Journal of Humanistic Mathematics

Volume 10 | Issue 1 January 2020

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Recommended Citation

Danesi, M. "Mathematical Cognition and the Arts," *Journal of Humanistic Mathematics*, Volume 10 Issue 1 (January 2020), pages 317-336. DOI: 10.5642/jhummath.202001.14 . Available at: https://scholarship.claremont.edu/jhm/vol10/iss1/14

Mathematical Cognition and the Arts

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Synopsis

This article revisits the study of mathematics in the arts, and vice versa, the arts in mathematics, with a view to connecting mathematical and artistic creativity to the same neural circuits—a proposition put forward for mathematics and language in a critical 2000 book by Lakoff and Núñez, *Where Mathematics Comes From.* This expanded perspective would open up suggestive avenues for connecting mathematics, language, and the arts as part of an imaginative blend that comes out in different forms but having the same underlying neural source. Whether or not this can be established empirically, it is plausible and highly interesting and, thus, needs to be explored seriously in order to see if equations to theorems are born of the same mental structures that produce music, poetry and drawing, as the philosopher Max Black [4] had anticipated before the advent of contemporary neuroscience in the early 1960s. The argument put forth here is that art can be studied through a mathematical lens, and that mathematics can be studied through an artistic lens, in order to glean what the common neural substratum is like. The approach is called *hermeneutic*, in line with critical approaches in the arts, from visual art to literature.

1. Introduction

When Pythagoras carried out the systematic investigation of numbers and their geometrical properties he laid the foundations for mathematics as an autonomous discipline. But he did not isolate it from philosophy or the arts, seeing in mathematical concepts a means to understand connections between phenomena such as the cosmos and music (Godwin [20], James [26]). According to the Pythagoreans, the proportions in the movements of celestial bodies and the ratios of the frequencies of strings that produced consonant sounds were mirrored in mathematical proportions. Mathematics was thus a means of studying the connection between numbers, cosmology, and the human arts. The Renaissance continued the tradition of connecting mathematics with the arts. This was broken by the time of the Enlightenment, when new social and educational realities encouraged a separation of the two, with mathematics and science on one side and the arts and humanities on the other. To this day, these are perceived as products of distinct, separate mental faculties in school, society, and many sciences of the mind, even though, as will be argued in this paper, this state of affairs is changing (Gardner [18]).

In fact, despite this artificial separation, mathematics, science and the arts are still felt to be interconnected by mathematicians, scientists, and artists themselves (Bell [3]). The study of "mathematics-in-the arts", or MIA for short, continues to produce many valuable insights. But the reverse has rarely been contemplated—namely, using the arts to study mathematics (Danesi [10]). This can be called instead, an "arts-in-mathematics", or AIM, mode of inquiry. Starting with Stanislas Dehaene [11] and Brian Butterworth in [5] the scientific search for the answer to whether or not mathematics is separate from other faculties, such as language and the arts, has become a central area of research with the cognitive and neurosciences. But these sciences have rarely contemplated using the arts as a channel for gaining access to the nature of mathematics.

This essay will take a look at the possibility of using an AIM approach to the study of mathematical cognition—an approach that is not separate from other approaches, including the MIA one. The groundwork for establishing such an approach was laid, indirectly, by George Lakoff and Rafael Núñez in their book *Where Mathematics Comes From: How the Embodied Mind Brings Mathematics into Being* [32], in which they discussed a coherent, albeit controversial, view of how mathematicians come to use and invent their proofs and theorems through the use of analogies and metaphor. If they are correct, then the same neural circuits are involved in mathematics and the arts, and this would open up suggestive investigative avenues for connecting mathematics, language, and the arts. Whether or not a neural interconnection can be established empirically, the point is that it is plausible and highly interesting and, thus, needs to be explored seriously if we are ever to come to an understanding of what mathematical cognition is.

If its basis is indeed analogical and metaphorical, then its artifacts, from equations to theorems, are born of the same mental structures that produce music, poetry and drawing, as the philosopher Max Black [4] had already anticipated before the advent of neuroscience in the early 1960s.

