Complete Bredon cohomology and its applications to hierarchically defined groups

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Abstract

By considering the Bredon analogue of complete cohomology of a group, we show that every group in the class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ of type Bredon-FP_∞ admits a finite dimensional model for $E_{\mathfrak{F}}G$.
We also show that abelian-by-infinite cyclic groups admit a 3-dimensional model for the classifying space for the family of virtually nilpotent subgroups. This allows us to prove that for $\mathfrak{F}$, the class of virtually cyclic groups, the class of $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$-groups contains all locally virtually soluble groups and all linear groups over $\mathbb{C}$ of integral characteristic.

1. *Introduction*

Classifying spaces with isotropy in a family have been the subject of intensive research, with a large proportion focussing on $\underline{E}G$, the classifying space with finite isotropy [19–21]. Classes of groups admitting a finite dimensional model for $\underline{E}G$ abound, such as elementary amenable groups of finite Hirsch length [9, 14], hyperbolic groups [28], mapping class groups [21] and $Out(F_n)$ [31]. Finding manageable models for $\underline{\underline{E}}G$, the classifying space for virtually cyclic isotropy, has been shown to be much more elusive. So far manageable models have been found for crystallographic groups [17], polycyclic-by-finite groups [24], hyperbolic groups [12], certain HNN-extensions [10], elementary amenable groups of finite Hirsch length [5, 6, 11] and groups acting isometrically with discrete orbits on separable complete CAT(0)-spaces [7, 22].
Let $\mathfrak{F}$ be a family of subgroups of a given group and denote by $E_{\mathfrak{F}}G$ the classifying space with isotropy in $\mathfrak{F}$. In this note we propose a method to decide whether a group has a finite dimensional model for $E_{\mathfrak{F}}G$ without actually providing a bound. This is closely related to Kropholler's Theorem that a torsion-free group in $\mathbf{LH}\mathfrak{F}$ of type FP_∞ has finite integral cohomological dimension [15]. To do this we consider groups belonging to the class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$, a class recently considered in [8]:
Let $\mathfrak{F}$ be a class of groups closed under taking subgroups. Let G be a group and set $\mathfrak{F}\cap G = \{H \le G \mid H$ is isomorphic to a subgroup in $\mathfrak{F}\}$. Let $\mathfrak{X}$ be a class of groups. Then $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$ is defined as the smallest class of groups containing the class $\mathfrak{X}$ with the property that if a group G acts cellularly on a finite dimensional CW-complex X with all isotropy subgroups in $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$, and such that for each subgroup $F \in \mathfrak{F}\cap G$ the fixed point set X^F is contractible, then G is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$. The class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$ is

defined to be the class of groups that are locally $\mathbf{H}^{\mathfrak{F}}\mathfrak{X}$-groups.

In this definition and throughout the paper, we always assume that a cellular action of group on a CW-complex is admissible. That is, if an element of group stabilises a cell, then it fixes it pointwise.

We generalise complete cohomology of a group to the Bredon setting and verify that some of the main results hold in this new context. This allows us to establish:

Theorem A. *Let G be group in $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ of type Bredon-FP_∞. Then G admits a finite dimensional model for $E_{\mathfrak{F}}G$.*

We consider the class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$, especially when $\mathfrak{F} = \mathfrak{X}$ is either the class of all finite groups or the class of all virtually cyclic groups. Note, that if $\mathfrak{F}$ contains the trivial group only, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$ is exactly Kropholler's class $\mathbf{LH}\mathfrak{X}$. If $\mathfrak{F}$ is the class of all finite groups, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ also turns out to be quite large. It contains all elementary amenable groups and all linear groups over a field of arbitrary characteristic (see [7], [8]). It is also closed under extensions, taking subgroups, amalgamated products, HNN-extensions, and countable directed unions. Here we show that similar closure operations hold when $\mathfrak{F}$ is the class of virtually cyclic groups.

In [8], it was shown that when $\mathfrak{F}$ is the class of finite groups, then $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ contains all elementary amenable groups. We show that when $\mathfrak{F} = \mathfrak{F}_{vc}$, the class of virtually cyclic groups, then $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ contains all locally virtually soluble groups. We also show that any countable subgroup of a general linear group $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic lies in $\mathbf{H}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$. Both of these results rely on the following:

Theorem B. *Let G be a semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group. Define $\mathfrak{H}$ to be the family of all virtually nilpotent subgroups of G. Then there exists a 3-dimensional model for $E_{\mathfrak{H}}G$.*

Another consequence of Theorem B is that any semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group lies in $\mathbf{H}_3^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$.

2. *Background on Bredon cohomology*

In this note, a family $\mathfrak{F}$ of subgroups of a group G is closed under conjugation and taking subgroups. The families most frequently considered are the family $\mathfrak{F}_{\mathrm{fin}}(G)$ of all finite subgroups of G and the family $\mathfrak{F}_{\mathrm{vc}}(G)$ of all virtually cyclic subgroups of G.

For a subgroup $K \leq G$ we consider:

$$\mathfrak{F} \cap K = \{H \cap K \mid H \in \mathfrak{F}\}.$$

Bredon cohomology has been introduced for finite groups by Bredon [3] and later generalised to arbitrary groups by Lück [19].

The orbit category $\mathscr{O}_{\mathfrak{F}}G$ is defined as follows: objects are the transitive G-sets G/H with $H \leq G$ and $H \in \mathfrak{F}$; morphisms of $\mathscr{O}_{\mathfrak{F}}G$ are all G-maps $G/H \to G/K$, where $H, K \in \mathfrak{F}$.

An $\mathscr{O}_{\mathfrak{F}}G$-module, or Bredon module, is a contravariant functor $M\colon\ \mathscr{O}_{\mathfrak{F}}G \to \mathfrak{Ab}$ from the orbit category to the category of abelian groups. A natural transformation $f\colon M \to N$ between two $\mathscr{O}_{\mathfrak{F}}G$-modules is called a morphism of $\mathscr{O}_{\mathfrak{F}}G$-modules.

The trivial $\mathscr{O}_{\mathfrak{F}}G$-module is denoted by $\mathbb{Z}_{\mathfrak{F}}$. It is given by $\mathbb{Z}_{\mathfrak{F}}(G/H) = \mathbb{Z}$ and $\mathbb{Z}_{\mathfrak{F}}(\varphi) = \mathrm{id}$ for all objects and morphisms φ of $\mathscr{O}_{\mathfrak{F}}G$.

