on the contrary that there exists a measure $\nu \neq \mu$ in the same class of μ and let γ be a path between them with finite cost. Then condition (1) or (2) of Theorem 2.13 are satisfied (note that in general $0 < W_p(\mu,\nu) \leq \mathrm{Var}(\gamma)$). Then μ is in the equivalence class of a Dirac mass, which is not possible. $\square$

By Theorem 3.4 of [BBS], for $r > 1-1/N$ every measure is reachable. If $r < 1-1/N$ not all measures are reachable. In this case the equivalence class of a Dirac mass is a set of first category.

Proposition 2.16. *The equivalence class of a Dirac mass is a set of first category in $\mathcal{W}_p(\overline{\Omega})$, if $0 \leq r \leq 1-1/N$.*

Proof. The equivalence class of a Dirac mass is the set

$$\{\mu \in \mathcal{W}_p(\overline{\Omega}) : \mathcal{G}_{r,p}(\mu,\delta_{x_1}) < +\infty\} = \bigcup_{n\geq 1} \{\mu \in \mathcal{W}_p(\overline{\Omega}) : \mathcal{G}_{r,p}(\mu,\delta_{x_1}) \leq n\}.$$

We now prove that the set $\{\mu \in \mathcal{W}_p(\overline{\Omega}) : \mathcal{G}_{r,p}(\mu,\delta_{x_1}) \leq n\}$ is closed and nowhere dense.

The set is closed by the semicontinuity of $\mathcal{G}_{p,r}(\cdot,\delta_{x_1})$.

Suppose on the contrary that some ball (w.r.t. W_p) $B_r(\mu)$ is contained in $\{\mu \in \mathcal{W}_p(\overline{\Omega}) : \mathcal{G}_{r,p}(\mu,\delta_{x_1}) \leq n\}$. First consider a measure $\tilde{\mu}$ discrete with finite support such that $\tilde{\mu} \in B_R(\mu)$ (thanks to Lemma 2.9). Let

$$\tilde{\mu} = \sum_{h=1}^{n} a_h \delta_{x_h}.$$

Consider now the measure

$$\hat{\mu} = \frac{a_1}{|B_{R_1}|}\chi_{B_{R_1}(x_1)}\mathcal{L}^N + \sum_{h=2}^{n} a_h \delta_{x_h}.$$

Choosing R_1 sufficiently small, we have $\hat{\mu} \in B_R(\mu)$ ($W_p(\hat{\mu},\tilde{\mu})$ is bounded by R_1 if $p \geq 1$ and R_1^p if $0 \leq p \leq 1$).

By (2.12) of Theorem 2.13, it follows that

$$\sum_i 2^{-i} g_{r,p,\mu}(2^{-i}) < +\infty.$$

However, it is easy to see from the analysis of [BBS] that

$$g_{r,p,\mu}(t) \geq \frac{C}{t}$$

for some fixed constant C, so that we reach a contradiction. $\square$

2.5. **Regularity.**

Theorem 2.17. *Let μ, ν be measures and let $\gamma \in \mathrm{Lip}([0,1], \mathcal{W}_p(\overline{\Omega}))$ such that $\gamma(0) = \mu, \gamma(1) = \nu$ and $\mathcal{G}_{r,p}(\gamma) < +\infty$. Then, fixed $\delta > 0$, there exists $\tilde{\gamma} \in \mathrm{Lip}([0,1], \mathcal{W}_p(\overline{\Omega}))$ such that $\tilde{\gamma}(0) = \mu, \tilde{\gamma}(1) = \nu$, $\sharp \operatorname{spt} \gamma(t) < +\infty$ for $t \in]0,1[$ and $\mathcal{G}_{r,p}(\tilde{\gamma}) < (2+\delta)\mathcal{G}_{r,p}(\gamma)$.*

Proof. We parametrize γ by arc length obtaining

$$\int_0^L G_r(\gamma(t))|\gamma'|(t)\mathrm{d}t = \int_0^L G_r(\gamma(t))\mathrm{d}t < +\infty.$$

Fix $d > 0$. We will construct a perturbed path between two atomic measure with finite support: in fact from Theorem 2.13, we can find a path connecting μ to some purely atomic measure $\tilde{\mu}$, and a path connecting ν to some purely atomic measure $\tilde{\nu}$. By Lemma 2.9, we can assume that there is a measure μ_1 and a measure ν_1 with finite support such that

$$\mathcal{G}_{r,p}(\mu, \mu_1) + \mathcal{G}_{r,p}(\nu, \nu_1) \leq d\mathcal{G}_{r,p}(\mu, \nu).$$

We conclude that there is a path γ_1 connecting $\tilde{\mu}$ to ν with $\mathcal{G}(\tilde{\mu}, \nu) \leq (1+d)\mathcal{G}(\mu, \nu)$. One then repeats the argument below from μ to $\tilde{\mu}$ and from ν to $\tilde{\nu}$.

Step 1. We use the following fact: if $d > 0$ and $f : [0, L] \to [0, +\infty)$ is an integrable function, then for all ε sufficiently small there exists points t_i, $t_{i+1} - t_i \in (\varepsilon/(1+d), (1+d)\varepsilon)$, such that

$$\left| \sum f(t_i)(t_{i+1} - t_i) - \int_0^L f(t)dt \right| \leq d \int_0^L f(t)dt.$$

The proof follows immediately from the definition of Lebesgue integral, by approximating f with simple functions whose level sets are made of finitely many connected components.

Applying this to $\mathcal{G}_{r,p}(\gamma_1)$, it follows that we can fix a sequence of increasing points x_i, $i = 1, \dots, I$, such that

$$\left| \mathcal{G}(\mu_1, \nu) - \sum_i G_r(\gamma_1(t_i)) W_p(\gamma_1(t_{i+1}), \gamma(t_i)) \right| \leq d\mathcal{G}(\mu_1, \nu).$$

and $W_p(\gamma_1(t_{i+1}), \gamma_1(t_i)) \in [\varepsilon/(1+d), (1+d)\varepsilon]$.

Step 2. Let m_i be measures with finite support such that $G_r(m_i) \leq G_r(\gamma_1(t_i))$, and such that

$$W_p(m_i, \gamma_1(t_i)) \leq d\varepsilon.$$

Consider the path $\gamma_{1,i}$ of Proposition 2.8 connecting m_i with m_{i+1}. It follows that

$$\begin{aligned} \mathcal{G}_{r,p}(m_1, m_{i+1}) &\leq W_p(m_1, m_{i+1})\big(G_r(m_i) + G_r(m_{i+1})\big) \\ &\leq (1+3d)\varepsilon\big(G_r(m_i) + G_r(m_{i+1})\big). \end{aligned}$$

Step 3. Piecing together all these paths, one obtains that μ_1 can be connected to ν with a path $\tilde{\gamma}_1$ such that $\tilde{\gamma}_1(t)$ has finite support for all t and

$$\mathcal{G}(\tilde{\gamma}_1) \leq (1+3d) \sum_i \varepsilon\big(G_r(m_i) + G_r(m_{i+1})\big) \leq 2(1+3d)^2 \mathcal{G}(\gamma_1). \qquad \square$$

Example 2.18. The curve provided by Theorem 2.17 is in general not optimal (at least on non compact Ω), as the following example shows. We consider the measures

$$\mu(0) = \sum_i a_i \delta_{x_i(0)}, \quad \mu(1) = \sum_i a_i \delta_{x_i(1)}$$

such that

$$\sum_{j>0} a_{i+j} < a_i, \quad D_i \leq D_{i+1}, \quad x_i(0) + d_i = x_i(1), \; x_i(1) + D_i = x_{i+1}(0).$$

To prove that the optimal transportation is only the trivial translation, i.e. the path

$$\mu(t) = \sum_i a_i \delta_{x_i(t)}, \quad x_i(0) \leq x_i(t) \leq x_i(1), \quad x_i(t) \geq x_i(s),$$

we assume first that there exists a $\bar{t} \in (0,1)$ such that

$$G_r(\mu(\bar{t})) = \sum_i b_i^r < \sum_i a_i^r,$$

and estimate the path as

$$\int G_r(\mu(t))|\gamma'|(t)dt \geq G_r(\mu(\bar{t}))\Big(W_1(\mu(0), \mu(\bar{t})) + W_1(\mu(\bar{t}), \mu(1)\Big).$$

Since we are in $\mathbb{R}$, the Wasserstein distance can be evaluated as the area among the two distribution functions.

Fixed the measure $\bar{\mu} = \sum_i b_i \delta_{y_i}$, we observe that the Wasserstein distance

$$W_1(\mu(0), \mu(\bar{t})) + W_i(\mu(\bar{t}), \mu(1))$$

decreases if $y_i \in [x_i(0), x_i(1)]$ (up to a re-parametrization of the y_i): in fact, one just needs to move towards the left if more mass comes from the left, or towards the right in the other case. Moreover, if for some i $b_i \neq a_i$, then the mass difference $|b_i - a_i|$ should arrive from a distance of at least D_i: the mass which leaves $x_i(0)$ and arrives in $x_i(1)$ has a cost of at least

$$|b_i - a_i|(D_i - d_i) + a_i d_i.$$

In fact, the difference in mass should come from some other point with distance $\geq D_i$, and the rest of the mass $(a_i - b_i)^+$ should move of at least d_i.

We thus have

$$\begin{aligned}
& G_r(\mu(\bar{t}))\Big(W_1(\mu(0), \mu(\bar{t})) + W_i(\mu(\bar{t}), \mu(1))\Big) - G_t(\mu(0))W_1(\mu(0), \mu(1)) \\
& \quad \geq \Big(\sum_i b_i^r\Big)\Big(\sum_i |b_i - a_i|(D_i - d_i) + \sum_i a_i d_i\Big) - \Big(\sum_i a_i^r\Big)\Big(\sum_i a_i d_i\Big) \\
& \quad \geq \sum_i |b_i - a_i|(D_i - d_i)\Big(\sum_i a_i d_i\Big) - \Big(\sum_i a_i^r - b_i^r\Big)\Big(\sum_i a_i d_i\Big) \\
& \quad \geq \sum_i |b_i - a_i|(D_i - d_i)\Big(\sum_i a_i d_i\Big) - \Big(\sum_i r a_i^{r-1}|b_i - a_i|\Big)\Big(\sum_i a_i d_i\Big) \\
& \quad = \sum_i |b_i - a_i|\Big[D_i - d_i - r a_i^r\Big]\Big(\sum_i a_i d_i\Big),
\end{aligned}$$

where we used the estimate

$$\sum_i (a_i^r - b_i^r) \leq \sum_i r a_i^{r-1} |b_i - a_i|.$$

If we choose

$$D_i > d_i + r a_i^{r-1},$$

then the optimal transference plans should have $b_i = a_i$. This concludes the example, if we can show an explicit case: for this, take

$$r = \frac{1}{2}, \quad a_i = 2 \cdot 3^{-i}, \quad d_i = 1, \quad D_i = 2 + \frac{3^{i/2}}{2^{3/2}}.$$

Note that

$$\sum_i 2 \cdot 3^{-i} \left(i + \sum_{j \leq i} D_j \right) \leq C \sum_i 3^{-i} 3^{(i+1)/2} < +\infty,$$

so that this measure can be connected to a δ.