2. Mathematics in the Arts

The Renaissance painters not only developed perspective drawing but also examined the geometry behind it. They also studied and utilized the Golden Ratio as part of their repertoire of tools for creating visual art works. The tradition of connecting mathematics explicitly to the arts by artists and mathematicians, or MIA, started essentially in that era. The intrinsic interconnection between the two—mathematics and art—reveals, arguably, that the brain is based on what the Italian philosopher Giambattista Vico called a "poetic logic" (Danesi [9]) and Edgar Allan Poe, a "bi-part" soul, that blends imagination and reason (Stade [47]). As Growney [21] has convincingly shown, many poets have used their bi-part soul to create some magnificent works of literary art, including poems with mathematical imagery (*Geometry* by Rita Dove, *Figures of Thought* by Howard Nemerov, *Pi* by Wislawa Szymborska, and others), and poems with mathematical structure, such as Lewis Carroll's *The Mouse's Tale*.

In effect, MIA has been a means, since the Renaissance, for unraveling the relationship between numbers and visual art, poetry, music, and other creative products, but it has not affected the way mathematics is studied in any significant way. It is a hermeneutic tool, similar to the ones used by literary critics, musicologists, and art critics for understanding the nature of creative texts and the creative impulse itself. The overall method involves identifying points of contact between the arts and mathematics. Mathematicians themselves have often sensed this connection, referring to "good mathematics" as "beautiful," much like a literary critic would designate a poem as aesthetically pleasing or beautiful. It is relevant to repeat some of these here:

> The mathematician's patterns, like the painter's or the poet's must be beautiful; the ideas, like the colors or the words must fit together in a harmonious way. Beauty is the first test: there is no permanent place in this world for ugly mathematics (G. H. Hardy [22, page 85]).

> Mathematics, rightly viewed, possesses not only truth, but supreme beauty—a beauty cold and austere, like that of sculpture, without appeal to any part of our weaker nature, without the gorgeous trappings of paintings or music, yet sublimely pure and capable of a stern perfection such as only the greatest art can show (Bertrand Russell [43, page 60]).
>
> It is impossible to be a mathematician without being a poet in soul (Sophia Kovalevskaya [31]).
>
> A mathematician who is not also something of a poet will never be a complete mathematician (Karl Weierstrass [50, page *xix*]).

The sense of beauty that, say, a mathematical proof can evoke does indeed have many features in common with a poetic or artistic text, which generates sense and aesthetic power through its coherence and internal symmetry. In effect, MIA is a hermeneutic tool connecting structural features of art forms and their aesthetic qualities to mathematical structures. This is likely why some artists will even incorporate mathematics directly into their works, starting with the perspective painters and a little later with polyphonic music. Among well-known examples of mathematically-based musical works, the polyphonic works of Johann Sebastian Bach, the atonal music of Arnold Schoenberg, which employs permutation theory as its structural framework, and the incorporation of the Golden Ratio by Debussy in *La mer*, are among the best known ones. In the visual arts, fractal theory, mathematical symmetry, and topology have informed the works of painters and sculptors, such as M. C. Escher who created pictures that explored the complex relationship between perception and mathematics. Beginning around 1936, Escher started drawing ambiguous patterns in which he interlocked repeated figures of stylized animals, birds, and fish, leaving no spaces between the figures (Schattschneider [44]). A little later, he began toying with visual perception itself, creating such "impossible figures" as staircases that appeared to lead both upward and downward in the same direction, and alligators that seemed to come to life, walking off the edge of the paper. In his famous 1960 lithograph, *Ascending and Descending,* the people in it are both climbers and descenders at once.

Perhaps the only way to grasp this illusion is to "slice the staircase," as Falletta [15, page 32] proposes. Doing so shows that the levels of the staircase do not lie in a horizontal plane, but rather move upward spirally. The steps, on the other hand, remain in the horizontal plane. Hence, it is a geometrical play on the representation of three-dimensions on a two-dimensional surface that produces the illusion. Another well-known producer of this kind of optical illusion art was the Swedish artist and art historian Oscar Reutersvärd. One of his most famous works is the "devil's triangle," which creates a jarring "devilish" sense of distortion and surreal unease. In 1958, the English biologist L. S. Penrose and his son Roger drew their own version of the devil's triangle, which has since come to be known as the Penrose Triangle (Fauvel, Flood, and Wilson [16]).

Works such as these are examples of what is known generally as "mathematical art," that is, art that incorporates some mathematical property or idea in its textual structure. One mathematical construct that has been particularly fertile in this genre of art is the Golden Ratio (see Livio [33]). The ratio is mentioned at the beginning of Book VI of Euclid's *Elements*.

