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Many-particle quantum graphs and Bose-Einstein condensation

Jens Bolte[1,a)] and Joachim Kerner[2,b)]
[1]Department of Mathematics, Royal Holloway, University of London, Egham TW20 0EX, United Kingdom
[2]Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, 70565 Stuttgart, Germany

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In this paper, we propose quantum graphs as one-dimensional models with a complex topology to study Bose-Einstein condensation and phase transitions in a rigorous way. We first investigate non-interacting many-particle systems on quantum graphs and provide a complete classification of systems that exhibit Bose-Einstein condensation. We then consider models of interacting particles that can be regarded as a generalisation of the well-known Tonks-Girardeau gas. Here, our principal result is that no phase transitions occur in bosonic systems with repulsive hardcore interactions, indicating an absence of Bose-Einstein condensation. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4879497]

I. INTRODUCTION

Bose-Einstein condensation (BEC) is a well established phenomenon in bosonic many-particle systems. In its original version, a system of non-interacting particles in a three-dimensional box was studied and, below a critical temperature, condensation into a joint state (the one-particle ground state) was found. This is a simple, exactly solvable model for a phase transition. The effect, however, depends on the spatial dimension and is absent in lower-dimensional, immediate analogues. Examples of BEC in one dimension were subsequently found when boundary conditions lead to negative eigenvalues that survive the thermodynamical limit (TL).[1]

BEC in non-ideal gases is a much harder problem, as the Penrose-Onsager criterion[2] requires a sufficient knowledge not only of the spectrum of the interacting many-particle Hamiltonian, but also of the eigenstates. As a consequence, results proving condensation in interacting systems and beyond mean-field approximations remain scarce.[3–6]

This paper is the third in a series of papers[7,8] devoted to the investigation of many-particle systems on general compact quantum graphs. Quantum graphs are models describing a quantum particle moving along the edges of a metric graph. They combine the simplicity of a one-dimensional model with the complexity of a graph and have become popular models in quantum chaos.[9,10] Their spectra display correlations that can be well described with random matrix models.

Models of many particles on a compact graph with singular interactions have been developed in the preceding papers, Refs. 7 and 8. Among them is an extension of the well-known Lieb-Liniger model,[11] which incorporates two-particle δ-interactions, to graphs. In this paper, our aim is to explore to what extent BEC can or cannot occur in a one-dimensional system with complex structure. In a first stage, we consider free Bose gases on graphs. Depending on the boundary conditions in the vertices the one-particle spectrum may or may not contain negative eigenvalues. Relevant for the occurrence of BEC are negative eigenvalues that remain negative in the thermodynamical limit, which we realise in terms of increasing edge lengths. We provide a complete classification of free Bose gases that

a)jens.bolte@rhul.ac.uk
b)Joachim.Kerner@mathematik.uni-stuttgart.de

display BEC. In a second step, we consider Tonks-Girardeau gases[12] on graphs as limits of Lieb-Liniger models. Our Tonks-Girardeau models describe bosons with repulsive hardcore interactions on the edges of a graph. We develop a Fermi-Bose map and prove that a Tonks-Girardeau gas is isospectral to a gas of free Fermions. This then finally proves the absence of BEC when hardcore interactions are switched on, even when BEC was present before.

The paper is organised as follows: In Sec. II, we review basic facts about BEC, quantum graphs, and many-particle systems on graphs. Section III is devoted to the classification of free Bose gases according to whether or not they display BEC. Tonks-Girardeau gases are then studied in Sec. IV via suitable Fermi-Bose maps, and the absence of a phase transition is proven.

II. PRELIMINARIES

In this section, we briefly summarise relevant concepts of BEC as well as of many-particle quantum graphs. For more details on BEC see Refs. 2, 13, and 14, on quantum graphs see Refs. 10 and 15–17 and on many-particle quantum graphs see Refs. 7 and 8.

A. Bose-Einstein condensation

For a gas of non-interacting bosons (a free gas), it is the macroscopic occupation of a one-particle eigenstate that implies the existence of BEC. This classical definition is meaningful since in non-interacting systems the eigenstates of the full Hamiltonian are (symmetrised) products of one-particle eigenstates. However, in the presence of interactions between particles, there is no preferred set of one-particle states and it is not immediately clear how to generalise this criterion for BEC. A suitable generalisation, introduced by Penrose and Onsager,[2] is based on the reduced one-particle density matrix ρ_1. At inverse temperature $\beta = \frac{1}{T}$ and fixed particle number N, the canonical thermal density matrix ρ_N of the full system is given by

$$\rho_N = \frac{1}{Z_N(\beta)} \sum_n e^{-\beta E_n} |\Psi_n\rangle\langle\Psi_n|, \tag{1}$$

where Ψ_n is the nth eigenvector of the N-particle system with eigenvalue E_n, and $Z_N(\beta) = \sum_n e^{-\beta E_n}$ is the canonical partition function. The reduced one-particle density matrix is obtained from (1) by tracing out $N - 1$ particles, i.e.,

$$\rho_1 = N \, \mathrm{Tr}_{2\ldots N}\,(\rho_N). \tag{2}$$

Definition 2.1 (Penrose and Onsager[2]). Let ρ_N be the canonical thermal state (1) of an N-particle system with one-particle reduced density matrix ρ_1 as in (2). The system is said to display BEC if the largest eigenvalue $\lambda_{\max}$ of ρ_1 satisfies

$$c_1 < \frac{\lambda_{\max}}{N} < c_2, \quad \forall N \geq N_0, \tag{3}$$

where $0 < c_1 \leq c_2$ and N_0 is sufficiently large.

Note that the limit $N \to \infty$ in Definition 2.1 is accompanied by the limit $V \to \infty$, where V is the volume of the one-particle configuration space, such that the particle density remains fixed. This is the standard thermodynamical limit of the canonical ensemble.[18] Unfortunately, although the criterion of Penrose and Onsager[2] is very general, it is usually difficult to establish BEC rigorously in the sense of Definition 2.1, since the eigenstates of the full system are hard to construct.[13] On the other hand, it is of fundamental interest to understand how particle-particle interactions affect the occurrence of BEC.[19–21] In this context, a more tractable approach to BEC aims at identifying phase transitions, which are expected to occur in any condensation process. This method has been used in Refs. 21–23, and it is this approach that will be used below to show the absence of BEC in the presence of repulsive hardcore contact interactions on graphs.

When studying condensation it is more natural to drop the requirement of a fixed particle number and to work in the grand-canonical ensemble. Then the free-energy density at finite volume is given by

$$f_V(\beta,\mu) = -\frac{1}{\beta V}\log Z(\beta,\mu), \tag{4}$$

where

$$Z(\beta,\mu) = \sum_{N=0}^{\infty} \mathrm{e}^{N\mu\beta} Z_N(\beta) \tag{5}$$

is the grand-canonical partition function, and μ is the chemical potential. In the grand-canonical ensemble, the thermodynamical limit is performed in terms of the limit $V \to \infty$ alone.[18] The chemical potential is then chosen such that the (fixed) particle density ρ satisfies

$$\rho = -\frac{\partial f}{\partial \mu}(\beta,\mu), \tag{6}$$

where $f(\beta,\mu) = \lim_{V\to\infty} f_V(\beta,\mu)$. Inverting the relation (6) yields a function $\mu = \mu(\rho)$ that then allows to replace the chemical potential by the particle density. It may, however, happen that the relation (6) is not invertible for all values $\rho \in \mathbb{R}^+$. This is the case in some well-known examples, including the three-dimensional free Bose gas and the one-dimensional Bose gas with a gap in the one-particle spectrum caused by boundary conditions. In such a case, it is necessary to choose a volume dependent sequence of chemical potentials such that the relation (6), with $f(\beta,\mu)$ replaced by $f_V(\beta,\mu_V)$, is fulfilled at any finite volume V, see Ref. 1.

