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Discrete Geometry and Covering Problems

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Contents

Abstract

This thesis explores several problems in discrete geometry, focusing on covering problems. We first go over some well known results, explaining Keith Ball's solution to the symmetric Tarski plank problem, as well as results of Alon and Füredi on covering all but vertices of a cube with hyperplanes. The former extensively utilizes techniques from matrix analysis, and the latter applies polynomial method. We state and explore the related problem, asking for the number of parallel hyperplanes required to cover a given discrete set of points in $\mathbb{Z}^d$ whose entries are bounded, and prove that there exist sets which are "difficult" to cover in every dimension for entries whose absolute values are bounded by 1 using a similar polynomial-based approach.

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1. Introduction

Covering problems appear naturally throughout mathematics, and generally ask how many instances, or how much of one object is required to cover another. One famous example is that of covering the plane using circles while minimizing how much the circles overlap–minimizing the density of the circles in the space. This problem could, for example, correspond to optimally placing cellular towers so as to minimize cost while making sure that there is adequate reception throughout a given space.

Tarski asked the question of whether or not a convex set could be covered by planks in such a way that the the sum of the widths of the planks is smaller than the width of the set. We define the width of a convex set S to be $\inf_{H_1,H_2}\{d(H_1,H_2)|H_1$ parallel to H_2 $\}$ where H_1, H_2 are distinct supporting hyperplanes of the set S. We define planks to be the intersection of two half-spaces associated with parallel hyperplanes. Precisely stated, he conjectured that for any convex set S, if $P_1, ..., P_n$ are planks with widths $w_1, ..., w_n$ which cover S, then $width(S) \leqslant \sum_{i=1}^{n} w_i$ [3].

Tarski proved this for the special case of the disk in two dimensions and Bang provided a solution in general. A follow-up conjecture asks about the relative widths of covering planks. That is, whether or not the inequality $\sum_{i=1}^{n} \frac{w_i}{h_i} \geqslant 1$ holds in general, where h_i denotes the width in the direction perpendicular to the hyperplane which defines P_i. Ball answers this in the affirmative for centrally symmetric sets in [2]. We present Ball's proof in Section 2.

Turning our attention to a more discrete problem, we study coverings of lattice points by hyperplanes. In [1], Alon and Füredi prove that any covering of the $2^n - 1$ points of the $n-$dimensional unit hypercube by hyperplanes which avoid the origin requires at least n hyperplanes. These results are proven using polynomial and linear algebraic methods–we go over some of these in Section 3.

We also study problems relating to number of parallel hyperplanes required to cover sets of lattice points. Define $S_T^d = \left\{x \in \mathbb{Z}^d : \|x\|_\infty \leqslant T\right\}$. It is obvious that S_T^d can always be covered using $2T + 1$ parallel hyperplanes by choosing hyperplanes orthogonal to one of the standard unit vectors. Further, it is easy to see that S_T^d cannot be covered in fewer than $2T + 1$ parallel hyperplanes. This leads to the natural question which asks how small can sets $X \subset S_T^d$ get while still requiring $2T + 1$ parallel hyperplanes to cover it? We study the existence of sets $X \subset S_T^d$ which cannot be covered using fewer than $2T + 1$ parallel hyperplanes and where $|X| = 2T + d$. This is optimal in the sense that any smaller set of points could be covered trivially by covering at least d points with the first hyperplane, and one with each of the rest. We construct such sets for $T = 1$ in every dimension, and use polynomial methods to prove that none of these sets can be covered by fewer than three parallel hyperplanes in Section 4.

Sets with this property – having bounded entries and requiring a maximal number of parallel hyperplanes to cover, have applications in compressed sensing. Specifically, they give us methods to generate sensing matrices for the purpose of sparse integer recovery. We prove and comment on these results in Section 5.

2. Plank Problem for Symmetric Bodies

Here, we present Keith Ball's solution to the symmetric plank problem from [2]. Given a symmetric convex body C in a Banach space X and n hyperplanes $H_1, ..., H_n$, there is a translate of a multiple of C which is at least $\frac{1}{n+1}$ times the size of C inside C which is not hit by any of the hyperplanes. In other words, there exists v such that

$$v + \frac{1}{n+1} C \subset C$$

and

$$\operatorname{int}\left(v + \frac{1}{n+1} C\right) \cap H_i = \varnothing.$$

We will soon see that this is equivalent to plank coverings. We define planks in the following way, using a unit norm functional ϕ, a real number m, and half width w:

$$P = \{x \in C \mid |\phi(x) - m| \leqslant w\}.$$

Theorem 1 (Ball). *Given unit functionals $(\phi_i)_{i=1}^n$, real numbers $(m_i)_{i=1}^n$, and positive real numbers $(w_i)_{i=1}^n$ such that $\sum w_i \leqslant 1$, there is some point x in the unit ball with respect to the norm associated with X such that*

$$|\phi_i(x) - m_i| \geqslant w_i$$

Assuming this theorem, we have the following corollary:

Corollary 2. *If C is a symmetric convex body in $\mathbb{R}^d$ and $(H_i)_{i=1}^n$ are hyperplanes, then there is a set of the form $x + \frac{1}{n+1} C$ inside C whose interior is not met by any of the hyperplanes H.*

Proof. We represent each of the hyperplanes with a unit functional ϕ_i and real m_i.

We choose the norm which corresponds to C as the unit ball under this norm. This can, in general, be constructed using the Minkowski gauge functional

$$f_C(x) = \inf_{\alpha \in \mathbb{R}^+} \left\{ \frac{x}{\alpha} \in C \right\},$$

which defines a norm whenever C is convex, 0-symmetric, bounded, and contains an open set.

We then apply Theorem 1, setting $w_i = \frac{1}{n+1}$. Then there exists a point x in $\frac{n}{n+1}C$ such that

$$|\phi_i(x) - m_i| \geqslant w_i$$

for each i. The assumption that $x \in \frac{n}{n+1}C$ gives us that

$$x + \frac{1}{n+1}C \subset C.$$

This follows from an elementary application of the triangle inequality: let $z \in \frac{1}{n+1}C$, then $\|z\| \leqslant \frac{1}{n+1}$ by homogeneity, so

$$\|x + z\| \leqslant \|x\| + \|z\| \leqslant \frac{n}{n+1} + \frac{1}{n+1} = 1.$$

To prove that the interior of $x + \frac{1}{n+1}C$ is hit by none of the hyperplanes, let $y \in x + \frac{1}{n+1}C$. Then

$$\|y - x\| \leqslant \frac{1}{n+1},$$

as $x + \frac{1}{n+1}C$ is a radius $\frac{1}{n+1}$ (open or closed) neighborhood around x. Then for each ϕ_i,

$$|\phi_i(x) - \phi_i(y)| = |\phi_i(x - y)| \leqslant \|\phi_i\|_{X*} \|x - y\|_X = \frac{1}{n+1},$$

so

$$|\phi_i(x) - m_i - (\phi_i(y) - m_i)\,| = |\phi_i(x) - \phi_i(y)| \leqslant \frac{1}{n+1}.$$

Since $|\phi_i(x)| \geqslant \frac{1}{n+1}$, $\phi_i(y) - m_i$ has the same sign as $\phi_i(x) - m_i$, giving us that it lies on the same side of the hyperplane. Thus, we conclude that for every $y \in x + \frac{1}{n+1}C$, y is on the same side of the hyperplane as x, and the hyperplane cannot cross the interior.

□

We now give a paraphrase of Theorem 1 before proving it.