The Golden Ratio also inspired the first true MIA study in 1509, by Luca Pacioli, called *De divina proportione,* a book illustrated by Leonardo da Vinci. In it, Pacioli showed that the Golden Ratio, which he renamed the "divine proportion," was an inherent mathematical property that constitutes a pattern in visual art and architecture. It is no coincidence that da Vinci's *La gioconda (Mona Lisa)* exhibits the divine proportion if a rectangle is drawn around the face of the Mona Lisa. No documentation exists to indicate that Leonardo consciously used the Golden Ratio in his painting, nor where precisely the golden rectangle should be drawn. Nevertheless, the fact that Leonardo was a close personal friend of Pacioli and illustrated his book leads one to strongly suspect that he incorporated the divine proportion in that work and in others. The number of art works that have since incorporated the Golden Ratio is truly astounding. One of the most famous is by Salvador Dalí, who deliberately included it in many of his works. For example, the ratio of the dimensions of *Sacrament of the Last Supper* is equal to the Golden Ratio.

The connection between the Golden Ratio, mathematics, and aesthetics was studied for the first time scientifically by the German and psychologist Gustav Theodor Fechner in the 1860s (Livio [33]). In one fascinating experiment, he presented ten rectangles varying in their length-to-width ratios to subjects, who were then asked to select the most pleasing one.

The results showed that 76% of all choices centered on the three rectangles having proportions that either exhibited the Golden Ratio (1.62), or else approached it (1.75 and 1.50). Subsequent research has shown a degree of variance with these findings, indicating that culture-specific perceptual styles probably guide the choice of aesthetic figures. But the fact remains that the Golden Ratio has been embedded in some of the greatest art works of history and is found across cultures and across time.

The exact same kind of story can be drafted for the presence of Fibonacci numbers, π, and other mathematical forms in the arts of virtually all cultures (for examples, see [45, 12, 7, 2, 1, 14, 38, 39]). Given the extensiveness of this kind of story, MIA can be considered a tool not only for decoding the presence of mathematical forms in art and music, but also for connecting the dots between them. In effect, MIA might be a valuable tool of cognitive science and the neurosciences aiming to investigate the nature of mathematical cognition via empirical research on the brain. Take, for example, the work of the mathematician Benoit Mandelbrot [35], who (as is well known) found that random fluctuations in Nature and in human affairs formed geometrical patterns, which he called fractals when they were reduced to smaller elements. As it turns out, fractals disclose a hidden pattern in shapes that would otherwise appear random to the naked eye—shapes that are not unlike contemporary modern art forms. Indeed today there is a whole field called "fractal art" which is generated by computer fractal algorithms. Solomon Marcus [36, page 179] makes the following appropriate observation on why fractal geometry is such a powerful cognitive tool:

> What art and poetry anticipated in the 19th century, together with some phenomena pointed out by Weierstrass, Peano and Koch, related to curves devoid of tangents in all their points, became explicit in the mathematics of the second half of the past century, when Benoit Mandelbrot invented the fractal geometry of nature. Its idea is that nature, in most of its aspects, is not at all simple and regular. Clouds, ocean coasts, Brownian motion, snowflakes, mountains, rivers don't fit with the regular objects of traditional geometry. Even celestial bodies, longtime considered models of regularity, prove to be less regular than they supposed to be. How to approach this world of high complexity? The answer proposed by Mandelbrot is the notion of a fractal object.

> Such objects are obtained as limits of some asymptotic processes, starting with some regular figures. If the first steps of these processes are visible objects and fit with the simplicity of traditional geometry, as soon as we go to next steps the new objects become less and less visible and regular. At the limit, we get completely invisible, however perfectly intelligible objects, the fractal ones. What makes them very attractive is their inner, hidden simplicity, in contrast with their outer complexity: in a fractal object, there is a remarkable phenomenon of self-similarity: it repeats at its different levels in the same structure As a matter of fact, everybody can test this fact looking carefully at the structure of a tree in the forest.

Fractal shapes were known long before Mandelbrot provided a mathematical framework for studying them. They turn up in Islamic art, in Celtic artifacts, and in ancient myths. In Mahayana Buddhism, for example, the fractal nature of reality is captured in the *Avatamsaka Sutra* by the god Indra's net, a vast network of precious gems hanging over Indra's palace, arranged in such a way that all the gems are reflected in each other. In recent times, both Dalí and Escher have exploited fractal techniques, creating shapes out of repeated copies of one another.