2.6. **Path functional dimension.** We now introduce a new definition of dimension recovered from path functionals. We assume that $p > 0$ to avoid pathological cases (only some purely atomic measures can be reached for $p = 0$). Set

$$d_r := \frac{1}{1-r}.$$

The map $r \mapsto d_r$ is monotone increasing on the interval $[0, 1[$, $d_r \geq d_0 = 1$

$$\lim_{r \to 1-} d_r = +\infty.$$

Definition 2.19 (Path dimension)**.** Let $\mu \in \mathcal{W}_p(\overline{\Omega})$ and define

$$S_p(\mu) = \big\{ 0 \leq r < 1 : \mu \text{ is reachable w.r.t. } r, p \big\}.$$

Note that $S_p(\mu)$ is an interval, since

$$r_1 < r_2 \Rightarrow G_{r_2}(\mu) \leq G_{r_1}(\mu). \tag{2.13}$$

Set $r^* := \inf S_p(\mu)$. We define then

$$\dim_{path,p}(\mu) := \min\{1, p\} d_{r^*} = \frac{\min\{1, p\}}{1 - r^*}.$$

For every measure μ, $\dim_{path,p}(\mu) \geq 1$. Moreover, by Theorem 3.4 of [BBS] we know that $]1 - 1/N, 1[\subseteq S_p(\mu)$ in the case $p \geq 1$, so that $r^* \leq 1 - 1/N$ and

$$\dim_{path,p}(\mu) \leq N.$$

With a slight modification of Theorem 3.4 of [BBS] (see (2.17)), in the case $0 \leq p < 1$, $]1 - p/N, 1[\subseteq S_p(\mu)$, so that $r^* \leq 1 - p/N$ and

$$\dim_{path,p}(\mu) \leq N.$$

We now enumerate some easy known inequalities.

(1) By Jensen inequality with $f(t) = t^{q/p}$ with $q \geq p$

$$\left[\int_{X\times X} d(x,y)^p \pi(dxdy)\right]^{q/p} \leq \int_{X\times X} [d(x,y)^p]^{q/p} \pi(dxdy).$$

Then,

$$\left[\int_{X\times X} d(x,y)^p \pi(dxdy)\right]^{1/p} \leq \left[\int_{X\times X} d(x,y)^q \pi(dxdy)\right]^{1/q}$$

and

$$W_p^{\max\{1,1/p\}}(\mu,\nu) \leq W_q^{\max\{1,1/q\}}(\mu,\nu). \tag{2.14}$$

(2) Suppose now the space X is bounded. Since

$$\frac{d(x,y)}{\operatorname{diam} X} \leq 1,$$

taking the power $q-p>0$,

$$\frac{[d(x,y)]^q}{[\operatorname{diam} X]^q} \leq \frac{[d(x,y)]^p}{[\operatorname{diam} X]^p}.$$

Then,

$$W_q^{\max\{1,q\}}(\mu,\nu) \leq (\operatorname{diam} X)^{q-p} W_p^{\max\{1,q\}}(\mu,\nu). \tag{2.15}$$

(3) Let $r \leq s \leq 1$. Then, we have

$$s = r\frac{1-s}{1-r} + 1\frac{s-r}{1-r}, \quad \sum_i a_i^s = \sum_i (a_i^r)^{\frac{1-s}{1-r}} (a_i)^{\frac{s-r}{1-r}}.$$

Applying Hölder inequality with $p = (1-r)/(1-s), q = (1-r)/(s-r)$, we get

$$\sum_i a_i^s \leq \left(\sum_i a_i^r\right)^{\frac{1-s}{1-r}} \left(\sum_i a_i\right)^{\frac{s-r}{1-r}} = \left(\sum_i a_i^r\right)^{\frac{1-s}{1-r}}.$$

This gives

$$G_s(\mu) \leq G_r(\mu)^{\frac{1-s}{1-r}}. \tag{2.16}$$

Note that since $(1-s)/(1-r) < 1$ and $G_r(\mu) \geq 1$, (2.16) is a better estimate than simply $G_s(\mu) \leq G_r(\mu)$.

Using the above inequalities, we deduce immediately the following theorem.

Theorem 2.20. *Let μ, ν be Borel probability measures.*
If $1 \leq p \leq q$, $r \leq s \leq 1$, then

$$\mathcal{G}_{s,p}(\mu,\nu) \leq \mathcal{G}_{r,q}(\mu,\nu).$$

If $p \leq q \leq 1$, $r \leq s \leq 1$, and $\operatorname{diam} \Omega < +\infty$*, then*

$$\mathcal{G}_{s,q}(\mu,\nu) \leq (\operatorname{diam} \bar{\Omega})^{q-p} \mathcal{G}_{r,p}(\mu,\nu).$$

Proof. Our setting is now $\mathcal{W}_q(\overline{\Omega})$ which is included in $\mathcal{W}_p(\overline{\Omega})$. The first inequality relies on the fact that for $1 \leq p \leq q$, $W_p(\mu,\nu) \leq W_q(\mu,\nu)$ which implies the $\mathrm{Lip}([0,1],\mathcal{W}_p(\overline{\Omega})) \subseteq \mathrm{Lip}([0,1],\mathcal{W}_q(\overline{\Omega}))$ and $|\gamma'|_p(t) \leq |\gamma'|_q(t)$ when $\gamma \in \mathrm{Lip}([0,1],\mathcal{W}_q(\overline{\Omega}))$ and finally on equation (2.13) for estimating $G_s(\mu)$:

$$\mathcal{G}_{s,p}(\mu,\nu) = \int_0^1 G_s(\gamma(t))|\gamma'|_p(t)dt \leq \int_0^1 G_r(\gamma(t))|\gamma'|_q(t)dt = \mathcal{G}_{r,q}(\mu,\nu).$$

The second inequality follows from the fact that for bounded domains in the case $p \leq q \leq 1$ we have $W_p(\mu,\nu) \leq (\operatorname{diam}\bar{\Omega})^{q-p} W_q(\mu,\nu)$ which implies the $\mathrm{Lip}([0,1],\mathcal{W}_q(\overline{\Omega})) \subseteq \mathrm{Lip}([0,1],\mathcal{W}_p(\overline{\Omega}))$ and $|\gamma'|_q(t) \leq (\operatorname{diam}\bar{\Omega})^{q-p}|\gamma'|_p(t)$ when $\gamma \in \mathrm{Lip}([0,1],\mathcal{W}_p(\overline{\Omega}))$ and finally on equation (2.13). ☐

Theorem 2.21. *Consider the function*

$$(0,+\infty) \ni p \mapsto d(p) = \dim_{path,p}(\mu) \in [1,+\infty)$$

for a fixed measure μ. Then $d(p)$ satisfies the following estimates:

(1) *for μ with bounded support*

$$1 \leq d(p) \leq N; \tag{2.17}$$

(2) *for $p \leq q$ then*

$$d(p) \leq d(q); \tag{2.18}$$

(3) *if μ has bounded support, then for $q \leq p$*

$$d(p) \leq d(q)\frac{p}{q}. \tag{2.19}$$

In particular we have the following corollary.

Corollary 2.22. *The function $p \mapsto d(p)$ is monotone and locally Lipschitz continuous when X is bounded.*

We prove Theorem 2.21: the first part reflects the estimate of Theorem 3.4 of [BBS] with a proof based on Theorem 2.13.

Proof. 1) Let Q_R be a cube of size R where the mass of μ is supported, and consider the cubes Q_j, $j = 1,\dots,2^{iN}$, of size $2^{-i}R_k$ and centered in x_j: let ν_i be the measure

$$\nu_i = \sum_{j=1}^{2^{iN}} \delta_{x_j}\mu(Q_j),$$

so that a direct computation of W_p yields

$$W_p(\mu,\nu_i) \leq (2^{-i}R_k)^{\min\{1,p\}} \sum_j \mu(Q_j) = (2^{-i}R_k)^{\min\{1,p\}}.$$

We next estimate the function $G_r(\nu_i)$: using the concavity of x^r and Jensen's inequality one obtains

$$\sum_{j=1}^{I} a_j^r = I\sum_{j=1}^{I} a_j^r\frac{1}{I} \leq I\biggl(\sum_{j=1}^{I} a_j\frac{1}{I}\biggr)^r = I^{1-r}\biggl(\sum_{j=1}^{I} a_j\biggr)^r,$$

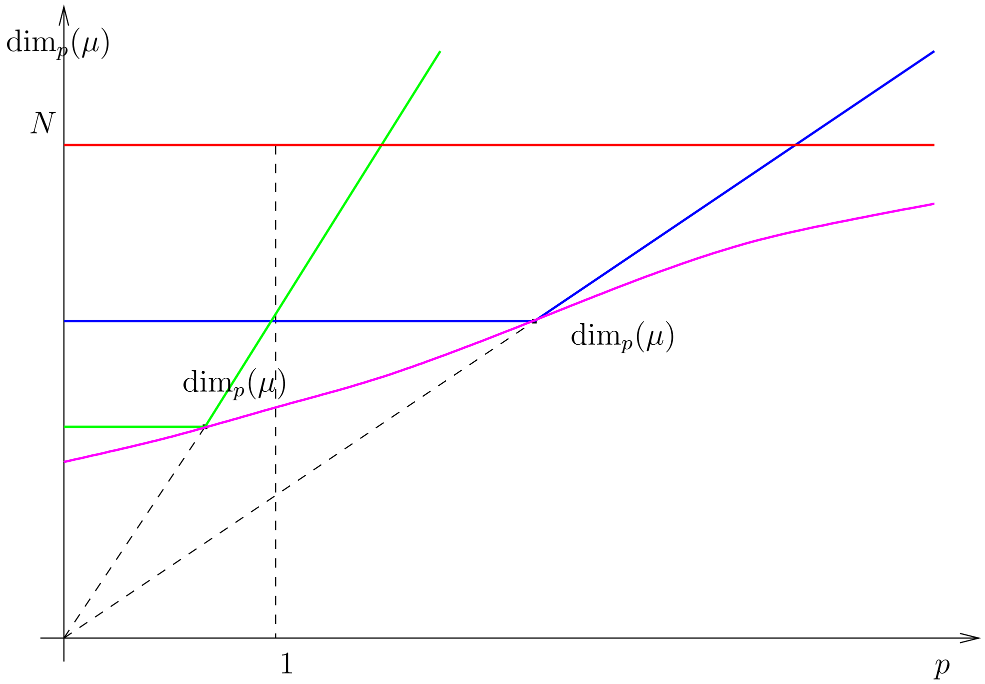

Figure 1. The estimates given by Theorem 2.21, in the case of bounded domain: the red curve is the estimate (2.17), while the blue and green ones correspond to estimates (2.18), (2.19) in two different points. The magenta curve is an admissible graph of the function $p \mapsto d(p)$ for a measure μ.

we obtain

$$G_r(\nu_i) = \sum_{j=1}^{2^{iN}} \mu(Q_j)^r \leq 2^{iN(1-r)}.$$

Hence, using Theorem 2.13, we conclude that

$$\sum_k W_p(\mu, \nu_i) G_r(\nu_i) \leq \sum_k (2^{-i} R_k)^{\min\{1,p\}} 2^{iN(1-r)},$$

which is convergent for $\min\{1,p\}/(1-r) > N$.