Theorem 3. *For an $n \times n$ matrix with diagonal entries equal to 1, reals $(m_i)_{i=1}^n$, nonnegative reals $(w_i)_1^n$ where $\sum_{i=1}^n w_i \leqslant 1$, there are reals $(\lambda_i)_{i=1}^n$ such that*

$$\sum_j |\lambda_j| \leqslant 1,$$

where for every i,

$$\left|\sum_j a_{i,j}\lambda_j - m_i\right| \geqslant w_i.$$

Theorem 2 implies Theorem 1. For a sequence $(\phi_i)_1^n$ of norm 1 functionals on X, choose a sequence of unit vectors $(x_i)_{i=1}^n$ such that $\phi_i(x_i) = 1$. Then we construct the matrix

$$A = (\phi_i(x_j))\,, i, j \in 1, 2, \ldots, n.$$

This will look like

$$A = \begin{bmatrix} \phi_1(x_1) & \ldots & \phi_1(x_n) \\ \vdots & \ddots & \vdots \\ \phi_n(x_1) & \ldots & \phi_n(x_n) \end{bmatrix} = \begin{bmatrix} 1 & \ldots & \phi_1(x_n) \\ \vdots & \ddots & \vdots \\ \phi_n(x_1) & \ldots & 1 \end{bmatrix}$$

by the above assumption, where the off-diagonal entries are less than 1 by the Hölder inequality. In the case of the Euclidean norm ball, we have

$$A = \Phi^T \Phi,$$

where the columns of Φ are unit vectors. An analogous matrix A is obtained with any inner-product induced norm. The insight here is that A is symmetric whenever X is also a Hilbert space.

Returning to the matter at hand, for linear combinations of these dual vectors $x = \sum_{i=1}^n \lambda_i x_i$ where $\sum_{i=1}^n |\lambda_i| \leqslant 1$, we have that

$$\|x\| = \|\sum_{i=1}^n \lambda_i x_i\| \leqslant \sum_i |\lambda_i| \|x_i\| = \sum_i |\lambda_i| \leqslant 1,$$

and more importantly, that

$$\phi_i(x) = \sum_j \lambda_j \phi_i(x_j) = \sum_j \lambda_j a_{i,j}.$$

Then, applying Theorem 2, we have that there exists some sequence of coefficients $(\lambda_i)_1^n$ with the restrictions above, and nonnegative reals $(w_i)_1^n$ which sum to no more than 1,

$$\left|\sum_j a_{i,j}\lambda_j - m_i\right| \geqslant w_i,$$

which means that

$$|\phi_i(x) - m_i| \geqslant w_i$$

for each i, where $x = \sum_{i=1}^n x_i\lambda_i$. Note that this even slightly stronger–we are taking a subset of of the unit ball, as we look only at the span of the dual vectors of the hyperplanes. Bang proved a version of Theorem 2 for symmetric matrices:

Lemma 4. *Let $H = (h_{i,j})$ be a real symmetric $n \times n$ matrix with ones on the diagonal, $(\mu_i)_1^n$ a sequence of reals and $(\theta_i)_1^n$ a sequence of nonnegative reals. Then there exists some sequence of signs $(\epsilon_j)_1^n$ such that for any i,*

$$\left|\sum_j h_{i,j}\epsilon_j\theta_j - \mu_i\right| \geqslant \theta_i.$$

Proof. Choose your signs (ϵ_i) to maximize

$$\sum_{i,j} h_{i,j}\epsilon_i\epsilon_j\theta_i\theta_j - 2\sum_i \epsilon_i\theta_i\mu_i.$$

Fix $k \in \{1, ..., n\}$ and define $(\delta_j)_1^n$ by

$$\delta_j = \begin{cases} \epsilon_j & j \neq k \\ -\epsilon_j & j = k. \end{cases}$$

Then we have

$$\sum_{i,j} h_{i,j}\delta_i\delta_j\theta_i\theta_j - 2\sum_i \delta_i\theta_i\mu_i \leqslant \sum_j h_{i,j}\epsilon_i\epsilon_j\theta_i\theta_j - 2\sum_i \epsilon_i\theta_i\mu_i$$

as $(\epsilon_i)_1^n$ maximized the expression. So

$$0 \leqslant \sum_j h_{i,j}\epsilon_i\epsilon_j\theta_i\theta_j - 2\sum_i \epsilon_i\theta_i\mu_i - \left(\sum_{i,j} h_{i,j}\delta_i\delta_j\theta_i\theta_j - 2\sum_i \delta_i\theta_i\mu_i\right)$$

$$= \sum_j h_{i,j}\left[\epsilon_i\epsilon_j - \delta_i\delta_j\right]\theta_i\theta_j - 2\sum_i \left[\epsilon_i - \delta_i\right]\theta_i\mu_i$$

$$= \sum_j h_{i,j} \left[\epsilon_i \epsilon_j - \delta_i \delta_j\right] \theta_i \theta_j - 2 \left[2\epsilon_k\right] \theta_k \mu_k$$

If $i \neq k$ and $j \neq k$, the first term evaluates to zero.

$$= \sum_i h_{i,k} \left[\epsilon_i \epsilon_k - \delta_i \delta_k\right] \theta_i \theta_k + \sum_j h_{k,j} \left[\epsilon_k \epsilon_j - \delta_k \delta_j\right] \theta_k \theta_j - 4\epsilon_k \theta_k \mu_k$$

By symmetry of H,

$$= 2 \left(\sum_i h_{i,k} \left[\epsilon_i \epsilon_k - \delta_i \delta_k\right] \theta_i \theta_k \right) - 4\epsilon_k \theta_k \mu_k$$

$$= 2 \sum_i h_{i,k} \left[\epsilon_i \epsilon_k - \delta_i (-\epsilon_k)\right] \theta_i \theta_k - 4\epsilon_k \theta_k \mu_k$$

$$= 2 \sum_i h_{i,k} \left[\epsilon_i \epsilon_k + \delta_i \epsilon_k)\right] \theta_i \theta_k - 4\epsilon_k \theta_k \mu_k$$

$$= 2\epsilon_k \sum_i h_{i,k} \left[\epsilon_i + \delta_i\right] \theta_i \theta_k - 4\epsilon_k \theta_k \mu_k$$

And $\epsilon_i + \delta_i = \begin{cases} 2\epsilon_i & i \neq k \\ 0 & i = k \end{cases}$

$$= 4\epsilon_k \theta_k \sum_{i \neq k} h_{i,k} \theta_i \epsilon_i - 4\epsilon_k \theta_k \mu_k$$

We may add and subtract $4\epsilon_k \theta_k h_{k,k} \theta_k \epsilon_k$, obtaining

$$= -4\theta_k^2 + 4\epsilon_k \theta_k \sum_i h_{i,k} \theta_i \epsilon_i - 4\epsilon_k \theta_k \mu_k$$

Thus,

$$4\theta_k^2 \leqslant 4\epsilon_k \theta_k \sum_i h_{i,k} \theta_i \epsilon_i - 4\epsilon_k \theta_k \mu_k$$

$$\theta_k^2 \leqslant \epsilon_k \theta_k \left[\sum_i h_{i,k} \theta_i \epsilon_i - \mu_k \right]$$

$$\left| \sum_i h_{i,k} \theta_i \epsilon_i - \mu_k \right| \geqslant \theta_k$$

Since we fixed an arbitrary k, the proof is complete. □

Note that in this case, the role of $(\lambda_i)_1^n$ as defined before is played by $(\theta_i \epsilon_i)_1^n$. Remarkably, given distances to each hyperplane $(\theta_i)_1^n$ that we wish for, we can achieve the desired inequality by choosing only the signs of the coefficients in the linear combination. Additionally, note that these dual vectors are, in some sense "orthogonal" to the hyperplanes. This is most evident in Euclidean space, defining a functional $\phi_y(x) = \langle y, x\rangle$, the dual vector to the functional will be y. y will be orthogonal to the differences of the vectors in the affine space $\{x \mid \phi_y(x) - m = 0\}$. Thus, given a functional ϕ_y, choosing a multiple of the point $x = y$ to be far away from the hyperplane defined by $\phi_y(x)$ will give you the "most bang for your buck" in getting away from the chosen hyperplane.