Perhaps an overarching explanatory model that can be enlisted to explain the connections between mathematics and art is Carl Jung's concept of archetypes [27]. To the best of my knowledge, cognitive scientists and neuroscientists have not used this Jungian construct to investigate mathematical cognition. The Golden Ratio, the Fibonacci sequence, and other mathematical constructs may, however, either be archetypes themselves or else derive from some archetypal substratum in the mind. This might explain why they crop up in music, poetry, and other creative activities (Emmer [13]). An "archetype theory" of mathematical cognition should thus be considered seriously in the scientific research agenda in order to grasp the meaning of such coincidences. If we look for archetypes in the arts, music, dance, poetry, and the like we will find them. In an interesting and relevant book, Stuart Isacoff [25] argues that the invention of western musical harmony traditions came about from an unconscious Pythagorean archetype. Stewart [48, page 9] makes the following relevant observation:

> The main empirical support for the Pythagorean concept of a numerical universe comes from music, where they had noticed some remarkable connections between harmonious sounds and simple numerical facts. Using simple experiments they discovered that if a plucked string produces a note with a particular pitch, then a string half as long produces an extremely harmonious note, now called the octave. A string two-thirds as long produces the next most harmonious note, and one three-quarters as long also produces a harmonious note. These two numerical aspects of music are traced to the physics of vibrating strings, which move in patterns of waves. The number of waves that can fit into a given length of string is a whole number, and these whole numbers determine the simple numerical ratios. If the numbers do not form a simple ratio then the corresponding notes interfere with each other, forming discordant 'beats' which are unpleasant to the ear. The full story is more complex, involving what the brain is accustomed to, but there is a definite physical rationale behind the Pythagorean discovery.

Remarkably, in 1865 the British chemist John Newlands discovered that by arranging elements according to atomic weight, those with similar properties occur at every eighth element like musical octaves. It came to be called, appropriately, the Law of Octaves, and it led to the development of the Periodic Law of chemical elements. This discovery strongly suggests that the structure of matter and music is likely to be one and the same, and that mathematics is the conceptual bridge between them. But the archetype hypothesis also poses a deep riddle. When ratios between certain string vibrations are set, other ratios are thrown off, making strict use of a single tuning system impossible because it produces dissonances. This means that a fatal defect haunts the Pythagorean model of harmonics, which the Pythagoreans knew but kept secret. To banish the dissonances, the keyboard was tempered by breaking the octave into equal parts, so that all harmonies sounded in tune. The most prominent example of this is Bach's *Well-Tempered Clavier*. This made the flourishing of western music a reality. In other words, it was a human invention that turned the Pythagorean archetype into a viable tradition. Dissonances exist and are even employed aesthetically by musicians; but they can be understood mainly in opposition to consonances.

In other words, dissonant music is an outgrowth of consonant traditions; the reverse would never have produced contemporary music.

3. The Arts in Mathematics

In a 2008 collection of scientific studies on mathematical cognition, edited by James Royer [42], the study of mathematical cognition is portrayed as an interdisciplinary enterprise, with computer science and neuroscience standing out as primary approaches. The studies published in the journal *Mathematical Cognition*, also show a wide range of approaches to mathematical cognition. But rarely is art seen as a complementary discipline for studying the mathematical mind included within the cognitive science paradigm. Whatever the reason for this, there is little doubt that if we are ever to make a true headway into the study of the mathematical mind, we cannot avoid an "arts-in-mathematics", or AIM, line of inquiry, as the Renaissance artists and mathematicians certainly understood. Immanuel Kant [28] also saw mathematics as the ability to use visual signs to grasp quantity and space in a symbolic-artistic way. His perspective found a correlative view in the ideas of Charles Peirce [37] and, especially, in his Existential Graph Theory, whereby he claimed that the diagrams that mathematicians make mirrored their thought processes.