2) Let $r > 1 - \min\{1,q\}/d(q)$ and let ν_i be a sequence of measures such that

$$W_q(\mu, \nu_i) \leq 2^{-i}, \qquad \sum_i 2^{-i} G_r(\nu_i) < +\infty.$$

Using (2.14), it follows that

$$W_p(\mu, \nu_i) \leq 2^{-i \min\{1,p\} \max\{1,1/q\}},$$

and using (2.16) one obtains for $r \leq s$

$$G_s(\nu_i) \leq \big(G_r(\nu_i)\big)^{\frac{1-s}{1-r}}.$$

Hence we conclude by Theorem 2.13 that

$$\begin{aligned}\mathcal{G}_{p,s}(\gamma) &\leq C\sum_i 2^{-i\min\{1,p\}\max\{1,1/q\}}G_s(\nu_i)\\ &\leq C\sum_i 2^{-i\min\{1,p\}\max\{1,1/q\}}G_r(\nu_i)^{\frac{1-s}{1-r}}\\ &= C\sum_i\left(2^{-i\min\{1,p\}\max\{1,1/q\}\frac{1-r}{1-s}}G_r(\nu_i)\right)^{\frac{1-s}{1-r}}.\end{aligned}$$

Since $G_r(\nu_i) \leq 2^i$ definitely by $\sum_i 2^{-i}G_r(\nu_i) < +\infty$, we conclude that the series converges for

$$\frac{\min\{1,p\}}{1-s} > \frac{\min\{1,q\}}{1-r}.$$

From the definition of $d(p)$ the estimate (2.18) follows.

3) In the case μ has bounded support, we use the same computation as in the second case by replacing (2.14) with (2.15): let ν_i be a sequence of measures such that

$$W_q(\mu,\nu_i) \leq 2^{-i}, \quad \sum_i 2^{-i}G_r(\nu_i) < +\infty.$$

Then from (2.15) it follows that for $p \geq q$

$$W_p(\mu,\nu_i) \leq CW_q(\mu,\nu_i)^{\max\{1,q\}\min\{1,1/p\}},$$

so that, using again $G_r(\mu_i) \leq C2^i$,

$$\begin{aligned}\mathcal{G}_{p,s}(\gamma) &\leq C\sum_i 2^{-i\max\{1,p\}\min\{1,1/q\}}G_s(\nu_i)\\ &\leq C\sum_i\left(2^{-i\max\{1,p\}\min\left\{1,\frac{1}{q}\right\}\frac{1-r}{1-s}}G_r(\nu_i)\right)^{\frac{1-s}{1-r}}\\ &\leq C\sum_i\left(2^{1-i\max\{1,p\}\min\left\{1,\frac{1}{q}\right\}\frac{1-r}{1-s}}\right)^{\frac{1-s}{1-r}} < \infty,\end{aligned}$$

if

$$\frac{1}{1-s} > \frac{1}{1-r}\frac{\max\{1,q\}}{\max\{1,p\}}.$$

This implies immediately (2.19). □

3. Other notions of dimension

In this section we consider the comparison of various definitions of dimension for a measure. In the next section we will compare these dimensions with $\dim_{path,p}$.

3.1. **Hausdorff measure and dimension.** We just recall some definitions for reader's convenience. Let $\alpha \geq 0$, and given $\delta > 0$ define:

$$\mathcal{H}^\alpha_\delta(A) := \inf\left\{\sum_{n=1}^{+\infty}(\operatorname{diam} A_n)^\alpha : A \subseteq \bigcup_{n=1}^{+\infty} A_n,\ \operatorname{diam} A_n < \delta\right\}. \tag{3.1}$$

In the previous definition one can always assume A_n closed. The *α-dimensional Hausdorff outer measure* is defined by:

$$\mathcal{H}^\alpha(A) := \sup_{\delta>0} \mathcal{H}^\alpha_\delta(A) = \lim_{\delta\to 0^+} \mathcal{H}^\alpha_\delta(A). \tag{3.2}$$

Let $\alpha < \beta$. Then, $\mathcal{H}^\beta_\delta(A) \le \delta^{\beta-\alpha}\mathcal{H}^\alpha_\delta(A)$, and

$$\mathcal{H}^\alpha(A) < +\infty \Longrightarrow \mathcal{H}^\beta(A) = 0, \quad \mathcal{H}^\beta(A) > 0 \Longrightarrow \mathcal{H}^\alpha(A) = +\infty.$$

The map $\alpha \mapsto \mathcal{H}^\alpha(A)$ is strictly positive and finite in at most one point. So, we define:

$$\dim_{\mathcal{H}}(A) := \inf\{\alpha \ge 0 : \mathcal{H}^\alpha(A) = 0\}. \tag{3.3}$$

Definition 3.1 (Hausdorff concentration dimension)**.** We define the *Hausdorff concentration dimension* of μ as

$$\dim_{\mathcal{H}}(\mu) = \inf\left\{\dim_{\mathcal{H}}(B) : \mu(B^c) = 0\right\}. \tag{3.4}$$

Trivially, the Hausdorff concentration dimension of μ is lower than the Hausdorff dimension of $\operatorname{spt}\mu$, and the infimum is assumed.

Lemma 3.2. *There exists B such that $\dim_{\mathcal{H}}(B) = \dim_{\mathcal{H}}(\mu)$.*

Proof. The definition of $\dim_{\mathcal{H}}(\mu)$ implies that for all $\beta > \dim_{\mathcal{H}}(\mu)$ there exists B such that $\mathcal{H}^\beta(B) = 0$ and $\mu(B) = 1$. Consider then a sequence $\beta_n \searrow \dim_{\mathcal{H}}(\mu)$ and sets B_n such that $\mu(B_n) = 1$ and $\mathcal{H}^{\beta_n}(B_n) = 0$: the set $B = \cap_n B_n$ satisfies $\mu(B) = 1$ and $\mathcal{H}^\beta(B) = 0$ for all $\beta > \dim_{\mathcal{H}}$. Hence $\dim_{\mathcal{H}}(B) = \dim_{\mathcal{H}}(\mu)$. □

The estimate of the Hausdorff measure implies the following lemma.

Lemma 3.3. *If $\dim_{\mathcal{H}}(\mu) = \beta$, then*

(1) *for all $\gamma < \beta$ there exists a set E of positive μ-measure such that*

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^\gamma} = 0 \tag{3.5}$$

for all $x \in E$.

(2) *for all $\gamma > \beta$*

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^\gamma} = +\infty \tag{3.6}$$

for μ-a.e. x.

Proof. This lemma can be obtained immediately by using the following estimates (Theorem 2.4.3 of [AT]): for all $x \in A$, A Borel,

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^\gamma} \le t \quad \Longrightarrow \quad \mu|_A \le 2^\gamma t \mathcal{H}^\gamma|_A,$$

and

$$\limsup_{r\to 0^+} \frac{\mu(B(x,r))}{r^\gamma} \ge t \quad \Longrightarrow \quad \mu|_A \ge t \mathcal{H}^\gamma|_A.$$

In fact, the first implies Point (2) for $\gamma > \beta$, while the second implies Point (1) by contradiction. □

In the next sections we will use the following elementary lemma, which will allow us to compute the Hausdorff concentration dimension.

Lemma 3.4. *Assume that for all δ there is a set Γ_δ of measure $\mu(A) > 1-\delta$ whose δ-Hausdorff outer measure $\mathcal{H}^\alpha_\delta(\Gamma_\delta)$ is less than δ. Then, there exists a set Γ of Hausdorff measure $\mathcal{H}^\alpha(\Gamma) = 0$ and $\mu(\Gamma) = 1$.*

Proof. One just considers the sequence $\delta_{nm} = 2^{-n-m}$, $n, m \in \mathbb{N}$, and sets $\Gamma_{\delta_{nm}}$ such that $\mathcal{H}^\alpha_{\delta_{nm}}(\Gamma_{nm}) \leq 2^{-n-m}$ and $\mu(\Gamma_{nm}) > 1 - 2^{-n-m}$. Define the set $\Gamma = \cup_n \cap_m \Gamma_{\delta_{nm}}$: the measure of $\cap_m \Gamma_{\delta_{nm}}$ is $> 1 - 2^{-n}$ and its α-Hausdorff measure is

$$\begin{aligned}
\mathcal{H}^\alpha(\Gamma_n) &= \lim_{\delta\to 0^+} \mathcal{H}^\alpha_\delta(\Gamma_n) = \lim_{m\to\infty} \mathcal{H}^\alpha_{2^{-n-m}}(\cap_m \Gamma_{nm}) \\
&\leq \lim_{m\to\infty} \mathcal{H}^\alpha_{2^{-n-m}}(\Gamma_{nm}) \leq \lim_{m\to\infty} 2^{-n-m} = 0.
\end{aligned}$$

Using the σ-additivity of $\mathcal{H}^\alpha$ and μ, it follows that $\mu(\Gamma) = 1$ and $\mathcal{H}^\alpha(\Gamma) = 0$. □

3.2. **Minkowski dimension.** Let A be a bounded subset of $\mathbb{R}^N$. Define $N(A, \varepsilon)$ as

$$N(A,\varepsilon) := \min\left\{k \in \mathbb{N} : A \subseteq \bigcup_{i=1}^k B_\varepsilon(x_i),\ x_i \in \mathbb{R}^N\right\}, \tag{3.7}$$

which is the least number of balls of radius ε whose union covers A.

Let $\alpha < \beta$. Then,

$$\begin{aligned}
\limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\alpha < +\infty \qquad &\Longrightarrow \qquad \limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\beta = 0, \\
\limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\beta > 0 \qquad &\Longrightarrow \qquad \limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\alpha = +\infty.
\end{aligned}$$

The map $\alpha \mapsto \limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\alpha$ is strictly positive and finite in at most one point. So, we define *Minkowski upper dimension*:

$$\overline{\dim}_M(A) := \inf\left\{\alpha \geq 0 : \limsup_{\varepsilon\to 0^+} N(A,\varepsilon)\varepsilon^\alpha = 0\right\}. \tag{3.8}$$

It is easy to see that Minkowski upper dimension is also given by:

$$\overline{\dim}_M(A) = \limsup_{\varepsilon\to 0^+} \frac{\log N(A,\varepsilon)}{-\log\varepsilon} = \limsup_{\varepsilon\to 0^+} \log_{1/\varepsilon} N(A,\varepsilon). \tag{3.9}$$

Minkowski dimension measures in terms of a power of $1/\varepsilon$ how fast $N(A,\varepsilon)$ grows as $\varepsilon \to 0^+$.

