Book Review: Prime Suspects: The Anatomy of Integers and Permutations by Andrew Granville and Jennifer Granville

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Book Review: *Prime Suspects:*
The Anatomy of Integers and Permutations
by Andrew Granville and Jennifer Granville

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Synopsis

Prime Suspects is a delightful graphic novel that performs what might, prima facie, seem almost like an impossible feat. Apart from the aesthetic pleasure it provides by being a well-illustrated and well thought out graphic novel, I would describe its major conceptual goal to be that of conveying to a reader not necessarily *au fait* with contemporary mathematics (or even just classical mathematics for that matters), a taste of the exciting nature of mathematical discovery. It does so by focusing on a set of results connecting the study of prime numbers, the study of cycles of permutations, irreducible polynomials in finite fields and more.

> **Prime Suspects: The Anatomy of Integers and Permutations**. Written by Andrew Granville and Jennifer Granville. Illustrated by Robert J. Lewis. Princeton University Press, 2019, 232 pp., \$22.95 (Paperback)

How to convey abstract mathematics in a graphic novel? The authors of *Prime Suspects: The Anatomy of Integers and Permutations* happily start with the idea that investigating mathematical phenomena is not essentially different from the sleuthing that detectives apply in solving their murder cases. On page 3 we are told that we are entering a world where "detectives work closely to mathematicians" (see Figure 1). Sherlock Holmes is often referred to in the graphic novel as a source of excellent advice on how to observe, theorize, and investigate.

Figure 1: This and all the remaining images in this review are excerpted from Prime Suspects: The Anatomy of Integers and Permutations by Andrew Granville and Jennifer Granville, illustrated by Robert J. Lewis. Copyright © 2019 by Princeton University Press. Reprinted by permission.

However, unlike other popular representations of mathematics in forensic contexts which abound in television (see Sklar and Sklar [14] for mathematics in popular culture), here the forensic context is the sleuthing after the anatomy (i.e. the structure) of mathematical objects which might well look very different on the surface. The "anatomical" metaphor is apt, first because comparative anatomy was born out of showing that very different looking organisms might share, surprisingly, the same anatomy but also because its application to mathematics has roots going back to the nineteenth century.

Our mathematicians are presented as a group of forensic mathematicians collaborating with detective Jack von Neumann in the solution of two gruesome murders that display some similar characteristics although the victims are apparently completely unrelated. The team of forensic mathematicians working on the case are Professor C. F. Gauss, and prospective Ph.D. students Emmy Germain and Sergei Langer. (The graphic novel is replete with more or less explicit references to mathematical theorems and people, as evidenced by the previous four names that refer to John von Neumann, Carl Friedrich Gauss, a combination of Sophie Germain and Emmy Noether, and to Serge Lang).

