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Ribbon graph minors and low-genus partial duals

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Abstract. We give an excluded minor characterisation of the class of ribbon graphs that admit partial duals of Euler genus at most one.

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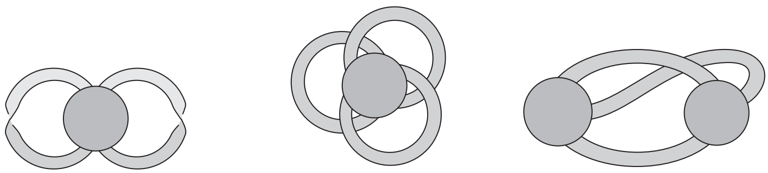
1. Statement of results

Here we are interested in minors of ribbon graphs. For ribbon graph minors it is necessary to contract loops and doing so can create additional vertices and components [1, 2]. Thus there is a fundamental difference between graph and ribbon graph minors. While it is known that every minor-closed family of graphs can be characterised by a finite set of excluded minors [9], the corresponding result for ribbon graphs and their minors is currently only a conjecture [7]. In this note we provide some support for this conjecture by giving an explicit list of excluded minors for an interesting class of ribbon graphs: those that admit partial duals of low genus. Our main result is the following.

Theorem 1.1. *Let X_1–X_3 be the ribbon graphs of Figure 1. Then a ribbon graph has a partial dual of Euler genus at most one if and only if it has no ribbon graph minor equivalent to X_1, X_2, or X_3.*

Although we assume a familiarity with ribbon graphs, ribbon graph minors and partial duals (see, for example, [4] for a leisurely overview of these topics, or Sections 2 and 3.2 of [7] for a brief review), we briefly recall that the partial dual G^A of a ribbon graph G is the ribbon graph obtained by, roughly speaking, forming the geometric dual of G but only with respect to a given set A of edges of G. Partial duality was defined by Chmutov in [2]. It appears to be a funtamental construction, arising as a natural operation in knot theory, topological graph theory, graph polynomials, matroid theory,



(A) X_1. (B) X_2. (C) X_3.

FIGURE 1. The excluded minors.

and quantum field theory. The *Euler genus* of a connected ribbon graph G is its genus if it is non-orientable, and twice its genus otherwise. For non-connected ribbon graphs it is defined as the sum of the Euler genera of its components. A connected ribbon graph of Euler genus 0 is a *plane ribbon graph*, and of Euler genus 1 is an $\mathbb{R}\mathrm{P}^2$ *ribbon graph.*

Our approach to Theorem 1.1 is as follows. We use a compatibility between partial duals and minors (Equation 2.1) to reduce the proof of the theorem to showing that a bouquet (i.e., a 1-vertex ribbon graph) whose partial duals are all of Euler genus at least two contains X_1 or X_2 as a minor. We then apply a 'rough structure theorem' (Theorem 2.1) that tells us that such a ribbon graph G does not have a particular type of decomposition into a set of plane graphs and one $\mathbb{R}\mathrm{P}^2$ graph. X_1 is an obstruction to finding an appropriate $\mathbb{R}\mathrm{P}^2$ graph ribbon subgraph in G for the decomposition, X_2 is an obstruction to finding an appropriate set of plane ribbon subgraphs for the decomposition, and, somewhat surprisingly, X_1 is also an obstruction to plane and $\mathbb{R}\mathrm{P}^2$ subgraphs fitting together in a way required by the rough structure theorem to allow for a low-genus partial dual.

2. Proof of the main theorem

The following constructions and results will be needed for the proof of Theorem 1.1. A vertex v of a ribbon graph G is a *separating vertex* if there are non-trivial ribbon subgraphs G_1 and G_2 of G such that $G = G_1 \cup G_2$ and $G_1 \cap G_2 = \{v\}$. Let G be a ribbon graph and $A \subseteq E(G)$. We use $G|_A$ to denote the restriction of G to A (i.e., the ribbon subgraph with edge set A and vertices incident to edges in A), and A^c to denote the complement of A in $E(G)$. Following [6], we say that A *defines a biseparation* of G if every vertex of G that is in both $G|_A$ and $G|_{A^c}$ is a separating vertex of G. If, in addition, every component of $G|_A$ and of $G|_{A^c}$ is plane then we say A defines a *plane-biseparation*; and if exactly one component of $G|_A$ or of $G|_{A^c}$ is $\mathbb{R}\mathrm{P}^2$ and all of the other components are plane then A defines an $\mathbb{R}\mathrm{P}^2$*-biseparation.*

We will use the following rough structure theorem for ribbon graphs with low genus partial duals from [6] (the plane case first appeared in [5]).