Since $\bar{A} \subseteq \cup_{i=1}^k \bar{B}_\varepsilon(x_i)$, then $\overline{\dim}_M(A) = \overline{\dim}_M(\bar{A})$.

Another way to introduce Minkowski dimensions is to use the *packing number* $P(A,\varepsilon)$ defined as the maximum number of balls of radius ε with center in A that are pairwise disjoint:

$$P(A,\varepsilon) := \min\left\{k \in \mathbb{N} : \{x_1,\dots,x_k\} \subseteq A, B(x_i,\varepsilon) \cap B(x_j,\varepsilon) = \emptyset\ \forall i \neq j\right\}. \tag{3.10}$$

Lemma 3.5. *We have the following bound:*

$$P(A, 2\varepsilon) \leq N(A,\varepsilon) \leq P(A, \varepsilon/2). \tag{3.11}$$

Proof. For the first inequality suppose on the contrary that $P(A,2\varepsilon) > N(A,\varepsilon)$ and let the balls $B_{2\varepsilon}(x_i)$ be maximizers for $P(A,2\varepsilon)$ and let the balls $B_\varepsilon(\tilde{x}_j)$ be minimizers for $N(A,\varepsilon)$. Since $\cup_j B_\varepsilon(\tilde{x}_j) = A$ for every i there exists $j(i)$ such

$$x_i \in B_\varepsilon(\tilde{x}_{j(i)}) \subseteq B_{2\varepsilon}(x_i).$$

The map $i \mapsto j(i)$ is clearly injective, so that we reach a contradiction.

For the second inequality, let $B_{\varepsilon/2}(\tilde{x}_i)$ be maximizers for $P(A,\varepsilon/2)$. The balls with the same centers and double radius cover A (if there were y out of their union, the ball with center in y and radius $\varepsilon/2$ does not intersect any of $B_{\varepsilon/2}(\tilde{x}_i)$ in contrast to the fact that they were maximizers). The definition of $N(A,\varepsilon)$ concludes the proof. $\square$

Now, thanks to (3.11) we can restate equations (3.8), (3.9) and (3.11) in terms of $P(A,\varepsilon)$ instead of $N(A,\varepsilon)$.

Consider the closed ε-neighborhood of A:

$$A_\varepsilon := \big\{x \in \mathbb{R}^N : d(x,A) \leq \varepsilon\big\}.$$

Thanks to the easy estimate:

$$P(A,\varepsilon)\omega_N\varepsilon^N \leq \mathcal{L}^N(A_\varepsilon) \leq N(A,\varepsilon)\omega_N(2\varepsilon)^N,$$

the definition of Minkowski upper dimension can be restated in terms of the *Minkowski contents*:

$$\overline{\mathcal{M}}^s(A) := \limsup_{\varepsilon\to 0^+} (2\varepsilon)^{s-N}\mathcal{L}^N(A_\varepsilon). \tag{3.12}$$

In fact, we have:

$$\overline{\dim}_M(A) := \inf\big\{\alpha \geq 0 : \overline{\mathcal{M}}^s(A) = 0\big\}. \tag{3.13}$$

Another equivalent definition following from (3.9), (3.12) and (3.13) is also:

$$\overline{\dim}_M(A) := N + \limsup_{\varepsilon\to 0^+} \frac{\log \mathcal{L}^N(A_\varepsilon)}{-\log\varepsilon}. \tag{3.14}$$

Finally, a definition for computer scientists (which turns out useful in this context). By a *dyadic cube of order* m in $\mathbb{R}^N$ we mean a Cartesian product of N intervals of the kind $[k2^{-m},(k+1)2^{-m}[$ for $k\in\mathbb{Z}, m\in\mathbb{N}$. Note that, fixed m, the dyadic cubes of order m cover $\mathbb{R}^N$ and they are pairwise disjoint.

Definition 3.6 (Minkowski box counting dimension)**.** Let $Q(A,m)$ be the cardinality of dyadic cubes of order m which meet A, and define

$$\overline{\dim}_B(A) = \limsup_{\varepsilon\to 0^+} \frac{\log Q(A,m)}{m\log 2} \tag{3.15}$$

Note that since we can find an estimate like (3.11) between $N(A,\cdot)$ and $Q(A,\cdot)$, we have $\overline{\dim}_M(A) = \overline{\dim}_B(A)$.

In the same way, replacing lim sup with lim inf, *Minkowski lower dimension* $\underline{\dim}_M(A)$ can be defined. Clearly

$$\underline{\dim}_M(A) \leq \overline{\dim}_M(A).$$

Proposition 3.7. *We have that*

$$\dim_{\mathcal{H}}(A) \leq \underline{\dim}_M(A) \leq \overline{\dim}_M(A).$$

Proof. In fact, if $\alpha > \underline{\dim}_M(A)$, then $\mathcal{H}^\alpha(A) = 0$, since $\alpha > \underline{\dim}_M(A)$ implies that there are $\varepsilon_i \to 0^+$ such that

$$\limsup_{i\to+\infty} N(A,\varepsilon_i)\varepsilon_i^\alpha = 0,$$

and

$$\mathcal{H}^\alpha_{\varepsilon_i}(A) \leq \sum_{i=1}^{N(A,\varepsilon_i)} (2\varepsilon_i)^\alpha = 2^\alpha \varepsilon_i^\alpha N(A,\varepsilon_i).$$

Finally,

$$\mathcal{H}^\alpha(A) \leq \lim_{i\to+\infty} 2^\alpha \varepsilon_i^\alpha N(A,\varepsilon_i) = 0,$$

concluding the proof. □

Definition 3.8. The *upper (lower) Minkowski dimension* of a measure μ is given by the infimum of upper (lower) Minkowski dimensions of the sets B on which μ is concentrated (or equivalently of the support of μ).

Hence there exists a set B such that $\mu(B) = 1$ and $\overline{\dim}_M(B) = \overline{\dim}_M(\mu)$ or $\underline{\dim}_M(B) = \underline{\dim}_M(\mu)$. From this fact and Proposition 3.7 the next proposition follows.

Proposition 3.9. *Let μ be a measure. Then,*

$$\dim_{\mathcal{H}}(\mu) \leq \underline{\dim}_M(\mu) \leq \overline{\dim}_M(\mu).$$

3.3. **Renyi dimension or q-dimension.** Let $q \in \mathbb{R} \setminus \{1\}$. The *$q$-entropy* is defined for a probability vector $\underline{a} = (a_1,\dots,a_n)$ by

$$H_q(\underline{a}) := \frac{1}{1-q}\log\left(\sum_{k=1}^n a_k^q\right). \tag{3.16}$$

If $q = 1$ we set

$$H_1(\underline{a}) := -\sum_{k=1}^n a_k \log a_k. \tag{3.17}$$

Lemma 3.10. *The functions H_q satisfy the following properties:*

$$\lim_{q\to 1} H_q(\underline{a}) = H_1(\underline{a}).$$

and

$$q_1 < q_2 \Longrightarrow H_{q_1}(\underline{a}) \geq H_{q_2}(\underline{a}).$$

Proof. The first one is straightforward. For the second we have for $q_2 < 1$:

$$\begin{aligned} H_{q_2}(\underline{a}) &= \frac{1}{1-q_2}\log\left(\sum_{k=1}^n a_k^{q_2}\right) = \frac{1}{1-q_2}\log\left(\sum_{k=1}^n a_k a_k^{q_2-1}\right) \\ &= \frac{1}{1-q_2}\log\left(\sum_{k=1}^n a_k (a_k^{q_1-1})^{\frac{q_2-1}{q_1-1}}\right) \leq \frac{1}{1-q_2}\frac{q_2-1}{q_1-1}\log\left(\sum_{k=1}^n a_k^{q_1}\right) \\ &= \frac{1}{1-q_1}\log\left(\sum_{k=1}^n a_k^{q_1}\right) = H_{q_1}(\underline{a}). \end{aligned}$$

The inequality follows from the concavity of the map $t \mapsto t^{\frac{q_2-1}{q_1-1}}$. The case $1 < q_1 < q_2$ is treated similarly.

The inequality between H_q and H_1 can be seen directly by the concavity of the log function:

$$\log\left(\sum_{k=1}^n a_k^{q_2}\right) = \log\left(\sum_{k=1}^n a_k a_k^{q_2-1}\right) \geq \sum_{k=1}^n a_k(q_2-1)\log a_k,$$

or using the fact that $H_q(\underline{a}) \to H_1(\underline{a})$. □

Let $q \in \mathbb{R}\setminus\{1\}$. Let $I_q(\mu,\varepsilon)$ defined by:

$$I_q(\mu,\varepsilon) = \inf\left\{\left[\sum_{i=1}^k \mu(S_i)^q\right]^{\frac{1}{1-q}} : \mu(\cup_i S_i) = 1, \operatorname{diam} S_i \leq 2\varepsilon\right\},$$

and set

$$\overline{\dim}_q(\mu) := \limsup_{\varepsilon\to 0^+} \frac{\log I_q(\mu,\varepsilon)}{-\log\varepsilon}, \quad \underline{\dim}_q(\mu) := \liminf_{\varepsilon\to 0^+} \frac{\log I_q(\mu,\varepsilon)}{-\log\varepsilon}. \tag{3.18}$$

If $q = 1$. Let $I_1(\mu,\varepsilon)$ defined by:

$$I_1(\mu,\varepsilon) = \inf\left\{\left[\prod_{i=1}^k \mu(S_i)^{-\mu(S_i)}\right] : \mu(\cup_i S_i) = 1, \operatorname{diam} S_i \leq 2\varepsilon\right\}.$$

We define now the so-called *information dimension*:

$$\overline{\dim}_1(\mu) := \limsup_{\varepsilon\to 0^+} \frac{\log I_1(\mu,\varepsilon)}{-\log\varepsilon}, \quad \underline{\dim}_1(\mu) := \liminf_{\varepsilon\to 0^+} \frac{\log I_1(\mu,\varepsilon)}{-\log\varepsilon}. \tag{3.19}$$

The following are easy remarks.

(1) It can be seen directly from the definition that the upper (lower) 0-Renyi dimension coincides with upper (lower) Minkowski dimension.