In general, a phase transition manifests itself in terms of points where a suitable thermodynamical function $g(\beta,\mu) = \lim_{V\to\infty} g_V(\beta,\mu)$ is not differentiable.[1,18,24] In the examples mentioned above, it is well-known that phase transitions occur in this sense and, independently, that BEC occurs.[1] In fact, the BEC induces the phase transitions. As a consequence, we adopt the point of view that an absence of a phase transition indicates an absence of BEC.

Below we shall consider systems of bosons on a graph, interacting via repulsive hardcore interactions. The models are built after the example of N particles in one dimension whose interactions are described by the Lieb-Liniger Hamiltonian[11]

$$H_N^\alpha = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + \alpha \sum_{i>j} \delta(x_i - x_j). \tag{7}$$

In this example, repulsive hardcore interactions are obtained by taking the limit $\alpha \to \infty$. This procedure leads to Dirichlet boundary conditions imposed whenever two coordinates coincide, $x_i = x_j$. Note that a Bose gas in one dimension with repulsive hardcore interactions is known as the Tonks-Girardeau gas.[14,25] The importance of this model lies in the fact that, although being a gas of bosons, it exhibits fermionic behaviour in various ways.[25,26] The origin of the Tonks-Girardeau gas is found in the calculation of the classical partition function of a one-dimensional gas of hard spheres with diameter $a > 0$ by Tonks.[27] Later, Girardeau[12] gave a quantum mechanical description of the gas considered by Tonks[27] and found a one-to-one mapping between a one-dimensional gas of bosons with hardcore interaction and a free gas of fermions. Considering the case where $a = 0$, Girardeau[12] showed that the spectrum of a gas of bosons with hardcore repulsion is the same as the spectrum of a gas of free fermions, and the eigenfunctions are related by simple algebraic manipulations. Let, e.g., $\psi_0^{(F)}(x_1,\ldots,x_N)$ be the ground state of the gas of free fermions whose one-particle configuration space is an interval. The ground state $\psi_0^{(B)}(x_1,\ldots,x_N)$ of a gas of bosons with hardcore point interactions is then obtained via the relation

$$\psi_0^{(B)}(x_1,\ldots,x_N) = |\psi_0^{(F)}(x_1,\ldots,x_N)|. \tag{8}$$

Despite the close connection between a gas of free fermions and a gas of bosons with hardcore interactions there are still some subtle differences in the eigenfunctions (8), see Ref. 25 for an overview of this so-called *Fermi-Bose mapping*.

For the original Tonks-Girardeau gas, BEC was previously investigated in various papers.[25,28–31] As shown in Ref. 2, the existence of BEC in the sense of Definition 2.1 is equivalent to the existence of a long-range order in the position representation $\rho_1(x, x')$ of the reduced one-particle density matrix. For the Tonks-Girardeau gas at zero temperature, it was shown that

$$\rho_1(x, 0) \sim \frac{K}{\sqrt{x}}, \quad x \to \infty, \tag{9}$$

where $K > 0$ is a constant.[25,31] This in turn implies that the maximal eigenvalue of (2) is of order $\sqrt{N}$ and, hence, at zero temperature no BEC occurs.[25] For finite temperature, however, in Ref. 30 an exponential decay of $\rho_1(x, x')$ was conjectured. This was subsequently established in Ref. 32 at low temperature.

It is interesting to note that studies of particles in one dimension interacting via repulsive hardcore interactions have a long history in statistical mechanics.[18,24,33] Van Hove, e.g., had shown (see Theorem 5.6.7 in Ref. 34, and Ref. 33) that a classical one-dimensional system with hardcore two-particle interactions, plus possibly a short-range contribution, shows no phase transitions.

B. Quantum graphs

The classical configuration space for a particle on a graph is a compact metric graph, i.e., a finite, connected graph $\Gamma = (\mathcal{V}, \mathcal{E})$ with vertex set $\mathcal{V} = \{v_1, \ldots, v_V\}$ and edge set $\mathcal{E} = \{e_1, \ldots, e_E\}$. The edges are identified with intervals $[0, l_e]$, $e = 1, \ldots, E$, thus assigning lengths to intervals; this then introduces a metric on the graph. Note that we do not exclude multiple edges or loops at this point.

1. One-particle quantum graphs

Functions on the graph are collections of functions on the edges, i.e.,

$$F = (f_1, \ldots, f_E), \quad \text{with} \quad f_e : [0, l_e] \to \mathbb{C}, \tag{10}$$

so that spaces of functions on Γ are (finite) direct sums of the respective spaces of functions on the edges. The most relevant space is the Hilbert space

$$\mathcal{H}_1 = L^2(\Gamma) := \bigoplus_{e=1}^{E} L^2(0, l_e), \tag{11}$$

and all other function spaces are constructed in a similar way.

Standard single particle quantum mechanics suggests to choose the one-particle Hilbert space $\mathcal{H}_1 = L^2(\Gamma)$, see (11). One-particle observables are then self-adjoint operators in $\mathcal{H}_1$. In the absence of external forces or gauge fields, the Hamiltonian should be a suitable realisation of a Laplacian. As a differential operator the Laplacian acts according to

$$-\Delta_1 F = (-f_1'', \ldots, -f_E'') \tag{12}$$

on $F \in C^2(\Gamma)$. We here use the index to indicate that this is a one-particle Laplacian.

Viewed as an operator in $L^2(\Gamma)$ with domain $C_0^\infty(\Gamma)$, this Laplacian is symmetric, but not (essentially) self-adjoint. Its self-adjoint extensions can be classified, and there are several ways to parametrise these extensions (see, e.g., Refs. 15 and 16). All of these parametrisations involve boundary values

$$F_{bv} = (f_1(0), \ldots, f_E(0), f_1(l_1), \ldots, f_E(l_E)) \tag{13}$$

of the functions in the domain of the operator, as well of derivatives,

$$F'_{bv} = (f_1'(0), \ldots, f_E'(0), -f_1'(l_1), \ldots, -f_E'(l_E)). \tag{14}$$

One (unique) characterisation of self-adjoint extensions uses quadratic forms,[15]

$$Q_1[F] = \sum_{e=1}^{E} \int_0^{l_e} |f'(x)|^2 \, \mathrm{d}x - \langle F_{bv}, L_1 F_{bv} \rangle_{\mathbb{C}^{2E}}, \tag{15}$$

with domains

$$\mathcal{D}_{Q_1} = \{F \in H^1(\Gamma);\, P_1 F_{bv} = 0\}. \tag{16}$$

Here, P_1 is a projector on the space of boundary values $\mathbb{C}^{2E}$ and L_1 is a self-adjoint map on $\ker P_1$. (The index indicates that these quantities relate to one-particle quantities.) This form is uniquely associated with a one-particle Laplacian $-\Delta_1$ on the domain

$$\mathcal{D}_1(P_1, L_1) = \{F \in H^2(\Gamma);\, P_1 F_{bv} = 0,\ Q_1 F'_{bv} + L_1 Q_1 F_{bv} = 0\}, \tag{17}$$

where $Q_1 = \mathbf{1}_{2E} - P_1$.

Any such one-particle Laplacian has a discrete spectrum, with eigenvalues only accumulating at infinity and following a Weyl asymptotic law (see, e.g., Ref. 17). Potentially, there are finitely many negative eigenvalues. Their number is bounded by the number of positive eigenvalues of L_1 (in each case including multiplicities).[16] Hence, a negative semi-definite map L_1 implies a non-negative Laplacian.