On page 3, we are also told, by a character who will be later revealed to be Count Nicolas Bourbaki, that we are entering a world of truth, of beauty, "a world where you don't have to understand everything in order to know something". A similar statement is found on page 29, where Emmy says: "But you don't really need to understand all the details to appreciate the big picture", see Figure 2.
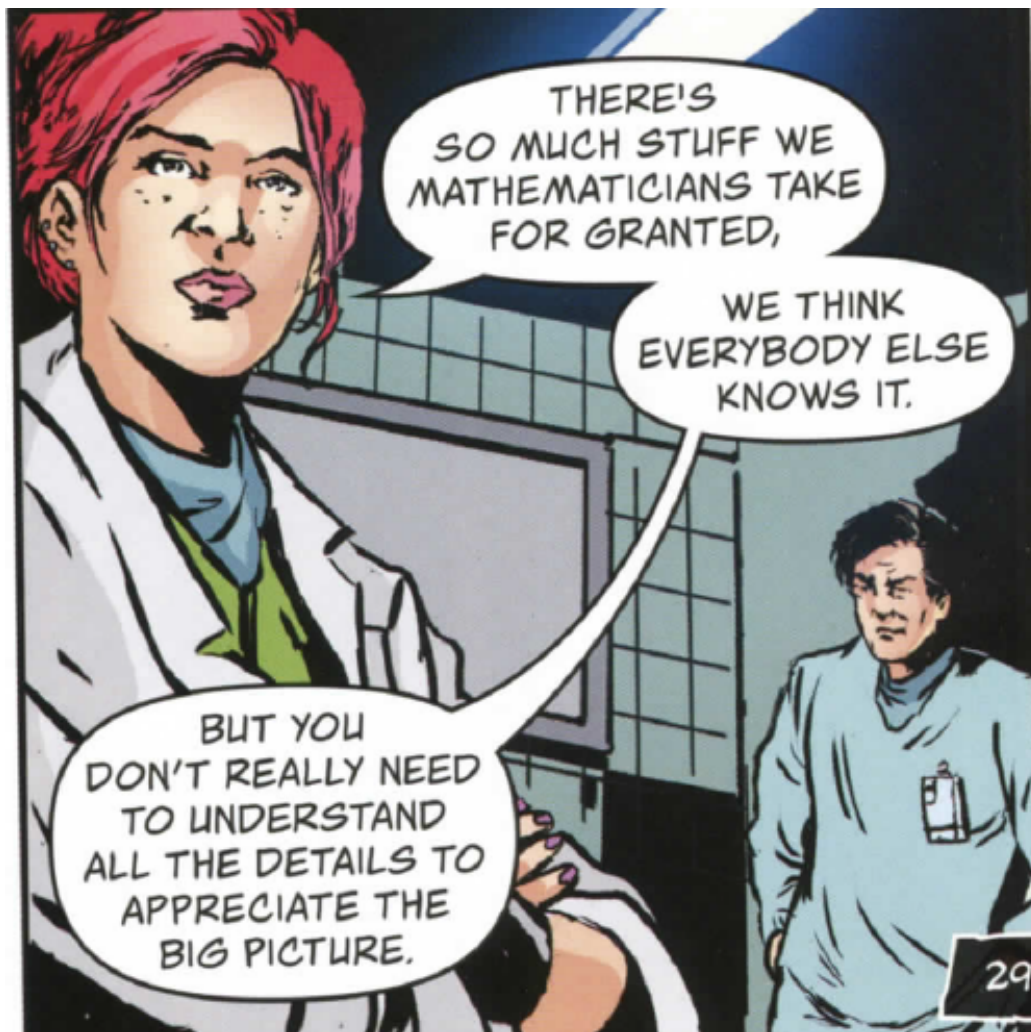

Figure 2: Excerpt from Granville and Granville, page 29. Reprinted with permission. See caption to Figure 1.

Indeed, while this is true of the mathematical experience tout court, I find in this sentence one of the keys to the success of the graphic novel. The exact nature of the results that form the mathematical subject matter of the graphic novel (resemblances of phenomena concerning apparently unconnected areas such as prime numbers, cycles of permutations, and irreducible polynomials in finite fields) is explained in an appendix of twenty-five dense but well-written pages (193–218) that outlines all the mathematics that constitutes the backbone of what the graphic novel refers to, per force, mostly allusively. And I think that the authors did quite well: "you don't really need to understand all the details to appreciate the big picture" can also be read as indicating that what the graphic novels wants to convey are some high-level facts about mathematical discovery that do not need a full understanding of the mathematical details to be fully appreciated. It would have been a terrible mistake, indeed a major flop, to have attempted to lead the reader through the complexities of the mathematical theorems in number theory and permutation cycles that are in any case well explained in the appendix. The result would have been a failed attempt at teaching very complex notions in an inappropriate set-up. The graphic novel only introduces a few key concepts in an informal and friendly way (for instance, the notions of a prime number, a factorial of a natural number, a permutation, a cycle of a permutation; see Figure 3) but the remaining mathematical details, mainly alluded to, go above the ordinary reader's head (Poisson point processes, Buchstab's function, etc.)

But it does not matter. One still manages to understand what is essential, namely the excitement of investigating mathematical coincidences that appear mysterious. Incidentally, for reasons of space I will not introduce any of the mathematical results as I can get my points across about what the graphic novel achieves without entering the details of the mathematics.