In proving Theorem 2, Ball proves the special case in which $w_i = \frac{1}{n}$ for each i, as we may simply increase the number of hyperplanes and tile the original planks of smaller planks which are each of the same size. He proves an even stronger version of the theorem, stating that

$$\sum_j \lambda_j^2 \leqslant \frac{1}{n}$$

This is stronger becuase by the Cauchy Schwartz inequality,,

$$\sum_j |\lambda_j| = 1 = \left\langle \vec{1}, \vec{\lambda} \right\rangle \leqslant \|\vec{1}\|\|\vec{\lambda}\| = \sqrt{n}\sqrt{\sum_i \lambda_i^2}$$

where we assume the first equality as we can always pad $\vec{\lambda}$ by scaling it up, and preserving the inequalities.

$$1 \leqslant \sqrt{n}\sqrt{\sum_i \lambda_i^2}$$

$$\sum_i \lambda_i^2 \leqslant 1$$

Ball then approaches the problem with Hilbert space methods to transform the problem into one which may be solved by Lemma 4. The idea is that if the theorem holds for AU, where U is orthogonal, then it holds for A. This is evident from the fact that Lemma 3 says that for matrix B with 1 on the diagonal which is symmetric, there exists $\vec{\lambda}$ with $\|\lambda\|_1 \leqslant 1$ such that

$$\left[B\vec{\lambda} - \vec{m}\right]_i \geqslant w_i$$

If we were to assume $B = AU$ for U orthogonal, then

$$\left[AU\vec{\lambda} - \vec{m}\right]_i \geqslant w_i$$

But then we would let

$$\vec{\lambda}' = U\lambda$$

obtaining

$$\left[B\vec{\lambda}' - \vec{m}\right]_i \geqslant w_i$$

where letting $w_i = \frac{1}{n}$, we have equality in 1 and 2 norms, so that $\|\lambda'\|_2 = \|\lambda\|_2 = \|\lambda\|_1 = 1$, thus proving the proposition.

However, this is the point where we find issues with the symmetry–despite the existence of U such that AU is symmetric (this is obvious from the polar decomposition) there does not, in general, exist U so that the diagonal elements of AU are all the same.

Thus, isometry is not enough–we must extend the transformation to isometry combined with a diagonal transformation,

Lemma 5. *Let A be an $n \times n$ matrix of reals, each of whose rows are non-null. Then there is a sequence $(\theta_i)_1^n$ of positive reals and an orthogonal matrix U so that*

$$H = (h_{i,j}) = \left(\theta_i \left(AU\right)_{i,j}\right)$$

is positive and has 1's on the diagonal.

In other words, there exists a positive definite diagonal matrix D_θ and an orthogonal matrix U such that $D_\theta AU$ is positive and has ones on the diagonal. In order to prove this fact, we introduce the nuclear norm $\|\cdot\|_{C_1}$ of a matrix, defined as

$$\|A\|_{C_1} = tr\left(\sqrt{A^*A}\right)$$

We require the following facts:[1]

$$\|B\|_{C_1} = \max\left\{tr(BU) \mid U^TU = I\right\}$$

$$\|BC\|_{C_1} \leqslant \sqrt{tr(B^TB)tr(C^TC)} = \|B\|_F\|C\|_F$$

Before proving this, we require a couple more lemmas.

[1]See [10] for proof

Lemma 6. *If $H = (h_{i,j})$ is a positive matrix with non-zero diagonal entries and U is orthogonal, then*

$$\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \leqslant \sum_i h_{ii}$$

Proof. For each i, let $\gamma_i = \frac{(HU)_{ii}^2}{h_{ii}}$ and let D be the diagonal matrix with γ_i on the diagonals. Let $T = \sqrt{H^T H}$. We have that

$$\sum_i \frac{(HU)_{ii}^2}{h_{ii}} = \sum_i \gamma_i \, (HU)_{ii}$$

$$= tr\,(DHU) \leqslant \sup_{O^T O = I} tr\,(DHO) \leqslant \|DH\|_{C_1}$$

by the first property above

$$= \|DTT\|_{C_1} = \|\,(DT)\,T\|_{C_1}$$

$$\leqslant \|DT\|_F \|T\|_F$$

$$= \sqrt{tr\,(DTT^* D)\, tr\,(T^* T)}$$

$$= \sqrt{tr(DHD) tr(H)}$$

And because H is positive,

$$= tr(H)^{1/2}\,(tr(DHD))^{1/2}$$

$$= \left[\sum_i h_{ii}\right]^{1/2} \left[\sum_{i=1}^n \gamma_i^2 h_{i,i}\right]^{1/2}$$

$$= \left[\sum_i h_{ii}\right]^{1/2} \left[\sum_{i=1}^n \frac{(HU)_{ii}^2}{h_{ii}}\right]^{1/2}$$

So

$$\sum_i \frac{(HU)_{ii}^2}{h_{ii}} \leqslant \left[\sum_i h_{ii}\right]^{1/2} \left[\sum_{i=1}^n \frac{(HU)_{ii}^2}{h_{ii}}\right]^{1/2}$$

Giving us that

$$\left[\sum_{i=1}^n \frac{(HU)_{ii}^2}{h_{ii}}\right]^{1/2} \leqslant \left[\sum_i h_{ii}\right]^{1/2}$$

$$\sum_{i=1}^n \frac{(HU)_{ii}^2}{h_{ii}} \leqslant \sum_i h_{ii}$$

□

Lemma 7. *If $H = (h_{ij})$ is a positive $n \times n$ matrix with nonzero diagonal entries, then*

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{i,j} \right) \right\|_{C_1} \leqslant \sqrt{n} \|H\|_{C_1}^{1/2}$$

Proof. Let $D_{\sqrt{h}}$ be the diagonal matrix with $\frac{1}{\sqrt{h_{ii}}}$ on the diagonal. Then

$$\left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{i,j} \right) \right\|_{C_1} = \|D_{\sqrt{h}} H\|_{C_1}$$

And by Property 1 above,

$$= tr \left(D_{\sqrt{h}} HU \right)$$

for some orthogonal matrix U.

$$= \sum_i \frac{(HU)_{ii}}{\sqrt{h_{ii}}}$$

We now apply Cauchy-Schwartz,

$$\leqslant \sqrt{n} \sqrt{\sum_i \frac{(HU)_{ii}^2}{h_{ii}}}$$

And by Lemma 5,

$$\leqslant \sqrt{n} \sqrt{\sum_i h_{ii}}$$

$$= \sqrt{n} \|H\|_{C_1}$$

because H was assumed to be positive. ☐

We now prove Lemma 5

Proof. (Lemma 5)

We wish to find $(\theta_i)_1^n$, and U orthogonal such that

$$\left(\theta_i \, (AU)_{i,j} \right) = D_\theta AU$$

where D_θ is the diagonal matrix with θ_i on the diagonal. Since A has no rows that are uniformly 0, there is some constant c so that

$$\|D_\theta A\|_{C_1} \geqslant c \max_i \theta_i$$

We apply the equivalence of norms in finite dimensions to note that there exists l such that

$$\|D_\theta A\|_{C_1} \geqslant l \|D_\theta A\|_F$$

$$\|D_\theta A\|_F^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j}^2 \theta_i^2$$

Let $M = \min_i \max_j |A_{i,j}|$. Then

$$\|D_\theta A\|_F^2 \geqslant \sum_{i=1}^{n} M^2 \theta_i^2$$

In producing this inequality, we are essentially replacing the norm of each row with a lower bound for its single largest component.

$$\sum_{i=1}^{n} M^2 \theta_i^2 \geqslant M^2 \max_i \theta_i^2$$

so

$$\|D_\theta A\|_F \geqslant M \max_i \theta_i$$

thus proving the desired inequality, setting $c = lM$.