(2) $\dim_2(\mu)$ coincides with the correlation dimension defined by

$$\overline{\dim}_{\mathrm{corr}}(\mu) = \limsup_{\varepsilon\to 0^+} \frac{-\log\Big(\mu\times\mu(\{(x,y) : |x-y| < \varepsilon\})\Big)}{-\log\varepsilon},$$

$$\underline{\dim}_{\mathrm{corr}}(\mu) = \liminf_{\varepsilon\to 0^+} \frac{-\log\Big(\mu\times\mu(\{(x,y) : |x-y| < \varepsilon\})\Big)}{-\log\varepsilon}.$$

In general for q integer ≥ 2 $\dim_q(\mu)$ coincides with the q-correlation dimension

$$\overline{\dim}_q(\mu) = \limsup_{\varepsilon\to 0^+} \frac{-\log\Big(\underbrace{\mu\times\cdots\times\mu}_{q\text{ times}}\big(\{(x_1,\dots,x_q) : |x_i-x_j| < \varepsilon\ \forall i,j\}\big)\Big)}{-\log\varepsilon},$$

$$\underline{\dim}_q(\mu) = \liminf_{\varepsilon\to 0^+} \frac{-\log\Big(\underbrace{\mu\times\cdots\times\mu}_{q\text{ times}}\big(\{(x_1,\dots,x_q) : |x_i-x_j| < \varepsilon\ \forall i,j\}\big)\Big)}{-\log\varepsilon}.$$

(3) Directly from Lemma 3.10 it follows that for $q_1 \geq q_2$

$$\underline{\dim}_{q_1}(\mu) \leq \underline{\dim}_{q_2}(\mu), \quad \overline{\dim}_{q_1}(\mu) \leq \overline{\dim}_{q_2}(\mu). \tag{3.20}$$

We observe that for $q \geq 1$ the q-dimension has peculiar behavior, as the next lemma shows.

Lemma 3.11. *The following holds:*

(1) *If $q = 1$ and $\mu = \mu_1 + \mu_2$ with disjoint supports, then*

$$\overline{\dim}_1(\mu) = \mu_1(X)\overline{\dim}_1\left(\frac{\mu_1}{\mu_1(X)}\right) + \mu_2(X)\overline{\dim}_1\left(\frac{\mu_2}{\mu_2(X)}\right).$$

(2) *If $q > 1$ and there exists $\tilde{\mu} \leq \mu$ such that $\underline{\dim}_q(\tilde{\mu}) \leq \alpha$ ($\overline{\dim}_q(\tilde{\mu}) \leq \alpha$), then*

$$\underline{\dim}_q(\mu) \leq \alpha \quad (\overline{\dim}_q(\mu) \leq \alpha).$$

Proof. Point (1). From the definition, we have that for $\varepsilon < \mathrm{dist}(\mathrm{spt}\,\mu_1, \mathrm{spt}\,\mu_2)$

$$I_1(\mu, \varepsilon) = \left[\mu_1(X) I_1\left(\frac{\mu_1}{\mu_1(X)}, \varepsilon\right)\right]^{\mu_1(X)} \left[\mu_2(X) I_1\left(\frac{\mu_2}{\mu_2(X)}, \varepsilon\right)\right]^{\mu_2(X)},$$

from which it follows

$$\limsup_{\epsilon \to 0^+} \frac{\log I_1(\mu, \varepsilon)}{-\log \varepsilon} \leq \mu_1(X) \limsup_{\epsilon \to 0^+} \frac{\log I_1\left(\frac{\mu_1}{\mu_1(X)}, \varepsilon\right)}{-\log \varepsilon} + \mu_2(X) \limsup_{\epsilon \to 0^+} \frac{\log I_1\left(\frac{\mu_2}{\mu_2(X)}, \varepsilon\right)}{-\log \varepsilon}.$$

Point (2). Take a disjoint family of sets B_j of diameter $\leq 2r$ such that

$$\sum_j \tilde{\mu}(B_j)^q \geq (1 + \varepsilon)^{1-q} I_q(\tilde{\mu}, r)^{1-q}.$$

Since $\tilde{\mu}(A) \leq \mu(A)$ for all measurable sets A, then

$$(1 + \varepsilon)^{1-q} I_q(\tilde{\mu}, r)^{1-q} \leq \sum_j \tilde{\mu}(B_j)^q \leq \sum_j \mu(B_j)^q \leq I_q(\mu, r)^{1-q}.$$

It follows from $q > 1$ that

$$\frac{\log I_q(\mu, r)}{\log 1/r} \leq \frac{\log I(q, r; \tilde{\mu}) + \log(1 + \varepsilon)}{\log 1/r}.$$

Taking the limit for $r \to 0^+$ we obtain the conclusion. $\square$

In particular, if $\mu = c\delta_x + \nu$, then $\overline{\dim}_q(\mu) = 0$ for all $q > 1$. To compare $\dim_q(\mu)$ with $\dim_{\mathcal{H}}(\mu)$, we use the following easy lemma.

Lemma 3.12. *If $\sum_i a_i = 1$, $\sum_i a_i^q = A$ and $q < 1$, then by erasing a set of measure $B \leq Ac^{1-q}$, we obtain a sum with only a finite number $1/c$ of elements $\geq c$. Similarly, if $-\sum_i a_i \log a_i = A$, one can remove a set of measure $B \leq A/\log(1/c)$.*

Proof. We have the estimate

$$A \geq \sum_{a_i < c} a_i^q = \sum_{a_i < c} a_i \frac{1}{a_i^{1-q}} \geq \frac{B}{c^{1-q}}, \quad \text{i.e.} \quad B \leq Ac^{1-q},$$

and the number of the remaining elements is $\leq 1/c$, because $\sum_i a_i = 1$.

With similar computations, we have

$$\sum_{a_i < c} a_i = \sum_{a_i < c} a_i \log(1/a_i) \frac{1}{\log(1/a_i)} \leq \frac{1}{\log(1/c)} \sum_i a_i \log(1/a_i). \qquad \square$$

We deduce the following proposition.

Proposition 3.13. *For all probability measures μ we have* $\dim_{\mathcal{H}}(\mu) \leq \underline{\dim}_q(\mu)$, $0 \leq q < 1$.

Proof. For any fixed $0 \leq q < 1$, we take $r_i \leq 2^{-i}$ and a disjoint covering B_i with $\operatorname{diam} B_i \leq 2r_i$ and such that

$$\sum \mu(B_i)^q \leq r_i^{-(\underline{\dim}_q(\mu)+\varepsilon)(1-q)}.$$

Let $\beta = \underline{\dim}_q(\mu) + \varepsilon$, and use the above lemma with $A = r_i^{-\beta(1-q)}$, $c = r_i^{\beta+\varepsilon}$ so that the mass we remove is $B \leq r_i^{\varepsilon(1-q)}$ and the number of the remaining elements in the sum is $\leq r_i^{-\beta-\varepsilon}$. We thus conclude that the Hausdorff outer measure $\mathcal{H}^{\beta+2\varepsilon}_{r_i}$ of the set $\cup_{\mu(B_i)\geq c} B_i$ is

$$\mathcal{H}^{\beta+2\varepsilon}_{r_i}\left(\bigcup_{\mu(B_i)\geq c} B_i\right) \leq \sum_{\mu(B_i)\geq c} r_i^{\beta} \leq r_i^{-\beta-\varepsilon} r_i^{\beta+2\varepsilon} = r_i^{\varepsilon}.$$

We now use Lemma 3.4, replacing δ with $2^{-i\varepsilon(1-q)}$ so that $\mathcal{H}^{\underline{\dim}_q(\mu)+3\varepsilon}(\Gamma) = 0$ for all $\varepsilon > 0$: then $\dim_{\mathcal{H}}(\mu) \leq \underline{\dim}_q(\mu)$. □

Since $\dim_0(\mu) = \dim_M(\mu)$, the following corollary follows.

Corollary 3.14. *For all probability measure μ we have* $\dim_{\mathcal{H}}(\mu) \leq \underline{\dim}_M(\mu)$.

3.4. **Resolution dimension.** The *resolution dimension* was introduced by Devillanova and Solimini in [DS2]. Let $\mu \in \mathcal{P}(\Omega)$. Consider the set D_n of discrete measures ν with $\sharp \operatorname{spt} \nu \leq n$ and the minimization problem

$$W_p(\mu, D_n) := \min_{\nu \in D_n} W_p(\mu, \nu). \tag{3.21}$$

It is well-known (see, for example, [BJR, BW]) that if μ has a lower semicontinuous density f w.r.t. $\mathcal{L}^N$ and $p \geq 1$, then

$$\lim_{n\to+\infty} W_p(\mu, D_n) n^{\frac{1}{N}} = \theta_{N,p}\left(\int_\Omega f(x)^{\frac{N}{p+N}} \mathrm{d}x\right)^{\frac{p+N}{N}} > 0,$$

where $\theta_{N,p}$ is constant depending only on the dimension. It is then reasonable to consider the quantity given by

$$-\left(\limsup_{n\to+\infty} \frac{\log W_p(\mu, D_n)}{\log n}\right)^{-1}.$$

which turns out to be equal to N.

Definition 3.15 (Resolution dimension)**.** Let $\mu \in \mathcal{P}(\Omega)$ and $p > 0$, then the *upper resolution dimension* of μ of index p is given by

$$\overline{\dim}_{\mathcal{W}_p}(\mu) := -\min\{1, p\}\left(\limsup_{n\to+\infty} \frac{\log W_p(\mu, D_n)}{\log n}\right)^{-1}. \tag{3.22}$$

Similarly, the *lower resolution dimension* of μ of index p is given by

$$\underline{\dim}_{\mathcal{W}_p}(\mu) := -\min\{1, p\}\left(\liminf_{n\to+\infty} \frac{\log W_p(\mu, D_n)}{\log n}\right)^{-1}. \tag{3.23}$$

The estimates of page 18 provide the following proposition (see also Proposition 5.3 in [DS2]).

Proposition 3.16. *Let $\mu \in \mathcal{P}(\overline{\Omega})$ and let $p \leq q$. Then*

$$\underline{\dim}_{\mathcal{W}_p}(\mu) \leq \underline{\dim}_{\mathcal{W}_q}(\mu), \quad \overline{\dim}_{\mathcal{W}_p}(\mu) \leq \overline{\dim}_{\mathcal{W}_q}(\mu),$$

and if Ω is bounded

$$\underline{\dim}_{\mathcal{W}_q}(\mu) \leq \underline{\dim}_{\mathcal{W}_p}(\mu)\frac{p}{q}, \quad \overline{\dim}_{\mathcal{W}_q}(\mu) \leq \overline{\dim}_{\mathcal{W}_p}(\mu)\frac{p}{q}.$$

In particular, for any measure μ the maps $p \mapsto \overline{\dim}_{\mathcal{W}_p}(\mu)$, $p \mapsto \underline{\dim}_{\mathcal{W}_p}(\mu)$ are Lipschitz continuous and monotone increasing.