A few more words about the imaginative way in which the graphic novel explores its conceptual themes before we get to the main points of the review. Having identified the victims as Arnie Int (a lieutenant of the Integer Crime Family: "came up the hard way to be Bombieri's consigliere; in the center of the ring") and Daisy Permutation (a ballet dancer: "she was odd, outside of the alternating group, her family's 'business'"), the inquiry begins by trying to connect the two cases. But the only similarities between the two victims, which are otherwise apparently unrelated, are the type of knife wound they display on their chests and some graffitis found in the vicinity of where the
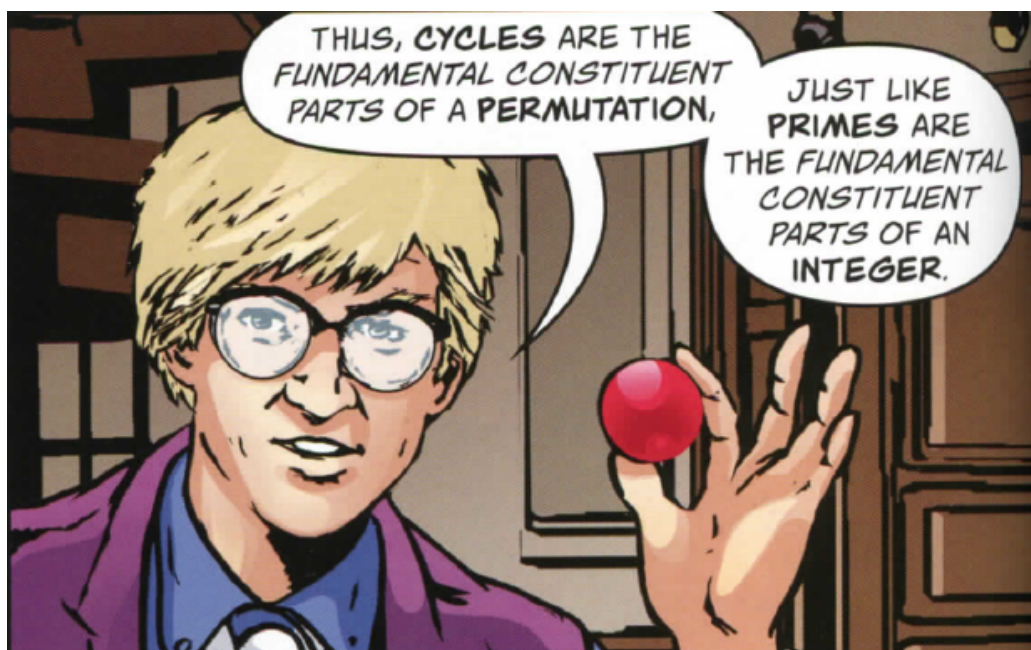
Figure 3: Excerpt from Granville and Granville, page 48. Reprinted with permission. See caption to Figure 1.

corpses were found (two different locations). Our forensic mathematicians start poking around the internal organs of Arnie Int and extract an organ that is a prime number (more primes will be found in his body). But connecting the integer ring and the permutation group (instantiated by Daisy Permutation and the cycles found in her body) seems prima facie like putting together "apples and i-phones" as Langer points out on page 34. From now on the story proceeds with the forensic mathematicians trying to come up with more and more similarities but not without running into dead ends and other conceptual obstacles. At the end of the story, the perpetrator will be found out, also thanks to Emmy Germain's discovery in the files of a third murder case: the victim, Polly Nomial, provides yet another link to the previous two (this refers to the similarity of patterns that polynomials in finite fields display with prime numbers and permutation cycles). All the victims, although very different on the surface, turn out to have, "for all intents and purposes", the same anatomy.

There would be much more to say about the entertaining aspects of the graphic novel. Indeed, it would be hard to convey in a short review just

how much fun this book is with the hilarious references to living and dead mathematicians and the mathematical puns. See for instance the descriptions of Arnie Int and Daisy Permutation that I quoted in parentheses above with their references to Enrico Bombieri (as Godfather of the Integers), rings, and groups. And I don't think I will be able to say "polynomial" again without smiling and thinking of "Polly Nomial". But I will forego all of that in order to discuss what seems to me an important contribution of this graphic novel to an appreciation of how mathematicians proceed in their research.

We can perhaps begin with a contrast. Lots of popular accounts of mathematics have wrongly presented the axiomatic method, pioneered by Hilbert, Peano and others, as the essence of mathematical activity. The axiomatic method presents mathematical theories as mechanisms for generating theorems: starting from some axioms one then painstakingly derives theorems through inferential rules of logic and mathematics. While the axiomatic method has very important foundational uses, many of the popularizers go wrong in (or at least mislead the general public into) thinking that it provides the essence of how mathematics is done. The first thing that this graphic novel achieves is conveying how the source of much mathematical research and discovery is puzzlement.