The category of $\mathscr{O}_{\mathfrak{F}}G$-modules, denoted Mod-$\mathscr{O}_{\mathfrak{F}}G$, is a functor category and therefore inherits

properties from the category $\mathfrak{Ab}$. For example, a sequence $L \to M \to N$ of Bredon modules is exact if and only if, when evaluated at every $G/H \in \mathscr{O}_{\mathfrak{F}}G$, we obtain an exact sequence $L(G/H) \to M(G/H) \to N(G/H)$ of abelian groups.
Since $\mathfrak{Ab}$ has enough projectives, so does Mod-$\mathscr{O}_{\mathfrak{F}}G$, and we can define homology functors in Mod-$\mathscr{O}_{\mathfrak{F}}G$ analogously to ordinary cohomology, using projective resolutions.

There now follow the basic properties of free and projective $\mathscr{O}_{\mathfrak{F}}G$-modules as described in [19, 9.16, 9.17]. An $\mathfrak{F}$-set Δ is a collection of sets $\{\Delta_K \,|\, K \in \mathfrak{F}\}$. For any two $\mathfrak{F}$-sets Δ and Ω, an $\mathfrak{F}$-map is a family of maps $\{\Delta_K \to \Omega_K \,|\, K \in \mathfrak{F}\}$. Hence we have a forgetful functor from the category of $\mathscr{O}_{\mathfrak{F}}G$-modules to the category of $\mathfrak{F}$-sets. One defines the free functor as the left adjoint to this forgetful functor. This satisfies the usual universal property.
There is a more constructive description of free Bredon-modules as follows: Consider the right Bredon-module: $\mathbb{Z}[-,G/K]_{\mathfrak{F}}$ with $K \in \mathfrak{F}$. When evaluated at G/H we obtain the free abelian group $\mathbb{Z}[G/H,G/K]_{\mathfrak{F}}$ on the set $[G/H,G/K]_{\mathfrak{F}}$ of G-maps $G/H \to G/K$. These modules are free, cf. [19, p. 167], and can be viewed as the building blocks of the free right Bredon-modules. Generally, a free module is one of the form $\mathbb{Z}[-,\Delta]_{\mathfrak{F}}$, where Δ is a G-set with isotropy in $\mathfrak{F}$. Projectives are now defined to be direct summands of frees.

Given a covariant functor $F\colon\ \mathscr{O}_{\mathfrak{F}_1}G_1 \to \mathscr{O}_{\mathfrak{F}_2}G_2$ between orbit categories, one can now define induction and restriction functors along F, see [19, p. 166]:

$$\begin{array}{rccc} \mathrm{Ind}_F: & \mathscr{O}_{\mathfrak{F}_1}G_1 & \to & \mathscr{O}_{\mathfrak{F}_2}G_2 \\ & M(-) & \mapsto & M(-)\otimes_{\mathfrak{F}_1}[--,F(-)]_{\mathfrak{F}_2} \end{array}$$

and

$$\begin{array}{rccc} \mathrm{Res}_F: & \mathscr{O}_{\mathfrak{F}_2}G_2 & \to & \mathscr{O}_{\mathfrak{F}_1}G_1 \\ & M(--) & \mapsto & M\circ F(--) \end{array}$$

Since these functors are adjoint to each others, Ind_F commutes with arbitrary colimits [26, pp. 118f.] and preserves free and projective Bredon modules [19, p. 169]. The case of particular interest is when F is given by inclusion of a subgroup of G.
For subgroup K of G we consider the following functor

$$\begin{array}{rccc} \iota_K^G: & \mathscr{O}_{\mathfrak{F}\cap K}K & \to & \mathscr{O}_{\mathfrak{F}}G \\ & K/H & \mapsto & G/H. \end{array}$$

and denote the corresponding induction and restriction functors by Ind_K^G and Res_K^G respectively.

LEMMA 2·1. [30, Lemma 2.9] *Let K be a subgroup of G. Then* Ind_K^G *is an exact functor.*

Symmond's [30] methods also yields; for a short account see also the proof of Lemma 3.5 in [14]:

LEMMA 2·2. *Let $K \le H \le G$ be subgroups. Then*

$$\mathrm{Ind}_K^G\,\mathbb{Z}_{\mathfrak{F}} \cong \mathbb{Z}[-,G/K]_{\mathfrak{F}},$$

and

$$\mathrm{Ind}_H^G\,\mathbb{Z}[-,H/K]_{\mathfrak{F}\cap H} \cong \mathbb{Z}[-,G/K]_{\mathfrak{F}}.$$

The Bredon cohomological dimension $\mathrm{cd}_{\mathfrak{F}}\,G$ of a group G with respect to the family $\mathfrak{F}$ of subgroups is the projective dimension $\mathrm{pd}_{\mathfrak{F}}\,\mathbb{Z}_{\mathfrak{F}}$ of the trivial $\mathscr{O}_{\mathfrak{F}}G$-module $\mathbb{Z}_{\mathfrak{F}}$. The cellular chain complex of a model for $E_{\mathfrak{F}}G$ yields a free resolution of the trivial $\mathscr{O}_{\mathfrak{F}}G$-module $\mathbb{Z}_{\mathfrak{F}}$ [19, pp. 151f.].

In particular, this implies that for the Bredon geometric dimension $\mathrm{gd}_{\mathfrak{F}} G$, the minimal dimension of a model for $E_{\mathfrak{F}}G$, we have

$$\mathrm{cd}_{\mathfrak{F}} G \leq \mathrm{gd}_{\mathfrak{F}} G.$$

Furthermore, one always has:

PROPOSITION 2·3. [23, Theorem 0.1 (i)] *Let G be a group. Then*

$$\mathrm{gd}_{\mathfrak{F}} G \leq \max(3, \mathrm{cd}_{\mathfrak{F}} G).$$

Next, suppose $\mathfrak{T}$ and $\mathfrak{H}$ are families of subgroups of a group G where $\mathfrak{T} \subseteq \mathfrak{H}$. In Section 5, we will need to adapt a model for $E_{\mathfrak{T}}G$ to obtain a model for $E_{\mathfrak{H}}G$. For this we will use a general construction of Lück and Weiermann (see [24, §2]). We recall the basics of this construction:
Suppose that there exists an equivalence relation $\sim$ on the set $\mathscr{S} = \mathfrak{H} \smallsetminus \mathfrak{T}$ that satisfies the following properties:

- $\forall H, K \in \mathscr{S} : H \subseteq K \Rightarrow H \sim K$;
- $\forall H, K \in \mathscr{S}, \forall x \in G : H \sim K \Leftrightarrow H^x \sim K^x$.