Theorem 2.1. *Let G be a connected ribbon graph and $A \subseteq E(G)$. Then G^A is a plane ribbon graph (respectively, $\mathbb{R}\mathrm{P}^2$ ribbon graph) if and only if A defines a plane-biseparation (respectively, $\mathbb{R}\mathrm{P}^2$-biseparation) of G.*

We will also need the following result from [7] that says the partial duals of minors are the minors of a partial dual. For a ribbon graph G and $A \subseteq E(G)$, we have

$$\{J^A \mid J \text{ is a minor of } G\} = \{H \mid H \text{ is a minor of } G^A\}. \qquad (2.1)$$

Here we use the convention that if H is a minor of G, and $A \subseteq E(G)$, then by H^A we mean $H^{A\cap E(H)}$.

A *bouquet* is a ribbon graph that has exactly one vertex. Two edges e and f in a bouquet are *interlaced* if their ends are met in the cyclic order $e\,f\,e\,f$ when travelling round the boundary of the unique vertex of G. The *intersection graph* $\mathcal{I}(G)$ of a bouquet G is the vertex weighted simple graph whose vertices set is $E(G)$ and which two vertices e and f of $\mathcal{I}(G)$ are adjacent if and only if the edges e and f are interlaced in G. A vertex e of $\mathcal{I}(G)$ has weight "+" if e is an orientable loop in G, and has weight "−" if it is non-orientable.

Proof of Theorem 1.1. For one implication observe that X_1–X_3 do not have partial duals of Euler genus less than 1. It was shown in [7] that for each $k \in \mathbb{N}_0$ the set of all ribbon graphs that have a partial dual of Euler genus at most k is minor-closed, and so X_1–X_3 cannot be minors of ribbon graphs that have partial duals of Euler genus less than 1.

For the converse we prove that if a ribbon graph does not admit a plane- or $\mathbb{R}\mathrm{P}^2$-biseparation, then it has an X_1–X_3-minor. The result then follows from Theorem 2.1. If G is an orientable ribbon graph that does not admit a plane-biseparation then, from [5], it contains X_2 or X_3 as a minor. Now suppose that G is a non-orientable ribbon graph that does not admit an $\mathbb{R}\mathrm{P}^2$-biseparation. To complete the proof we need to show that G has an X_1–X_3-minor. By Equation (2.1) it is enough to show that when G is a bouquet it has X_1 or X_2 as a minor. (To see why, suppose G has no $\mathbb{R}\mathrm{P}^2$-biseparation, and let T be the edge set of a spanning tree of G. Then G^T is a bouquet. If G^T has X_1 or X_2 as a minor, by Equation 2.1, $G = (G^T)^T$ has X_1^T or X_2^T as a minor. Finally, $X_1^T = X_1$, and $X_2^T = X_2$ or $X_2^T = X_3$.) Assume that G is a non-orientable bouquet that does not admit an $\mathbb{R}\mathrm{P}^2$-biseparation. We can write $G = G_O \sqcup G_N$ where G_O is the ribbon subgraph of G consisting of all of the orientable loops, and G_N the non-orientable loops. If G_N has two edges that are not interlaced then these two edges induce an X_1-minor and we are done. Otherwise G_N consists of $q \geq 1$ non-orientable loops that all interlace each other (i.e., the edges are met in the cyclic order $1\,2\cdots q\,1\,2\cdots q$). Assume this is the case.

Next consider G_O. Suppose the intersection graph $\mathcal{I}(G_O)$ is not bipartite. It then contains an odd cycle of length at least 3. This odd cycle corresponds to a ribbon subgraph of G that is an orientable bouquet with $p \geq 3$ edges, for p odd, that meet the vertex in the cyclic order $2\,1\,3\,2\,4\,3\cdots p\,(p-$

1) 1 p. Contracting the edges corresponding to $4, 5, \ldots, p$ results in a copy of X_2 (see the example in [7]).