Mathematicians are puzzled by certain similarities and start investigating what might be behind them (cf. Figure 4). Some similarities are so striking that the appropriate reaction is: "surely it cannot just be mere coincidence!" (As forensic mathematician Gauss says when the same function pops up
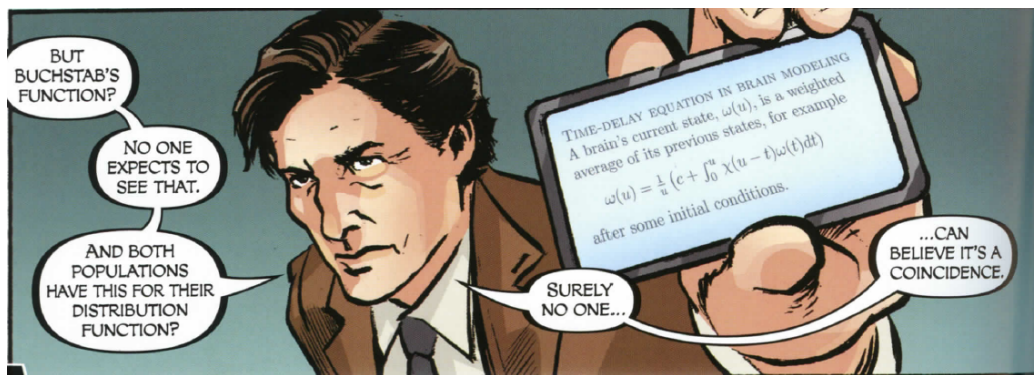

Figure 4: Excerpt from Granville and Granville, page 130. Reprinted with permission. See caption to Figure 1.

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in several different contexts: "This is **great**. These can't be simple coincidences", page 133). As evidence of the similarities pile up, one needs to persuade fellow mathematicians that there is really something going on.

On page 161 Gauss finds himself frustrated by von Neumann's skepticism concerning the deep similarities between the anatomies of integers and permutations. But von Neumann is not happy with the mere descriptive level of the analogies (*that* the analogies hold); he wants to know *why* the anatomies of integers and permutations are so similar (page 162). Gauss retorts that his team has provided von Neumann with two explanations (one from probability and one from analytic combinatorics) of why such similarities are there. In the end it will be Emmy Germain's discovery of the Polly Nomial case (i.e. the further similarities displayed by polynomials in finite fields to integers and permutations) that will convince von Neumann and will lead to the arrest of the perpetrator of the murders.

Figure 5: Excerpt from Granville and Granville, page 163. Reprinted with permission. See caption to Figure 1.

What the Granvilles have offered us is a wonderful informal discussion of a case study in contemporary mathematics that highlights some crucial aspects of mathematical research. I would like now to provides a philosophical meta-commentary on the case study by indicating how many aspects of this story lead naturally to recent discussions in philosophy of mathematics, especially in the direction that goes under the name "philosophy of mathematical practice" (Mancosu [10]). But first, I would like to start with a broader point in epistemology.

Faith, evidence, and mathematics. Consider the tenaciousness with which our forensic mathematicians go after the hunch that there must be something quite similar in the two cases. One of the most striking facts about mathematical research is that this attitude has something in common with faith. I don't mean faith in a religious sense specifically but rather faith as it encompasses both a religious and a more mundane meaning (see Buchak [1, 2]). This is a case of propositional faith, "faith in p" where p might be a specific proposition or the more general "there must be some underlying cause between two mathematical facts". As epistemologists have pointed out the commitment shown by "faith in p" is often rational despite the fact that at various moments the evidence that comes in might lower our credences in p. And yet this steadfastness in the face of counterevidence is not irrational and it is what allows the pursuit of ambitious programs that we would perhaps otherwise abandon if we were simply subject to the vagaries of evidence coming in that might prima facie speak against the original project. In other words, the steadfastness protects us against information that might seem to speak against our overall goal.

In the graphic novel this is beautifully illustrated by Emmy Germain's desperation at her initial inability to find any meaningful connection between integers and permutations. The evidence speaking against the connection is related to lowest primes and smallest cycles, which just does not seem to tell in favor of an underlying structure. But, as the graphic novel points out, one needs to learn what is relevant and what can be dropped as peripheral. Emmy Germain reaches a desperate point in her research on page 112 (see Figure 6); but she does not abandon the project and with Gauss' help she manages to find a breakthrough.