2. Many-particle quantum graphs

An N-particle quantum system on a graph requires the tensor product of one-particle Hilbert spaces, $\mathcal{H}_N = \mathcal{H}_1 \otimes \ldots \otimes \mathcal{H}_1$. For a quantum graph, this means that

$$\mathcal{H}_N = \Big(\bigoplus_{e=1}^{E} L^2(0, l_e)\Big) \otimes \ldots \otimes \Big(\bigoplus_{e=1}^{E} L^2(0, l_e)\Big), \tag{18}$$

such that vectors $\Psi \in \mathcal{H}_N$ are collections $\Psi = (\psi_{e_1 e_2 \ldots e_N})$ of E^N functions defined on the hyper-rectangles $D_{e_1 e_2 \ldots e_N} = (0, l_{e_1}) \times (0, l_{e_2}) \times \ldots \times (0, l_{e_N})$. Their disjoint union is denoted as

$$D_\Gamma^N = \dot{\bigcup_{e_1 e_2 \ldots e_N}} D_{e_1 e_2 \ldots e_N}, \tag{19}$$

so that one may view $\mathcal{H}_N$ as

$$L^2(D_\Gamma^N) = \bigoplus_{e_1 e_2 \ldots e_N} L^2(D_{e_1 e_2 \ldots e_N}). \tag{20}$$

The corresponding Sobolev spaces are introduced in the same way and are denoted as $H^m(D_\Gamma^N)$.

A free Hamiltonian for N particles is a lift of a one-particle Hamiltonian $-\Delta_1$ to $\mathcal{H}_N$, i.e.,

$$-\Delta_N^{free} = \sum_{j=1}^{N} \mathbf{1} \otimes \ldots \otimes (-\Delta_1) \otimes \ldots \mathbf{1}, \tag{21}$$

where the (one-particle) Laplacian acts on the coordinates of the jth particle. As a differential expression, this operator is a Laplacian,

$$(-\Delta_N \Psi)_{e_1 \ldots e_N} = -\left(\frac{\partial^2}{\partial x_{e_1}^2} + \ldots + \frac{\partial^2}{\partial x_{e_N}^2}\right) \psi_{e_1 \ldots e_N}, \tag{22}$$

and it can be realised on a suitable domain in (20). Hence, any free Hamiltonian (21) is some extension of the symmetric operator $-\Delta_N$ defined on the domain $C_0^\infty(D_\Gamma^N)$.

However, there are extensions of $(-\Delta_N, C_0^\infty(D_\Gamma^N))$ that are not of the form (21); these operators necessarily contain interactions among the particles. These interactions are induced by boundary conditions imposed on the functions in their domain and hence are of a singular type, acting when a particle hits a vertex. A certain class of such extensions was introduced in Ref. 7.

Another class of interactions, introduced in Ref. 8, consists of two-particle contact interactions, acting whenever two particles are in the same position along an edge. They can be constructed as rigorous versions of the formal Hamiltonian

$$H_N^\alpha = -\Delta_N + \alpha \sum_{i<j} \delta(x_i - x_j). \tag{23}$$

This requires to add additional boundaries to the configuration space (19) along diagonal hyperplanes characterised by $x_i = x_j$.

A bosonic many-particle system has to be realised on the symmetric tensor product of N one-particle Hilbert spaces, $\mathcal{H}_{N,B} = \Pi_B \mathcal{H}_N$, where

$$(\Pi_B \Psi)_{e_1 \ldots e_N} := \frac{1}{N!} \sum_{\pi \in S_N} \psi_{\pi(e_1)\ldots\pi(e_N)}(x_{\pi(e_1)}, \ldots, x_{\pi(e_N)}) \tag{24}$$

is the projector onto the totally symmetric states under particle exchange.

In order to realise the contact interactions indicated in (23), one has to introduce jump conditions across hyperplanes $x_i = x_j$ on the normal derivatives of the functions in the operator domain.[8] To achieve this one dissects the hyper-rectangles $D_{e_1 e_2 \ldots e_N}$ describing configurations with at least two particles on the same edge (i.e., with at least a pair $i \neq j$ such that $e_i = e_j$) into polyhedral sub-domains by cutting $D_{e_1 e_2 \ldots e_N}$ along the hyperplanes $x_i = x_j$. We denote the dissected version of $D_{e_1 e_2 \ldots e_N}$ by $D^*_{e_1 e_2 \ldots e_N}$, and the union of all these polyhedral domains by D_Γ^{N*}. The corresponding L^2-spaces are

$$L^2(D_\Gamma^{N*}) = \bigoplus_{e_1 e_2 \ldots e_N} L^2(D^*_{e_1 e_2 \ldots e_N}), \tag{25}$$

and similarly for Sobolev spaces. Given a dissected hyper-rectangle $D^*_{e_1 e_2 \ldots e_N}$, the union of all hyperplanes that form the internal boundaries will be denoted as $\partial D^{int}_{e_1 \ldots e_N}$, whereas the external boundaries $\partial D^{ext}_{e_1 e_2 \ldots e_N}$ are the boundaries of the undissected hyper-rectangle $D_{e_1 e_2 \ldots e_N}$. We then set

$$\partial D_\Gamma^{N, ext/int} = \bigcup_{e_1 \ldots e_N} \partial D^{ext/int}_{e_1 \ldots e_N}. \tag{26}$$

Implementing jump conditions across $\partial D_\Gamma^{N,int}$ will then yield the δ-interactions (23).

One can also introduce boundary conditions on $\partial D_\Gamma^{N,ext}$ that prevent the resulting N-particle Laplacian from being a free Hamiltonian.[7] These boundary conditions cause the interactions mentioned above to arise when one particle hits a vertex. In order to implement such interactions, one needs the boundary values

$$\Psi_{bv}(\boldsymbol{y}) = \begin{pmatrix} \sqrt{l_{e_2} \ldots l_{e_N}}\, \psi_{e_1 \ldots e_N}(0, l_{e_2} y_1, \ldots, l_{e_N} y_{N-1}) \\ \sqrt{l_{e_2} \ldots l_{e_N}}\, \psi_{e_1 \ldots e_N}(l_{e_1}, l_{e_2} y_1, \ldots, l_{e_N} y_{N-1}) \end{pmatrix}, \tag{27}$$

and

$$\Psi'_{bv}(\boldsymbol{y}) = \begin{pmatrix} \sqrt{l_{e_2} \ldots l_{e_N}}\, \psi_{e_1 \ldots e_N, x^1_{e_1}}(0, l_{e_2} y_1, \ldots, l_{e_N} y_{N-1}) \\ -\sqrt{l_{e_2} \ldots l_{e_N}}\, \psi_{e_1 \ldots e_N, x^1_{e_1}}(l_{e_1}, l_{e_2} y_1, \ldots, l_{e_N} y_{N-1}) \end{pmatrix}, \tag{28}$$

where $\boldsymbol{y} = (y_1, \ldots, y_{N-1}) \in [0, 1]^{N-1}$, of functions $\Psi \in H^1_B(D_\Gamma^N)$ and the normal derivatives. Acting on these (vertex related) boundary values are the bounded and measurable maps $P_N, L_N : [0, 1]^{N-1} \to \mathrm{M}(2E^N, \mathbb{C})$ such that for a.e. $\boldsymbol{y} \in [0, 1]^{N-1}$ the linear map $P_N(\boldsymbol{y})$ is an orthogonal projector and $L_N(\boldsymbol{y})$ is a self-adjoint endomorphism on ker $P_N(\boldsymbol{y})$. These maps define two bounded, self-adjoint multiplication operators Π_N and Λ_N, respectively, on $L^2(0, 1) \otimes \mathbb{C}^{2E^N}$, see Ref. 8. Note that in Ref. 7 it was shown that actual interactions are obtained whenever P_N, L_N are not all independent of $\boldsymbol{y}$.