An equivalence relation that satisfies these properties is called a *strong equivalence relation*. Let $[H]$ be an equivalence class represented by $H \in \mathscr{S}$ and denote the set of equivalence classes by $[\mathscr{S}]$. The group G acts on $[\mathscr{S}]$ via conjugation, and the stabiliser group of an equivalence class $[H]$ is

$$\mathrm{N}_G[H] = \{x \in \Gamma \mid H^x \sim H\}.$$

Note that $\mathrm{N}_G[H]$ contains H as a subgroup. Let $\mathscr{I}$ be a complete set of representatives $[H]$ of the orbits of the conjugation action of G on $[\mathscr{S}]$. Define for each $[H] \in \mathscr{I}$ the family

$$\mathfrak{T}[H] = \{K \leq \mathrm{N}_G[H] \mid K \in \mathscr{S}, K \sim H\} \cup \Big(\mathrm{N}_G[H] \cap \mathfrak{T}\Big)$$

of subgroups of $\mathrm{N}_G[H]$.

PROPOSITION 2·4 (Lück-Weiermann, [24, 2.5]). *Let $\mathfrak{T} \subseteq \mathfrak{H}$ be two families of subgroups of a group G such that $S = \mathfrak{H} \smallsetminus \mathfrak{T}$ is equipped with a strong equivalence relation. Denote the set of equivalence classes by $[\mathscr{S}]$ and let $\mathscr{I}$ be a complete set of representatives $[H]$ of the orbits of the conjugation action of G on $[\mathscr{S}]$. If there exists a natural number d such that $\mathrm{gd}_{\mathfrak{T} \cap \mathrm{N}_G[H]}(\mathrm{N}_G[H]) \leq d-1$ and $\mathrm{gd}_{\mathfrak{T}[H]}(\mathrm{N}_G[H]) \leq d$ for each $[H] \in \mathscr{I}$, and such that $\mathrm{gd}_{\mathfrak{T}}(G) \leq d$, then $\mathrm{gd}_{\mathfrak{H}}(G) \leq d$.*

3. *Complete Bredon cohomology*

Since Mod-$\mathscr{O}_{\mathfrak{F}}G$ is an abelian category, we can just follow the approaches of Mislin [29] and Benson-Carlson [2]. We will, however, include the main steps of the construction. We will begin by describing the Satellite construction due to Mislin [29]. The methods used there can be carried over to the Bredon-setting by applying [25, XII.7-8.].
Let M be an $\mathscr{O}_{\mathfrak{F}}G$-module and denote by FM the free $\mathscr{O}_{\mathfrak{F}}G$-module on the underlying $\mathfrak{F}$-set of M. Let $\Omega M = ker(FM \twoheadrightarrow M)$, and inductively $\Omega^n M = \Omega(\Omega^{n-1} M)$. Let T be an additive functor from Mod-$\mathscr{O}_{\mathfrak{F}}G$ to the category of abelian groups. Then the left satellite of T is defined as

$$S^{-1}T(M) = ker(T(\Omega M) \to T(FM)).$$

Furthermore, $S^{-n}T(M) = S^{-1}(S^{-n+1}T(M))$, and the family $\{S^{-n} \mid n \geq 0\}$ forms a connected sequence of functors where $S^{-n}T(P) = 0$ for all projective $\mathscr{O}_{\mathfrak{F}}G$-modules P and $n \geq 1$. Following the approach in [29] further, we call a connected sequence of additive functors $T^* = \{T^n \,|\, n \in \mathbb{Z}\}$ from Mod-$\mathscr{O}_{\mathfrak{F}}G$ to the category of abelian groups a $(-\infty,+\infty)$-Bredon-cohomological functor, if for every short exact sequence $M' \rightarrowtail M \twoheadrightarrow M''$ of $\mathscr{O}_{\mathfrak{F}}G$-modules the associated sequence

$$\cdots \to T^n M' \to T^n M \to T^n M'' \to T^{n+1} M' \to \cdots$$

is exact. Obviously, Bredon-cohomology $\mathrm{H}^*_{\mathfrak{F}}(G,-)$ is such a functor with the convention that $\mathrm{H}^n_{\mathfrak{F}}(G,-) = 0$ whenever $n < 0$.

DEFINITION 3·1. A $(-\infty,+\infty)$-Bredon-cohomological functor $T^* = \{T^n \,|\, n \in \mathbb{Z}\}$ is called P-complete if $T^n(P) = 0$ for all $n \in \mathbb{Z}$ and every projective $\mathscr{O}_{\mathfrak{F}}G$-module P.
A morphism $\varphi^* : U^* \to V^*$ of $(-\infty,+\infty)$-Bredon-cohomological functors is called a P-completion, if V^* is P-complete and if every morphism $U^* \to T^*$ into a P-complete $(-\infty,+\infty)$-Bredon-cohomological functor T^* factors uniquely through $\varphi^* : U^* \to V^*$.

The following theorem is now the exact analogue to [29, Theorem 2.2].

THEOREM 3·2. *Every $(-\infty,+\infty)$-Bredon-cohomological functor T^* admits a unique P-completion $\widehat{T}^*$ given by*

$$\widehat{T}^j(M) = \varinjlim_{k \geq 0} S^{-k} T^{j+k}(M)$$

for any $M \in$ Mod-$\mathscr{O}_{\mathfrak{F}}G$. □

In particular, we have, for every $\mathscr{O}_{\mathfrak{F}}G$-module M that

$$\widehat{\mathrm{Ext}}^j_{\mathfrak{F}}(M,-) = \varinjlim_{k \geq 0} S^{-k}\,\mathrm{Ext}^{j+k}_{\mathfrak{F}}(M,-).$$

We have immediately:

LEMMA 3·3. *Let M and N be $\mathscr{O}_{\mathfrak{F}}G$-modules. If either of these has finite projective dimension, then*

$$\widehat{\mathrm{Ext}}^*_{\mathfrak{F}}(M,N) = 0.$$

We can also mimic Benson and Carlson's approach [2]. For any two $\mathscr{O}_{\mathfrak{F}}G$-modules we denote by $[M,N]_{\mathfrak{F}}$ the quotient of $\mathrm{Hom}_{\mathfrak{F}}(M,N)$ by the subgroup of those homomorphisms factoring through a projective module. Then it follows that there is a homomorphism $[M,N]_{\mathfrak{F}} \to [\Omega M, \Omega N]_{\mathfrak{F}}$ and it can be shown analogously to [29, Theorem 4.4] that

$$\widehat{\mathrm{Ext}}^n_{\mathfrak{F}}(M,M) = \varinjlim_{k,k+n \geq 0} [\Omega^{k+n} M, \Omega^k N]_{\mathfrak{F}}.$$

This now allows us to deduce the following Lemma, which is an analogue to [15, 4.2].

