COHOMOLOGICAL FINITENESS CONDITIONS AND CENTRALISERS IN GENERALISATIONS OF THOMPSON'S GROUP V.

C. MARTÍNEZ-PÉREZ, F. MATUCCI, AND B. E. A. NUCINKIS

ABSTRACT. We consider generalisations of Thompson's group V, denoted $V_r(\Sigma)$, which also include the groups of Higman, Stein and Brin. We show that, under some mild hypotheses, $V_r(\Sigma)$ is the full automorphism group of a Cantor-algebra. Under some further minor restrictions, we prove that these groups are of type F_∞ and that this implies that also centralisers of finite subgroups are of type F_∞.

1. INTRODUCTION

Thompson's group V is defined as a homeomorphism group of the Cantor-set. The group V has many interesting generalisations such as the Higman-Thompson groups $V_{n,r}$, [10], Stein's generalisations [14] and Brin's higher dimensional Thompson groups sV [3]. All these groups contain any finite group, contain free abelian groups of infinite rank, are finitely presented and of type FP_∞ (see work by several authors in [4, 7, 9, 11, 14]). The first and third authors together with Kochloukova [11, 13] further generalise these groups, denoted by $V_r(\Sigma)$ or $G_r(\Sigma)$, as automorphism groups of certain Cantor-algebras. We shall use the notation $V_r(\Sigma)$ in this paper. We show in Theorem 2.5 that they are the full automorphism groups of these algebras.

Fluch, Marschler, Witzel and Zaremsky [7] use Morse-theoretic methods to prove that Brin's groups sV are of type F_∞. By adapting their methods we show, Theorem 3.1, that under some restrictions on the Cantor-algebra, which still comprehend all families mentioned above, $V_r(\Sigma)$ is of type F_∞. We also give some constructions of further examples.

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Bleak *et al.* [2] and the first and the third authors [13] show independently that centralisers of finite subgroups Q in $V_{n,r}$ and $V_r(\Sigma)$ can be described as extensions

$$K \rightarrowtail C_{V_r(\Sigma)}(Q) \twoheadrightarrow V_{r_1}(\Sigma) \times \ldots \times V_{r_t}(\Sigma),$$

where K is locally finite and $r_1, ..., r_t$ are integers uniquely determined by Q. It was conjectured in [13] that these centralisers are of type F_∞ if the groups $V_r(\Sigma)$ are. In Section 4 we expand the description of the centralisers given in [2, 13], which allows us to prove that the conjecture holds true. This also implies that any of the generalised $V_r(\Sigma)$ which are of type F_∞ admit a classifying space for proper actions that is a mapping telescope of cocompact classifying spaces for smaller families of finite subgroups. In other words, these groups are of Bredon type quasi-F_∞. For definitions and background the reader is referred to [13].

We conclude with giving a description of normalisers of finite subgroups in Section 5. These turn up in computations of the source of the rationalised Farrell-Jones assembly map, where one needs to compute not only centralisers, but also the Weyl-groups $W_G(Q) = N_G(Q)/C_G(Q)$. For more detail see [12], or [8] for an example where these are computed for Thompson's group T.

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2. Background on generalised Thompson groups

2.1. **Cantor-algebras.** We shall follow the notation of [13, Section 2] and begin by defining the Cantor algebras $U_r(\Sigma)$. Consider a finite set of colours $S = \{1, \ldots, s\}$ and associate to each $i \in S$ an integer $n_i > 1$, called arity of the colour i. Let U be a set on which, for all $i \in S$, the following operations are defined: an n_i-ary operation $\lambda_i : U^{n_i} \to U$, and n_i 1-ary operations $\alpha_i^1, \ldots, \alpha_i^{n_i}; \alpha_i^j : U \to U$. Denote $\Omega = \{\lambda_i, \alpha_i^j\}_{i,j}$ and call U an Ω-algebra. For detail see [5] and [11]. We write these operations on the right. We also consider, for each $i \in S$ and $v \in U$, the map $\alpha_i : U \to U^{n_i}$ given by $v\alpha_i := (v\alpha_i^1, v\alpha_i^2, \ldots, v\alpha_i^{n_i})$. The maps α_i are called descending operations, or expansions, and the maps λ_i are called ascending operations, or contractions. Any word in the descending operations is called a descending word.

A *morphism* between Ω-algebras is a map commuting with all operations in Ω. Let $\mathfrak{B}_0$ be the category of all Ω-algebras for some Ω. An object $U_0(X) \in \mathfrak{B}_0$ is a *free object* in $\mathfrak{B}_0$ with X as a *free basis*, if for any $S \in \mathfrak{B}_0$ any mapping $\theta : X \to S$ can be extended in a unique way to a morphism $U_0(X) \to S$.

For every set X there is an Ω-algebra, free on X, called the Ω-*word algebra on* X and denoted by $W_\Omega(X)$ (see [11, Definition 2.1]). Let $B \subset W_\Omega(X)$, $b \in B$ and i a colour of arity n_i. The set

$$(B \smallsetminus \{b\}) \cup \{b\alpha_i^1, \dots, b\alpha_i^{n_i}\}$$

is called a simple expansion of B. Analogously, if $b_1, \dots, b_{n_i} \subseteq B$ are pairwise distinct,

$$(B \smallsetminus \{b_1, \dots, b_{n_i}\}) \cup \{(b_1, \dots, b_{n_i})\lambda_i\}$$

is a simple contraction of B. A chain of simple expansions (contractions) is an expansion (contraction). A subset $A \subseteq W_\Omega(X)$ is called *admissible* if it can be obtained from the set X by finitely many expansions or contractions.
We shall now define the notion of a Cantor-algebra. Fix a finite set X and consider the variety of Ω-algebras satisfying a certain set of identities as follows:

Definition 2.1. [13, Section 2] We denote by $\Sigma = \Sigma_1 \cup \Sigma_2$ the following set of laws in the alphabet X.

i) A set of laws Σ_1 given by

$$u\alpha_i\lambda_i = u,$$
$$(u_1, \dots, u_{n_i})\lambda_i\alpha_i = (u_1, \dots, u_{n_i}),$$

for every $u \in W_\Omega(X)$, $i \in S$, and n_i-tuple: $(u_1, \dots, u_{n_i}) \in W_\Omega(X)^{n_i}$.

ii) A second set of laws

$$\Sigma_2 = \bigcup_{1 \le i < i' \le s} \Sigma_2^{i,i'}$$

where each $\Sigma_2^{i,i'}$ is either empty or consists of the following laws: consider first i and fix a map $f : \{1, \dots, n_i\} \to \{1, \dots, s\}$. For each $1 \le j \le n_i$, we see $\alpha_i^j \alpha_{f(j)}$ as a set of length 2 sequences of descending operations and let $\Lambda_i = \cup_{j=1}^{n_i} \alpha_i^j \alpha_{f(j)}$. Do the same for i' (with a corresponding map f') to get $\Lambda_{i'}$. We need to assume that f, f' are chosen so that $|\Lambda_i| = |\Lambda_{i'}|$ and fix a bijection $\phi : \Lambda_i \to \Lambda_{i'}$. Then $\Sigma_2^{i,i'}$ is the set of laws

$$u\nu = u\phi(\nu) \quad \nu \in \Lambda_i, u \in W_\Omega(X).$$

Factor out of $W_\Omega(X)$ the fully invariant congruence $\mathfrak{q}$ generated by Σ to obtain an Ω-algebra $W_\Omega(X)/\mathfrak{q}$ satisfying the identities in Σ.
The algebra $W_\Omega(X)/\mathfrak{q} = U_r(\Sigma)$, where $r = |X|$, is called a *Cantor-Algebra.*