Following Ref. 8, the quadratic form for N bosons on a graph subject to any of the interactions introduced above is

$$\begin{aligned} Q_B^{(N)}[\Psi] = N \sum_{e_1 \ldots e_N} \int_0^{l_{e_1}} \ldots \int_0^{l_{e_N}} |\psi_{e_1 \ldots e_N, x_{e_1}}(x_{e_1}, \ldots, x_{e_N})|^2 \, \mathrm{d}x_{e_N} \ldots \mathrm{d}x_{e_1} \\ - N \int_{[0,1]^{N-1}} \langle \Psi_{bv}, L_N(\boldsymbol{y}) \Psi_{bv} \rangle_{\mathbb{C}^{2E^N}} \, \mathrm{d}\boldsymbol{y} \\ + \frac{N(N-1)}{2} \sum_{e_2 \ldots e_N} \int_{[0,1]^{N-1}} \alpha(y_1) \, |\sqrt{l_{e_2} \ldots l_{e_N}} \psi_{e_2 e_2 \ldots e_N}(l_{e_2} y_1, \boldsymbol{l\,y})|^2 \, \mathrm{d}\boldsymbol{y}, \end{aligned} \tag{29}$$

where $\boldsymbol{l\,y} = (l_{e_2} y_1, l_{e_3} y_2, \ldots, l_{e_N} y_{N-1})$, with domain

$$\mathcal{D}_{Q_B^{(N)}} = \{\Psi \in H_B^1(D_\Gamma^{N*}); \; P_N(\boldsymbol{y}) \Psi_{bv}(\boldsymbol{y}) = 0 \text{ for a.e. } \boldsymbol{y} \in [0,1]^{N-1}\}. \tag{30}$$

The second line on the right-hand side of (29) implies the vertex-related interactions, whereas the third line yields contact interactions with (bounded) variable strength α. The latter can be turned into a repulsive hardcore interaction by taking the limit $\alpha \to \infty$. Taking this limit one has to amend the domain (30) in that the functions have to vanish on the internal boundaries $\partial D_\Gamma^{N,int}$. This subspace of $H_B^1(D_\Gamma^{N*})$ shall be denoted as $H_{0,int,B}^1(D_\Gamma^{N*})$.

Under these conditions it was shown in Ref. 8 that the quadratic form (29) is closed and semi-bounded. Hence, there exists a unique, self-adjoint, and semi-bounded operator associated with this form. This operator is a self-adjoint realisation of the N-particle Laplacian which will, in the case of hardcore interactions, be denoted by $(-\Delta_N, \mathcal{D}_N^{\alpha=\infty}(P_N, L_N))$. It has a discrete spectrum and the eigenvalue asymptotics follows a Weyl law.[8]

III. BEC IN NON-INTERACTING BOSE GASES

As described in (21), a self-adjoint realisation of the free N-particle Laplacian follows from a tensor product construction based on a given one-particle Laplacian $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$. The eigenfunctions $\{\Psi_n\}_{n \in \mathbb{N}^N}$ of the N-particle Laplacian are then given as symmetrised products of the one-particle eigenfunctions $\{\phi_n\}_{n \in \mathbb{N}}$, i.e.,

$$\Psi_n = \Pi_B(\phi_{n_1} \otimes \ldots \otimes \phi_{n_N}), \tag{31}$$

where $n = (n_1, \ldots, n_N)$. The N-particle eigenvalues are $\lambda_n = k_{n_1}^2 + \ldots + k_{n_N}^2$, where $\{k_n^2\}_{n \in \mathbb{N}}$ are the corresponding one-particle eigenvalues.

We shall mainly work in the grand-canonical ensemble where the thermodynamical limit is taken as the limit of an infinite volume of the one-particle configuration space (see Sec. II). For a graph the volume of the one-particle configuration space is the total length of the graph,

$$\mathcal{L} = \sum_{e=1}^{E} l_e. \tag{32}$$

In principle, the infinite-volume limit can be achieved by either increasing the number of edges, or by scaling the lengths of the edges. As we do not want to change the topology of the graph, we here choose to leave the number of edges fixed and only increase the edge lengths by a common factor.

Definition 3.1. Let Γ be a compact, metric graph with edge lengths $l_1, \ldots, l_E$. The *TL* consists of the scaling

$$l_e \mapsto \eta l_e, \tag{33}$$

and taking the limit $\eta \to \infty$.

We shall also use the notation

$$\lim_{TL} F(\mathcal{L}) \tag{34}$$

for the thermodynamical limit of a function $F(\mathcal{L})$.

According to general folklore, in the absence of disorder a free gas of bosons in one dimension shows no BEC at finite temperature. This, however, is only true if the one-particle spectrum has no gap separating a finite number of eigenvalues at the bottom of the spectrum from the (quasi-) continuum of states above. In order to generate such a gap in a quantum graph, it is necessary for the Laplacian $-\Delta_1$ to possess negative eigenvalues.

An upper bound for the number $n_-(-\Delta_1(P_1, L_1))$ of negative eigenvalues of the one-particle Laplacian was proved in Ref. 16. The exact number was later determined in Ref. 35, where the matrix

$$M_0(l_1, \ldots, l_E) := \begin{pmatrix} m_1(l_1) & & 0 \\ & \ddots & \\ 0 & & m_E(l_E) \end{pmatrix}, \tag{35}$$

with

$$m_e(l_e) := \frac{1}{l_e}\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}, \tag{36}$$

was introduced. It was then shown in Ref. 35 that

$$n_-(-\Delta_1(P_1, L_1)) = n_+(L_1 + Q_1 M_0 Q_1), \tag{37}$$

where the right-hand side denotes the number of positive eigenvalues of the linear map $L_1 + Q_1 M_0 Q_1$ on $\ker P_1 \subseteq \mathbb{C}^{2E}$ (and $Q_1 = \mathbf{1}_{2E} - P_1$). Therefore, when the edge lengths tend to infinity in the TL, the quantity $n_-(-\Delta_1(P_1, L_1))$ approaches $n_+(L_1)$. This, however, does not imply that there are $n_+(L_1)$ negative Laplace-eigenvalues in the TL as some of the negative eigenvalues could approach zero in the limit. Nevertheless, for the question of BEC the number of positive eigenvalues of the map L_1 is still relevant, as we shall show below.

In the following proposition, we first prove absence of BEC when the domain of the one-particle Laplacian is defined in terms of a negative semi-definite map L_1, and therefore the Laplacian has no negative eigenvalues.

Proposition 3.2. Let $-\Delta_N$ be a bosonic, self-adjoint realisation of the free N-particle Laplacian. If the corresponding one-particle Laplacian $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$ is such that L_1 is negative semi-definite, no Bose-Einstein condensation occurs at finite temperature in the thermodynamical limit.

Proof. It is well known that there is no BEC in a gas of free bosons on an interval of finite length with either Dirichlet or Neumann boundary conditions. In both cases, the eigenvalues are known explicitly and the standard argument applies. The same is true for a compact quantum graph with Dirichlet or Neumann boundary conditions: the eigenvalue equations on the edges decouple and, again, the eigenvalues are known explicitly. The spectrum, therefore, is the union of the spectra for each edge.

Let

$$\mathcal{N}(K) = \#\{n;\ k_n^2 \le K^2\} \tag{38}$$

be the eigenvalue counting function for $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$, where the eigenvalues k_n^2 are counted with their multiplicities, and denote by $\mathcal{N}_{D/N}(K)$ the respective counting functions for the Dirichlet- and Neumann-Laplacian. A bracketing argument then implies

$$\mathcal{N}_D(K) \le \mathcal{N}(K) \le \mathcal{N}_N(K). \tag{39}$$

This follows, e.g., from the proof of Proposition 4.2 in Ref. 17, taking into account that L_1 is negative semi-definite.