We now prove that there exists a $\vec{\theta}$ which minimizes $\|D_\theta A\|_{C_1}$subject to $\prod_i \theta_i = 1$. Suppose that $\max_i \theta_i \geqslant \frac{1}{c} \|A\|_{C_1}$. Then $\|D_\theta A\|_{C_1} \geqslant \|A\|_{C_1}$. We therefore have that

$$\inf_{\prod \theta_i = 1} \|D_\theta A\|_{C_1} = \inf_{\prod \theta_i = 1, \|\vec{\theta}\|_\infty \leqslant \frac{1}{c}\|A\|_{C_1}} \|D_\theta A\|_{C_1}$$

We can therefore constrain our maximization problem to a compact set, and since $\|D_\theta A\|_{C_1}$ is continuous, the minimum must be attained. Choose $\vec{\theta}$ which minimizes $\|D_\theta A\|_{C_1}$subject to $\prod_i \theta_i = 1$.

Let $H = \sqrt{D_\theta A A^T D_\theta}$ given this $\vec{\theta}$. Since A is full rank, H is positive definite. Therefore H has strictly positive diagonal entries. By polar decomposition, we may write

$$H = D_\theta A U$$

for some U. For each i, let

$$\gamma_i = \frac{1}{\sqrt{h_{ii}}} \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right)^{1/n}$$

Then $\prod \gamma_i = 1$.

$$\prod_i \gamma_i = \prod_i \left(\frac{1}{\sqrt{h_{ii}}} \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right)^{1/n} \right)$$

$$= \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right) \left(\prod_i \frac{1}{\sqrt{h_{ii}}} \right) = 1$$

Because $(\theta_i)_1^n$ was chosen as a minimizer of $\|D_\theta A\|_{C_1}$, we have

$$\|H\|_{C_1} \leqslant \| (\gamma_i \theta_i a_{i,j}) \|_{C_1}$$

$$= \| (\gamma_i h_{i,j}) \|_{C_1}$$

We may factor out a constant $\left(\prod_{j=1}^{n} \sqrt{h_{jj}}\right)^{1/n}$,

$$= \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right)^{1/n} \left\| \left(\frac{1}{\sqrt{h_{ii}}} h_{i,j} \right) \right\|_{C_1}$$

By Lemma 6,

$$\leqslant \sqrt{n} \|H\|_{C_1}^{1/2} \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right)^{1/n}$$

So

$$\|H\|_{C_1} \leqslant \sqrt{n} \|H\|_{C_1}^{1/2} \left(\prod_{j=1}^{n} \sqrt{h_{jj}} \right)^{1/n}$$

$$\|H\|_{C_1}^{1/2} \leqslant \sqrt{n} \left(\prod_{j=1}^{n} h_{jj} \right)^{1/2n}$$

$$\frac{\|H\|_{C_1}}{n} \leqslant \left(\prod_{j=1}^{n} h_{jj} \right)^{1/n}$$

But H is positive, so $\|H\|_{C_1} = tr(H)$.

$$\frac{\sum_i h_{ii}}{n} \leqslant \left(\prod_{j=1}^{n} h_{jj} \right)^{1/n} \leqslant \frac{\sum_i h_{ii}}{n}$$

where the second inequality follows from the AM-GM inequality.

□

We may finally prove Theorem 2.

Proof. Using Lemma 4, choose $(\theta_i)_1^n$ and an orthogonal matrix U such that $H = D_\theta AU = \left(\theta_i \left(AU\right)_{i,j}\right)$ has 1 on the diagonal and is positive. By Lemma 3,

we may choose signs $(\epsilon_i)_1^n$ such that for each i,

$$\left|\sum_j h_{i,j}\epsilon_j\theta_j - n\theta_i m_i\right| \geqslant \theta_i$$

This is allowed becase the $(\theta_i)_1^n$ were first fixed, and the m_i mentioned in lemma were arbitrary, so we replace them with $n\theta_i m_i$.

$$\left|\theta_i \sum_j \theta_j \left[AU\right]_{i,j} \epsilon_j - n\theta_i m_i\right| \geqslant \theta_i$$

$$\left|\sum_j \theta_j \left[AU\right]_{i,j} \epsilon_j - n m_i\right| \geqslant 1$$

$$\left|\sum_j \frac{\theta_j \left[AU\right]_{i,j} \epsilon_j}{n} - m_i\right| \geqslant \frac{1}{n}$$

$$\left|\left[A\left(U\frac{\left(\vec{\theta}\odot\vec{\epsilon}\right)}{n}\right)\right]_i - m_i\right| \geqslant \frac{1}{n}$$

So set $\vec{\lambda} = (\lambda_i)_1^n = U\frac{(\vec{\theta}\odot\vec{\epsilon})}{n}$ so that

$$\left|\left[A\lambda\right]_i - m_i\right| \geqslant \frac{1}{n}$$

$$\left|\sum_j a_{i,j}\lambda_j - m_i\right| \geqslant \frac{1}{n}$$

We need only to show that $\sum_j \lambda_j^2 \leqslant \frac{1}{n}$. U is orthogonal, so

$$\sum_j \lambda_j^2 = \sum_j \left[\frac{\left(\vec{\theta}\odot\vec{\epsilon}\right)}{n}\right]_j^2 = \frac{1}{n^2}\sum_j \theta_j^2$$

We need to show that $\sum_j \theta_j^2 \leqslant n$. We know by definition that

$$\left(\theta_i \left(AU\right)_{i,j}\right) = D_\theta AU = H$$

So

$$D_\theta A = HU^*$$

$$\theta_i a_{i,j} = (HU^*)_{i,j}$$

Assuming that A has 1 on the diagonals, $a_{ii} = 1$.

$$\theta_i = (HU^*)_{i,i}$$

Now we take

$$\sum_{i=1}^{n} \theta_i^2 = \sum_i (HU^*)_{i,i}^2$$
$$= \sum_i \frac{(HU^*)_{ii}^2}{h_{ii}}$$

because all of the h_{ii} are 1. We apply lemma 5 to obtain that

$$\leqslant \sum_i h_{ii} = n$$

Thus,

$$\sum_{i=1}^{n} \theta_i^2 \leqslant n$$

□

3. Covering the cube except for one point

We explore some results of Alon and Füredi in related to covering all of the points of the unit hypercube $\{0,1\}^n$ except for the origin, as well as some extensions.

Theorem 8 (Alon and Füredi)**.** *Suppose that the hyperplanes $H_1, ..., H_m \subset \mathbb{R}^n$ avoid $\vec{0}$, but cover the other $2^n - 1$ vertices of the unit cube $C = \{0,1\}^n$. Then $m \geqslant n$.*

This theorem states that in order to cover the n dimensional hypercube with hyperplanes without covering the origin, we require at least n hyperplanes.

Before proving this, we require a lemma.