Now suppose $\mathcal{I}(G_O)$ is bipartite. Consider the graph $\mathcal{I}'(G)$ which is obtained from $\mathcal{I}(G)$ by identifying all of the negatively weighted vertices into a single negatively weighted vertex v, and deleting any loops created in this process (so all of the edges in G_N are represented by a single vertex v). $\mathcal{I}'(G)$ cannot be bipartite since otherwise G would admit an $\mathbb{R}\mathrm{P}^2$-biseparation (if $\mathcal{I}'(G)$ is 2-coloured then the set A of all edges of G of a single colour defines an $\mathbb{R}\mathrm{P}^2$-biseparation). We therefore have that $\mathcal{I}'(G)$ is not bipartite and therefore contains an odd cycle. Let C be a minimal odd cycle of $\mathcal{I}'(G)$. Since $\mathcal{I}(G_O) = \mathcal{I}'(G) \setminus v$ is bipartite, this odd cycle must contain v. Furthermore, by minimality, the subgraph of $\mathcal{I}'(G)$ induced by C must also be C (since adding any other edge between the vertices of C will create a smaller cycle). Suppose the vertices of C are $1, 2, \cdots, 2m, v$ in that order. Consider the ribbon subgraph H of G corresponding to C. Since C is minimal, H consists of $2m$ orientable loops, which we name $1, \ldots, 2m$, whose ends appear in the order $1\,2\,1\,3\,2\,4\,3 \cdots (2m-1)\,(2m-2)\,(2m)\,(2m-1)\,(2m)$, together with some number of non-orientable loops such that a non-orientable loop interlaces with 1, a (not necessarily distinct) orientable loop interlaces with $2m$, and all of the non-orientable loops interlace with each other (since the non-orientable part is G_N). See Figure 2. With respect to this order, we name the arcs on the boundary of the unique vertex of H that are bounded by the orientable edges on H as follows:

$$\varepsilon\, 1\, \alpha\, 2\, \gamma_1\, 1\, \delta_1\, 3\, \gamma_2\, 2\, \delta_2\, 4\, \gamma_3 3 \cdots (2m-1)\, \gamma_{(2m-2)} \\ (2m-2)\, \delta_{(2m-2)}\, (2m)\, \gamma_{(2m-1)}\, (2m-1)\, \beta\, (2m)\, \varepsilon,$$

as in Figure 2. The ends of the non-orientable edges of H lie on these arcs, and we will use the names of these arcs to discuss the possible ways that the non-orientable edges in H can be interlaced with its orientable edges, and for each case to describe how the find the required ribbon graph minor.

First suppose that $m = 1$. If there is a single non-orientable loop e interlacing both 1 and 2 then the ribbon subgraph with edges $1, 2, e$ is given by the order of the ends $1\,e\,2\,1\,e\,2$ or $1\,2\,e\,1\,2\,e$ around the vertex. In either case contracting e results in an X_1-minor. So suppose that no non-orientable loop interlaces both 1 and 2. Then there is a non-orientable loop e interlacing 1 but not 2, and a non-orientable loop f interlacing 2 but not 1. Then the ends of e are therefore on α and ε, or γ_1 and β (otherwise it interlaces 2). Similarly, the ends of f are therefore on β and ε, or γ_1 and α. Using the fact that all non-orientable loops interlace, H must therefore contain one of the four ribbon subgraphs with edges $1, 2, e, f$ and their positions given by one of $1\,e\,2\,1\,f\,2\,e\,f$, or $1\,f\,e\,2\,f\,1\,2\,e$, or $1\,2\,e\,1\,f\,e\,2\,f$, or $1\,f\,2\,e\,f\,1\,e\,2$. In all of these cases contracting the edges 1 and 2 results in a copy of X_1, as required. Thus we can obtain the required minor when $m = 1$.