Remark 3.17. By Definition 3.15 it easily follows that given d, there exists N such that

$$W_p(\mu, D_n) \leq n^{-\frac{1}{d}} \text{ for all } n \geq N, \tag{3.24}$$

if $\min\{1,p\}d > \overline{\dim}_{\mathcal{W}_p}(\mu)$. On the other side, if $\min\{1,p\}d < \overline{\dim}_{\mathcal{W}_p}(\mu)$, for infinitely many n,

$$W_p(\mu, D_n) \geq n^{-\frac{1}{d}}.$$

Similar conditions hold when comparing $\min\{1,p\}d$ with $\underline{\dim}_{\mathcal{W}_p}(\mu)$.

The next proposition is contained in [DS2].

Proposition 3.18. *For all probability measures μ we have*

$$\dim_{\mathcal{H}}(\mu) \leq \underline{\dim}_{\mathcal{W}_p}(\mu) \leq \underline{\dim}_M(\mu), \quad \overline{\dim}_{\mathcal{W}_p}(\mu) \leq \overline{\dim}_M(\mu). \tag{3.25}$$

Moreover for $p = \infty$ the resolution dimension coincides with the Minkowski dimension.

We now compare the resolution dimension with the q-dimension for $q > 1$.

Lemma 3.19. *For $q > 1$ it holds*

$$\underline{\dim}_q(\mu) \leq \underline{\dim}_{\mathcal{W}_p}(\mu), \quad \overline{\dim}_q(\mu) \leq \overline{\dim}_{\mathcal{W}_p}(\mu).$$

Proof. We prove this lemma for the upper dimensions, the proof for the lower one being completely similar.

Assume $p \leq 1$ (for $p > 1$ the resolution dimension is greater than $p = 1$ by Proposition 3.16), and let $W_p(\mu, D_n) \leq n^{-\frac{(1-\varepsilon)p}{\overline{\dim}_{\mathcal{W}_p}(\mu)}}$. Let $B_r(x_{i,n})$, $i = 1, \dots, n$, be the balls centered at the atoms of $\mu_n \in D_n$. The definition of $I_q(\mu, r)$ and Jensen's inequality imply that for $q > 1$

$$\begin{aligned} I_q(\mu, r) &\leq \bigg(\sum_{i=1}^n \mu(B_r(x_{i,n}))^q \bigg)^{\frac{1}{1-q}} = \bigg(n \sum_{i=1}^n \mu(B_r(x_{i,n}))^q \frac{1}{n} \bigg)^{\frac{1}{1-q}} \\ &\leq \bigg(n^{1-q} \bigg(\sum_{i=1}^n \mu(B_r(x_{i,n})) \bigg)^q \bigg)^{\frac{1}{1-q}} = \mu\big(\cup_i B_r(x_{i,n}) \big)^{-\frac{q}{q-1}} n. \end{aligned}$$

The estimate on the Wasserstein distance implies that for

$$r \in \bigg[n^{-\frac{(1-2\varepsilon)}{\overline{\dim}_{\mathcal{W}_p}(\mu)}}, (n-1)^{-\frac{(1-2\varepsilon)}{\overline{\dim}_{\mathcal{W}_p}(\mu)}} \bigg)$$

the mass outside the balls $B_r(x_{i,n})$ is bounded by $n^{-\frac{\varepsilon p}{\overline{\dim}_{\mathcal{W}_p}(\mu)}}$, so that

$$\frac{\log I_q(\mu, r)}{\log 1/r} \leq \frac{-\frac{q}{q-1} \log \big(\mu\big(\cup_i B_r(x_{i,n}) \big) \big) + \log n}{\frac{(1-2\varepsilon)}{\overline{\dim}_{\mathcal{W}_p}(\mu)} \log(n-1)} \leq \frac{\frac{q}{q-1} n^{-\frac{\varepsilon p}{\overline{\dim}_{\mathcal{W}_p}(\mu)}} + \log n}{\frac{(1-2\varepsilon)}{\overline{\dim}_{\mathcal{W}_p}(\mu)} \log(n-1)}.$$

Taking first the limit for $n \to +\infty$ and then for $\varepsilon \to 0^+$ we obtain the conclusion. □

3.5. **Irrigation dimension.** For the definition of irrigation functional and irrigation dimension, we refer to [MMS, BCM1, DS1, DS2].

Consider $([0,1], \mathcal{B}, \mathcal{L}|_{[0,1]})$ and let $S \in \mathbb{R}^N$ be a given point of R^N.

Definition 3.20 (Set of fibers)**.** A *set of fibers* is a mapping

$$\chi : [0,1] \times [0,+\infty) \to \mathbb{R}^N$$

such that:

(1) for μ-almost-every $\omega \in [0,1]$, the curve given by χ_ω

$$t \mapsto \chi_\omega(t) := \chi(\omega, t)$$

is Lipschitz continuous and $\mathrm{Lip}(\chi_\omega) \le 1$;

(2) χ is measurable.

We will denote by $\mathbf{P}$ the set of such functions.

A *set of fibers with source* S is a set of fiber χ such that

(3) $\chi_\omega(0) = S$ for all $\omega \in [0,1]$.

We will denote by $\mathbf{P}_S$ the set of such functions.

Definition 3.21 (χ-vessels at time t)**.** Given $t \in [0,+\infty)$, the χ-vessels at time t will be the equivalence classes of the equivalence relation defined by:

$$\omega_1 \simeq_t \omega_2 \iff \chi_{\omega_1} = \chi_{\omega_2} \text{ on } [0,t].$$

Definition 3.22 (Absorption time)**.** Given $\chi \in \mathbf{P}$, the function $\sigma_\chi : [0,1] \to [0,+\infty)$ given by

$$\sigma_\chi(\omega) := \inf\{t \in [0,+\infty) \ : \ \chi_\omega \text{ constant on } [t,+\infty]\}$$

is the *absorption time*. A point $\omega \in [0,1]$ is *absorbed* if $\sigma(\omega) < +\infty$, while it is *absorbed at time* t if $\sigma(\omega) \le t$. We will denote by $A_t(\chi)$ the set of absorbed points at time t:

$$A_t(\chi) := \{\omega \in [0,1] \ : \ \sigma_\chi(\omega) \le t\}, \tag{3.26}$$

and by $M_t(\chi)$ its complementary:

$$M_t(\chi) := [0,1] \setminus A_t(\chi) = \{\omega \in [0,1] \ : \ \sigma_\chi(\omega) > t\}. \tag{3.27}$$

Set $A_\chi = \cup_t A_t(\chi) \subseteq [0,1]$ as the *set of absorbed points*, and define the *irrigation function* $i_\chi : A_\chi \mapsto \mathbb{R}^N$ as $i_\chi(p) = \chi(p, \sigma_\chi(p))$. The *irrigated measure* is defined as $\mu = (i_\chi)_\sharp \mathcal{L}|_{[0,1]}$.

Let $\alpha \in [0,1]$. The *irrigation cost* $I_\alpha(\chi)$ is the functions

$$I_\alpha(\chi) := \int_0^{+\infty} \left[\int_{M_t(\chi)} [\mathcal{L}([\omega]_t)]^{\alpha-1} \, \mathrm{d}\omega \right] \mathrm{d}t. \tag{3.28}$$

We say that a measure ν is *α-irrigable* if there exists a set of fiber $\chi \in \mathbf{P}_S$ such that $I_\alpha(\chi) < \infty$ and $(i_\chi)_\sharp \mu = \nu$. As before, this definition does not depend on the point S.

Definition 3.23. The irrigation dimension $\dim_{irr}(\mu)$ is

$$1 - \frac{1}{\dim_{irr}(\mu)} = \inf\big\{\alpha : \mu \text{ is } \alpha \text{ irrigable}\big\}. \tag{3.29}$$

We recall the following result for irrigation dimension (actually it is a simple extension of Theorem 1.1 of [DS2]).

Proposition 3.24. *The following estimates hold:*

1. $\dim_{\mathcal{H}}(\mu) \leq \dim_{irr}(\mu) \leq \max\{\overline{\dim}_M(\mu), 1\}$;
2. $\dim_{irr}(\mu) = \overline{\dim}_{\mathcal{W}_p}(\mu)$ *for the* $p \in [1,\infty]$ *solution to* $\overline{\dim}_{\mathcal{W}_p}(\mu) = p/(p-1)$;
3. $\dim_{irr}(\mu) \geq \overline{\dim}_q(\mu)$ *for all* $q > 1$.

Proof. Point (1) is Theorem 1.1 and Point (2) is Theorem 8.1 of [DS2], and Point (3) follows from Point (2) and Lemma 3.19. □

4. Upper and lower bounds for path dimension

In this section we compare the path dimension with the dimensions introduced in the previous section. As one can expect, the first result is that $\dim_{\mathcal{H}}(\mu) \leq \dim_{path,p}(\mu)$ for all $p > 1$.

Proposition 4.1. *If* $\mathcal{G}_{p,r}(\mu) < \infty$, *then there exists* B *such that* $\mathcal{H}^{\frac{\min\{1,p\}}{1-r}}(B) = 0$ *and* μ *is concentrated on* B.

In particular we conclude that $\dim_{\mathcal{H}}(\mu) \leq \dim_{path,p}(\mu)$.

Proof. As usual we assume that $p \leq 1$.

Step 1. We first observe that if $\mathcal{G}_{p,r}(\mu) < \infty$, $p \leq 1$, it follows that there are μ_i, $i \in \mathbb{N}$, such that

$$W_p(\mu, \mu_i) \leq 2^{-i} \quad \text{and} \quad \sum_{i=1}^{+\infty} 2^{-i} G_r(\mu_i) \leq +\infty.$$

Hence we can choose indexes $i_k \geq 2k$ such that $\mu_{i_k} = \sum_j \mu_{i_k j} \delta_{x_{i_k j}}$ and

$$2^{-i_k} \sum_j \mu_{i_k j}^r \leq 2^{-k} \quad \text{and} \quad W_p\Big(\mu, \sum \mu_j \delta_{x_j}\Big) \leq 2^{-i_k},$$

because $2^{-i} G_r(\mu_i) \to 0$.

Step 2. By Lemma 3.12, it follows that the total mass B_k of atoms of μ_{i_k} with mass $\leq c$ is bounded by

$$B_k = \sum_{\mu_{i_k j} < c} \mu_{i_k j} < c^{1-r} \sum_j \mu_j^r \leq c^{1-r} 2^{i_k - k} \leq 2^{-\frac{k}{3}}$$

when $c = c_k = 2^{\frac{k-i_k}{1-r}} 2^{-\frac{k}{3(1-r)}}$.

Step 3. Since $W_p(\mu, \nu_{i_k}) \leq 2^{-i_k}$, the measure outside the balls of radius $r_k = 2^{-\frac{i_k}{p} + \frac{k}{3p}}$ is bounded by $2^{-\frac{k}{3}}$. Hence, by Step 1., the mass of μ outside the balls of radius $2^{-\frac{i_k}{p} + \frac{k}{3p}}$ centered at the points with mass greater than $c_k = 2^{\frac{k-i_k}{1-r} - \frac{k}{3(1-r)}}$ is bounded by $2^{-2k/3}$.