An AIM approach would allow cognitive scientists and neuroscientists a means for casting a wider net to the study of mathematical cognition, without excluding the extant interdisciplinary focus of the whole enterprise. The interdisciplinary study of mathematical cognition really took off after the publication of Lakoff and Núñez's 2000 book, mentioned above [32]. In it, Lakoff and Núñez claimed that the ability to do math is not separate from other faculties such as language—an hypothesis that has been corroborated by various neuroscientific studies since and which are beyond the scope of the present discussion. One concrete verification of this is the fact that we use language to learn math and that math has many structural properties that are linguistic (Danesi [10]). What unites mathematical and linguistic reasoning is metaphor. In a lecture given at the Fields Institute of Mathematics at the University of Toronto in 2011, titled "The Cognitive and Neural Foundation of Mathematics: The Case of Gödel's Metaphors," Lakoff elaborated upon the work he carried out with Nüñez, arguing essentially that mathematicians devise their proofs by resorting to metaphorical (analogical) strategies.

He looked specifically at Gödel's famous indeterminacy theorem and how Gödel likely came to conceptualize it—by analogy with Georg Cantor's diagonal method of proof. Gödel found a statement in a set of statements that could be extracted by going through them in a diagonal fashion—now called Gödel's diagonal lemma. The statement, S, like Cantor's C, simply does not exist in the set of statements. Cantor's diagonalization proof is a conceptual metaphor, as Lakoff calls it; it is the result of linking different domains of knowledge into a new one that is a "blend" of these domains. The blend produces new insights. It led Gödel to imagine three metaphors of his own. The first one, called the "Gödel Number of a Symbol," is evident in the argument that a symbol in a system is the corresponding number in the Cantorian one-to-one matching system (whereby any two sets of symbols can be put into a one-to-one relation). The second one, called the "Gödel Number of a Symbol in a Sequence," consists in Gödel's demonstration that the n^{th} symbol in a sequence is the n^{th} prime raised to the power of the Gödel Number of the Symbol. And the third one, called "Gödel's Central Metaphor," was his proof that a symbol sequence is the product of the Gödel numbers of the symbols in the sequence. Gödel's metaphors, Lakoff argued, come from neural circuits linking a *number* source to a *symbol* target. In each case, there is a blend, with a single entity composed of both a *number* and a *symbol sequence*. When the symbol sequence is a formal proof, a new mathematical entity appears—a "proof number." The underlying premise in this whole line of argumentation is that metaphorical blends in the brain produce knowledge and insights. This applies to the creation of new language as it does to the creation of new mathematics.

Myriad treatises have been written about proof. However, rarely before Lakoff has it been considered to be a kind of metaphorical process that reveals a need to organize mathematical information into textual forms that make sense to us in the same ways that all kinds of texts do—poems, drawings, novels, songs, and so on. Proof is thus a type of text-making governed by principles of analogy and blending that make sense to the human brain. So is art, poetry, and other creative products. The statements by mathematicians that mathematics is beautiful, when considered from Lakoff's perspective, and AIM more broadly, suggest that their sense of beauty derives from the parts of a proof forming blends that produce insights along the pathways of the proof. A similar approach can be seen in a groundbreaking 1962 study by the American philosopher Max Black [4], a study that has clearly influenced the work of Lakoff and others subsequently. Black argued that mathematics was not solely the result of deducing theorems from observations, but also, and primarily, by making inferences and connections between facts, other theories, and even everyday experience. Indirectly, Black laid the foundations for AIM long before the advent of cognitive science in the mid-1980s (Gardner [18]).

The AIM and MIA perspectives were unconscious frames of mind in the ancient world. Plato devised the first known school curriculum for his academy, dividing it into four fields—arithmetic, geometry, astronomy, and music (called the *quadrivium*). In combination with the liberal arts of grammar, rhetoric, and logic (known as the *trivium*), the basis for integrated learning was laid by the Greek philosopher. The implicit objective of Plato's curriculum was a Pythagorean one—to interconnect mathematics with other mental faculties and visual artifacts. To this day, the most powerful form of investigation is to envision mathematical forms as having visual artistry. In a fascinating study, Louis H. Kauffman [41] illustrates how this type of inquiry can be easily carried out, showing how knot theory might overlap with visualization strategies. In effect, knots are de facto art structures when they are viewed through the mind's eye at the same time that they are hidden mathematical structures. The same can be said about mathematics and rhythm, as Luis Radford [41] has cogently demonstrated. He defines rhythm as "a complex of conflicting 'components,' each one exploring and expressing our experience of the world in a different manner." Radford explored this idea with an ingenious experiment, whereby a class of Grade 9 students, divided in groups of three, were asked to figure out the algebraic pattern inherent in a sequence of circle figures by first drawing the next figure in the sequence, then calculating the number of circle figures, and finally devising an algebraic formula that describes the pattern. The key result of the experiment showed that the students arrived at the deduction through corporeal movement, that is, by the use of bodily movements that were connected rhythmically with thought—pointing, gesturing, and so on. Using voice analysis software to record the tones of the speech of one of the subjects, Radford also found that the tone patterns matched the rhythms of the gestures. The main components of rhythm (meter, grouping, theme, and prolongation) manifest themselves in both the gestural and vocal components of the thought processes manifested by the subjects.