In the grand-canonical ensemble, the expected number of particles can be expressed as

$$N(\beta, \mu) = \sum_{n=0}^{\infty} \frac{1}{e^{\beta(k_n^2 - \mu)} - 1} = \int_0^{\infty} \frac{1}{\mathrm{e}^{\beta(k^2 - \mu)} - 1}\, \mathrm{d}\mathcal{N}(k). \tag{40}$$

The relation (39) hence implies that

$$N_D(\beta, \mu) \leq N(\beta, \mu) \leq N_N(\beta, \mu). \tag{41}$$

In the TL, the expected particle density is

$$\rho(\beta, \mu) = \lim_{TL} \frac{N(\beta, \mu)}{\mathcal{L}}, \tag{42}$$

and (41) implies

$$\rho_D(\beta, \mu) \leq \rho(\beta, \mu) \leq \rho_N(\beta, \mu). \tag{43}$$

Due to the explicit knowledge of the eigenvalues it is known that $\rho_D(\beta, \mu) = \rho_N(\beta, \mu)$. Thus, (42) yields

$$\rho(\beta, \mu) = \rho_{D/N}(\beta, \mu). \tag{44}$$

Therefore, as the Dirichlet- and Neumann case shows no BEC, the same holds for any gas of free bosons satisfying the conditions of the proposition. □

It is known[1] that in dimension less than three, despite the general folklore a free Bose gas may show BEC, if the spectrum of the one-particle Hamiltonian has a gap below zero. More precisely, if the one-particle spectrum is such that there are infinitely many positive eigenvalues and, say, one negative eigenvalue that remains negative after having taken the TL, infinitely many particles will occupy the eigenstate corresponding to the negative eigenvalue (the ground state). An example for this mechanism is given by an attractive delta-potential on the real axis. This has exactly one negative eigenvalue and undergoes BEC at low temperatures.[36] Furthermore, the condensate is spatially localised around the location of the delta-potential. It is worth mentioning that in the current context the real axis with a delta-potential at the origin is a quantum graph with two edges of infinite length and one vertex, at which appropriate boundary conditions are imposed.

We now determine a class of quantum graphs that maintain a spectral gap below zero in the thermodynamical limit. Our main tool will be a Rayleigh quotient,

$$R[\Psi] = \frac{Q_1[\Psi]}{\|\Psi\|^2_{L^2(\Gamma)}}, \qquad \Psi \in \mathcal{D}_{Q_1}, \tag{45}$$

which is an upper bound for the ground state eigenvalue.

Proposition 3.3. Let Γ *be a compact, metric graph with a self-adjoint one-particle Laplacian* $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$. *Assume that* L_1 *has at least one positive eigenvalue and denote the largest eigenvalue by* $L_{\max}$. *Then the ground state eigenvalue* $-\kappa^2_{\min} < 0$ *of the one-particle Laplacian converges to* $-L^2_{\max} < 0$ *in the thermodynamical limit.*

Proof. As L_1 is assumed to possess at least one positive eigenvalue, $n_+(L_1) \geq 1$, the relation (37) implies that the Laplacian has at least one negative eigenvalue as long as the edge lengths are finite and sufficiently large. Hence, for any $\Phi \in \mathcal{D}_{Q_1}$,

$$-s^2 \leq -\kappa^2_{\min} \leq R[\Phi]. \tag{46}$$

Here, $-s^2$ is the lower bound for the spectrum of the one-particle Laplacian proved in Ref. 16, where s a solution of

$$s \tanh\left(\frac{s l_{\min}}{2}\right) = L_{\max}, \tag{47}$$

and is $l_{\min}$ the shortest edge-length. In the TL, where $l_{\min} \to \infty$, the lower bound in (46) converges to $-L^2_{\max}$. To find an upper bound in (46), we need to determine the Rayleigh quotient of a suitable trial function.

We assume that $P_1 \neq \mathbf{1}_{2E}$ as this would correspond to Dirichlet boundary conditions in the vertices, where it is known that there are no negative eigenvalues. Hence, there exists a non-trivial vector

$$v := (c_1, \ldots, c_E, c_{E+1}, \ldots, c_{2E})^T \in \ker P_1. \tag{48}$$

Using the components of such a vector, we now define a trial function Φ with components

$$\phi_e(x) = \begin{cases} c_e\left(1 - \frac{x}{\lambda}\right)^\alpha, & x \le \lambda \\ 0, & \lambda \le x \le l_e - \lambda, \qquad \alpha \ge 1. \\ c_{e+E}\left(\frac{x}{\lambda} + 1 - \frac{l_e}{\lambda}\right)^\alpha, & x \ge l_e - \lambda \end{cases} \tag{49}$$

As we shall take the TL, given any value for λ we can arrange that $l_e \ge 2\lambda$ for all $e = 1, \ldots, E$. The boundary values of this function, therefore, are

$$\Phi_{bv} = (c_1, \ldots, c_E, c_{E+1}, \ldots, c_{2E})^T = v \in \ker P_1, \tag{50}$$

hence this function is in the domain (16) of the quadratic form.

We now intend to estimate the Rayleigh quotient of Φ, noting that we are free to choose $v \in \ker P_1$. The optimal choice for our purpose is to let $v = \Phi_{bv}$ be an eigenvector of L_1 corresponding to its maximal eigenvalue $L_{\max} > 0$. Then,

$$\begin{aligned} Q_1[\Phi] &= \sum_{e=1}^{E} \int_0^{l_e} |\phi_e'(x)|^2 \, \mathrm{d}x - \langle \Phi_{bv}, L_1 \Phi_{bv} \rangle \\ &= \frac{\alpha^2}{(2\alpha - 1)\lambda} \sum_{e=1}^{E} \left(|c_e|^2 + |c_{e+E}|^2\right) - L_{\max} \|\Phi_{bv}\|^2_{\mathbb{C}^{2E}} \\ &= \left(\frac{\alpha^2}{(2\alpha-1)\lambda} - L_{\max}\right) \|\Phi_{bv}\|^2_{\mathbb{C}^{2E}}. \end{aligned} \tag{51}$$

Moreover,

$$\|\Phi\|^2 = \sum_{e=1}^{E} \int_0^{l_e} |\phi_e(x)|^2 \, \mathrm{d}x = \frac{\lambda}{2\alpha + 1} \sum_{e=1}^{E} \left(|c_e|^2 + |c_{e+E}|^2\right) = \frac{\lambda}{2\alpha+1} \|\Phi_{bv}\|^2_{\mathbb{C}^{2E}}, \tag{52}$$

so that

$$R[\Phi] = \left(\frac{\alpha^2}{(2\alpha-1)\lambda} - L_{\max}\right) \frac{2\alpha+1}{\lambda}. \tag{53}$$

The right-hand side is negative when $\lambda > \frac{\alpha^2}{(2\alpha-1)L_{\max}}$ and has a minimum at $\lambda_{\min} = \frac{2\alpha^2}{(2\alpha-1)L_{\max}}$. With this optimal choice we find that

$$R[\Phi] = -\frac{4\alpha^2 - 1}{4\alpha^2} L^2_{\max}. \tag{54}$$

As $\alpha \ge 1$ can be chosen arbitrarily large in the TL, the optimal upper bound in (46) approaches $-L^2_{\max}$. Hence, $-\kappa^2_{\min}$ converges to $-L^2_{\max}$ in the TL. □

We are now in a position to state our main result of this section.