Lemma 9. *If $Q(\vec{x}) \in Z\left[x_1, ..., x_n\right]$ is a multilinear polynomial with $Q(\vec{0}) = c \neq 0$, and $Q(\vec{x}) = 0$ for $x \in \{0,1\}^n \setminus \left\{\vec{0}\right\}$, then $Q(\vec{x}) = c(x_1 - 1)(x_2 - 1) \cdots (x_n - 1)$, which is degree n.*

Proof. First, note that

$$c(x_1-1)(x_2-1)\cdots(x_n-1)=c\sum_{I\subset\{1,\ldots,n\}}(-1)^{|I|}\,x_I$$

where we define $x_I=\prod_{i\in I}x_i$ and $x_\varnothing=1$. We write

$$Q\left(\vec{x}\right)=\sum_{I\subset\{1,\ldots,n\}}c_I x_I$$

and prove that $c_I=c\,(-1)^{|I|}$. We proceed by induction on $|I|$. We know that $c_\varnothing=Q(\vec{0})=c$. Now suppose that $c_I=(-1)^{|I|}$ for all $J\subset I$ where $J\neq I$ and $|I|\geqslant 1$. Let $\vec{e}_I\in\{0,1\}^n$ be the vector with coordinates one for the indices contained in I, and 0 elsewhere. Since $|I|\neq 0$, $Q(\vec{e}_I)=0$. For all $J\subset I$, we therefore have that $\prod_{i\in J}\left(\vec{e}_I\right)_i=1$. Thus,

$$Q(\vec{e}_I)=\sum_{J\subset I}c_J=c_I+\sum_{J\subset I,J\neq I}c\,(-1)^{|J|}$$

there are $\binom{|I|}{j}$ subsets of size j of $|I|$, so

$$c_I+\sum_{J\subset I,J\neq I}c\,(-1)^{|J|}=c_I+c\left(\sum_{0\leqslant j<|I|}\binom{|I|}{j}(-1)^j\right)$$

$$=c_I+c\left(\sum_{0\leqslant j\leqslant|I|}\binom{|I|}{j}(-1)^j-(-1)^{|I|}\right)$$

$$=c_I+c\left((1-1)^{|I|}-(-1)^{|I|}\right)$$

$$=c_I-c\,(-1)^{|I|}$$

so

$$0=c_I-c\,(-1)^{|I|}$$

giving the desired result, that $Q(\vec{x})=c(x_1-1)(x_2-1)\cdots(x_n-1)$. □

Proof. (Theorem 8) We define each hyperplane H_i with an equation $\langle\vec{a}_i,\vec{x}\rangle=b_i$, where $\vec{a}_i,\vec{x}\in\mathbb{R}^n$ and $b_i\in\mathbb{R}$. Consider the polynomial

$$P(\vec{x})=\prod_{i=1}^{m}\left(\langle\vec{a}_i,\vec{x}\rangle-b_i\right)$$

It is clear that $P(\vec{x}) = 0$ for all $\vec{x} \in C\backslash\vec{0}$, and $P(\vec{0}) = \prod_{i=1}^{m} b_i \neq 0$. We also have that $deg(P) = m$, which gives us another way of viewing this problem. We wish to lower bound the degree of polynomials of this form which satisfy $P(\vec{x}) = 0$ for all $\vec{x} \in C\backslash\vec{0}$, and $P(\vec{0}) = \prod_{i=1}^{m} b_i \neq 0$. Further, we may replace each instance of x_i^d with $d \geqslant 2$ with x_i, without changing the behavior of the polynomial on C. Let $Q(\vec{x})$ be the polynomial obtained through this procedure. Note that $deg(Q) \leqslant deg(P)$. $Q(\vec{x})$ is multilinear, $Q(\vec{0}) = c \neq 0$, and $Q(\vec{x}) = 0$ for $x \in \{0,1\}^n \backslash\vec{0}$. Thus, applying the lemma above, we obtain that

$$m = deg(P) \geqslant deg(Q) = n$$

$\square$

Remarkably, this theorem can be extended to the covering arbitrary rectangles except for a single point using similar techniques.

Let $V = V(h_1, ..., h_n)$ be the set of lattice points $(y_1, ..., y_n)$ such that $0 \leqslant y_i \leqslant h_i$. Let $\vec{v} \in V$ and define $U = V\backslash\vec{v}$. In a similar manner to the result above, U cannot be covered by fewer than $\sum h_i$ hyperplanes while avoiding $\vec{v}$.

Theorem 10 (Alon and Füredi)**.** *Suppose that the hyperplanes $H_1, H_2, ..., H_m \subset \mathbb{R}^n$ avoid $\vec{v}$ but $H_1 \cup ... \cup H_m$ contains $V(h_1, ..., h_n)\backslash\vec{v}$. Then $m \geqslant h_1 + ... + h_n$.*

For $\vec{p} = (p_1, ..., p_n) \in V$, we define the polynomial $B_{\vec{p}}(\vec{x}) \in Z(x_1, ..., x_n)$ to be the following:

$$B_{\vec{p}}(\vec{x}) = \left(\prod_{0 \leqslant j_1 \leqslant h_1, j_1 \neq p_1} (x_1 - j_1)\right)\left(\prod_{0 \leqslant j_2 \leqslant h_2, j_2 \neq p_2} (x_2 - j_2)\right) \cdots \left(\prod_{0 \leqslant j_n \leqslant h_n, j_n \neq p_n} (x_n - j_n)\right)$$

For each $\vec{p}$, $deg(B_{\vec{p}}(\vec{x})) = \sum h_i$.

Lemma 11. *The polynomials $B_{\vec{p}}(\vec{x})$ for $\vec{p} \in V$ for a basis for the subspace Z spanned by the polynomials $\{x_1^{a_1} x_2^{a_2} \cdots x_n^{a_n} : 0 \leqslant a_i \leqslant h_i\}$.*

Proof. $B_{\vec{p}}(\vec{x}) \in Z$, and $dim(Z) = \prod (h_i + 1)$ so we need only to prove that the polynomials $B_{\vec{p}}(\vec{x})$ are linearly independent. Suppose that $\vec{x} = \vec{p}$.

$$B_{\vec{p}}(\vec{p}) = \left(\prod_{0 \leqslant j_1 \leqslant h_1, j_1 \neq p_1} (p_1 - j_1)\right)\left(\prod_{0 \leqslant j_2 \leqslant h_2, j_2 \neq p_2} (p_2 - j_2)\right) \cdots \left(\prod_{0 \leqslant j_n \leqslant h_n, j_n \neq p_n} (p_n - j_n)\right)$$

so since $p_i \neq j_i$ for each of the terms, $B_{\vec{p}}(\vec{p}) \neq 0$. However, if $\vec{x}' = (x'_1, ..., x'_n) \neq \vec{p}$, then

$$B_{\vec{p}}(\vec{x}') = \left(\prod_{0 \leqslant j_1 \leqslant h_1, j_1 \neq p_1} (x'_1 - j_1)\right)\left(\prod_{0 \leqslant j_2 \leqslant h_2, j_2 \neq p_2} (x'_2 - j_2)\right) \cdots \left(\prod_{0 \leqslant j_n \leqslant h_n, j_n \neq p_n} (x'_n - j_n)\right)$$

Since $x'_i \neq \vec{p}_i$ for some i, $(x'_i - j_i) = 0$ for some i, j_i. Thus, evaluating these polynomials only V, it is obvious that polynomials $B_{\vec{p}}(\vec{x})$ are linearly independent. Intuitively, these form a polynomial basis for interpolation. □