Step 4. If follows that the set

$$\Gamma_K = \bigcap_{k \geq K} \bigcup_{j : \mu_{i_k j} \geq c_k} B(x_{i_k j}, r_k)$$

measures $\mu(\Gamma) \geq 1 - \mathcal{O}(1) 2^{-2K/3}$ and its $\mathcal{H}^\alpha_{r_k}$ outer measure is

$$\mathcal{H}^\alpha_{r_k}(\Gamma) \leq \frac{1}{c_k} r_k^\alpha = 2^\alpha 2^{-\frac{k-i_k}{1-r} + \frac{k}{3(1-r)}} 2^{(-\frac{i_k}{p} + \frac{k}{3p})\alpha}$$

which tends to 0 for $\alpha \geq \frac{p}{1-r}$. We the use Lemma 3.4 to conclude. □

We then have the following proposition for the comparison with Minkowski dimension.

Proposition 4.2. *The following holds:*
(1) $\dim_{path,p}(\mu) \le \max\{1, \overline{\dim}_M(\mu)\}$;
(2) $\underline{\dim}_q(\mu) \le \dim_{path,p}(\mu)$ *for* $q > 1$.
(3) *for* $p = \infty$ *we have* $\dim_{path,\infty}(\mu) = \overline{\dim}_q(\mu)$ *for the* q *such that*

$$\overline{\dim}_q(\mu) = \frac{1}{1-q}.$$

Proof. Point (1). From the definition of Minkowski dimension we have that the number n_i of ball $B_{2^{-i}}(x_j)$, $j = 1, \dots, n_i$, covering $\operatorname{spt}\mu$ is bounded by

$$n_i \le 2^{i(\overline{\dim}_M(\mu)+\varepsilon)}$$

for all $\varepsilon > 0$ for $i \ge \bar{i}$ sufficiently large.

Let $\nu_i = \sum_{j=1}^{n_i} \nu_j \delta_{x_j}$ be an atomic measure such that $W_\infty(\mu, \nu_i) \le 2^{-i}$ (i.e. the mass travels at most of 2^{-i}). Theorem 2.13 yields that for a large constant

$$\begin{aligned}\mathcal{G}_{p,r}(\mu) &\le \mathcal{O}(1) + 2\sum_{i=\bar{i}}^{+\infty} 2^{-i\min\{1,p\}} 2^{i(\overline{\dim}_M+\varepsilon)(1-r)} \\ &= \mathcal{O}(1) + 2\sum_{i=\bar{i}}^{+\infty} 2^{-i(\min\{1,p\}-(\overline{\dim}_M+\varepsilon)(1-r))}.\end{aligned}$$

The conclusion follows because the series is converging for

$$\frac{\min\{1,p\}}{1-r} \ge \overline{\dim}_M(\mu)(1+\varepsilon)$$

and ε is arbitrary.

Point (2). By Theorem 2.13, consider a sequence ν_i, $i \in \mathbb{N}$, of measures such that for $p \le 1$

$$W_p(\mu, \nu_i) \le 2^{-i}, \quad \sum_i 2^{-i} G_r(\nu_i) < +\infty.$$

The measure ν_i is clearly purely atomic, and the estimate on the Wasserstein distance yields that the measure outside the balls centered at the atoms of $\nu_i = \sum_j \nu_{ij}\delta_{x_{ij}}$ and of radius $2^{-i(1-\varepsilon)/p}$ is bounded by $2^{-i\varepsilon}$.

Restricting the measure μ to the set

$$\Gamma = \bigcap_i \bigcup_j B(x_{ij}, 2^{-i(1-\varepsilon)/p}),$$

we thus remove a total mass $\sum_i 2^{-i\varepsilon} < 1$ and we have the estimate

$$I_r(\mu|_\Gamma, 2^{-i(1-\varepsilon)/p}) \le \left[\sum_j \nu_{ij}^r\right]^{\frac{1}{1-r}} = \big(G_r(\nu_i)\big)^{\frac{1}{1-r}} \le C 2^{\frac{i}{1-r}},$$

where we used the fact that $\mu(B_{2^{-i(1-\varepsilon)/p}}(x_j)) \le \nu_{ij}$.

The definition of r-dimension yields

$$\dim_r(\mu|_\Gamma) \le \frac{p}{(1-\varepsilon)(1-r)},$$

and letting $\varepsilon \to 0^+$ one conclude that $\dim_r(\mu|_\Gamma) \le \dim_{path,p}(\mu)$. By Lemma 3.11 and the monotonicity of $\dim_q(\mu)$ w.r.t. q, the conclusion follows.

Point (3). For $p = \infty$, we do not need to remove the mass outside the balls of radius 2^{-i}. Then

$$\inf \big\{ G_q(\nu), W_\infty(\mu,\nu) \leq r \big\} = I_q(\mu, r)^{1-q}$$

and thus $I_q(\mu, r) \leq \frac{C}{r^{1/(1-q)}}$ if μ is (q,∞)-reachable or there exists ν_i, $i \in \mathbb{N}$, such that $G_q(\nu_i) \leq 2^{i(1-\varepsilon)}$ if $\overline{\dim}_q(\mu) < \frac{1-\varepsilon}{1-q}$.

In both cases we conclude that $\max\{1, \overline{\dim}_q(\mu)\} = \dim_{path,\infty}(\mu)$. □

We now compare with the resolution dimension.

Proposition 4.3. *We have* $\dim_{path,p}(\mu) \leq \max\{1, \overline{\dim}_{\mathcal{W}_p}(\mu)\}$. *Moreover, if it is reachable for* $r = 0$, *then the lower resolution dimension is* ≤ 1.

Proof. Let $\nu_n \in D_n$ be the atomic measure minimizing $W_p(\mu, D_n)$. From the concavity of G_r it follows that $G_r(\nu_n) \leq n^{1-r}$, and the definition of resolution dimension yields that for any $\varepsilon > 0$ $W_p(\mu,\nu_n) \leq n^{-\frac{1}{\dim_{\mathcal{W}_p}(\mu)+\epsilon}}$ for $n \geq N(\varepsilon)$.

Hence we conclude with

$$\sum_i 2^{-i} G_r(\nu_i) \leq \sum_i 2^{-i} 2^{i(\overline{\dim}_{\mathcal{W}_p}+\varepsilon)(1-r)} < \infty$$

for all $r > 1 - \frac{1}{\max\{1, \overline{\dim}_{\mathcal{W}_p}(\mu)+\varepsilon\}}$.

The last part follows because $G_0(\mu) = \sharp\,\mathrm{spt}(\mu)$ counts exactly the number of Dirac masses, so that if there exists ν_i such that

$$W_p(\mu,\nu_i) \leq 2^{-i}, \quad \sum_i 2^{-i} G_0(\nu_i) < +\infty,$$

then $\sharp\,\mathrm{spt}\,\nu_i \leq 2^i$ and

$$\underline{\dim}_{\mathcal{W}_p}(\mu) \leq \lim_{i\to+\infty} \frac{\log 2^i}{\log 1/2^{-i}} = 1.$$

□

We can consider the path dimension as an average of the upper and lower resolution dimension.

We finally compare with the irrigation dimension.

Proposition 4.4. *If the measure is* (r,p)*-reachable with* $p > 1$, *then it is irrigable, and the following estimates holds*

$$\dim_{irr}(\mu) \leq \frac{p}{p-1} \dim_{path,p}(\mu). \tag{4.1}$$

Proof. Let n_i be a sequence of measures such that

$$W_p(\mu,\nu_i) \leq 2^{-i}, \quad \sum_i 2^{-i} G_r(\nu_i) < +\infty.$$

The measures ν_i are clearly purely atomic.

We estimate the irrigation cost of the optimal transport considered in Proposition 2.8: the irrigation cost $I_{r'}(\nu_i, \nu_{i+1})$ from ν_i to ν_{i+1} is bounded by

$$\begin{aligned} I_{r'}(\nu_i,\nu_{i+1}) &\le \sum_{ij} \pi_{ij}^{r'} |x_i - y_j| \le \Big(\sum_{ij} \pi_{ij}^{\frac{r'-\alpha}{1-\alpha}} \Big)^{1-\alpha} \Big(\sum_{ij} \pi_{ij} |x_i - y_j|^{\frac{1}{\alpha}} \Big)^{\alpha} \\ &\le \big(G_r(\nu_i) + G_r(\nu_{i+1}) \big)^{1-\alpha} W_{\frac{1}{\alpha}}(\nu_i, \nu_{i+1}) \le C 2^{i(1-\alpha)} W_p(\nu_i,\nu_{i+1})^{\alpha p} \\ &\le C 2^{-i(\alpha(1+p)-1)} \end{aligned}$$

when α satisfies

$$\alpha \ge \frac{1}{p}, \quad \frac{r'-\alpha}{1-\alpha} = r.$$

Hence we conclude that the irrigation cost of μ (obtained by adding all the irrigation costs of the path $[\nu_i, \nu_{i+1}]$) is bounded if

$$\frac{1}{1-r'} = \frac{1}{1-\alpha} \frac{1}{1-r} \quad \text{and} \quad \alpha \ge \frac{1}{p} > \frac{1}{1+p}.$$

□

4.1. **Special distributed measures.** As one can see from the proofs, the main difficulty in comparing the various dimensions arises from the fact that the measures do not need to be uniformly distributed. A great simplification is to consider measures so that for μ-almost-all points x

$$\frac{1}{C} r^{\bar\beta} \le \mu(B_r(x)) \le C r^{\underline{\beta}} \tag{4.2}$$

with C independent on x.

We first define the local dimension.

Definition 4.5. The upper/lower local dimension of μ at x are given by

$$\underline{\dim}_{loc}(\mu,x) = \limsup_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}, \quad \overline{\dim}_{loc}(\mu,x) = \liminf_{r\to 0^+} \frac{\log \mu(B(x,r))}{\log r}. \tag{4.3}$$

Under the assumption (4.2), the local dimensions satisfy uniform estimates for almost all points.

Theorem 4.6. *Under the assumption* (4.2)*, the following estimates hold:*

(1) $\underline{\beta} \le \mathcal{H}(\mu) \le \bar\beta$*;*
(2) $\underline{\beta} \le \underline{\dim}_q(\mu) \le \overline{\dim}_q(\mu) \le \bar\beta$ *for all* $q \in [0,+\infty]$*;*
(3) $\underline{\beta} \le \underline{\dim}_{\mathcal{W}_p}(\mu) \le \overline{\dim}_{\mathcal{W}_p}(\mu) \le \bar\beta$*;*
(4) $\underline{\beta} \le \dim_{irr}(\mu) \le \bar\beta$*;*
(5) $\underline{\beta} \le \dim_{path,p}(\mu) \le \bar\beta$*.*

Proof. Using Proposition 3.18, Proposition 3.24 and Proposition 4.2, Points (3), (4) and (5) are a consequence of Points (1), (2).