As in [11] we say that Σ is *valid* if for any admissible $Y \subseteq W_\Omega(X)$, we have $|Y| = |\overline{Y}|$, where $\overline{Y}$ is the image of Y under the epimorphism $W_\Omega(X) \twoheadrightarrow U_r(\Sigma)$. In particular this implies that $U_r(\Sigma)$ is a free object on X in the class of those Ω-algebras which satisfy the identities Σ above. In other words, this implies that X is a basis. If the set Σ used to define $U_r(\Sigma)$ is valid, we also say that $U_r(\Sigma)$ is valid. As done for $W_\Omega(X)$, we say that a subset $A \subset U_r(\Sigma)$ is *admissible* if it can be obtained by a finite number of expansions or contractions from $\overline{X}$, where expansions and contractions mean the same as before. We shall, from now on, not distinguish between X and $\overline{X}$. If A can be obtained from a subset B by expansions only, we will say that A is an expansion or a descendant of B and we will write $B \leq A$. If A can be obtained from B by applying a single descending operation, i.e., if

$$A = (B \smallsetminus \{b\}) \cup \{b\alpha_i^1, \dots, b\alpha_i^{n_i}\}$$

for some colour i of arity n_i, then we will say that A is a simple expansion of B.

Remark 2.2. Let B be a basis in a valid $U_r(\Sigma)$, and let $A \leq B$. The fact that A is also a basis implies that for any element $b \in B$ there is a single $A(b) \in A$ such that $A(b)w = b$ for some descending word w. In this case we say that $A(b)$ is a prefix of b.

Definition 2.3. [13, Definition 2.12] Let $U_r(\Sigma)$ be a valid Cantor algebra. $V_r(\Sigma)$ denotes the group of all Ω-algebra automorphisms of $U_r(\Sigma)$, which are induced by a map $V \to W$, where V and W are admissible subsets of the same cardinality.

Throughout we shall denote group actions on the left.

Remark 2.4. For any basis $A \geq X$ and any $g \in V_r(\Sigma)$, there is some B with $A \leq B, gB$. To see it, take B such that $A, g^{-1}A \leq B$, which exists by [13, Lemma 2.8].

We now explore the relation between admissible subsets and bases.

We say that $U_r(\Sigma)$ is *bounded* (see [13, Definition 2.7]) if for all admissible subsets Y and Z such that there is some admissible $A \leq Y, Z$, there is a unique least upper bound of Y and Z. By a unique least upper bound we mean an admissible subset T such that $Y \leq T$ and $Z \leq T$, and whenever there is an admissible set S also satisfying $Y \leq S$ and $Z \leq S$, then $T \leq S$.

Theorem 2.5. *Let $U_r(\Sigma)$ be a valid and bounded Cantor algebra. Then $V_r(\Sigma)$ is the full group of Ω-algebra automorphisms of $U_r(\Sigma)$.*

Proof. Any Ω-algebra automorphism of $U_r(\Sigma)$ is induced by a bijective map between two bases V and W with the same cardinality. Thus, from the definition of $V_r(\Sigma)$, we need to show that, under our hypotheses, a subset of $U_r(\Sigma)$ is admissible if and only if it is a basis.

Since every admissible subset is a basis of $U_r(\Sigma)$, [11, Lemma 2.5], we only need to show that any basis of $U_r(\Sigma)$ is admissible. Let $Y = \{y_1, ..., y_n\}$ be an arbitrary basis. Since X is a basis, it generates all of $U_r(\Sigma)$. Hence, for each $y_i \in Y$ there exists some admissible subset T_i of $U_r(\Sigma)$ containing y_i. Now let Z be a common upper bound of the T_i, $i = 1, ...n$. This exists by [13, Lemma 2.8], using the argument of [11, Proposition 3.4]. The set Z is an admissible subset containing a set $\widehat{Y}$ whose elements are obtained by performing finitely many descending operations in Y. Denote by $\widehat{Y}_i$

the subsets of $\widehat{Y}$ given by the following: $\{y_i\} \leq \widehat{Y}_i$ and $\widehat{Y} = \cup \widehat{Y}_i$. Since Y and Z are bases and $Y \leq Z$, then Remark 2.2 implies that $\widehat{Y}_i \cap \widehat{Y}_j = \varnothing$, for $i \neq j$. By Remark 2.6, since $\widehat{Y}$ is admissible, it is a basis. Remark 2.6 also implies that Z is a basis. It follows from the definition of free basis, see for example [11, Page 3], that no proper subset of a basis is a basis. Hence $\widehat{Y} = Z$ is admissible, thus Y is as well. □

Remark 2.6. Any set obtained from a basis by performing expansions or contractions is also a basis. Furthermore, the cardinality m of every admissible subset satisfies $m \equiv r \bmod d$ for $d := \gcd\{n_i - 1 \mid i = 1, \ldots, s\}$. In particular, any basis with m elements can be transformed into one of r elements. Hence $U_r(\Sigma) = U_m(\Sigma)$ and we may assume that $r \leq d$.

2.2. **Brin-like groups.** In this section we give some examples of the groups $V_r(\Sigma)$, which generalise both Brin's groups sV [3] and Stein's groups $V(l, A, P)$ [14]. Furthermore, these groups satisfy the conditions of Definition 2.14 below, and we show in Section 3 that they are of type F_∞.

Example 2.7. (i) We begin by recalling the definition of the Brin-algebra [11, Section 2] and [13, Example 2.4]: Consider the set of s colours $S = \{1, \ldots, s\}$, all of which have arity 2, together with the relations: $\Sigma := \Sigma_1 \cup \Sigma_2$ with

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; l, t = 1, 2\}.$$

Then $V_r(\Sigma) = sV$ is Brin's group.

(ii) Furthermore one can also consider s colours, all of arity $n_i = n \in \mathbb{N}$, for all $1 \leq i \leq s$. Let

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; 1 \leq l, t \leq n\}.$$

Here $V_r(\Sigma) = sV_n$ is Brin's group of arity n.
It was shown in [13, Example 2.9] that in this case $U_r(\Sigma)$ is valid and bounded.

(iii) We can also mix arities. Consider s colours, each of arity $n_i \in \mathbb{N}$ $(i = 1, ..., s)$, together with $\Sigma := \Sigma_1 \cup \Sigma_2$ where

$$\Sigma_2 := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq i \neq j \leq s; 1 \leq l \leq n_i; 1 \leq t \leq n_j\}.$$

We denote these mixed-arity Brin-groups by $V_r(\Sigma) = V_{\{n_1\}, \ldots, \{n_s\}}$.
The same argument as in [11, Lemma 3.2] yields that the Cantor-algebra $U_r(\Sigma)$ in this case is also valid and bounded.

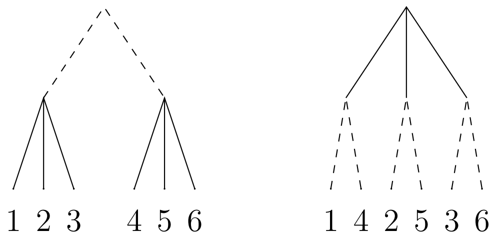

Figure 1: Visualising the identities in Σ_2 for $V_{\{2\},\{3\}}$.