Now suppose that $m > 1$. First consider the case where there is a single non-orientable loop e interlacing both 1 and $2m$. Then e cannot have ends on

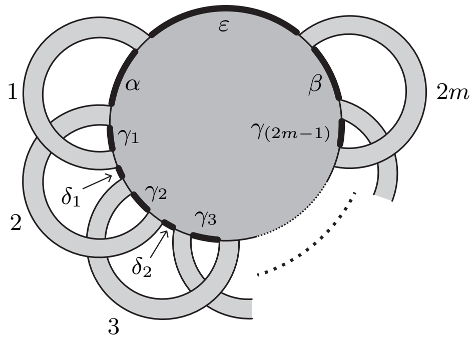


FIGURE 2. H without its non-orientable loops, and the labelled arcs on its vertex.

either of the arcs γ_1 or $\gamma_{(2m-1)}$, as this could mean that C is not a minimal odd cycle in $\mathcal{I}'(G)$ (given by the edges $1, 2, e$ or $(2m-1), (2m), e$). Thus e has ends on α and β. Deleting all of the non-orientable loops of H except for e, then contracting the edges $e, 2, 3, \ldots, (2m-1)$ gives an X_1-minor.

Now consider the case when no non-orientable loop interlaces both 1 and $2m$. Then there is a non-orientable loop e interlacing 1 but not $2m$, and a non-orientable loop f interlacing $2m$ but not 1, e has an end on α or γ_1, and f has an end on β or $\gamma_{(2m-1)}$.

First suppose that e has an end on γ_1. Then its other end must be on δ_1 or γ_2. (This is since either e would not be interlaced with 1, or the edges $1, 2, e$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$.) Consider f. Since all non-orientable loops interlace and f does not interlace with 1, we have that if e has an end on δ_1 then f has an end on δ_1, and if e has an end on γ_2 then f has an end on δ_1 or γ_2. In all three cases the edges $1, 2, e, f$ induce an odd cycle smaller that C in $\mathcal{I}'(G)$. Thus e cannot have an end on γ_1, and must have an end on α. A similar argument shows that f cannot have an end on $\gamma_{(2m-1)}$, and must have an end on β. We now examine where the other ends of e and f can lie.

If e has an end on some γ_i for $i \in \{2, \ldots, (2m-1)\}$ (we have shown that it cannot have an end on γ_1) then the edges $e, i, i+1$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$. Similarly, if f has an end on some γ_i for $i \in \{1, \ldots, (2m-2)\}$ (we have shown that it cannot have an end on $\gamma_{(2m-1)}$) then the edges $f, i, i+1$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$. Thus e has one end on α, and one on ε or on some δ_i; and f has one end on β and one on ε or on some δ_j. If one of e or f has an end on δ_i, then, by interlacement, the other has an end on some δ_j. If e has an end on δ_{2k-1}, for some k, then the edges $e, 1, \ldots, 2k$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$. Similarly, if f has an end on δ_{2k}, for some k, then the edges $2k+1, \ldots, 2m$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$. It follows that if e has an end on δ_{2k} then f has an end on δ_{2j-1} for some $j \leq k$, but then the edges $e, f, 2j, \ldots, (2k+1)$ will induce an odd cycle smaller that C in $\mathcal{I}'(G)$. Thus we have that e has ends on α and ε and f has ends on β and ε, with e and f interlaced. Deleting all of the non-orientable loops of H except for e and f, and contracting all of the orientable loops results in a copy of X_1. Thus we have shown in all cases that H, and therefore G has an X_1-minor. This completes the proof of the theorem. □

We conclude with an application of Theorem 1.1 to knot theory. Dasbach, Futer, Kalfagianni, Lin and Stoltzfus, in [3], described how every alternating link can be represented by a ribbon graph. This construction readily extends to link diagrams on other surfaces. Using, from [8], that a ribbon graph represents a checkerboard colourable diagram of a link in real projective space, $\mathbb{R}\mathrm{P}^3$, if and only if it has a plane or $\mathbb{R}\mathrm{P}^2$ partial dual immediately gives the following corollary.

Corollary 2.2. *A ribbon graph G represents a checkerboard colourable diagram of a link in real projective space if and only if it has no minor equivalent to X_1, X_2, or X_3.*

Finding the corresponding result for non-checkerboard colourable diagrams is an interesting open problem.

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