4. AIM-Oriented Research

The claim made here is that mathematical cognition can be studied as scientifically, not just speculatively, through the arts—a challenge that has actually been taken up, in a fledgling way, by several pioneering neuroscientists and educators. As an example, the Dana Foundation, a philanthropic organization, sponsored several major research projects in 2004 to study the hypothesis that learning mathematics can be bolstered by the arts. Under the guidance of the well-known neuroscientist, Michael S. Gazzaniga, a group of cognitive and neuroscientists from seven universities were assembled to develop a research agenda to examine the possibility that the visual arts, music, dance, and theater might affect other areas of learning, including and especially mathematics.

The first results were published in 2008 under the rubric of *Learning, Arts, and the Brain*. One of these, by psychologists Michael Posner and Brenda Patoine [40], who observed the brain activity of children 4-6 years of age while they worked on exercises intended to simulate the attention-focusing aspects of art, found that such simulation enhanced the children's attention spans, thus improving cognition. Another study, by neuroscientist Elizabeth Spelke [46], is of special relevance to AIM, since it examined the relationship between music and mathematical abilities. Specifically, she sought to determine if engaging students in music will activate brain systems that also enable them to understand representations of number and geometry. If music training fosters mathematical ability, it does so by activating and enhancing one or more cognitive systems, as blending theory, mentioned above, also postulates. This study thus established a tentative link between mathematical cognition and music. More significantly, the music-trained students outperformed those with little or no music training at detecting geometrical properties.

The various studies of the Dana report, however, were not received positively by other neuroscientists and educators, who claimed that they did not establish that arts training directly boosts mathematical cognition. This might be somewhat true, but the studies did tighten up the connections that had been noted anecdotally before, as discussed above, laying the foundations for an AIM-based research paradigm into a causality between the arts and mathematics. In his introduction to the reports, Gazzaniga [19] also saw the project as a first step that will undoubtedly open up neuroscience, at long last, "to discover how the performance and appreciation of the arts enlarge cognitive capacities will be a long step forward in learning how better to learn." The negative reception of the Dana reports is due, as Gazzaniga also seems to imply, to a kind of residue intransigence in cognitive science and neuroscience towards using the arts as channels for understanding science and mathematics. It may also be simply part of a myth that sees the need for specialized learning. For the present purposes, it is sufficient to state that the kind of work initiated by the Dana Foundation is likely to catch on, colloquially speaking, as more and more neuroscientists and mathematicians are starting to look to the arts in order to understand mathematics or, more generally, how the two are related neurologically. In other words, AIM is an unconscious "force" in both neuroscience and education that probably cannot be stemmed. Already in 2007, Hetland, Winner, Veneema, Sheridan, and Perkins [24] published a persuasive study, which found that arts programs activate a special set of cognitive skills that they call "studio habits of mind" that enhance thinking skills generally. These "habits" predispose students to understand the inherent connectivity among the disciplines, giving them more confidence to handle all kinds of complex learning tasks.

It is useful here to list the eight general findings of the Dana Project as discussed by Gazzaniga [19], since they strongly support the claim that AIM is likely to become an important line of inquiry within the cognitive and neurosciences. These can be paraphrased as follows:

1. Learning some art skill motivates a sustained attention required to improve performance in other areas of cognition.
2. Genetic studies might explain individual differences in children's and adult's interest in the arts.
3. There are links between music training and memory; these extend beyond music into mathematics.
4. In children, there seem to be specific links between engagement with music and geometrical representation.
5. Music training and literacy are intertwined; this general finding can, presumably, be extended to numeracy.

6. Training in the theater arts seems to lead to memory improvement.
7. Interest in aesthetics leads to an "openness" of mind, which in turn is influenced by dopamine-related genes.
8. Learning to dance enhances the neural substrate that supports the organization of complex actions.