Example 2.8. We now recall the laws Σ_2 for Stein's groups [14]: Let $P \subseteq \mathbb{Q}_{>0}$ be a finitely generated multiplicative group. Consider a basis of P of the form $\{n_1, \dots, n_s\}$ with all $n_i \geq 1$ integers $(i = 1, ..., s)$. Consider s colours of arities $\{n_1, \dots, n_s\}$ and let $\Sigma = \Sigma_1 \cup \Sigma_2$ with Σ_2 the set of identities given by the following order preserving identification:

$$\{\alpha_i^1\alpha_j^1, \dots, \alpha_i^1\alpha_j^{n_j}, \alpha_i^2\alpha_j^1, \dots, \alpha_i^2\alpha_j^{n_j}, \dots, \alpha_i^{n_i}\alpha_j^1, \dots, \alpha_i^{n_i}\alpha_j^{n_j}\} =$$
$$\{\alpha_j^1\alpha_i^1, \dots, \alpha_j^1\alpha_i^{n_i}, \alpha_j^2\alpha_i^1, \dots, \alpha_j^2\alpha_i^{n_i}, \dots, \alpha_j^{n_j}\alpha_i^1, \dots, \alpha_j^{n_j}\alpha_i^{n_i}\},$$

where $i \neq j$ and $i, j \in \{1, ..., s\}$.

The resulting Brown-Stein algebra $U_r(\Sigma)$ is valid and bounded, see, for example [13, Lemma 2.11]. We denote the resulting groups $V_r(\Sigma) = V_{\{n_1,\dots,n_s\}}$.

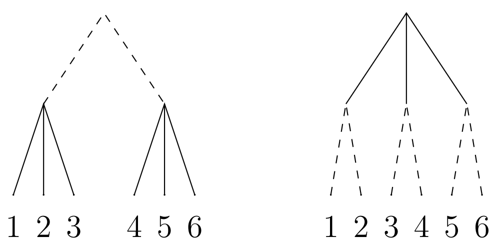

Figure 2: Visualising the identities in Σ_2 for $V_{\{2,3\}}$.

Definition 2.9. Let S be a set of s colours together with arities n_i for each $i = 1, ..., s$. Suppose S can be partitioned into m disjoint subsets S_k such that for each k, the set $\{n_i \,|\, i \in S_k\}$ is a basis for a finitely generated multiplicative group $P_k \subseteq \mathbb{Q}_{>0}$.

Consider Ω-algebras on s colours with arities as above and the set of identities $\Sigma = \Sigma_1 \cup \Sigma_2$, where $\Sigma_2 = \Sigma_{2_1} \cup \Sigma_{2_2}$ is given as follows:

Σ_{2_1} is given by the following order-preserving identifications (as in the Brown-Stein algebra in Example 2.8): for each $k \leq m$ we have

$$\{\alpha_i^1\alpha_j^1, \dots, \alpha_i^1\alpha_j^{n_j}, \alpha_i^2\alpha_j^1, \dots, \alpha_i^2\alpha_j^{n_j}, \dots, \alpha_i^{n_i}\alpha_j^1, \dots, \alpha_i^{n_i}\alpha_j^{n_j}\} =$$
$$\{\alpha_j^1\alpha_i^1, \dots, \alpha_j^1\alpha_i^{n_i}, \alpha_j^2\alpha_i^1, \dots, \alpha_j^2\alpha_i^{n_i}, \dots, \alpha_j^{n_j}\alpha_i^1, \dots, \alpha_j^{n_j}\alpha_i^{n_i}\},$$

where $i \neq j$ and $i, j \in S_k$.

Σ_{2_2} is given by Brin-like identifications (as in Example 2.7): for all $i \in S_k$ and $j \in S_l$ such that $S_k \cap S_l = \varnothing$ ($k \neq l, k, l \leq m$), we have

$$\Sigma_{2_2} := \{\alpha_i^l \alpha_j^t = \alpha_j^t \alpha_i^l \mid 1 \leq l \leq n_i; 1 \leq t \leq n_j\}.$$

We call the resulting Cantor algebra $U_r(\Sigma)$ Brin-like and denote the generalised Higman-Thompson group by $V_r(\Sigma) = V_{\{n_i \,|\, i \in S_1\}, \dots, \{n_i \,|\, i \in S_m\}}$.

Example 2.10. From Definition 2.9 we notice the following examples:

(i) If $m = s$, we have the Brin-groups as in Example 2.7 (iii).
(ii) If $m = 1$, we have Stein-groups as in Example 2.8.
(iii) Suppose we have that $\{n_i \,|\, i \in S_k\} = \{n_i \,|\, i \in S_l\}$ for each $l, k \leq m$. Then the resulting group can be viewed as a higher dimensional Stein-group $mV_{\{n_i \,|\, i \in S_m\}}$.

Question 2.11. Suppose $m \notin \{1, s\}$. What are the conditions on the arities for the groups $V_{\{n_i \,|\, i \in S_1\}, \dots, \{n_i \,|\, i \in S_m\}}$ not be isomorphic to any of the known generalised Thompson groups such as the Higman-Thompson groups, Stein's groups or Brin's groups? More generally, when are two of these groups non-isomorphic? See [6] for some special cases.

Remark 2.12. We can view these groups as bijections of m-dimensional cuboids in the m-dimensional Cartesian product of the Cantor-set, similarly to the description given for sV, the Brin-Thompson groups. In each direction, we get subdivisions of the Cantor-set as in the Stein-Brown groups given by Σ_{2_1}.

Lemma 2.13. *The Brin-like Cantor-algebras are valid and bounded.*

Proof. Using the description given in Remark 2.12 we can apply the same argument as in [11, Lemma 3.2]. □

The groups defined in this subsection all satisfy the following condition on the relations in Σ, and hence satisfy the conditions needed in Section 3.

Definition 2.14. Using the notation of Definition 2.1, suppose that for all $i \neq i'$, $i, i' \in S$ we have that $\Sigma_2^{i,i'} \neq \varnothing$ and that $f(j) = i'$ for all $j = 1, \dots, n_i$ and $f'(j') = i$ for all $j' = 1, \dots, n_{i'}$. Then we say that Σ (or equivalently $U_r(\Sigma)$) is *complete*.

Remark 2.15. The Brin-like Cantor-algebras are complete.

3. Finiteness conditions

In this section we prove the following result:

Theorem 3.1. *Let Σ be valid, bounded and complete. Then $V_r(\Sigma)$ is of type* F_∞.

We closely follow [7], where it is shown that Brin's groups sV are of type F_∞. We shall use a different notation, which is more suited to our set-up, and will explain where the original argument has to be modified to get the more general case. Throughout this section $U_r(\Sigma)$ denotes a valid, bounded and complete Cantor-algebra.

Definition 3.2. Let $B \leq A$ be admissible subsets of $U_r(\Sigma)$. We say that the expansion $B \leq A$ is *elementary* if there are no repeated colours in the paths from leaves in B to their descendants in A. Since Σ is complete, this condition is preserved by the relations in Σ. We denote an elementary expansion by $B \preceq A$. We say that the expansion is *very elementary* if all paths have length at most 1. In this case we write $B \sqsubseteq A$.

Remark 3.3. If $A \preceq B$ is elementary (very elementary) and $A \leq C \leq B$, then $A \preceq C$ and $C \preceq B$ are elementary (very elementary)).