LEMMA 3·4. $\widehat{\mathrm{Ext}}^0_{\mathfrak{F}}(M,M) = 0$ *if and only if M has finite projective dimension. In particular,*

$$\widehat{\mathrm{H}}^0_{\mathfrak{F}}(G, \mathbb{Z}_{\mathfrak{F}}) = 0 \iff \mathrm{cd}_{\mathfrak{F}}\, G < \infty.$$

4. *Proof of Theorem A*

The proof of Theorem A is analogous to the proof of the main result in [15]. We begin by recording two easy lemmas, which have their analogues in [15, 3.1] and [15, 4.1] respectively.

LEMMA 4·1. *Let*

$$0 \to M_n \to M_{n-1} \to \ldots \to M_1 \to M_0 \to L \to 0$$

be an exact sequence of $\mathcal{O}_{\mathfrak{F}}G$*-modules and* i *be an integer such that* $\mathrm{H}^i_{\mathfrak{F}}(G,L) \neq 0$. *Then there exists an integer* $0 \leq j \leq n-1$ *such that* $\mathrm{H}^{j+i}_{\mathfrak{F}}(G,M_j) \neq 0$..

Proof. This is an easy dimension shifting argument. □

LEMMA 4·2. *Let* G *be a group such that* $\mathrm{H}^k_{\mathfrak{F}}(G,-)$ *commutes with direct limits for infinitely many* k*, then* $\widehat{\mathrm{H}}^k_{\mathfrak{F}}(G,-)$ *commutes for all* $k \in \mathbb{Z}$.

Proof. This follows from the fact that direct limits commute with each other. □

The proof of Theorem A now relies on the fact that one can hierarchically decompose the class $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ in exactly the same way as Kropholler's decomposition, see [8, 15]:

- $\mathbf{H}^{\mathfrak{F}}_0\mathfrak{F} = \mathfrak{F}$;
- For an ordinal $\alpha > 0$, we let $\mathbf{H}^{\mathfrak{F}}_{\alpha}\mathfrak{F}$ be the class of groups acting cellularly on a finite dimensional complex X such that each stabiliser subgroup lies in $\mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ for some $\beta < \alpha$ and such that X^K is contractible for all $K \in \mathfrak{F}$.

A group G now lies in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ if and only if it lies in some $\mathbf{H}^{\mathfrak{F}}_{\alpha}\mathfrak{F}$ for some ordinal α.
In particular, $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is subgroup closed.

LEMMA 4·3. *Let* G *be a group and* G_λ, $\lambda \in \Lambda$ *its finitely generated subgroups. Then we have the following isomorphism:*

$$\varinjlim_{\lambda \in \Lambda} \mathbb{Z}[-,G/G_\lambda]_{\mathfrak{F}\cap G_\lambda} \cong \mathbb{Z}_{\mathfrak{F}}.$$

Proof. This follows directly from Lemma 2·2. □

THEOREM 4·4. *Let* G *be a group in* $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ *and suppose that* $\widehat{\mathrm{H}}^*_{\mathfrak{F}}(G,-)$ *commutes with direct limits. Then* $\mathrm{cd}_{\mathfrak{F}}\, G < \infty$.

Proof. We prove this by contradiction and suppose that $\mathrm{cd}_{\mathfrak{F}}\, G = \infty$. Hence, by Lemma 3·4, we have that $\widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathbb{Z}_{\mathfrak{F}}) \neq 0$. We claim that then there exists a group $H \in \mathfrak{F}$ and an integer $i \geq 0$ such that $\widehat{\mathrm{H}}^i_{\mathfrak{F}}(G,\mathrm{Ind}^G_H \mathbb{Z}_{\mathfrak{F}\cap H}) \neq 0$. By Lemma 2·2, we have $\mathrm{Ind}^G_H \mathbb{Z}_{\mathfrak{F}\cap H} \cong \mathbb{Z}[-,G/H]$, which is projective, giving us the desired contradiction.
It now remains to prove the claim: Let $\mathscr{S}$ be the set of ordinals β such there exists a $i \geq 0$ and $H \leq G$ lying in $\mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ and such that $\mathrm{H}^i_{\mathfrak{F}}(G,\mathrm{Ind}^G_H \mathbb{Z}_{\mathfrak{F}\cap H}) \neq 0$. If we can prove that $0 \in \mathscr{S}$, we are done.
(1) We show that $\mathscr{S}$ is not empty: Let $\{G_\lambda \,|\, \lambda \in \Lambda\}$ be the family of all finitely generated subgroups of G. Hence, applying Lemma 4·3 and the fact that $\widehat{\mathrm{H}}^*_{\mathfrak{F}}(G,-)$ commutes with direct limits, we get

$$\widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathbb{Z}_{\mathfrak{F}}) \cong \widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\varinjlim_{\lambda\in\Lambda} \mathbb{Z}[-,G/G_\lambda]_{\mathfrak{F}\cap G_\lambda}) \cong \varinjlim_{\lambda\in\Lambda} \widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathbb{Z}[-,G/G_\lambda]_{\mathfrak{F}\cap G_\lambda}).$$

Since $\widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathbb{Z}_{\mathfrak{F}}) \neq 0$, there exists a finitely generated subgroup G_λ such that, see also Lemma 2·2,

$$\widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathbb{Z}[-,G/G_\lambda]_{\mathfrak{F}\cap G_\lambda}) \cong \widehat{\mathrm{H}}^0_{\mathfrak{F}}(G,\mathrm{Ind}^G_{G_\lambda} \mathbb{Z}_{\mathfrak{F}\cap G_\lambda}) \neq 0.$$

Since $G \in \mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$, and $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is subgroup closed, $G_\lambda \in \mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ and in particular, there is an ordinal β such that $G_\lambda \in \mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$. Hence $\beta \in \mathscr{S}$.