Theorem 3.4. *Let a free Bose gas be given on a quantum graph with a one-particle Laplacian* $(-\Delta_1, \mathcal{D}_1(P_1, L_1))$ *such that* L_1 *has at least one positive eigenvalue. Then, in the thermodynamical limit, there is a critical temperature* $T_c > 0$ *such that Bose-Einstein condensation occurs below* T_c.

Proof. We denote the non-negative eigenvalues of the one-particle Laplacian (counted with their multiplicities) as $k_0^2 \le k_1^2 \le k_2^2 \le k_3^2 \le \cdots$. In the grand-canonical ensemble, the expected particle number occupying states of non-negative energy is

$$N_+(\beta, \mu) = \sum_{n=0}^{\infty} \frac{1}{e^{\beta(k_n^2 - \mu)} - 1}. \tag{55}$$

Recall that by Proposition 3.3 the one-particle ground state eigenvalue is a distance $L^2_{\max} > 0$ below zero. Hence, the chemical potential μ has to satisfy $\mu \leq -L^2_{\max}$. The density of particles in states of non-negative energy in the TL then is

$$\rho_+(\beta,\mu) = \lim_{TL} \frac{N_+(\beta,\mu)}{\mathcal{L}}. \tag{56}$$

In order to evaluate this expression, we employ Proposition 5.2 of Ref. 17, which provides a preliminary form of the trace formula and can be rearranged as

$$\begin{aligned}\sum_{n=0}^{\infty} h(k_n) = \frac{\mathcal{L}}{2\pi}\int_{-\infty}^{\infty} h(k)\,\mathrm{d}k + \gamma h(0) - \frac{1}{4\pi}\int_{-\infty}^{\infty} h(k)\,s(k)\,\mathrm{d}k \\ + \sum_{l\neq 0}\frac{1}{4\pi i}\int_{-\infty}^{\infty}\mathrm{Tr}[\Lambda(k)U^l(k)]\,h(k)\,\mathrm{d}k.\end{aligned} \tag{57}$$

Here, γ is a constant related to the multiplicity of the eigenvalue zero, Λ, U are matrix-valued functions involving the boundary conditions, and s is another function related to the boundary conditions. In this trace formula, h is a test function from a suitable test function space.[17]

Now, choosing $h(k) = \frac{1}{\mathrm{e}^{\beta(k^2-\mu)}-1}$, the left-hand side of (57) is $N_+(\beta, \mu)$, and the right-hand side provides four separate contributions to $N_+(\beta, \mu)$. It is obvious that the second and the third terms give no contributions to $\rho_+(\beta, \mu)$. An estimate of the fourth term can be found in the proof of Theorem 5.4 in Ref. 17,

$$\sum_{l\neq 0}\left|\int_{-\infty}^{\infty}\mathrm{Tr}[\Lambda(k)U^l(k)]\,h(k)\,\mathrm{d}k\right| = O\big(\mathrm{e}^{-\sigma l_{\min}}\big), \tag{58}$$

where $l_{\min}$ is the shortest edge-length and $\sigma > 0$. Hence, as in the thermodynamical limit $l_{\min} \to \infty$, this term, too, gives no contribution to $\rho_+(\beta, \mu)$. Therefore, the only non-vanishing contribution comes from the first term (which also provides the Weyl term in the asymptotics of the eigenvalue count),

$$\rho_+(\beta,\mu) = \frac{1}{\pi}\int_0^{\infty}\frac{1}{\mathrm{e}^{\beta(k^2-\mu)}-1}\,\mathrm{d}k = \frac{1}{\sqrt{4\pi\beta}}\,g_{\frac{1}{2}}(\mathrm{e}^{\beta\mu}). \tag{59}$$

Here,

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^{\infty}\frac{x^{\nu-1}}{z^{-1}e^x-1}\,\mathrm{d}x = \sum_{k=1}^{\infty}\frac{z^k}{k^\nu} \tag{60}$$

is the well-known Bose-Einstein function (a polylogarithm). The series converges for $|z| < 1$ and has a finite limit as $z \to 1$ when $\nu > 1$. Here, however, $z = \mathrm{e}^{\beta\mu} \leq \mathrm{e}^{-\beta L^2_{\max}} < 1$ as $\mu \leq -L^2_{\max}$. Hence, $\rho_+(\beta, \mu)$ is finite for all $\beta > 0$ and tends to zero as $\beta \to \infty$ (i.e., $T \to 0$).

The total particle density, $\rho(\beta, \mu)$, also has a contribution from particles occupying states with negative energy,

$$\rho(\beta,\mu) = \rho(\beta,\mu)_- + \rho_+(\beta,\mu). \tag{61}$$

Given that the limiting particle density has a fixed value, ρ_0, below a certain critical temperature $T_c = \frac{1}{\beta_c}$ the negative energy states must be populated because $\rho_+(\beta, \mu) < \rho_0$ when $\beta > \beta_c$. This critical temperature is implicitly defined by

$$\rho_0 = \frac{1}{\sqrt{4\pi\beta_c}}\,g_{\frac{1}{2}}(e^{-\beta_c L^2_{\max}}). \tag{62}$$

More explicitly, when $T \leq T_c$,

$$\rho_- = \rho_0 - \rho_+ = \rho_0\left(1 - \frac{1}{\rho_0}\frac{1}{\sqrt{4\pi\beta}}\,g_{\frac{1}{2}}(e^{-\beta L^2_{\max}})\right). \tag{63}$$

Hence, below the critical temperature the relative occupation of the negative energy states is

$$\frac{\rho_-}{\rho_0} = 1 - \sqrt{\frac{\beta_c}{\beta}}\, \frac{g_{\frac{1}{2}}(e^{-\beta L_{\max}^2})}{g_{\frac{1}{2}}(e^{-\beta_c L_{\max}^2})} > 0. \tag{64}$$

Thus, BEC occurs for $T < T_C$, with the relative occupation of the negative energy states approaching one as $T \to 0$. □

IV. BEC IN BOSE GASES WITH CONTACT INTERACTIONS

We now consider bosons on a graph with many-particle interactions described by the quadratic form (29). As explained in Sec. II B 2, these interactions consists of two types: they are either located in the vertices of the graph, or they are contact interactions. Introducing the latter type of interactions amounts to generalising the original Lieb-Liniger model[11] to graphs. In the limit of an infinite strength such a model turns into a gas with hardcore repulsion and can be viewed as a generalisation of a Tonks-Girardeau gas to the graph setting. We shall now investigate under what circumstances BEC on graphs may or may not occur in the presence of repulsive hardcore interactions.

A. Fermi-Bose mapping on general quantum graphs

Our first goal is to generalise the Fermi-Bose mapping introduced by Girardeau[12,25] to general quantum graphs. This requires us to introduce systems of N free fermions, and to relate them to systems of N bosons interacting via repulsive hardcore interactions. The fermionic states are vectors in the fermionic N-particle Hilbert space $L^2_F(D^N_\Gamma) = \Pi_F L^2(D^N_\Gamma)$, where

$$(\Pi_F \Psi)_{e_1\ldots e_N} := \frac{1}{N!}\sum_{\pi\in S_N}(-1)^{\operatorname{sgn}\pi}\,\psi_{\pi(e_1)\ldots\pi(e_N)}(x_{\pi(e_1)},\ldots,x_{\pi(e_N)}). \tag{65}$$