Lemma 3.4. *Let Σ be complete, valid and bounded. Then any admissible basis A has a unique maximal elementary admissible descendant, denoted by $\mathcal{E}(A)$.*

Proof. Let $\mathcal{E}(A)$ be the admissible subset of $n_1 \dots n_s|A|$ elements obtained by applying all descending operations exactly once to every element of A. □

3.1. **The Stein subcomplex.** Denote by $\mathcal{P}_r$ the poset of of admissible bases in $U_r(\Sigma)$. The same argument as in [11, Lemma 3.5 and Remark 3.7] shows that its geometric realisation $|\mathcal{P}_r|$ is contractible, and that $V_r(\Sigma)$ acts on $\mathcal{P}_r$ with finite stabilisers. In [11, 13] this poset was denoted by $\mathfrak{A}$, but here we will follow the notation of [7]. This poset is essentially the same as the poset of [7] denoted $\mathcal{P}_r$ there as well.

We now construct the Stein complex $\mathcal{S}_r(\Sigma)$, which is a subcomplex of $|\mathcal{P}_r|$. The vertices in $\mathcal{S}_r(\Sigma)$ are given by the admissible subsets of $U_r(\Sigma)$. The k-simplices are given by chains of expansions $Y_0 \leq \ldots \leq Y_k$, where $Y_0 \preceq Y_k$ is an elementary expansion.

Lemma 3.5. *Let $A, B \in \mathcal{P}_r$ with $A < B$. There exists a unique $A < B_0 \leq B$ such that $A \prec B_0$ is elementary and for any $A \leq C \leq B$ with $A \preceq C$ elementary we have $C \preceq B_0$.*

Proof. Let $\mathcal{E}(A)$ be as in the proof of Lemma 3.4. Let $B_0 = \mathrm{glb}(\mathcal{E}(A), B)$ which exists by [11, Lemma 3.14]. If $A \preceq C \leq B$, then $C \leq \mathcal{E}(A)$ and so $C \leq B_0$. □

Lemma 3.6. *For every r and every valid, bounded and complete Σ, the Stein-space $\mathcal{S}_r(\Sigma)$ is contractible.*

Proof. By [11, Lemma 3.5], $|\mathcal{P}_r|$ is contractible. Now use the same argument of [7, Corollary 2.5] to deduce that $\mathcal{S}_r(\Sigma)$ is homotopy equivalent to $|\mathcal{P}_r|$. Essentially, the idea is to use Lemma 3.5 to show that each simplex in $|\mathcal{P}_r|$ can be pushed to a simplex in $\mathcal{S}_r(\Sigma)$. □

Remark 3.7. Notice that the action of $V_r(\Sigma)$ on $\mathcal{P}_r$ induces an action of $V_r(\Sigma)$ on $\mathcal{S}_r(\Sigma)$ with finite stabilisers.

Consider the Morse function $t(A) = |A|$ in $\mathcal{S}_r(\Sigma)$ and filter the complex with respect to t, i.e.

$$\mathcal{S}_r(\Sigma)^{\leq n} := \text{ full subcomplex supportet on } \{A \in \mathcal{S}_r(\Sigma) \mid t(A) \leq n\}.$$

By the same argument as in [11, Lemma 3.7] $\mathcal{S}_r(\Sigma)^{\leq n}$ is finite modulo the action of $V_r(\Sigma)$. Let $\mathcal{S}_r(\Sigma)^{<n}$ be the complex given by the vertex set $\{A \in \mathcal{S}_r(\Sigma) \mid t(A) < n\}$.

Provided that

(1) the connectivity of the pair $(\mathcal{S}_r(\Sigma)^{\leq n}, \mathcal{S}_r(\Sigma)^{<n})$ tends to ∞ as $n \to \infty$,

Brown's Theorem [4, Corollary 3.3] implies that $V_r(\Sigma)$ is of type F_∞, thus proving Theorem 3.1. The rest of this section is devoted to proving (1).

3.2. **Connectivity of descending links.** Recall that for any $A \in \mathcal{S}_r(\Sigma)$ the descending link $L(A) := \mathrm{lk}{\downarrow}^t(A)$ with respect to t is defined to be the intersection of the link $\mathrm{lk}(A)$ with $\mathcal{S}_r(\Sigma)^{<n}$, where $t(A) = n$. To show (1), we proceed as in [7]. Using Morse theory, the problem is reduced to showing that for A as before, the connectivity of $L(A)$ tends to ∞ when $t(A) = n \to \infty$. Whenever this happens, we will say that $L(A)$ is *n-highly connected.* More generally: assume we have a family of complexes $(X_\alpha)_{\alpha\in\Lambda}$ together with a map $n : \Lambda \to \mathbb{Z}_{>0}$ such that the set $\{n(\alpha)_{\alpha\in\Lambda}\}$ is unbounded. Assume further that whenever $n(\alpha) \to \infty$, the connectivity of the associated X_αs tends to ∞. In this case we will say that the family is n-highly connected.

Note that $L(A)$ is the subcomplex of $\mathcal{S}_r(\Sigma)$ generated by

$$\{B \mid B \prec A \text{ is an elementary expansion}\}.$$

Following [7], define a height function h for $B \in L(A)$ as follows:

$$h(B) := (c_s, \ldots, c_2, b)$$

where $b = |B|$ and c_i $(i = 2, \ldots, s)$ is the number of elements in A whose length as descendants of their parent in B is i. We order these heights lexicographically. Let $c(B) = (c_s, \ldots, c_2)$, which are also ordered lexicographically. Denote by $L_0(A)$ the subcomplex of $\mathcal{S}_r(\Sigma)$ generated by $\{B \mid B \sqsubset A \text{ is a very elementary expansion}\}$. Then for any $B \in L(A)$, $B \in L_0(A)$ if and only if $h(B) = (0, \ldots, 0, |B|)$.

Lemma 3.8. *The set of complexes of the form $L_0(A)$ is $t(A)$-highly connected.*

Proof. For any $n \geq 0$, we define a complex denoted K_n as follows. Start with a set A with n elements. The vertex set of K_n consists of labelled subsets of A where the possible labels are the colours $\{1, \dots, s\}$, and where a subset labelled i has precisely n_i elements. Recall that n_i is the arity of the colour i. A k-simplex $\{\sigma_0, \dots, \sigma_k\}$ in K_n is given by an unordered set of pairwise disjoint σ_js. This complex is isomorphic to the barycentric subdivision of $L_0(A)$ for $n = t(A)$. To prove that K_n is n-highly connected, proceed as in the proof of [4, Lemma 4.20]. □

Now consider descending links in $L(A)$ with respect to the height function h, i.e. for $B \in L(A)$ let $\mathrm{lk}{\downarrow}^h(B)$ be the subcomplex of $L(A)$ generated by $\{C \in L(A) \mid h(C) \leq h(B) \text{ and either } B < C \text{ or } C > B\}$. Consider the following two cases:

i) $B \in L(A) \smallsetminus L_0(A)$ and there is at least one element of B that is expanded precisely once to obtain A.
ii) $B \in L(A) \smallsetminus L_0(A)$ and no element of B is expanded precisely once to obtain A.

The next two Lemmas show that in either case $\mathrm{lk}{\downarrow}^h(B)$ is $t(A)$-highly connected.