Point (1). We can use the estimates in the proof of Lemma 3.3 to conclude immediately.

Point (2). For the lower bound, we consider $q = \infty$, so that

$$I(\infty, r) \ge \frac{1}{C r^{\underline{\beta}}}.$$

This implies that $\dim_\infty(\mu) \geq \underline{\beta}$, and the monotonicity of $\dim_q$ yields $\dim_q(\mu) \geq \underline{\beta}$ for all $q \in [0, \infty]$.

For the upper estimate, we consider the case $q = 0$, and we use the Besicovich covering theorem (Theorem 2.18 of [AFP]) to find a finite number ξ of disjoint finite families of balls $B_r(x_{ij})$ such that they cover the support of μ. It follows that the number of these balls is at most

$$\xi \geq \sum_{j=1}^{\xi} \sum_i \mu(B_i) \geq \frac{r^{\bar{\beta}}}{C} \sum_j \sharp_i \{B_r(x_{ij})\} = \frac{r^{\bar{\beta}}}{C} \sharp_{ij} \{B_r(x_{ij})\},$$

i.e. we need less than $\mathcal{O}(1) r^{-\bar{\beta}}$ balls of radius r to cover $\operatorname{spt}\mu$, so that $\dim_0(\mu) = \dim_M(\mu) \leq \bar{\beta}$. $\square$

In particular we have that for *Ahlfors regular measures* $\beta = \underline{\beta} = \bar{\beta}$.

Corollary 4.7. *For Ahlfors regular measures we have that all dimensions coincide.*

To end this section, we give a summarizing table: the dimension on each line is compared to the dimension on each column.

	$\dim_{\mathcal{H}}(\mu)$	$\dim_q(\mu)$	$\dim_{\mathcal{W}_p}(\mu)$	$\dim_{irr}(\mu)$	$\dim_{path,p}(\mu)$
$\dim_{\mathcal{H}}(\mu)$		$\leq$ for $q < 1$	$\leq$	$\leq$	$\leq$
$\dim_q(\mu)$	$\geq$ for $q < 1$		$\geq$ for $q = 0$ $\leq$ for $q > 1$	$\geq$ for $q = 0$ $\leq$ for $q > 1$	$\geq$ for $q = 0$ $\leq$ for $q > 1$
$\dim_{\mathcal{W}_p}(\mu)$	$\geq$	$\leq$ for $q = 0$ $\geq$ for $q > 1$		$= \frac{p}{p-1}$	$\geq$
$\dim_{irr}(\mu)$	$\geq$	$\leq$ for $q = 0$ $\geq$ for $q > 1$	$= \frac{p}{p-1}$		$\leq \frac{p}{p-1} \dim_{path,p}$
$\dim_{path,p}(\mu)$	$\geq$	$\leq$ for $q = 0$ $\geq$ for $q > 1$	$\leq$	$\geq \frac{p-1}{p} \dim_{irr}$	

We conclude this section with a simple example, where not all computations are given.

Example 4.8. In the space $\mathbb{R}^N$, consider the atomic measure

$$\mu(\{x\}) = (1 - a^{-1})(2Na)^{-i} \quad x = \sum_{1 \leq j \leq i} b^{-j}(\pm e_k), \quad a > 1, b > 2, \tag{4.4}$$

where e_k are the unit vector of the coordinate axis. To be more clear, we write the first elements of μ:

$$\begin{aligned} \mu = (1 - a^{-1})\delta(x) &+ \frac{1 - a^{-1}}{2Na} \sum_{k=1}^N \big[\delta(x - be_k) + \delta(x + be_k)\big] \\ &+ \frac{1 - a^{-1}}{4N^2a^2} \sum_{k=1}^N \sum_{\ell=1}^N \Big[\delta(x - be_k - b^2 e_\ell) + \delta(x - be_k + b^2 e_\ell) \\ &\qquad + \delta(x + be_k - b^2 e_\ell) + \delta(x + be_k + b^2 e_\ell)\Big] + \dots \end{aligned}$$

We consider now the measure ν obtained by truncating the above sum at a index j and rescaling the measure in order to obtain a probability. The cost of transporting the remaining mass to μ is given by

$$W_p(\mu,\nu)^{\max\{1,p\}} = \sum_{i>j}(1-a^{-1})a^{-i}\Big|\sum_{j<k\le i} \pm e_k b^{-k}\Big|^p \simeq a^{-j}b^{-jp}.$$

The cost function is then

$$G_r(\nu) = \sum_{i\le j}(1-a^{-1})^r(2Na)^{-ir}(2N)^i \simeq \sum_{i\le j}\left(\frac{(2N)^{1-r}}{a^r}\right)^i \simeq \left(\frac{(2N)^{1-r}}{a^r}\right)^j,$$

where we assumed that $a \le (2N)^{1/r-1}$, otherwise the sum is converging and the measure μ is (r,p)-reachable. The condition of Theorem 2.13 is then

$$\sum_j \left(\frac{(2N)^{1-r}}{b^{\min\{1,p\}}a^{\min\{1,1/p\}+r}}\right)^j < \infty,$$

so that one obtains

$$r > 1 - \frac{\min\{1,p\}\log b + (1+\min\{1,1/p\})\log(a)}{\log(2Na)}.$$

It follows that the **path functional dimension** is

$$\dim_{path,p}(\mu) = \frac{\min\{1,p\}\log(2Na)}{\min\{1,p\}\log b + (1+\min\{1,1/p\})\log a},$$

and we have in particular that $p = 0$ $\dim_{path,0}(\mu) = 0$ and for $p = +\infty$ $\dim_{path,\infty}(\mu) = \log(2Na)/\log(ab)$.

We next estimate various dimensions of this measure.

Hausdorff dimension: $\dim_{\mathcal{H}} = 0$.

Renyi dimension: we need the same number of balls $\simeq (2N)^i$ of radius b^{-i},

$$\sum \mu_i^q \simeq (2N)^{i(1-q)}a^{-iq} \quad \text{for} \quad q < \frac{\log(2N)}{\log(2Na)}.$$

The dimension is then

$$\dim_q(\mu) = -\frac{1}{1-q}\lim_i \frac{\log((2N)^{i(1-q)}a^{-iq})}{\log b^{-i}} = \frac{\log(2N) - \frac{q}{1-q}\log a}{\log b}.$$

For $q = 0$ we reduce to the Minkowski dimension

$$\dim_M(\mu) = \frac{\log(2N)}{\log b}.$$

Note that for $q \ge \frac{\log(2N)}{\log(2Na)}$ one has $\dim_q(\mu) = 0$.

Resolution dimension: we need $(2N)^i$ balls if we want to be $(a^{1/p}b)^{-i}$ close in Wasserstein distance p. Thus the dimension is

$$\lim_i \log_{(a^{1/p}b)^i}(2N)^i = \frac{\min\{1,p\}\log(2N)}{\min\{1,1/p\}\log a + \min\{1,p\}\log b} = \dim_{\mathcal{W}_p}(\mu).$$

Irrigation dimension: by considering the tree in the definition of the support of μ, we have that each branch of length b^{-i} costs $\simeq (2Na)^{-i\alpha}$, so that since there are $2N$ branches the reachability condition is

$$\sum \left(\frac{(2Na)^{1-\alpha}}{ab} \right)^i < \infty,$$

which gives $\alpha > 1 - \log(ab)/\log(2Na)$, and the dimension is

$$\dim_{irr}(\mu) = \frac{\log(2Na)}{\log(ab)}.$$

This coincides with the $p = \infty$ path dimension.

References

[AT] L. Ambrosio and P. Tilli. *Topics on analysis in metric spaces.* Oxford Lecture Series in Mathematics and its Applications, 25. Oxford University Press, Oxford, 2004.

[AFP] L. Ambrosio, N. Fusco and D. Pallara, *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs, 2000.

[B] L. Brasco. *Curves of minimal action in metric spaces.* Preprint.

[BBS] A. Brancolini, G. Buttazzo and F. Santambrogio. *Path functionals over Wasserstein spaces.* J. Eur. Math. Soc. (JEMS) 8 (2006), no. 3, 415–434.

[BCM1] M. Bernot, V. Caselles and J.M. Morel. *Traffic plans*, Publ. Mat. 49 (2005), no. 2, 417–451.

[BCM2] M. Bernot, V. Caselles and J.M. Morel. *The structure of branched transportation networks.* Calc. Var. Partial Differential Equations 32 (2008), no. 3, 279–317.

[BCM3] M. Bernot, V. Caselles and J.M. Morel. *Optimal transportation networks. Models and theory.* Lecture Notes in Mathematics, 1955. Springer-Verlag Berlin, 2009.

[BJR] G. Bouchitté, C. Jimenez and M. Rajesh. *Asymptotique d'un problème de positionnement optimal.* C. R. Acad. Sci. Paris, Ser. I, 335: 1–6, 2002.

[BW] J. A. Bucklew, G. L. Wise. *Multidimensional asymptotic quantization theory with rth power distortion measures.* IEEE Trans. Inform. Theory 28 (1982), no. 2, 239–247.

[DT1] G. B. Dantzig and M. N. Thapa. *Linear Programming, 1: Introduction.*

[DT2] G. B. Dantzig and M. N. Thapa. *Linear Programming, 2: Theory and Extensions.*

[DS1] G. Devillanova and S. Solimini. *Elementary properties of optimal irrigation patterns*, Calc. Var. Partial Differential Equations 28 (2007), no. 3, 317–349.

[DS2] G. Devillanova and S. Solimini. *On the dimension of an irrigable measure.* Rend. Semin. Mat. Univ. Padova 117 (2007), 1–49.

[HW] K. Hestir and S.C. Williams. *Supports of double stochastic measures.* Bernoulli 3 (1995), 217–243.

[LE] A. Lewis. *Extreme points of infinite transportation problems.* Methods Operating Research 53 (1986), 115–126.

[MMS] F. Maddalena, S. Solimini and J.M. Morel. *A variational model of irrigation patterns*, Interfaces Free Bound. 5 (2003), no. 4, 391–415.

[V] C. Villani, *Topics in Optimal Transportation.* Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.

[X1] Q. Xia, *Optimal Path related to Transport Problems.* Commun. Contemp. Math. 5 (2003), no. 2, 251–279.

[X2] Q. Xia, *Interior regularity of optimal transport paths.* Calc. Var. Partial Differential Equations 20 (2004), no. 3, 283–299.

SISSA, Functional Analysis Sector, Via Beirut 2-4, 34014 Trieste
E-mail address: bianchin@sissa.it

Politecnico di Bari, Dipartimento di Matematica, Via Amendola 126/b, 70126 Bari
E-mail address: a.brancolini@poliba.it