An analogous notation is used for Sobolev spaces. Interactions will be described in terms of a quadratic form

$$\begin{aligned} Q_F^{(N)}[\Psi] = N\sum_{e_1\ldots e_N}\int_0^{l_{e_1}}\ldots\int_0^{l_{e_N}} |\psi_{e_1\ldots e_N,x_{e_1}}(x_{e_1},\ldots,x_{e_N})|^2\,\mathrm{d}x_{e_N}\ldots\mathrm{d}x_{e_1} \\ - N\int_{[0,1]^{N-1}}\langle\Psi_{bv}, L_{F,N}(\boldsymbol{y})\Psi_{bv}\rangle_{\mathbb{C}^{2E^N}}\,\mathrm{d}\boldsymbol{y} \end{aligned} \tag{66}$$

with domain

$$\mathcal{D}_{Q_F^{(N)}} = \{\Psi \in H^1_F(D^N_\Gamma);\ P_{F,N}(\boldsymbol{y})\Psi_{bv}(\boldsymbol{y}) = 0 \text{ for a.e. } \boldsymbol{y}\in[0,1]^{N-1}\}. \tag{67}$$

Here, we use the same notation as in Sec. II B 2, however with an additional index F indicating the fermionic nature of the form. Using the methods of Ref. 8, one can readily show that the form (66) is closed and semi-bounded, hence it corresponds to a unique self-adjoint operator, $(-\Delta_N, \mathcal{D}^N_F(P_{F,N}, L_{F,N}))$. Furthermore, a standard bracketing argument implies the discreteness of the spectrum of this self-adjoint operator as well as a Weyl law for its eigenvalue asymptotics. We denote the set of all such fermionic N-particle Laplacians by $\mathcal{M}_{F,N}$.

On the other hand, N Bosons with repulsive hardcore interactions are described in terms of a quadratic form

$$\begin{aligned} Q_B^{(N)}[\Psi] = N\sum_{e_1\ldots e_N}\int_0^{l_{e_1}}\ldots\int_0^{l_{e_N}} |\psi_{e_1\ldots e_N,x_{e_1}}(x_{e_1},\ldots,x_{e_N})|^2\,\mathrm{d}x_{e_N}\ldots\mathrm{d}x_{e_1} \\ - N\int_{[0,1]^{N-1}}\langle\Psi_{bv}, L_{B,N}(\boldsymbol{y})\Psi_{bv}\rangle_{\mathbb{C}^{2E^N}}\,\mathrm{d}\boldsymbol{y} \end{aligned} \tag{68}$$

on the domain

$$\mathcal{D}_{Q_B^{(N)}} = \{\Psi \in H^1_{0,int,B}(D_\Gamma^{N*});\ P_{B,N}(\boldsymbol{y})\Psi_{bv}(\boldsymbol{y}) = 0 \text{ for a.e. } \boldsymbol{y} \in [0,1]^{N-1}\}, \tag{69}$$

see the paragraph below (30). We recall that $H^1_{0,int,B}(D_\Gamma^{N*}) \subset H^1_B(D_\Gamma^{N*})$ consists of the functions vanishing along the internal boundary $\partial D_\Gamma^{N,int}$ of the dissected configuration space. Again, we added an index B to reflect the bosonic nature of the form. Consequently, we denote the associated operator by $(-\Delta_N, \mathcal{D}_{B,N}^{\alpha=\infty}(P_{B,N}, L_{B,N}))$, and the set of all such operators by $\mathcal{M}_{B,N}^{\alpha=\infty}$.

Theorem 4.1. *There exists a bijective map*

$$\sigma : \mathcal{M}_{F,N} \to \mathcal{M}_{B,N}^{\alpha=\infty}, \tag{70}$$

such that the operators $(-\Delta_N, \mathcal{D}_F^N(P_{F,N}, L_{F,N}))$ and $\sigma[(-\Delta_N, \mathcal{D}_F^N(P_{F,N}, L_{F,N}))]$ are isospectral. This map can be constructed explicitly.

Proof. In order to introduce the Fermi-Bose map (70), we first note that the fermionic symmetry implies the vanishing of functions in $H_F^1(D_\Gamma^N)$ along the internal boundary $\partial D_\Gamma^{N,int}$, see (26). Therefore, we seek to relate $H_F^1(D_\Gamma^N)$ to the bosonic Sobolev space $H^1_{0,int,B}(D_\Gamma^{N*})$. More explicitly, we define a bijective map

$$T_\sigma : H_F^1(D_\Gamma^N) \to H^1_{0,int,B}(D_\Gamma^{N*}) \tag{71}$$

as follows: Let $\Phi_F = (\varphi^F_{e_1\ldots e_N}) \in H_F^1(D_\Gamma^N)$, and divide the components into classes such that representatives have the same set of edge indices $e_1, \ldots, e_N$, up to permutations. Given a fixed representative $\varphi^F_{e_1\ldots e_N}$, let $n(e)$ be the number of times an edge $e \in \mathcal{E}$ occurs among the edge indices and introduce (particle) labels $\zeta(1), \ldots, \zeta(n(e))$. Then define a subdomain $\Omega \subset D_{e_1\ldots e_N}$ such that all $x = (x_e^{\zeta(1)}, \ldots, x_e^{\zeta(n(e))}) \in \Omega$ fulfil

$$x_e^{\zeta(1)} < \ldots < x_e^{\zeta(n(e))}. \tag{72}$$

This is used to define the component $\varphi^B_{e_1\ldots e_N}$ of a bosonic state by setting

$$\varphi^B_{e_1\ldots e_N}(x) := \varphi^F_{e_1\ldots e_N}(x) \tag{73}$$

for $x \in \Omega$, and extending this to all of $D_{e_1\ldots e_N}$ using the bosonic symmetry. Finally, by permuting the edge indices of $\varphi^B_{e_1\ldots e_N}$ and assigning the same values (73) to each representative we obtain all other components, defining a symmetric function $\Phi_B = T_\sigma(\Phi_F) \in H^1_{0,int,B}(D_\Gamma^{N*})$. This construction can be reversed in an obvious way so that the map T_σ is invertible.

Based on the map T_σ we introduce a diagonal matrix $\Sigma(\boldsymbol{y})$ with non-vanishing entries $\Sigma_{ee}(\boldsymbol{y}) \in \{1, -1\}$ that take account of the possible sign changes introduced by σ. Note that this matrix is such that $\Sigma(\boldsymbol{y})^2 = \mathbf{1}$. More explicitly, if $\Phi \in \mathcal{D}_{Q_F^{(N)}}$ is a function with boundary values Φ_{bv}, the function $T_\sigma(\Phi) \in H^1_{0,int,B}(D_\Gamma^{N*})$ has boundary values

$$[T_\sigma(\Phi)]_{bv}(\boldsymbol{y}) = \Sigma(\boldsymbol{y})\Phi_{bv}(\boldsymbol{y}). \tag{74}$$

Furthermore, we set

$$P_N^\sigma(\boldsymbol{y}) = \Sigma(\boldsymbol{y})P_N(\boldsymbol{y})\Sigma(\boldsymbol{y}) \qquad \text{and} \qquad L_N^\sigma(\boldsymbol{y}) = \Sigma(\boldsymbol{y})L_N(\boldsymbol{y})\Sigma(\boldsymbol{y}), \tag{75}$$

which are, for every $\boldsymbol{y} \in [0,1]^{N-1}$, projectors and self-adjoint maps on $\ker P_N^\sigma$, respectively. Hence, given a fermionic quadratic form $(Q_F^{(N)}, \mathcal{D}_{Q_F^{(N)}})$ as in (66) and (67), we can associate to it a unique bosonic form via