As in [7] the descending link $\mathrm{lk}{\downarrow}^h(B)$ of some $B \in L(A)$ with respect to h can be viewed as the join of two subcomplexes, the down-link and the up-link. The downlink consists of those elements C such that $C < B$ and $h(C) < h(B)$. Hence $c(B) = c(C)$. The uplink consists of those C that $B < C$, $h(C) < h(B)$, and therefore $c(B) > c(C)$.

Lemma 3.9. *Let $B \in L(A)$ as in i). Then $\mathrm{lk}{\downarrow}^h(B)$ is contractible.*

Proof. It suffices to follow the proof of [7, Lemma 3.7]. We briefly sketch this proof using our notation: let $b \in B$ be an element that is expanded precisely once to obtain A. given $B \prec A$ and let $b \in B$, which is expanded precisely once to get to A, then there is an M such that $B \preceq M \sqsubset A$ and $b \in M$. The existence of M follows from a variation of Lemma 3.5. Now, for any $C \in \mathrm{lk}{\downarrow}^h(B)$ lying in the uplink we let $B \prec C_0 \sqsubseteq C$, where C_0 is obtained by performing all expansions in B needed to get C, except the one of b.

One easily checks that $C_0 \leq M$, that C_0 and M lie in $\mathrm{lk}{\downarrow}^h(B)$ and that both C_0 and M lie in the uplink. Hence $M \geq C_0 \leq C$ provides a contraction of the uplink. As $\mathrm{lk}{\downarrow}^h(B)$ is the join of the downlink and the uplink we get the result. □

Lemma 3.10. *Let B be as in ii). Then $\mathrm{lk}{\downarrow}^h(B)$ is $t(A)$-highly connected.*

Proof. As before, we follow the proof of [7, Lemma 3.8] with only minor changes. With our notation, we let k_s be the number of elements in B that are also leaves of A and let k_b be the remaining leaves. Then one checks that the up-link in $\mathrm{lk}{\downarrow}^h(B)$ is k_b-highly connected and that the down-link is k_s-highly connected. As $t(A) = n \leq k_b n_1 \dots n_s + k_s$, we get the result. □

Finally, using Morse theory as in [7], we deduce that the pair $(L(A), L_0(A))$ is $t(A)$-highly connected. As a result, $L(A)$ is also $t(A)$-highly connected, establishing (1) and hence Theorem 3.1.

Some time after a preprint of this work was posted, we learned of Thumann's work [15, 16], where he provides a generalised framework of groups defined by operads to apply the techniques introduced in [7]. We believe that automorphism groups of valid, bounded and complete Cantor algebras might be obtained making a suitable choice of cube cutting operads, see [15, Subsection 4.2]. Therefore Theorem 4.1 could also be seen as a special case of [16, Subsection 10.2].

4. Finiteness conditions for centralisers of finite subgroups

From now on, unless mentioned otherwise, we assume that the Cantor-algebra $U_r(\Sigma)$ is valid and bounded.

Definition 4.1. Let L be a finite group. The set of bases in $U_r(\Sigma)$ together with the expansion maps can be viewed as a directed graph. Let $(U_r(\Sigma), L)$ be the following diagram of groups associated to this graph: To each basis A we associate $\mathrm{Maps}(A, L)$, the set of all maps from A to L. Each simple expansion $A \leq B$ corresponds to the diagonal map $\delta : \mathrm{Maps}(A, L) \to \mathrm{Maps}(B, L)$ with $\delta(f)(a\alpha_i^j) = f(a)$, where $a \in A$ is the expanded element, i.e. $B = (A \smallsetminus \{a\}) \cup \{a\alpha_i^1, \dots, a\alpha_i^{n_i}\}$ for some colour i of arity n_i. To arbitrary expansions we associate the composition of the corresponding diagonal maps.

Centralisers of finite subgroups in $V_r(\Sigma)$ have been described in [13, Theorem 4.4] and also in [2, Theorem 1.1] for the Higman-Thompson groups $V_{n,r}$. This last description is more explicit and makes use of the action of $V_{n,r}$ on the Cantor set (see Remark 4.3 below).

We will use the following notation, which was used in [13]: let $Q \leq V_r(\Sigma)$ be a finite subgroup and let t be the number of transitive permutation representations $\varphi_i : Q \to S_{m_i}$ of Q. Here, $1 \leq i \leq t$, m_i is the orbit length and S_{m_i} is the symmetric group of degree m_i. Also let $L_i = C_{S_{m_i}}(\varphi_i(Q))$.
There is a basis Y setwise fixed by Q and which is of minimal cardinality. The group Q acts on Y by permutations. Thus there exist integers $0 \leq r_1, \dots, r_t \leq d$ such that $Y = \bigcup_{i=1}^t W_i$ with W_i the union of exactly r_i Q-orbits of type φ_i. See Remark 2.6 for the definition of d.

The next result combines the descriptions in [13, Theorem 4.4] and [2, Theorem 1.1] giving a more detailed description of the centralisers of finite subgroups in $V_r(\Sigma)$.

Theorem 4.2. *Let Q be a finite subgroup of $V_r(\Sigma)$. Then*

$$C_{V_r(\Sigma)}(Q) = \prod_{i=1}^{t} G_i$$

where $G_i = K_i \rtimes V_{r_i}(\Sigma)$ and $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$. Here, $V_r(\Sigma)$ acts on K_i as follows: let $g \in V_{r_i}(\Sigma)$ and let A be a basis in $U_{r_i}(\Sigma)$. The action of g on K_i is induced, in the colimit, by the map $Maps(A, L) \to Maps(gA, L)$ obtained contravariantly from $gA \overset{g^{-1}}{\to} A$.

Proof. The decomposition of $C_{V_r(\Sigma)}(Q)$ into a finite direct product of semidirect products was shown in [13, Theorem 4.4]. Hence, for the first claim, all that remains to be checked is that $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$. We use the same notation as in the proof of [13, Theorem 4.4].

Fix $\varphi = \varphi_i$, $l := r_i$, $L := L_i$, $m := m_i$ and $K := K_i = \operatorname{Ker}\tau$. Let $x \in K = \operatorname{Ker}\tau$, where $\tau : C_{V_r(\Sigma)}(Q) \twoheadrightarrow V_l(\Sigma)$ is the split surjection of the proof of [13, Theorem 4.4]. With Y as above, there is a basis $Y_1 \geq Y$ with $xY_1 = Y_1$ and Y_1 is also Q-invariant. Then the basis Y_1 decomposes as a union of l Q-orbits (all of them of type φ), and x fixes these orbits setwise. We denote these orbits by $\{C_1, \dots, C_l\}$. In each of the C_j there is a marked element. Since φ is transitive this can be used to fix a bijection $C_j \to \{1, \dots, m\}$ corresponding to φ. Then the action of x on C_j yields a well defined $l_j \in L$. This means that we may represent x as $(l_j)_{1 \leq j \leq l}$. Let A be the basis of $U_l(\Sigma)$ obtained from Y_1 by identifying all elements in the same Q-orbit, i.e. $A = \tau^{\mathfrak{U}}(Y_1)$ with the notation of [13]. Denote $A = \{a_1, \dots, a_l\}$ with a_j coming from C_j. Then the element x described before can be viewed as the map $x : A \to L$ with $x(a_j) = l_j$. Suppose we chose a different basis Y_2 fixed by x. It is a straightforward check to see that there is a basis Y_3 also fixed by x, such that $Y_1, Y_2 \leq Y_3$, and that this representation is compatible with the associated expansion maps.