We now prove Theorem 10

Proof. (Theorem 10) Consider the polynomial

$$P(\vec{x}) = \prod_{i=1}^{m} (\langle \vec{a}_i, \vec{x} \rangle - b_i)$$

and let $Q(\vec{x}) \in Z$ be the polynomial obtained by replacing each instance of $x_i^{h_i+1}$ with $x_i^{h_i+1} - \left(\prod_{0 \leqslant j \leqslant h_i} (x_i - j)\right)$, which is degree at most h_i, as the $x_i^{h_i+1}$ terms cancel. For any $\vec{x} \in V$, $x_i \in \{0, 1, ..., h_i\}$ so $\left(\prod_{0 \leqslant j \leqslant h_i} (x_i - j)\right) = 0$. Thus, $Q(\vec{x}) = P(\vec{x})$ for $\vec{x} \in V$ and $deg(Q) \leqslant deg(P)$. We therefore have that $Q(\vec{x}) = 0$ for all $\vec{x} \in U$ and $Q(\vec{v}) \neq 0$. Writing

$$Q(\vec{x}) = \sum_{\vec{p} \in V} \alpha_{\vec{p}} B_{\vec{p}}(\vec{x})$$

and evaluating both sides for each $\vec{x} \in V$, it is clear that $Q(\vec{x}) = cB_{\vec{p}}(\vec{x})$ for some $\vec{p}$ and $c \in \mathbb{R}$. Thus,

$$\sum h_i = deg(Q) \leqslant deg(P) = m$$

□

4. Sets which are difficult to cover by parallel hyperplanes

We move to counting the number of parallel hyperplanes required to cover sets of lattice points. Define $S_T^n = \{x \in \mathbb{Z}^n : \|x\|_\infty \leqslant T\}$. For $T = 1$, we construct such sets in every dimension inductively and utilize techniques inspired by Alon and Füredi's work to prove that these cannot be covered in fewer than three parallel hyperplanes. Because our hyperplanes are parallel, they are all defined by a single functional, and each hyperplane may be identified with the intercept b_i in the equation $\langle \vec{a}, \vec{x} \rangle = b_i$ defining the hyperplane. Further, this provides a different geometric angle from which we may view the problem. Given a vector $\vec{a}$, the number of hyperplanes perpendicular to $\vec{a}$ required to cover X is equal to the number of distinct orthogonal projections of points of X onto $\vec{a}$.

Such a set in dimension 2 is given by $X_2 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$. The construction in general is relatively simple. We extend our specific example in two dimensions to arbitrary dimensions inductively in a manner similar to the construction of simplexes in higher dimensions. Given our set X_2, we demonstrate the construction of X_3 before proceeding to the general case. First, we place each of the nonzero points of X_2 into the intersection of the plane defined by the equation $\langle (0,0,1), \vec{x} \rangle = 1$ as a subset of S_1^3, preserving the first two coordinates. This gives us the points. Giving us the points $\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We leave keep the zero vector, and simply make it into the zero vector in three dimensions. We now have have to add one more point, which we choose to be $\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$. This gives us that $X_3 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. In general, given X_n, we construct X_{n+1} in the same way. We append a 1 to the end of each of the nonzero vectors, change the point $\vec{0} \in \mathbb{R}^n$ to $\vec{0} \in \mathbb{R}^{n+1}$, and add the point $-e_{n+1}$ to the set. Following this construction, $X_4 = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$. Note that for dimension 1, $X_1 = S_1^1 = \{[-1], [0], [1]\}$We now prove that these sets cannot be covered using fewer than two parallel hyperplanes.

Theorem 12. *X_n cannot be covered in 2 hyperplanes for any dimension n.*

Proof. We index the $n+1$ nonzero vectors of X_n as $\vec{x}_1, ..., \vec{x}_{n+1}$ so that $\vec{x}_i = -\vec{e}_{n-i+1} + \sum_{k=n-i+2}^{n} \vec{e}_k$ for $i \in \{1, ..., n\}$ and $\vec{x}_{n+1} = \vec{1}$. Suppose that X_n can be covered using two parallel hyperplanes H_1, H_2. We may, without loss of generality, represent them with the equations $\langle \vec{v}, \vec{x} \rangle = 0, \langle \vec{v}, \vec{x} \rangle = 1$. We know that one of the intercepts will be 0 becuase $\vec{0} \in X_n$. Thus, we define $n+1$ multivariate polynomials $P_1^n, ..., P_{n+1}^n$ with the following:

$$P_i^n(\vec{v}) = \langle v, \vec{x}_i \rangle (\langle v, \vec{x}_i \rangle - 1) = \langle v, \vec{x}_i \rangle^2 - \langle v, \vec{x}_i \rangle$$

for $v = (v_1, ..., v_n)$.

It is evident that finding a pair of parallel hyperplanes which cover X_n is equivalent to finding a nonzero system of polynomial equations $P_i^n(\vec{v}) = 0$ for $i = 1, ..., n$.

We proceed by induction. The case $n = 1$ is trivial. Suppose $n = 2$, so that $x_1 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, x_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. We obtain that

$$P_1^2(v_1, v_2) = v_2^2 - v_2 = v_2 (v_2 + 1)$$

$$P_2^2(v_1, v_2) = (-v_1 + v_2)^2 + v_1 - v_2$$

$$P_3^2(v_1, v_2) = (v_1 + v_2)^2 - v_1 - v_2$$

Setting $P_1(v_1, v_2) = 0$, we have that v_2 must be either 0 or -1.

Suppose that $v_2 = 0$. Plugging this into the other two equations, we have

$$P_2^2(v_1, 0) = v_1^2 + v_1 = v_1(v_1 + 1)$$

$$P_3^2(v_1, 0) = v_1^2 - v_1 = v_1(v_1 - 1)$$

This implies that $v_1 = 0$, which means that $v_2 \neq 0$ in any nontrivial solution.

Suppose that $v_2 = -1$. Then

$$P_2^2(v_1, -1) = (-v_1 - 1)^2 + v_1 + 1$$

$$= v_1^2 + 1 + 2v_1 + v_1 + 1 = (v_1 + 2)(v_1 + 1)$$

$$P_3^2(v_1, -1) = (v_1 - 1)^2 - v_1 + 1 = v_1^2 - 2v_1 + 1 - v_1 + 1$$

$$= v_1^2 - 3v_1 + 2$$

$$= (v_1 - 2)(v_1 - 1)$$

Thus, there is no solution to all three polynomial equations simultaneously. Now suppose that the system of polynomials $P_i^{n-1}(\vec{v}) = 0$, $i = 1, ..., n$ has no nonzero solution. We prove that the system of polynomials $P_i^n(\vec{v}) = 0$, $i = 1, ..., n+1$ also has no nonzero solution. First, we examine $P_1^n(\vec{v})$. $P_1^n(\vec{v}) = v_n(v_n + 1)$ which implies that either $v_n = 0$ or $v_n = -1$.

For the first case, suppose that we fix $v_n = 0$. Then $P_1^n(\vec{v}) = 0$ is solved, and examining the remaining n equations,

$$P_i^n(v_1, ..., v_{n-1}, 0) = \langle (v_1, ..., v_{n-1}), (x_{i,1}, ..., x_{i,n-1}) \rangle \left(\langle (v_1, ..., v_{n-1}), (x_{i,1}, ..., x_{i,n-1}) \rangle - 1 \right)$$

Now notice that because

$$X_n = \left\{ \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ \vdots \\ -1 \\ 1 \end{bmatrix}, ..., \begin{bmatrix} -1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} \right\}$$

Taking each of the nonzero nonzero elements of X_n and deleting the last coordinate produces X_{n-1}. Thus, solving $P_i^n(\vec{v}) = 0, i = 1, ..., n+1$ after fixing $v_n = 0$ is equivalent to solving $P_i^{n-1}(\vec{v}) = 0, i = 1, ..., n$ which has no nonzero solution.