(2) We now show that, if $0 \neq \beta \in \mathscr{S}$, then there is an ordinal $\gamma < \beta$ such that $\gamma \in \mathscr{S}$: Let $0 \neq \beta \in \mathscr{S}$. Then there is a $H \in G$ and $i \geq 0$ such that $H \in \mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ and

$$\widehat{\mathrm{H}}^{i}_{\mathfrak{F}}(G, \mathrm{Ind}_H^G \mathbb{Z}_{\mathfrak{F}\cap H}) \neq 0.$$

Hence H acts cellularly on a finite dimensional contractible space X such that each isotropy group lies in some $\mathbf{H}^{\mathfrak{F}}_{\gamma}\mathfrak{F}$ for $\gamma < \beta$ and such that X^K is contractible if $K \in \mathfrak{F}$. Hence we have an exact sequence of free $\mathscr{O}_{\mathfrak{F}}H$-modules:

$$0 \to C_n(X^{(-)}) \to C_{n-1}(X^{(-)}) \to \dots C_1(X^{(-)}) \to C_0(X^{(-)}) \to \mathbb{Z}_{\mathfrak{F}\cap H} \to 0.$$

Each

$$C_k(X^{(-)}) \cong \mathbb{Z}[-, \bigoplus_{\sigma_k \in \Delta_k} H/H_{\sigma_k}],$$

where Δ_k is the set of orbit representatives for the k-cells of X. Furthermore, by Lemma 2·2, upon induction, we obtain an exact sequence of $\mathscr{O}_{\mathfrak{F}}G$-modules as follows:

$$0 \to \bigoplus_{\sigma_n \in \Delta_n} \mathrm{Ind}^G_{H_{\sigma_n}} \mathbb{Z}_{\mathfrak{F}\cap H_{\sigma_n}} \to \dots \to \bigoplus_{\sigma_1 \in \Delta_1} \mathrm{Ind}^G_{H_{\sigma_1}} \mathbb{Z}_{\mathfrak{F}\cap H_{\sigma_1}} \to \bigoplus_{\sigma_0 \in \Delta_0} \mathrm{Ind}^G_{H_{\sigma_0}} \mathbb{Z}_{\mathfrak{F}\cap H_{\sigma_0}} \to \mathrm{Ind}^G_H \mathbb{Z}_{\mathfrak{F}\cap H} \to 0.$$

Now, by Lemma 4·1, there is a $k \geq 0$ such that

$$\widehat{\mathrm{H}}^{j+k}_{\mathfrak{F}}(G, \bigoplus_{\sigma_k \in \Delta_k} \mathrm{Ind}^G_{H_{\sigma_k}} \mathbb{Z}_{\mathfrak{F}\cap H_{\sigma_k}}) \neq 0.$$

Since $\widehat{\mathrm{H}}^{j+k}_{\mathfrak{F}}(G, -)$ commutes, in particular, with direct sums, there is a $\sigma_k \in \Delta_k$ such that

$$\widehat{\mathrm{H}}^{j+k}_{\mathfrak{F}}(G, \mathrm{Ind}^G_{H_{\sigma_k}} \mathbb{Z}_{\mathfrak{F}\cap H_{\sigma_k}}) \neq 0,$$

thus proving the claim.

□

COROLLARY 4·5. *Let G be a group in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ and suppose that $\widehat{\mathrm{H}}^{*}_{\mathfrak{F}}(G, -)$ commutes with direct sums. Then* $\mathrm{cd}_{\mathfrak{F}}\, G < \infty$.

Proof. The proof is analogous to the proof of Theorem 4·4. To show that $\mathscr{S}$ is not empty, we can use the fact that $G \in \mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ for some β. Then follow step (2) as above. □

Theorem A now follows directly from Theorem 4·4, as, for groups of type Bredon-FP_∞ it follows that $\widehat{\mathrm{H}}^{*}_{\mathfrak{F}}(G, -)$ commutes with direct limits, see Lemma 4·2 and [27, Theorem 5.3].

5. *Some properties of* $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$

We consider containment and closure properties of the class $\mathbf{LH}^{\mathfrak{F}}\mathfrak{X}$ especially when $\mathfrak{F}$ either the class of finite groups or the class of virtually cyclic groups.

Let A be an abelian group and $\mathbb{Z} = \langle t \rangle$. Consider the semi-direct product $G = A \rtimes \mathbb{Z}$ with t acting on A by conjugation. To shorten the notation, wherever necessary, we will identify A with its image in G. Fix an arbitrary integer $k > 0$. For each integer $i \geq 0$, we define the subgroups P_i^k of A inductively as follows:

- $P_0^k = \langle 1 \rangle$,
- $P_{i+1}^k = \{x \in A \mid t^k(x)x^{-1} \in P_i^k\}$ for $i \geq 0$.

An easy induction on i shows that each P_i^k is a normal subgroup of G. We set $P^k = \cup_{i \geq 0} P_i^k$. Note that P^k is also a normal subgroup of G and it has the property that if $t^k(x)x^{-1} \in P^k$ and $x \in A$ then $x \in P^k$. In fact, P^k can be defined as the smallest subgroup of G with this property.

LEMMA 5·1. *Let $a \in A$. For each $i \geq 0$, consider the subgroup $G_i^k = \langle P_i^k, (a,t^k) \rangle$ of G. Then $P_i^k = G_i^k \cap A$ and G_i^k is nilpotent of nilpotency class at most $i+1$.*

Proof. For the first part one only needs to check that $G_i^k \cap A$ is in P_i^k as the reverse inclusion is trivially satisfied. But this follows immediately from the fact that $G_i^k \cong P_i^k \rtimes \langle (a,t^k) \rangle$.
For the second claim, note that $[G_i^k, G_i^k]$ lies in A. Let $0 \leq m \leq i$. The only possibly nontrivial m-fold commutators starting with an element $x \in P_i^k$ are of the form

$$y_m = [(a_1, t^{kn_1}), [(a_2, t^{kn_2}), \dots [(a_m, t^{kn_m}), x] \dots]]$$

for $a_1, \dots, a_m \in P_i^k$ where we denote $y_0 = x$. We claim that y_m is in P_{i-m}^k. Assuming the claim, we have that y_i is trivial and hence G_i^k is nilpotent of nilpotency class at most $i+1$.
To prove the claim we use induction on m. The case $m = 0$ is trivially satisfied. Now, suppose $m > 0$. Then, by induction, the $(m-1)$-fold commutator

$$z = [(a_2, t^{kn_2}), \dots [(a_m, t^{kn_m}), x] \dots] \in P_{i-m+1}^k.$$

But then

$$y_m = [(a_1, t^{kn_1}), z] = t^{kn_1}(z)z^{-1} \in P_{i-m}^k$$

because $z \in P_{i-m+1}^k$. This finishes the claim. □

LEMMA 5·2. *For a given integer $i > 0$, let N be a nilpotent subgroup of G of nilpotency class i, which is not contained in A. Then $N = \langle B, (a,t^k) \rangle$ where $B = P_i^k \cap N$ for some $a \in A$ and $k > 0$. In particular, N is contained in $G_i^k = \langle P_i^k, (a,t^k) \rangle$.*