To prove the second claim, consider an element $g \in V_l(\Sigma)$ viewed as an element in $C_{V_r(\Sigma)}(Q)$ using the splitting τ above. This means that g maps Q-fixed bases to Q-fixed bases and that g preserves the set of marked elements. Let Y_1, A and $x \in K$ be as above. Then the basis gY_1 is the union of the Q-orbits $\{gC_1, \dots, gC_l\}$ and $\tau^{\mathfrak{U}}(gY_1) = gA$. Also, for any $c_i \in C_i$, $gxg^{-1}gc_i = gxc_i$ which means that if the action of x on C_i is given by $l_i \in L$, then the action of x^g on gC_i is given also by l_i. Therefore the map $gA \to L$ which represents x^g is the composition of the maps $g^{-1} : gA \to A$ and the map $A \to L$ which represents x. □

Remark 4.3. In [2], where the ordinary Higman-Thompson group $V_r(\Sigma) = V_{n,r}$ is considered, the subgroups K_i are described as $\mathrm{Map}^0(\mathfrak{C}, L)$, where $\mathfrak{C}$ denotes the Cantor set, and Map^0 the set of continuous maps. Here the Cantor set is viewed as the set of right infinite words in the descending operations.

It is a straightforward check to see that both descriptions are equivalent in this case. In fact $x : A \to L$ corresponds to the element in $\mathrm{Map}^0(\mathfrak{C}, L)$ mapping each $\varsigma \in \mathfrak{C}$ to $x(a)$ for the only $a \in A$ which is a prefix of ς. Similarly, one can describe K_i when $V_{r_i}(\Sigma) = sV$ is a Brin-group, using the fact that these groups act on $\mathfrak{C}^s$, see [6].

We shall now show that for each i the action of $V_{r_i}(\Sigma)$ on K_i^n has finitely many orbits for any n.

Notation 4.4. Any element of $U_r(\Sigma)$ which is obtained from the elements in X by applying descending operations only is called a *leaf*. We denote by $\mathcal{L}$ the set of leaves. Observe that $\mathcal{L}$ depends on X. Note also that for any leaf l there is some basis $A \geq X$ with $l \in A$. Let $l \in \mathcal{L}$, we put:

$$l(\mathcal{L}) := \{b \in \mathcal{L} \mid lw = bw' \text{ for descending words } w, w'\}$$

and for a set of leaves $B \subseteq \mathcal{L}$ we also put

$$B(\mathcal{L}) = \bigcup_{b \in B} b(\mathcal{L}).$$

Let

$$\Omega := \{B(\mathcal{L}) \mid B \subset \mathcal{L} \text{ finite}\} \cup \{\varnothing\}.$$

We also denote

$$\Omega^n := \Omega \times \overset{n}{\cdots} \times \Omega = \{(\omega_1, \ldots, \omega_n) \mid \omega_i \in \Omega\},$$

$$\Omega^n_c := \{(\omega_1, \ldots, \omega_n) \in \Omega^n \mid \cup_{i=1}^n \omega_i = \mathcal{L}\}.$$

Note that the Ω here has no connection to the Ω of Ω-algebra used in Section 2.1.

Lemma 4.5. i) *Let $B \geq A \geq X$ be bases and $B_1 \subseteq B$. Let $A_1 := \{a \in A \mid a$ is a prefix of an element in $B_1\}$. Then $A_1(\mathcal{L}) = B_1(\mathcal{L})$.*

ii) *Let $A \geq X$ be a basis, then $A(\mathcal{L}) = \mathcal{L}$.*

iii) *For any $(\omega_1, \ldots, \omega_n) \in \Omega^n$ there is some basis A with $X \leq A$ and some $A_i \subseteq A$, $1 \leq i \leq n$ such that $\omega_i = A_i(\mathcal{L})$.*

iv) *Let $A \geq X$ be a basis, $A_1, A_2 \subseteq A$ and $\omega_i = A_i(\mathcal{L})$ for $i = 1, 2$. Then $\omega_1 = \omega_2$ if and only if $A_1 = A_2$.*

v) *Let $A, B \geq X$ be two bases and $\omega \in \Omega$ be such that for some $A_1 \subseteq A$, $B_1 \subseteq B$ we have $\omega = A_1(\mathcal{L}) = B_1(\mathcal{L})$. Then $|A_1| \equiv |B_1|$ mod d and $|A_1| = 0$ if and only if $|B_1| = 0$.*

vi) *Let $A, B \geq X$ be two bases and $A_1, A_2 \subseteq A$, $B_1, B_2 \subseteq B$ with $A_1(\mathcal{L}) = B_1(\mathcal{L})$ and $A_2(\mathcal{L}) = B_2(\mathcal{L})$. Then $A_1 \cap A_2 = \varnothing$ if and only if $B_1 \cap B_2 = \varnothing$.*

Proof. It suffices to prove i) in the case when B is obtained by a simple expansion from A. Moreover, we may assume that $A_1 = \{a\}$ and $B_1 = \{a\alpha_i^1, \dots, a\alpha_i^{n_i}\}$ for some colour i of arity n_i. Then obviously $B_1(\mathcal{L}) \subseteq a(\mathcal{L})$. Denote $b_j = a\alpha_i^j$ and let $u \in a(\mathcal{L})$. Then $uv = ac$ for descending words v and c. Performing the descending operations given by c on the basis A, we obtain a basis C with $ac \in C$. Let D be a basis with $C, B \leq D$. Then there is some element $d \in D$ which can be written as $d = acc'$ for some descending word c'. Moreover, Remark 2.2 also implies that $d = b_j b'$ for some j and descending word b'. As $uvc' = acc' = b_j b'$ we get $u \in b_j(\mathcal{L})$. Now ii) follows from i).
To prove iii), suppose that $\omega_i = \{a_i^1, \dots, a_i^{l_i}\}(\mathcal{L})$. For each a_i^j we may find a basis $T_i^j \geq X$ containing a_i^j. Now let A be common descendant of the T_i^j and use i).
To establish iv), it suffices to check that if $\widehat{a} \in A$, $\widehat{a} \notin A_i$, then $\widehat{a} \notin A_i(\mathcal{L})$. Suppose $\widehat{a} \in A_i(\mathcal{L})$. Then there are descending words v, u and some $a \in A_i$, such that $\widehat{a}v = au = b$. Performing the descending operations given by v and u on $\widehat{a}$ and a respectively, we get a basis $A \leq B$ and $b \in B$ contradicting Remark 2.2.
In v), since there is a basis C with $A, B \leq C$, we may assume $A \leq B$. Then v) is a consequence of i) and iv).
Finally, for vi) we may also assume $A \leq B$. Then we only have to use Remark 2.2. □

Notation 4.6. Let $\omega \in \Omega$, $X \leq A$ and $B \subseteq A$ such that $\omega = B(\mathcal{L})$. We put

$$\|\omega\| = \begin{cases} 0 \text{ if } \omega = \varnothing \\ t \text{ for } |B| \equiv t \bmod d \text{ and } 0 < t \leq d \text{ otherwise.} \end{cases}$$

This is well defined by Lemma 4.5 v). Take $B' \subseteq A$ and $\omega' = B'(\mathcal{L})$. If $B \cap B' = \varnothing$, we put $\omega \wedge \omega' = \varnothing$. Note that by Lemma 4.5 vi) this is well defined.