For the second case, suppose that we fix $v_n = -1$. Then for $i = 2, ..., n$,

$$P_i^n(v_1, ..., v_{n-1}, -1) = \left\langle (v_1, ..., v_{n-1}, -1), -e_{n-i+1} + \sum_{n-i+2}^{n} e_i \right\rangle \left(\left\langle (v_1, ..., v_{n-1}, -1), -e_{n-i+1} + \sum_{n-i+2}^{n} e_i \right\rangle - 1 \right)$$

$$= \left(\left\langle (v_1, ..., v_{n-1}), -e_{n-i+1} + \sum_{n-i+2}^{n-1} e_i \right\rangle - 1 \right) \left(\left\langle (v_1, ..., v_{n-1}), -e_{n-i+1} + \sum_{n-i+2}^{n-1} e_i \right\rangle - 3 \right)$$

where the e_i representing standard basis vectors become one dimension smaller. This gives us a system of $n-1$ polynomials in $n-1$ variables, as we would require $P_i^n(v_1, ..., v_{n-1}, -1) = 0$ for $i = 2, ..., n$. Now notice that $\left\{ -e_{n-i+1} + \sum_{n-i+2}^{n-1} e_i \right\}_{i=2}^{n} = X_{n-1} \backslash \vec{0}$. What this means, is that for any solution $\tilde{v} = (v_1, ..., v_{n-1})$ where $P_i^n(v_1, ..., v_{n-1}, -1) = 0$ for $i = 2, ..., n$, we will have either $\langle \tilde{v}, \vec{x} \rangle = 1$ or $\langle \tilde{v}, \vec{x} \rangle = 3$ for all $\vec{x} \in X_{n-1}$. Suppose that such a solution $\tilde{v}$ exists. Then $\tilde{v}$ gives a pair of hyperplanes defined by the pair of equa tions $\langle \tilde{v}, \vec{x} \rangle = 1$ and $\langle \tilde{v}, \vec{x} \rangle = 3$ above which together, cover $X_{n-1} \backslash \vec{0}$. This implies that $X_{n-1} \backslash \vec{0}$ is contained in the halfspace $B = \{\vec{x} : \langle \tilde{v}, \vec{x} \rangle > 0\}$. Further, the convex hull $conv\left(X_{n-1} \backslash \vec{0}\right)$ is contained in H. This, however, produces a contradiction as $conv\left(X_{n-1} \backslash \vec{0}\right)$ contains an open ball around the origin, and can therefore not lie on one side of a hyperplane which passes through the origin. Thus, there is no solution $\vec{v}$ to the system of polynomial equations $P_i^n(\vec{v}) = 0$ for $i = 1, ..., n$. By the induction hypothesis, X_n cannot be covered in two parallel hyperplanes for any n. □

5. Applications to compressed sensing

Sets of vectors with the above property have applications in compressive sensing. In particular, sets of k vectors in $\mathbb{R}^n$ which cannot be covered by fewer than $k - n + 1$ parallel hyperplanes can be used to construct certain types of sensing matrices for which are useful for sparse signal recovery. An $n \times d$ real matrix A is said to be a sensing matrix for ℓ-sparse signals, $1 \leqslant \ell \leqslant n$, if, for every nonzero vector $\vec{x} \in \mathbb{R}^d$ with no more than ℓ nonzero coordinates, $A\vec{x} \neq \vec{0}$. This is equivalent to saying that every combination of ℓ columns of A are linearly independent. Such matrices $A = (a_{ij})$ are extensively used in the area of compressive sensing, where the goal is to have $|A| := \max |a_{ij}|$ small and d as large as possible with respect to n.

Indeed, given such a matrix A and two vectors $\vec{x}$ and $\vec{y}$ with no more than $\ell/2$ nonzero coordinates each, then it is easy to see that $A\vec{x} = A\vec{y}$ if and only if $\vec{x} = \vec{y}$. Integer $n \times d$ matrices A with $d > n$ and all nonzero minors were recently studied in [4], [5], [6] in the context of integer sparse recovery. In this situation, the advantage to using integer matrices and integer signals is that if $A\vec{x} \neq \vec{0}$ then $\|A\vec{x}\| \geqslant 1$, which allows for robust error correction. We provide a first example of a construction of some sensing matrices.

Theorem 13. *Let $k > n$ and $\vec{x}_1, \ldots, \vec{x}_{k-1} \in \mathbb{R}^n$ be distinct nonzero vectors. Let*

$$S = \left\{\vec{0}, \vec{x}_1, \ldots, \vec{x}_{k-1}\right\} \subset \mathbb{R}^n$$

and A be the $n \times (k-1)$ matrix, whose columns are these vectors, i.e.

$$A = \begin{pmatrix} \vec{x}_1 & \ldots & \vec{x}_{k-1} \end{pmatrix}.$$

If S cannot be covered by fewer than $k-n+1$ parallel hyperplanes, then A is a sensing matrix for n-sparse signals.

Proof. Arguing towards a contradiction, suppose that some minor of A is zero. This means that the corresponding n vectors are linearly dependent, without loss of generality assume it is $\vec{x}_1, \ldots, \vec{x}_n$. Hence they all lie in some subspace of dimension $m \leqslant n-1$, call this subspace V. Naturally, $\vec{0}$ also lies in V, since V is a subspace. If all of the points $\vec{x}_{n+1}, \ldots, \vec{x}_{k-1}$ also lie in some $(n-1)$-dimensional subspace V' containing V, then $\vec{x}_1, \ldots, \vec{x}_{k-1}$ all project to one point on the line orthogonal to V', which is a contradiction. Hence assume that

$$\operatorname{span}_{\mathbb{R}}\{V, \vec{x}_{n+1}, \ldots, \vec{x}_{k-1}\} = \mathbb{R}^n.$$

Then there exists some $(n-1)-m$ points among $\vec{x}_{n+1}, \ldots, \vec{x}_{k-1}$ which do not lie in V. Let V' be the $(n-1)$-dimensional subspace spanned by V and these points. This means that V' contains a total of

$$n + (n-1) - m + 1 \geqslant n + 1$$

points of the set S. Let L be the line through the origin orthogonal to V', then all of these points project to one point on L. Since the number of remaining points in our collection is $k-(n+1)$, the total number of distinct projections of points of S onto L is at most $k-n$. However, the number of hyperplanes required to cover a set is equal to the minimum number (over the set of lines) of distinct projections of the set onto a line, as lines define functionals, and therefore hyperplanes. This produces a contradiction because S cannot be covered by fewer than $k-n+1$ parallel hyperplanes. Thus, all minors of A must be nonzero. □

This is not a particularly strong result. Because there can be at most $n+k+1$ points in a set which cannot be covered by $k-n$ hyperplanes, this does not produce particularly useful sensing matrices. Rather than using the point sets themselves, we can increase the size of the matrices by taking difference sets at the expense of decreasing the sparsity level ℓ. For a set of k points $S = \{\vec{x}_1, \ldots, \vec{x}_k\} \subset \mathbb{R}^n$ define a partition of S into two disjoint subsets

$$I_m = \{\vec{x}_{i_1}, \ldots, \vec{x}_{i_m}\},\ J_l = \{\vec{x}_{j_1}, \ldots, \vec{x}_{j_l}\} = S \backslash I_m, \tag{1}$$

so that $I_m \cap J_l = \varnothing$ and $S = I_m \cup J_l$, where $m, l \geqslant 1$ are such that $k = m + l$. For this partition, define the corresponding set of pairwise difference vectors

$$\mathcal{D}(I_m, J_l) = \{\vec{x}_i - \vec{x}_j : \vec{x}_i \in I_m, \vec{x}_j \in J_l\},$$

so $|\mathcal{D}(I_m, J_l)| = ml = m(k - m)$. For a subset $D \subseteq \mathcal{D}(I_m, J_l)$ define support of D to be the set of all distinct vectors $\vec{x}_i$ that appear in the differences in D. For instance, support of the difference set

$$\{\vec{x}_1 - \vec{x}_2, \vec{x}_3 - \vec{x}_2, \vec{x}_1 - \vec{x}_4, \vec{x}_3 - \vec{x}_4\}$$

is $\{\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4\}$. Let us write $c(D)$ for the cardinality of the support of D. Let us also write $A(D)$ for the matrix whose columns vectors are elements of the set D. We can now state our main result of this section.