In conclusion, mathematical activity, as other activities in which we commit to a certain plan, calls for steadfastness in the face of counterevidence. Of course, this does not mean that it is rational to hold on to "faith in p" no matter what comes in. If p is the Goldbach's conjecture and we come up with a straight counterexample to it then "faith in p" should be abandoned. This leads to the deep question of how evidence works in mathematical research. Epistemologists study such issues of rationality and all I am suggesting is that mathematics is just as good a field of application for such reflections as any other field of enquiry. But mathematical evidence poses special problems.

Figure 6: Excerpt from Granville and Granville, page 112. Reprinted with permission. See caption to Figure 1.

Some excellent work in this direction has been focused on the ways in which we choose axioms in set theory (the role of evidence in intrinsic versus extrinsic justifications etc.; see [8, 9] but much still needs to be done in this area, for instance on evidential support for conjectures etc).

Discovery. It is a striking fact about mathematics that areas that appear completely unrelated turn out to share deep commonalities. An initial analogy often works as a catalyst and as a propeller for achieving deeper results. This is of course well known and an appreciation of the beauty of math-

ematics cannot begin until one encounters such examples, for instance the striking connections evidenced by Galois theory. One of the most insightful discussions by a mathematician concerning the role of analogies in discovery is given in André Weil's letter to his sister, Simone, published in English in Weil [15].

The graphic novel emphasizes how one is led from an initial analogy to investigate what might be behind it, and by doing so it provides the reader with an understanding of what drives much mathematical research and discovery. How discovery works and what is rational in pursuing the analogies (even as a matter of faith, as pointed out above) is of course a very tricky issue. But the topic has been discussed in the philosophy of mathematical practice. For instance, Grosholz [6, 7] has written on the problem of the (partial) unification of autonomous domains. She asks the question "how does both the mutual autonomy and the rational relatedness of domains make the growth of mathematical knowledge possible?" [6, page 82], and develops an account of the role of "hybrids" in the growth of mathematics and discovery. It would be interesting to analyze the case study provided in the graphic novel by means of Grosholz's approach. Another tantalizing topic is the role of proof, or lack thereof, in discovery (see Giaquinto [4, 5] on visualization and discovery).

Coincidences and explanations. The graphic novel brings out the importance of the observation of surprising facts (including striking analogies). In some cases we get a strong intuition (and intuition can of course be the result of much educated training) that there must be a connection and we look for an underlying cause. This leads to the topic of mathematical explanation. It is only in the last two decades that this issue has become a topical area of investigations for philosophers of mathematics. With roots going back as far as Aristotle (distinction between theorems *that* show something to be the case and theorems that in addition to proving *that* also give us *why* the result holds), the topic has been dealt with by influential philosophers of mathematics such as Bolzano and Cournot. But in the analytic tradition, overemphasis on the explanation of physical phenomena through causal processes led to the mistaken idea that explanation could not be found in mathematics (as there are no causal processes among mathematical facts). But of course, this should be a reduction ad absurdum of an account of explanation that makes causality the key element of all purported explanations.

Mathematical explanations are a basic fact in the phenomenology of mathematical research and this graphic novel leaves no doubt about the correctness of this claim. We are indeed presented with a wonderful case study exemplifying the search for a mathematical explanation. Not only do the characters in the graphic novel speak about explanations but in his informative appendix on the mathematics at issue, Andrew Granville explicitly addresses the issue of mathematical explanation with the following remarks:

> Why are the anatomies of integers and permutations so similar? There are two proposed explanations for why their anatomies are so similar, one from probability theory, the other from analytic combinatorics (as handed out on page 162), though I find neither explain "why", but rather "how" (page 205).

And while Andrew Granville himself expresses doubts as to whether in this case we might not simply have descriptions as opposed to explanations (the same doubt is conveyed on page 162 of the graphic novel; see Figure 7 below), he obviously sees the importance of looking for explanations as motivating factors in mathematical investigation.

Figure 7: Excerpt from Granville and Granville, page 162. Reprinted with permission. See caption to Figure 1.

In the wake of interest on mathematical explanation there has also been some literature on mathematical coincidences. I refer here to the survey article on mathematical explanation in the Stanford Encyclopedia of Philosophy (Mancosu [12]) where one can also find references to the literature on mathematical coincidences.