Finally, let

$$\Omega^n_{c,\mathrm{dis}} := \{(\omega_1, \dots, \omega_n) \in \Omega^n_c \mid \mathcal{L} = \bigcup_{i=1}^n \omega_i \text{ and } \omega_i \wedge \omega_j = \varnothing \text{ for } i \neq j\}.$$

The group $V_r(\Sigma)$ does not act on the set of leaves. It does, however, act on Ω as we will see in Lemma 4.7. Nevertheless there is a partial action of $V_r(\Sigma)$ on the set of leaves as follows: if l is a leaf such that $l \in A$ for a certain basis $A \geq X$ and g is a group element such that $gA \geq X$, then we will denote by gl the leaf of gA to which l is mapped by g.

Lemma 4.7. *The group $V_r(\Sigma)$ acts by permutations on Ω and on $\Omega^n_{c,dis}$. There are only finitely many $V_r(\Sigma)$-orbits under the latter action. Furthermore, the stabiliser of any element in $\Omega^n_{c,dis}$ is of the form*

$$V_{k_1}(\Sigma) \times \ldots \times V_{k_n}(\Sigma)$$

for certain integers $k_1, \dots, k_n$.

Proof. To see that $V_r(\Sigma)$ acts on Ω, it suffices to check that if $\omega = l(\mathcal{L})$ for some leaf $l \in \mathcal{L}$, we have $g\omega \in \Omega$ for any $g \in V_r(\Sigma)$. Let $X \leq A$ be a basis with $l \in A$. By Remark 2.4 there is some $A \leq B$ with $A \leq gB$. Note that by Lemma 4.5 i) ω can also be written as

$$\omega = B_1(\mathcal{L})$$

where $B_1 = \{l_1, \dots, l_k\}$ is the set of leaves in B obtained from l. Therefore $gB_1 = \{gl_1, \dots, gl_k\} \subseteq gB$ and $g\omega = gB_1(\mathcal{L})$.
That this action induces an action on $\Omega^n_{c,\text{dis}}$ is a consequence of the easy fact that for any $g \in V_r(\Sigma)$ and any $(\omega_1, \dots, \omega_n) \in \Omega^n_{c,\text{dis}}$ we have $g\omega_i \wedge g\omega_j = \varnothing$ and $\mathcal{L} = \cup_{i=1}^n g\omega_i$.

Let $(\omega_1, \dots, \omega_n), (\omega'_1, \dots, \omega'_n) \in \Omega^n_{c,\text{dis}}$ be such that $\|\omega_i\| = \|\omega'_i\|$ for $1 \leq i \leq n$. There are bases $X \leq A, A'$ and subsets $A_1, \dots, A_n \subseteq A$, $A'_1, \dots, A'_n \subseteq A'$ such that for each $1 \leq i \leq n$, $\omega_i = A_i(\mathcal{L})$, $\omega'_i = A'_i(\mathcal{L})$ and $|A_i| = |A'_i|$. Hence we may choose a suitable element $g \in V_r(\Sigma)$ such that $gA = A'$ and $gA_i = A'_i$ for each $i = 1, \dots, n$. Then $g(\omega_1, \dots, \omega_n) = (\omega'_1, \dots, \omega'_n)$. Since the number of possible n-tuples of integers modulo d having the same number of zeros is finite, it follows that there are only finitely many $V_r(\Sigma)$-orbits.

Finally consider $\mathcal{W} = (\omega_1, \dots, \omega_n) \in \Omega^n_{c,\text{dis}}$ as before, i.e. with $X \leq A$ and $A_1, \dots, A_n \subseteq A$ such that $\omega_i = A_i(\mathcal{L})$ for $1 \leq i \leq n$. An element $g \in V_r(\Sigma)$ fixes $\mathcal{W}$ if and only if $g\omega_i = \omega_i$ for each $i = 1, \dots, n$. We may choose a basis B with $A \leq B, gB$ and then, by using Lemma 4.5 i) and iv), we see that g fixes $\mathcal{W}$ if and only if it maps those leaves of B, which are of the form av for some $a \in A_i$ and some descending word v, to the analogous subset in gB. Considering the subalgebra of $U_r(\Sigma)$ generated by the A_i, we see that g can be decomposed as $g = g_1 \dots g_n$ with $g_i \in V_{k_i}(\Sigma)$ for $k_i = |A_i|$. □

Let K be a group and denote by $Y = K * K * \dots$ the infinite join of copies of K viewed as a discrete CW-complex, i.e. Y is the space obtained by Milnor's construction for K. Then Y has a CW-complex decomposition whose associated chain complex yields the standard bar resolution. For detail see, for example, [1, Section 2.4].

Obviously, if a group H acts on K by conjugation, this action can be extended to an action of H on Y and to an action of $G = K \rtimes H$ on Y.

Lemma 4.8. *Let H and K be groups and let H act on K via $\varphi : H \to \mathrm{Aut}K$. Assume that H is of type* F_∞*, and that for every $n \in \mathbb{N}$ the induced action of H on K^n has finitely many orbits and has stabilisers of type* F_∞*. Then $G = K \rtimes_\varphi H$ is of type* F_∞*.*
The same statement holds if F_∞ *is replaced with* FP_∞*.*

Proof. Let $Y_n = K^{*n}$ and let Y be as above. Consider the action of G on Y induced by the diagonal action. Note that this preserves the individual join factors. Since the action of K on Y is free, the stabiliser of a cell in G is isomorphic to its stabiliser in H. The stabiliser of an $(n-1)$-simplex is the stabiliser of n elements of K, thus F_∞ by assumption. Maximal simplices in Y_n correspond to elements of K^n and every simplex of Y_n is contained in a maximal simplex. This, together with the fact that the action of G on K^n has only finitely many orbits, implies that the action of G on Y_n is cocompact. Finally, the connectivity of the filtration $\{Y_n\}_{n\in\mathbb{N}}$ tends to infinity as $n \to \infty$. Hence the claim follows from [4, Corollary 3.3(a)]. □

Theorem 4.9. *Assume that for any $t > 0$, the group $V_t(\Sigma)$ is of type F_∞. Then the groups $G_i = K_i \rtimes V_{r_i}(\Sigma)$ of Theorem 4.2 are of type F_∞.*

The same statement holds if F_∞ is replaced with FP_∞.

Proof. Put $V := V_{r_i}(\Sigma)$, $K := K_i$ and $G := G_i$. We claim that for every n there is some $\overline{n}$ big enough such that there is an injective map of V-sets

$$\phi_n : K^n \to \Omega_{c,\mathrm{dis}}^{\overline{n}}.$$

Let $x \in K$ be given by a map $x : A \to L$, where A is a basis with $X \le A$. The element x is determined uniquely by a map which, by slightly abusing notation, we also denote $x : L \to \Omega$. This x maps any $s \in L$ to $\omega_s := A_s(\mathcal{L})$ with $A_s = \{a \in A \mid x(a) = s\}$. Obviously $\cup_{s\in L}\omega_s = \mathcal{L}$. This means that fixing an order in L yields an injective map of V-sets

$$\xi_n : K^n \to \Omega_c^{n|L|}.$$

Consider any $(\omega_1, \dots, \omega_m) \in \Omega_c^m$ for $m = n|L|$. Let $X \le A$ with $A_1 \dots, A_m \subseteq A$ and $\omega_i = A_i(\mathcal{L})$ for $1 \le i \le m$. Let $\overline{n} := 2^m - 1$, i.e. the number of non-empty subsets $\varnothing \ne S \subseteq \{1, \dots, m\}$. For any such S let

$$A_S := \bigcap_{i\in S} A_i \smallsetminus \cup\{\bigcap_{j\in T} A_j \mid S \subset T \subseteq \{1, \dots, m\}\}.$$

Then one easily checks that the A_S are pairwise disjoint and that their union is $\mathcal{L}$. Let $\omega_S := A_S(\mathcal{L})$. The preceding paragraph means that fixing an ordering on the set of non-empty subsets of $\{1, \dots, m\}$ yields an injective map of V-sets

$$\rho_m : \Omega_c^m \to \Omega_{c,\mathrm{dis}}^{\overline{n}}.$$

Composing ξ_n and ρ_m we get the desired ϕ_n.