Proof. Clearly, $N = \langle B, (a,t^k) \rangle$ where $B = A \cap N$ for some $a \in A$ and $k > 0$. It is left to show that $B \leq P_i^k$. Let $0 \leq m \leq i$ and consider $(i-m)$-fold commutator

$$y_{(i-m)} = [(a,t^k), [(a,t^k), \dots [(a,t^k), x] \dots]]$$

where we denote $y_0 = x \in B$. We will prove by induction that $y_{(i-m)} \in P_m^k$. Since N has nilpotency class i, $y_i = 1 \in P_0^k$. So, assume $m > 0$. Consider $z = [(a,t^k), y_{(i-m)}]$. By induction, $z \in P_{m-1}^k$. But $z = t^k(y_{(i-m)})y_{(i-m)}^{-1}$. So, by the definition of P_m^k, we have $y_{(i-m)} \in P_m^k$.
Now, taking $m = i$, gives us that each $x \in B$ lies in P_i^k. □

PROPOSITION 5·3. *Define $P = \cup_{k>0} P^k$ in A. Then*

(a) *P is a normal subgroup of G.*

(b) *P is the smallest subgroup of G defined by the property that if $t^k(x)x^{-1} \in P$ for some $k > 0$ and $x \in A$, then $x \in P$.*

(c) *Let $N = \langle B, (a,t^l) \rangle$ where $B \leq P$, $a \in A$ and $l \geq 1$. Then N is locally virtually nilpotent.*

(d) *Let N be a locally nilpotent subgroup of G not contained in A. Then $N \cap A$ is contained in P.*

Proof. (a). Given any integers $k_1, k_2 > 0$ such that k_1 divides k_2, it follows that $P_{k_1} \subseteq P_{k_2}$. This shows that the set P is a subgroup of A. Since each P_k is a normal subgroup of G, their union P is also a normal in G.
(b). Let P' be the smallest subgroup of G defined by the property stated in (b); denote this property by $(*)$. Note that $P = \cup_{i\geq 0} P_i$ where the subgroups P_i are defined inductively by:

- $P_0 = \langle 1 \rangle$,
- $P_{i+1} = \{x \in A \mid \exists k > 0, t^k(x)x^{-1} \in P_i\}$ for $i \geq 0$.

An easy induction on i shows that each P_i is a subgroup of P'. Hence, $P \leq P'$. But since P has the property $(*)$ and P' is the smallest subgroup of G with the property $(*)$, we deduce that $P = P'$.
(c). Let $H = \langle b_1, \ldots, b_s, (a, t^l) \rangle$, for some $b_1, \ldots, b_s \in P$, $a \in A$, and $l, s \geq 1$. It suffices to show that H is virtually nilpotent. Since $P = \cup_{i,k>0} P_i^k$, we conclude that for each $j \in \{1, ..., s\}$, we have $b_j \in P_{i_j}^{k_j}$ for some $i_j, k_j > 0$. Set $k = \prod_{j=1}^{s} k_{i_j}$ and $i = \sup\{i_j \mid 1 \leq j \leq s\}$. It follows that the group $H' = \langle b_1, \ldots, b_s, (a, t^l)^k \rangle$ is a finite index subgroup of H, and $H' \leq \langle P_i^{kl}, (a, t^l)^k \rangle$. So, by Lemma 5·1, H' is nilpotent.
(d). This is a direct consequence of Lemma 5·2. □

THEOREM 5·4. *Let G be a semi-direct product $A \rtimes \mathbb{Z}$ where A is a countable abelian group. Define $\mathfrak{H}$ to be the family of all virtually nilpotent subgroups of G. Then there exists a 3-dimensional model for $E_{\mathfrak{H}}G$.*

Proof. Let $\mathfrak{T}$ be the subfamily of $\mathfrak{H}$ consisting of all countable subgroups of A. We will use the construction of Lück and Weiermann that adapts the model for $E_{\mathfrak{T}}G$ to a model for the larger family $\mathfrak{H}$.
First, we need a strong equivalence relation on the set

$$\mathscr{S} = \mathfrak{H} \smallsetminus \mathfrak{T} = \{H \leq G \mid H \nleq A \text{ and } H \text{ is virtually nilpotent}\}.$$

Let $\overline{}: G \to G/P$ denote the quotient homomorphism. By Proposition 5·3, we have that if $H \in \mathfrak{H}$, then $\overline{H}$ is virtually cyclic.
Now, for $H, S \in \mathscr{S}$, we say that there is a relation $H \sim S$ if $|\overline{H} \cap \overline{S}| = \infty$. It is not difficult to show that this indeed defines a strong equivalence relation on the set $\mathscr{S}$. Our group G acts by conjugation on the set of equivalence classes $[\mathscr{S}]$ and the stabiliser of an equivalence class $[H]$ is

$$N_G[H] = \{x \in G \mid H^x \sim H\}.$$

Note that $H \sim Z$ if $Z = \langle h \rangle$, $h \in H$, $h \notin A$. Hence $N_G[H] = N_G[Z]$. Clearly, Z is a subgroup of $N_G[Z]$ and $N_G[Z] = \langle B, Z \rangle$ for some subgroup $B \leq A$. But for each $b \in B$, we have $Z^b \sim Z$. Writing $h = (a, t^k)$ for some $a \in A$ and $k > 0$, this implies that $\overline{b^{-1}(a,t^k)^n b} = \overline{(a,t^k)^n}$ in G/P for some nonzero integer n. A quick computation then shows that $\overline{t^{kn}(b)} = \overline{b}$ in G/P. This means that $t^{kn}(b)b^{-1} \in P$. Then, by Proposition 5·3(b), $b \in P$. Hence, by part (c) of Proposition 5·3, we have that every finitely generated subgroup K of $N_G[Z]$ that contains Z is virtually nilpotent. Thus $K \in \mathscr{S}$ and $K \sim Z$ and hence it is in the family

$$\mathfrak{T}[H] = \{K \leq \mathrm{N}_G[H] \mid K \in \mathscr{S}, K \sim H\} \cup \Big(\mathrm{N}_G[H] \cap \mathfrak{T}\Big)$$

of subgroups of $\mathrm{N}_G[H]$. It follows that $N_G[H]$ is a countable directed union of subgroups that are in $\mathfrak{T}[H]$ but are not in $\mathrm{N}_G[H] \cap \mathfrak{T}$. Denote by T the tree on which $\mathrm{N}_G[H]$ acts with stabilisers as such subgroups. Note that the action of G on $\mathbb{R}$ via the natural projection of G onto $\mathbb{Z}$ makes $\mathbb{R}$ into a model for $E_{\mathfrak{T}}G$. Restricting this action to $\mathrm{N}_G[H]$ and considering the induced action on the join $T*\mathbb{R}$ gives us a 3-dimensional model for $E_{\mathfrak{T}[H]}\mathrm{N}_G[H]$. Invoking Proposition 2·4 entails a 3-dimensional model for $E_{\mathfrak{H}}G$, as was required to prove. □

REMARK 5·5. Since finitely generated nilpotent groups lie $\mathbf{H}_1^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$, it follows that countable virtually nilpotent groups are in $\mathbf{H}_2^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$. We obtain that the group $G=A\rtimes\mathbb{Z}\in\mathbf{H}_3^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$.