Unification. As a reaction against the models of explanation based on causality and natural laws, some alternative models of explanation based on the concept of unification were developed in the 1980s by Michael Friedman and Philip Kitcher. One of the major assets of a theory of explanation as unification, according to Kitcher, was that it could be applied to areas such as mathematics, which was excluded from the realm of explanatoriness in the other accounts of explanation. The basic idea, but the model is much more complex, is that by reducing through unification the number of basic brute facts, one achieves explanatory dividends. But that explanation and unification go together has been questioned. For instance, Margaret Morrison [13] has argued that unification in the natural sciences quite often occurs by ignoring the mechanisms that explain the physical phenomena and that in so doing unification takes us away from explanation. Morrison might be right about physical science but in mathematics the idea that unification and explanation are twins is deeply rooted. In his commentary (page 207), Andrew Granville speaks of certain mathematical phenomena as "begging for a unifying explanation" and there is no question that the search for an underlying cause of the similarities which are the backbone of the mathematics treated in the graphic novel is just the search for a unified account of the phenomena.

This desire for unification only increases when one realizes that there are many other areas that display the same anatomy. This is alluded to in the graphic novel on page 163 with reference to Anatoly Vershik's work (see Figure 8), and explained in the mathematical appendix on pages 206–207 (revealingly, the title of the section is "An underlying cause?"). But to get to the underlying cause one might need to dig quite deep.

Depth and styles. One of the entertaining aspects of the graphic novel, and one which might likely be ignored by those readers who have only limited acquaintance with mathematics, is that there are different styles of mathematical research. The style of applied mathematicians is not the same as the style of algebraic geometers. But even within pure mathematics,

Figure 8: Excerpt from Granville and Granville, page 163. Reprinted with permission. See caption to Figure 1.

there are very different styles of treating the same subject; for instance, until the breakdown of the Soviet Union, the Russian school of algebraic geometry had a very different style form that pursued in the West. Andrew and Jennifer Granville refer to this phenomenon when discussing the biographical details of Serge Lang (the real mathematician) and the reaction by number theorist C. L. Siegel to Lang's treatment of algebraic number theory, which Siegel refers to disparagingly as "senseless abstraction" (page 189). The graphic novel alludes repeatedly to such differences in style. It is in fact Sergei Langer that first taunts Professor Gauss, who has just lectured on a result by Hardy and Ramanujan, to offer something "more up-to-date and challenging, a little more abstract" (page 58) and in reply Gauss tells him that "Professor Joe Ten-Dieck [the fictional counterpart of Alexander Grothendieck] would have been an ideal mentor for you in your quest for the perfect abstraction" (page 59; see Figure 9).

Already in the middle of the graphic novel we discover that Sergei Langer is under the spell of the Bourbaki approach to mathematics and later he says: "Bourbaki explained to me that to get results you cannot be afraid,

Figure 9: Excerpt from Granville and Granville, page 59. Reprinted with permission. See caption to Figure 1.

that you have to dig deeper" (page 171; also see Figure 10). In an outburst on page 176, Langer accuses Professor Gauss not to be able to dig as deep as Joe Ten-Dieck.

The conflict between Gauss and Langer, which will explode at the end of the graphic novel, is one among two different styles of mathematics. Professor Gauss likes to pay proper attention to the surface phenomena (while of course going after the common patterns) and this leads to Jack von Neumann's telling him at one point that he [Gauss] needs to bring him more evidence, namely more evidence of what deeper connection underlies the similarities. He does so by scathingly remarking on page 143: "Stop trying to get down and dirty. And leave the applied mathematics to the people who can handle it." The end of the graphic novel sees a reconciliation between Gauss and Count Nicolas Bourbaki. Perhaps they have both come to see the positive aspects of each other's styles.

The graphic novel is thus effective in bringing to the reader's attention the issue of styles in mathematics and to allude to the problem of depth.

Figure 10: Excerpt from Granville and Granville, page 171. Reprinted with permission. See caption to Figure 1.

Both aspects have been at the focus of recent work in the philosophy of mathematical practice; for an overview of work on styles in mathematics, see Mancosu [11], and for an overview of depth, see the special issue of *Philosophia Mathematica*, edited in 2015 by Ernst *et al.* [3], devoted to mathematical depth.

Conclusion. The above discussion, necessarily limited in extent, was only meant to convey that this very enjoyable graphic novel does a terrific job at presenting readers with a fascinating and realistic picture of how mathematical research is conducted. It does so in a deep way and yet with a light hand without falling into the trap of transforming the novel into a lecture on advanced mathematics or on methodology. Both the story and the illustrations are a delight. It is a splendid success.

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