Now, applying Lemma 4.7 we deduce that K^n has only finitely many orbits under the action of $V_{r_i}(\Sigma)$ and that every cell stabiliser is isomorphic to a direct product of copies of $V_t(\Sigma)$ for suitable indices t. It now suffices to use Lemma 4.8. □

This implies that [13, Conjecture 7.5] holds.

Corollary 4.10.

(i) *$V_r(\Sigma)$ is quasi-$\underline{\mathrm{FP}}_\infty$ if and only if $V_k(\Sigma)$ is of type FP_∞ for any k.*
(ii) *$V_r(\Sigma)$ is quasi-$\underline{\mathrm{F}}_\infty$ if and only if $V_k(\Sigma)$ is of type F_∞ for any k.*

Proof. The "only if" part of both items is proven in [13, Remark 7.6]. The "if" part is a consequence of [13, Definition 6.3, Proposition 6.10] and Theorem 4.9 above. □

Theorem 4.9 also implies that the Brin-like groups of Section 3 are of type quasi-F_∞ :

Corollary 4.11. *Suppose $U_r(\Sigma)$ is valid, bounded and complete. Then $V_r(\Sigma)$ is of type quasi-F_∞.*

In particular, centralisers of finite groups are of type F_∞.

5. Normalisers of finite subgroups

Let Y be any basis. We denote

$$S(Y) := \{g \in V_r(\Sigma) \mid gY = Y\}.$$

Observe that this is a finite group, isomorphic to the symmetric group of degree $|Y|$.

Theorem 5.1. *Let $Q \leq V_r(\Sigma)$ be a finite subgroup. Let Y, t, r_i, l_i, φ_i, and $1 \leq i \leq t$ be as in the proof of Theorem 4.2. Then*

$$N_{V_r(\Sigma)}(Q) = C_{V_r(\Sigma)}(Q) N_{S(Y)}(Q)$$

and $N_{V_r(\Sigma)}(Q)/C_{V_r(\Sigma)}(Q) \cong N_{S(Y)}(Q)/C_{S(Y)}(Q)$.

Proof. Let $g \in N_{V_r(\Sigma)}(Q)$ and $Y_1 = gY$. Then for any $q \in Q$, $qY_1 = qgY = gq^gY = gY = Y_1$. Therefore Y_1 is also fixed setwise by Q. Let r'_i denote the number of components of type φ_i in Y_1. Then, by [13, Proposition 4.2] $r_i \equiv r'_i \mod d$, and $r_i = 0$ if and only if $r'_i = 0$.

We claim that Y and Y_1 are isomorphic as Q-sets, in other words, that $r_i = r'_i$ for every $1 \leq i \leq t$. Note that since g normalises Q, it acts on the set of Q-permutation representations $\{\varphi_1, \ldots, \varphi_t\}$, via $\varphi_i^g(x) := \varphi_i(x^{g^{-1}})$. Let i with $r_i \neq 0$ and let $g(i)$ be the index such that $\varphi_i^g = \varphi_{g(i)}$. The fact that $g : Y \to Y_1$ is a bijection implies that $r_i = r'_{g(i)}$. We may do the same for $g(i)$ and get an index $g^2(i)$ with $r_{g(i)} = r'_{g^2(i)}$. At some point, since the orbits of g acting on the sets of permutation representations are finite, we get $g^k(i) = i$ and $r_{g^{k-1}(i)} = r'_i$. As $r'_i \equiv r_i \mod d$ we have $r_{g^{k-1}(i)} \equiv r_i \mod d$, and since $0 < r_i, r_{g^{k-1}(i)} \leq d$ we deduce that $r'_i = r_{g^{k-1}(i)} = r_i$ as claimed.

Now, we can choose an $s \in V_r(\Sigma)$ mapping Y_1 to Y and such that $s : Y_1 \to Y$ is a Q-map, i.e., commutes with the Q-action. Therefore, $s \in C_{V_r(\Sigma)}(Q)$ and $sgY = Y$ thus $sg \in N_{S(Y)}(Q)$. □

Remark 5.2. We can give a more detailed description of the conjugacy action of $N_{S(Y)}(Q)$ on the group $C_{V_r(\Sigma)}(Q)$. Recall that, by Theorem 4.2 this last group is a direct product of groups $G_1, \ldots, G_t$. We use the same notation as in Theorem 4.2. Let $g \in N_{S(Y)}(Q)$ and put $\varphi_{g(i)} = \varphi_i^g$ as before. Denote by $Z_{g(i)}, Z_i \subseteq Y$ the subsets of Y which are unions of Q-orbits of types $\varphi_{g(i)}$ and φ_i respectively. Then one easily checks that $gZ_{g(i)} = Z_i$ and $G_{g(i)} = G_i^g$. Moreover, recall that $G_i = K_i \rtimes V_{r_i}(\Sigma)$ with $K_i = \varinjlim(U_{r_i}(\Sigma), L_i)$ and $L_i = C_{S_{l_i}}(\varphi_i(Q))$. Then $r_{g(i)} = r_i$ and g maps the subgroup $V_{r_i}(\Sigma)$ of G_i to the same subgroup of $G_{g(i)}$ and K_i to $K_{g(i)}$. We also notice that g acts diagonally on the system $(U_{r_i}(\Sigma), L_i)$ mapping it to $(U_{r_{g(i)}}(\Sigma), L_{g(i)})$ In particular, the action of g on L_i is the restriction of its action on $C_{S(V)}(Q)$ and this action yields taking to the colimit the conjugation action $K_i^g = K_{g(i)}$.

Remark 5.3. Using [17, Theorem 5], one can also give a more detailed description of the groups L_i above:

$$L_i = N_{\varphi_i(Q)}(\varphi_i(Q)_1)/\varphi_i(Q)_1$$

where $\varphi_i(Q)_1$ is the stabiliser of one letter in $\varphi_i(Q)$. Of course, if Q is cyclic, then so is $\varphi_i(Q)$ and we get $\varphi_i(Q)_1 = 1$ and $L_i = \varphi_i(Q)$.

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Conchita Martínez-Pérez, Departamento de Matemáticas, Universidad de Zaragoza, 50009 Zaragoza, Spain
E-mail address: conmar@unizar.es

Francesco Matucci, Département de Mathématiques, Faculté des Sciences d'Orsay, Université Paris-Sud 11, Bâtiment 425, Orsay, France
E-mail address: francesco.matucci@math.u-psud.fr

Brita E. A. Nucinkis, Department of Mathematics, Royal Holloway, University of London, Egham, TW20 0EX, UK.
E-mail address: brita.nucinkis@rhul.ac.uk