REMARK 5·6. In the statement of Theorem 5·4, one could enlarge $\mathfrak{H}$ to be the family of all locally virtually nilpotent subgroups of G. Then its proof together with Proposition 5·3(c)-(d) would imply that $\mathrm{N}_G[H]$ is $\mathfrak{T}[H]$. So, a point with the trivial action of $\mathrm{N}_G[H]$ would then be a model for $E_{\mathfrak{T}[H]}\mathrm{N}_G[H]$ for each $H\in\mathscr{S}$. Applying Proposition 2·4 would give us a 2-dimensional model for $E_{\mathfrak{H}}G$.

In the next example, we illustrate that the family $\mathfrak{H}$ of all virtually nilpotent subgroups of G can contain nilpotent subgroups of G of arbitrarily high nilpotency class.

EXAMPLE 5·7. Consider the unrestricted wreath product $W=\mathbb{Z}\wr\mathbb{Z}$. Rewriting this group as a semi-direct product, we have that $W=A\rtimes\mathbb{Z}$ where $A=\prod_{i\in\mathbb{Z}}\mathbb{Z}$ and the standard infinite cyclic subgroup of W is generated by t and acts on A by translations. Define G to be the subgroup of W given by $G=P\rtimes\mathbb{Z}$. For each $k>0$, note that P_1^k is the subgroup of A of all k-periodic sequences of integers and hence $P_1^k\cong\mathbb{Z}^k$. Since $P_1=\cup_{k>0}P_1^k$, it is countable of infinite rank. Similarly, one can argue that P_2/P_1 is countable of infinite rank and hence P_2 is also countable. Continuing in this manner, one obtains that P_i is countable for each $i>0$ and since P is a countable union of these groups it is itself countable. This shows that the group G satisfies the hypothesis of Theorem 5·4.
Now, it is not difficult to see, that for each $i>0$, the subgroup $P_i^1\rtimes\mathbb{Z}$ of G is nilpotent of nilpotency class i.

THEOREM 5·8. *Let $\mathfrak{F}$ be a class of subgroups of finitely generated groups. Then $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is closed under countable directed unions. If $\mathfrak{F}$ is the class of all virtually cyclic groups, then $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is closed under finite extensions and under extensions with virtually soluble kernels. In particular, $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$ contains all locally virtually soluble groups.*

Proof. The proof of the first fact is the same as for the class of finite groups $\mathfrak{F}$ given in Proposition 5.5 in [8]. That is, let G be a countable directed union of groups that are in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Then G acts on a tree with stabilisers exactly the subgroups that comprise this union. It is now easy to see that the action of G on the tree satisfies the stabiliser and the fixed-point set conditions of the definition of $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$-groups. This shows that G is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$.
For the second part, first note that by the Serre's Construction, $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$ is closed under finite extensions (see the proof of [20, 2.3(2)]).
Let G be a countable group that fits into an extension $K\rightarrowtail G\twoheadrightarrow Q$ such that K is virtually soluble and $Q\in\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Suppose K is finite. Then an easy transfinite induction on the ordinal associated to the class containing Q shows G lies in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. In general, since K is virtually soluble, it contains a soluble characteristic subgroup of finite index, which must be normal in G. In view of these facts, without loss of generality, we can assume that K is soluble.
Next, we proceed by the induction on the derived length of K to prove that $G\in\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. When K is the trivial group, then $G=Q\in\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Suppose K is nontrivial. Since $[K,K]$ is a characteristic subgroup of K, it is a normal subgroup of G. So, there are extensions

$$[K,K]\rightarrowtail G\twoheadrightarrow G/[K,K]\quad\text{and}\quad K/[K,K]\rightarrowtail G/[K,K]\twoheadrightarrow Q.$$

We claim that $G/[K,K] \in \mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Then by induction applied to the first extension $G \in \mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Let us now prove the claim.
In view of the second extension, it suffices to show that given an extension

$$A \rightarrowtail S \twoheadrightarrow Q$$

where A is abelian and $Q \in \mathbf{H}^{\mathfrak{F}}_{\alpha}\mathfrak{F}$, then $S \in \mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. We use transfinite induction on the ordinal α. When $\alpha = 0$, then S is virtually a semi-direct product $A \rtimes \mathbb{Z}$. Hence, by Theorem 5·4, it is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Suppose $\alpha > 0$, then there is a finite dimensional Q-CW-complex X such that each stabiliser subgroup lies in $\mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ for some $\beta < \alpha$ and such that X^H is contractible for all $H \in \mathfrak{F}$. The group S also acts on X via the projection onto Q. Each stabiliser of this action is abelian-by-$\mathbf{H}^{\mathfrak{F}}_{\beta}\mathfrak{F}$ and hence by transfinite induction is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. Therefore, $S \in \mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. This finishes the claim and the proof. □

Recall that a subgroup G of $\mathrm{GL}_n(\mathbb{C})$ is said to be of *integral characteristic* if the coefficients of the characteristic polynomial of every element of G are algebraic integers. It follows that G has integral characteristic if and only if the characteristic roots of every element of G are algebraic integers (see [1, §2]).