The findings thus support the main claim of this paper, namely that mathematicians and artists have always understood the connectivity between their crafts. Studying the works of Escher is, in effect, a study of the geometrical part of the brain that produces everything from proofs to topological models of thought. Ideas such as π, imaginary numbers, and the like are related to visual images, rhythms, and so on that are felt to be intuitively valid. AIM intends to make these intuitions factually obvious.

One hidden myth that research of this kind may finally eradicate is the idea that "mathematical cognition" is unique, and thus separate from all other kinds of cognition. To the contrary, mathematics and the arts may, in fact, be different manifestations of the same faculty—a bi-part soul, as Poe called it. Mathematics is certainly seen by science as its natural language; but the many excursions into the mathematical structure of art forms, discussed above, suggests that it may also be the language of the arts.

5. Concluding Remarks

The AIM approach to how mathematics occurs in the mind, as exemplified by the research projects above is a scientific one, alerting traditional cognitive science and neuroscience that studying mathematics through the arts is a fertile and meaningful area of investigation. The MIA approach, whereby art is studied through a mathematical lens, is also a meaningful mode of inquiry. It has been called hermeneutic here. Most artistic fields have a hermeneutic discipline—literature has literary criticism, music has musicology, and so on. Mathematics has started to have such a critical discipline, as the foundation of the *Journal of Mathematics and the Arts* in 2007 certainly brings out. The articles in this journal indicate, in effect, that mathematics can also be studied hermeneutically. Together with AIM, which provides the empirical side to the question of what mathematics is, MIA can shed significant light on the phenomenon of mathematical cognition, in the same way that a literary critic might illuminate the cognitive source of literary works. Chaitin [6] has introduced the term, "meta math," to provide a conceptual frame for looking at mathematics as an abstract ability that produces actual mathematical ideas. It could well be that there is also a "meta art" which overlaps with meta math to produce blended artifacts, from equations and formulas to art forms and poems that are manifestations of this overarching faculty. The study of abstract structure is what mathematics is all about. So is the abstract study of art, music, and poetry (Hersh [23]). This is because the brain is likely to be a "parallel distributing organ," as implied by work on so-called *Parallel Distributed Processing* (PDP) theory, which is based on writing computer programs designed to show how, potentially, brain networks interconnect with each other in the processing of information. The PDP model appears to perform the same kinds of tasks and operations that language and problem-solving do in tandem (MacWhinney [34]).

It is relevant to note that *What Is Mathematics?* [8] was the title of a significant book written for the general public by Courant and Robins in 1941. Their answer to their question was an indirect one—that is, they illustrate what mathematics looks like and what it does, allowing us to come to our own conclusions as to what mathematics is. Similarly, the only meaningful way to answer *What is music?* is to play it, sing it, or listen to it. A year before, in 1940, Kasner and Newman published another important popular book titled *Mathematics and the Imagination* [29]. The authors also illustrate how mathematics is tied to imaginative thought and its many products in art and elsewhere. We come away grasping intuitively that mathematics is both a system of thought and an art. AIM aims (no pun intended) to turn the approach in these popular treatises into an investigative tool. By illustrating mathematics as an art form and, vice versa, art as a mathematical form, we may be assembling valuable new pieces to the puzzle of mathematical cognition.

The notion of AIM and MIA as research tools within cognitive science and neuroscience may not, in the end, penetrate the substance of the enigma of what mathematics is beyond interesting anecdotal comparisons and inferences. But as Lynne Gamwell [17] has shown, mathematicians and artists have been on a common quest since antiquity to understand the physical world they see before them. Their visions are complementary. Indeed, perhaps the only way we can understand what mathematics and art are is by comparison, as Kasner and Newman also showed. An MIA-AIM paradigm would likely also be comparison-based. This paradigm is starting to gain a foothold within the brain sciences and mathematics itself, with new ventures such as The CogSci Network of the Fields Institute for Research in Mathematical Sciences and the Springer series "Mathematics in Mind," published under the aegis of the Network, which aim to introduce all kinds of disciplinary approaches to the study of mathematical cognition, including and especially the arts. As literary critic John William Navin Sullivan [49] so aptly put it, mathematics is perhaps itself best defined as an art: "The significance of mathematics resides precisely in the fact that it is an art; by informing us of the nature of our own minds it informs us of much that depends on our minds."

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