$$
\begin{aligned}
Q_F^{(N)}[\Phi] &= \sum_{e_1\ldots e_N} \int_0^{l_{e_1}} \cdots \int_0^{l_{e_N}} |\nabla \varphi_{e_1\ldots e_N}|^2 \, \mathrm{d}x_{e_1}^1 \ldots \mathrm{d}x_{e_N}^N \\
&\qquad - N \int_{[0,1]^{N-1}} \langle \Phi_{bv}, L_N(\boldsymbol{y})\Phi_{bv}\rangle_{\mathbb{C}^{2E^N}} \, \mathrm{d}\boldsymbol{y} \\
&= \sum_{e_1\ldots e_N} \int_0^{l_{e_1}} \cdots \int_0^{l_{e_N}} |\nabla T_\sigma(\varphi)_{e_1\ldots e_N}|^2 \, \mathrm{d}x_{e_1}^1 \ldots \mathrm{d}x_{e_N}^N \\
&\qquad - N \int_{[0,1]^{N-1}} \langle [T_\sigma(\Phi)]_{bv}, L_N^\sigma(\boldsymbol{y})[T_\sigma(\Phi)]_{bv}\rangle_{\mathbb{C}^{2E^N}} \, \mathrm{d}\boldsymbol{y} \\
&= Q_B^{(N)}[T_\sigma(\Phi)]
\end{aligned}
\tag{76}
$$

on the domain

$$
T_\sigma(\mathcal{D}_{Q_F^{(N)}}) = \{\Phi \in H^1_{0,int,B}(D_\Gamma^{N*});\; P_N^\sigma(\boldsymbol{y})\Phi_{bv}(\boldsymbol{y}) = 0 \text{ for a.e. } \boldsymbol{y} \in [0,1]^{N-1}\}. \tag{77}
$$

The associated self-adjoint operator is of hardcore type, see (68) and (69).

The above explicit construction defines the Fermi-Bose map (70) through

$$
\sigma[(-\Delta_N, \mathcal{D}_F^N(P_{F,N}, L_{F,N}))] = (-\Delta_N, \mathcal{D}_{B,N}^{\alpha=\infty}(P_N^\sigma, L_N^\sigma)). \tag{78}
$$

By construction it is bijective, and due to (76) the operators are isospectral. □

B. Bose-Einstein condensation in a gas of bosons interacting via repulsive hardcore interactions

In a second step, we use the Fermi-Bose mapping established in Theorem 4.1 in order to study BEC in a system of particles interacting via repulsive hardcore interactions. With this goal in mind we first consider the fermionic realisations of the N-particle Laplacian described in Subsection IV A and compare their free-energy densities (4) with those of free fermion gases. For the latter, we choose two comparison operators.

The first reference model is that of free fermions with Dirichlet boundary conditions in the vertices. For every $N \in \mathbb{N}$ we hence choose $P_{F,N}^D = \mathbf{1}_{2E^N}$ and $L_{F,N}^D = 0$ with corresponding operator $(-\Delta_N, \mathcal{D}_F^N(\mathbf{1}_{2E^N}, 0))$. This is a textbook example (see, e.g., Ref. 37) for which the free-energy density (4) is well known to be

$$
\begin{aligned}
f_{F,D}(\beta,\mu) &= -\lim_{TL} \frac{1}{\beta\mathcal{L}} \sum_{n=0}^{\infty} \log\left(1 + e^{-\beta(k_n^2-\mu)}\right) \\
&= -\frac{1}{\pi\beta} \int_0^\infty \log\left(1 + e^{-\beta(k^2-\mu)}\right) \mathrm{d}k.
\end{aligned}
\tag{79}
$$

Here, $\{k_n^2\}_{n\in\mathbb{N}_0}$ are the one-particle eigenvalues. Note that this function is smooth, $f_{F,D} \in C^\infty((0,\infty)\times\mathbb{R})$, hence there is no phase transition in a gas of free fermions.

The second reference model also describes free fermions, however with standard Robin boundary conditions in the vertices. Here, $P_{F,N}^R = 0$ and $L_{F,N}^R = M\mathbf{1}_{2E^N}$, and the corresponding operator is $(-\Delta_N, \mathcal{D}_F^N(0, M\mathbf{1}_{2E^N}))$, where $M > 0$ is a suitable constant.

Proposition 4.2. Let $(-\Delta_N, \mathcal{D}_F^N(P_{F,N}, L_{F,N}))_{N\in\mathbb{N}}$ be a family of fermionic Laplacians indexed by the particle number N as introduced above. Assume that for this family there exists $M > 0$ such that

$$
\|\Lambda_{F,N}\|_{op} \le M, \qquad \forall N. \tag{80}
$$

Then the grand-canonical free-energy density $f_F(\beta, \mu)$ coincides with the free-energy density (79) of free fermions with Dirichlet boundary conditions in the vertices.

Proof. Using the min-max principle[38] in the same way is in Proposition 3.2, we conclude that

$$f^{\mathcal{L}}_{F,R}(\beta, \mu) \leq f^{\mathcal{L}}_{F}(\beta, \mu) \leq f^{\mathcal{L}}_{F,D}(\beta, \mu) \tag{81}$$

holds for any family of the fermionic Laplacians that we allow. Since $f^{\mathcal{L}}_{F,R}(\beta, \mu)$ is the free-energy density of a free fermion gas, it can be reduced to

$$f^{\mathcal{L}}_{F,R}(\beta, \mu) = -\frac{1}{\beta\mathcal{L}} \sum_{K_n^2 \leq 0} \log\left(1 + e^{-\beta(K_n^2-\mu)}\right) - \frac{1}{\beta\mathcal{L}} \sum_{K_n^2 > 0} \log\left(1 + e^{-\beta(K_n^2-\mu)}\right), \tag{82}$$

where $\{K_n^2\}_{n\in\mathbb{N}_0}$ are the one-particle eigenvalues.

The number of negative eigenvalues of the one-particle Laplacian is finite so that the first term on the right-hand side of (82) does not contribute in the TL. The second term can be evaluated using the trace formula in the same way as in the proof of Theorem 3.4. This then gives

$$\lim_{TL} f^{\mathcal{L}}_{F,R}(\beta, \mu) = f_{F,D}(\beta, \mu). \tag{83}$$

Together with the bracketing (81) this completes the proof. □

Remark 4.3. Note that condition (80) can be understood as a stability condition for the interaction potential similar to what is often required in statistical mechanics.[34,39] More precisely, one requires a self-adjoint N-particle Hamiltonian $\hat{H}_N$ to be bounded from below by $-NB$ where $B \geq 0$ is some constant.

Finally, we can state the main results of this section.

Theorem 4.4. *Let $(-\Delta_N, \mathcal{D}_N^{\alpha=\infty}(P_{B,N}, L_{B,N}))_{N\in\mathbb{N}}$ be a family of bosonic Laplacians with repulsive hardcore interactions, indexed by the particle number N. Assume that for this family there exists $M > 0$ such that*

$$\|\Lambda_{B,N}\|_{op} \leq M, \qquad \forall N. \tag{84}$$

Then the associated bosonic grand-canonical free-energy density $f_B(\beta, \mu)$ coincides with the free-energy density (79) of free fermions with Dirichlet boundary conditions in the vertices,

$$f_B(\beta, \mu) = -\frac{1}{\pi\beta} \int_0^\infty \log\left(1 + e^{-\beta(k^2-\mu)}\right) \mathrm{d}k. \tag{85}$$

This function is smooth and, hence, there occurs no phase transition.

Proof. Using the inverse Fermi-Bose map as described in Theorem 4.1, we associate to the family $(-\Delta_N, \mathcal{D}_N^{\alpha=\infty}(P_{B,N}, L_{B,N}))_{N\in\mathbb{N}}$ of bosonic Laplacians an isospectral family of fermionic Laplacians. According to Proposition 4.2 the resulting family of fermionic Laplacians has a free-energy density $f_F(\beta, \mu) = f_{F,D}(\beta, \mu)$, and due to the isospectrality with the bosonic family one immediately finds that $f_B(\beta, \mu) = f_{F,D}(\beta, \mu)$. □

Remark 4.5. Theorem 4.4 can be regarded as a quantum statistical version of the theorem by van Hove.[33]

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