THEOREM 5·9. *Let G be a countable subgroup of some $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic. Then G lies in $\mathbf{H}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$.*

Proof. Since the class $\mathbf{H}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ is closed under countable directed unions, it is enough to prove the claim when G is finitely generated. Note that under a standard embedding of $\mathrm{GL}_n(\mathbb{C})$ into $\mathrm{SL}_{n+1}(\mathbb{C})$ the image of G is still of integral characteristic. So, we can assume that G is a subgroup of $\mathrm{SL}_n(\mathbb{C})$ of integral characteristic. Let A be the finitely generated subring of $\mathbb{C}$ generated by the matrix entries of a finite set of generators of G and their inverses. Then G is a subgroup of $\mathrm{SL}_n(A)$.
Let $\mathbb{F}$ denote the quotient field of A. Proceeding as in the proof of Theorem 3.3 of [1], there is an epimorphism $\rho : G \to H_1 \times \cdots \times H_r$ such that the kernel U of ρ is a unipotent subgroup of G and for each $1 \le i \le r$, H_i is a subgroup of some $\mathrm{GL}_{n_i}(A)$ of integral characteristic where the canonical action of H_i on $\mathbb{F}^{n_i}$ is irreducible and $\sum n_i = n$. So, by the proof of Theorem B in [4], each group H_i admits a finite dimensional model for $E_{\mathfrak{F}_{\mathrm{vc}}\cap H_i}H_i$. Applying [24, 5.6], one immediately sees that the product $Q = H_1 \times \cdots \times H_r$ admits a finite dimensional model for $E_{\mathfrak{F}_{\mathrm{vc}}\cap Q}Q$. So, Q is in $\mathbf{H}^{\mathfrak{F}_{\mathrm{vc}}}_1\mathfrak{F}_{\mathrm{vc}}$. By Theorem 5·8, it follows that G lies in $\mathbf{H}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$. □

COROLLARY 5·10. *Let $\mathfrak{F}$ be either the class of all finite groups or the class of all virtually cyclic groups and let G be a group such that $\widehat{\mathrm{H}}^*_{\mathfrak{F}}(G,-)$ commutes with direct limits. If G is a subgroup of some $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic or if G is a subgroup of some $\mathrm{GL}_n(\mathbb{F})$ where $\mathbb{F}$ is a field of positive characteristic, then $\mathrm{cd}_{\mathfrak{F}}(G) < \infty$.*

Proof. Suppose H is a finitely generated subgroup of G. If G is a subgroup of $\mathrm{GL}_n(\mathbb{C})$ of integral characteristic, then by [1] when $\mathfrak{F}$ is the class of finite groups or or by the previous theorem when $\mathfrak{F}$ is the class of virtually cyclic groups, we know that H in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. If G embeds into $\mathrm{GL}_n(\mathbb{F})$ for some field $\mathbb{F}$ of positive characteristic, then by [7, Corollary 5], H has finite Bredon cohomological dimension and hence it is in $\mathbf{H}^{\mathfrak{F}}\mathfrak{F}$. This shows that G is in $\mathbf{LH}^{\mathfrak{F}}\mathfrak{F}$. The result now follows from Theorem 4·4. □

6. *Change of family*

In this section we discuss the question when the functor $\widehat{\mathrm{H}}^*_{\mathfrak{F}}(G,-)$ commutes with direct limits. By the above, it is obvious that groups of finite Bredon cohomological dimension as well as

groups of Bredon-type FP_∞ satisfy this condition. It would be interesting to see whether there are groups a priori satisfying neither, that also have continuous $\widehat{\mathrm{H}}^*_{\mathfrak{F}}(G,-)$.
Considering Lemma 4·2, we see that it is enough to require that $\mathrm{H}^k_{\mathfrak{F}}(G,-)$ commutes with direct limits for infinitely many k. This, for example holds for groups, for which the trivial Bredon-module $\mathbb{Z}_{\mathfrak{F}}$ has a Bredon-projective resolution, which is finitely generated from a certain point onwards.
As mentioned in the introduction, the families of greatest interest are the families $\mathfrak{F}_{\mathrm{fin}}$ of finite subgroups and $\mathfrak{F}_{\mathrm{vc}}$ of virtually finite subgroups. In light of Juan-Pineda and Leary's conjecture [12], which asserts that no non-virtually cyclic group is of type $\underline{\underline{\mathrm{FP}}}_\infty$, the question above is of particular interest for the family $\mathfrak{F}_{\mathrm{vc}}$.
Let us begin with the following:

QUESTION 6·1. *Does $\widehat{\mathrm{H}}^*(G,-)$ being continuous imply that $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{fin}}}(G,-)$ is continuous?*

The converse of this question is obviously not true. Take any group G with $\mathrm{cd}_{\mathfrak{F}_{\mathrm{fin}}} G < \infty$, which is not of type FP_∞ and which has no bound on the orders of the finite subgroups. It follows from [15] that groups with $\mathrm{cd}_{\mathbb{Q}} G < \infty$ and continuous $\widehat{\mathrm{H}}^*(G,-)$ have a bound on the orders of their finite subgroups. Locally finite groups and Houghton's groups satisfy this condition. On the other hand [16, Theorem 2.7], any group G in $\mathbf{LH}\mathfrak{F}$, for which $\widehat{\mathrm{H}}^*(G,-)$ is continuous has finite $cd_{\mathfrak{F}_{\mathrm{fin}}} G$, hence $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{fin}}}(G,-)$ is continuous.
Also note that there are examples of groups of type FP_∞, which are not of type Bredon-FP_∞ for the class of finite subgroups [18]. These groups, however, satisfy $\mathrm{cd}_{\mathfrak{F}_{\mathrm{fin}}} G < \infty$, hence have continuous $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{fin}}}(G,-)$.

QUESTION 6·2. *Is $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{fin}}}(G,-)$ being continuous equivalent to $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{vc}}}(G,-)$ being continuous?*

Any group of type $\underline{\underline{\mathrm{FP}}}_\infty$ is of type $\underline{\mathrm{FP}}_\infty$ (see [13]) and any group with $\mathrm{cd}_{\mathfrak{F}_{\mathrm{vc}}} G < \infty$ also has $\mathrm{cd}_{\mathfrak{F}_{\mathrm{fin}}} G < \infty$ (see [24]). Hence we may ask:

QUESTION 6·3. *Suppose* $\mathrm{cd}_{\mathfrak{F}_{\mathrm{fin}}} G < \infty$. *Does this imply that $\widehat{\mathrm{H}}^*_{\mathfrak{F}_{\mathrm{vc}}}(G,-)$ is continuous?*

If this question has a positive answer, Theorem A would imply that any group in $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ with $\mathrm{cd}_{\mathfrak{F}_{\mathrm{fin}}} G < \infty$ satisfies $\mathrm{cd}_{\mathfrak{F}_{\mathrm{vc}}} G < \infty$.

We end with two questions on the family $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$.

QUESTION 6·4. *Is the class $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ closed under extensions?*

This reduces to asking whether an infinite cyclic extension of group in $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ is also in $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$.

QUESTION 6·5. *Does the class $\mathbf{LH}^{\mathfrak{F}_{\mathrm{vc}}}\mathfrak{F}_{\mathrm{vc}}$ contain all elementary amenable groups?*

Note that a positive answer to Question 6·4 implies a positive answer to this question.

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