Theorem 14. *Let $S = \{\vec{x}_1, \dots, \vec{x}_k\} \subset \mathbb{R}^n$ be a collection of $k > n$ points, $m, l \geqslant 1$ integers such that $k = m + l$, $S = I_m \sqcup J_l$ partition of S, and $D \subseteq \mathcal{D}(I_m, J_l)$. Let $1 \leqslant \ell \leqslant n - 1$. The following two statements are true:*

(1) If S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes and for every subset D' of ℓ vectors of D, $c(D') > \ell$, then $A(D)$ is a sensing matrix for ℓ-sparse vectors.

(2) If for every $m + l = k$ and partition $S = I_m \sqcup J_l$, $A(D(I_m, J_l))$ is a sensing matrix for n-sparse vectors, then S cannot be covered by fewer than $k - n + 1$ parallel hyperplanes.

Proof. First, suppose that at least $k - n + 1$ parallel hyperplanes are required to cover S, and for every subset D' of $\ell \leqslant n - 1$ vectors from D, $c(D') > \ell$. To prove that $A(D)$ is a sensing matrix for ℓ-sparse vectors, we simply need to establish that no ℓ vectors of D lie in the same $(\ell - 1)$-dimensional subspace of $\mathbb{R}^n$. Suppose they do, say some ℓ vectors

$$\vec{y}_1 = \vec{x}_{i_1} - \vec{x}_{j_1}, \dots, \vec{y}_\ell = \vec{x}_{i_{n-1}} - \vec{x}_{j_\ell} \tag{2}$$

are in the same $(\ell - 1)$-dimensional subspace V, where $\vec{x}_{i_1}, \dots, \vec{x}_{i_\ell} \in I_m$ and $\vec{x}_{j_1}, \dots, \vec{x}_{j_\ell} \in J_l$. Assume that $s \geqslant 1$ out of the $\vec{x}_{i_u}$ vectors are distinct and $p \geqslant 1$ of the $\vec{x}_{j_u}$ vectors are distinct: let S_1 be the set of these $s + p$ distinct vectors. Without loss of generality assume that $s \leqslant p$. Let U be the $(n - \ell + 1)$-dimensional subspace of $\mathbb{R}^n$ orthogonal to V, then each pair $\vec{x}_{i_r}, \vec{x}_{j_r}$ lies in the same parallel translate of V along U. So if, for instance, $\vec{x}_1 - \vec{x}_2$, $\vec{x}_1 - \vec{x}_3$ and $\vec{x}_4 - \vec{x}_2$ are in V, then $\vec{x}_1, \vec{x}_2, \vec{x}_3, \vec{x}_4$ all must lie in the same parallel translate of V along U. Hence the number of parallel translates of V along U needed to cover the set S_1 is at most

$$t := s - (\ell - p) \geqslant 1,$$

since for every subset D' of $\ell \leqslant n-1$ vectors from D, $c(D') > \ell$, and so $s + p > \ell$.

Let V_1 be the parallel translate of V along U containing the pair $\vec{x}_{i_1}, \vec{x}_{j_1}$. Since $k - n + 1 \geqslant 2$, S cannot be covered completely by any single $(n-1)$-dimensional hyperplane containing V_1. Since dimension of V_1 is $\ell - 1$, there must exist a set $Z \subset S \backslash V_1$ consisting of $n - \ell$ points in general position. Let H_1 be an $(n-1)$-dimensional hyperplane in $\mathbb{R}^n$ through Z and V_1 and let $L \subset U$ be the line through the origin orthogonal to H_1. Let us write $Z = Z_1 \sqcup Z_2$, where $Z_1 = Z \cap S_1$: here it is possible for Z_1 or Z_2 to be empty. Then H_1 covers all the points of S_1 in V_1 plus at least $|Z_1|$ more, and so H_1 together with at most $t - |Z_1| - 1$ additional parallel translates of H_1 along L cover S_1. Now at most $k - (s+p) - |Z_2|$ additional parallel translates of H_1 along L will cover the rest of S. Hence a total of at most

$$\begin{aligned} & (t - |Z_1|) + (k - (s+p) - |Z_2|) = t - |Z| + k - (s+p) \\ = \; & s - (\ell - p) - (n - \ell) + k - (s+p) = k - n < k - n + 1 \end{aligned}$$

parallel hyperplanes covers S. This is a contradiction, and hence $A(D)$ is a sensing matrix for ℓ-sparse vectors.

In the opposite direction, suppose that every $A(I_m, J_l)$ is a sensing matrix for n-sparse vectors, so no n vectors in the set $D(I_m, J_l)$ are linearly dependent. Suppose S can be covered by some collection of $t \leqslant k - n$ parallel hyperplanes. Out of these hyperplanes, let $H_1, \dots, H_s$ be those that contain more than one point of S, then the remaining $t - s$ hyperplanes $H_{s+1}, \dots, H_t$ (if any) contain just one point of S each, $1 \leqslant s \leqslant t$. Then

$$\left| S \cap \left(\bigcup_{i=1}^{s} H_i \right) \right| = k - (t - s) \geqslant k - (k - n - s) = n + s.$$

For each $1 \leqslant i \leqslant s$, let

$$S \cap H_i = \{\vec{x}_{i,1}, \dots, \vec{x}_{i,j_i}\},$$

hence $\sum_{i=1}^{s} j_i \geqslant n+s$. Let I_t be the set consisting of all the vectors $\vec{x}_{i,1}$ for $1 \leqslant i \leqslant s$, and all the vectors from $S \cap H_j$ for $s+1 \leqslant j \leqslant t$. Let $l = k-t$, and let $J_l = S \backslash I_t$. Consider the set of difference vectors

$$D' = \{\vec{x}_{i,1} - \vec{x}_{i,2}, \ldots, \vec{x}_{i,1} - \vec{x}_{i,j_i} : 1 \leqslant i \leqslant s\} \subseteq D(I_t, J_l).$$

Since all of the vectors $\vec{x}_{i,1}, \ldots, \vec{x}_{i,j_i}$, $1 \leqslant i \leqslant s$ lie in parallel hyperplanes, all the vectors of D' lie in the same $(n-1)$-dimensional subspace of $\mathbb{R}^n$. The total number of these vectors is

$$|D'| = \sum_{i=1}^{s} (j_i - 1) \geqslant n + s - s = n,$$

hence they are linearly dependent. This is a contradiction, so S cannot be covered by any collection of fewer than $k-n+1$ parallel hyperplanes. □

While this theorem gives a construction of sensing matrices, it is still not at all clear as to the potential size these can be. The question then becomes the following: given a set S which cannot be covered by fewer than $k-n+1$ parallel hyperplanes, how large can the cardinality of $D \subset \mathcal{D}(I_m, J_\ell)$ be, subject to the constraint that $c(D') > \ell$ for all subsets D' of D which are size ℓ. The following corollary provides the answer.

Corollary 15. *For all sufficiently large n, there exist $n \times d$ integer sensing matrices A for ℓ-sparse vectors, $1 \leqslant \ell \leqslant n-1$, such that $|A| = 2$ and*

$$d \geqslant \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}}.$$

If $\ell \leqslant (\log n)^{\varepsilon}$ for any $\varepsilon \in (0,1)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is superlinear in n.

The problem is most naturally phrased in the setting of bipartite graphs. In particular, given such a set $D \subset \mathcal{D}(I_m, J_\ell)$, we may construct an associated bipartite graph $\Gamma(D)$ with vertices corresponding to $c(D)$, and two vertices connect to form an edge if their associated vectors appear together as a difference vector in D. Returning to the problem at hand, we wish for all subsets D' of D with cardinality ℓ to have $c(D') > \ell$. This can occur if and only if there are no cycles of length less than ℓ in $\Gamma(D)$. The construction of such graphs has been extensively studied by various authors. See [11] for a survey of known results in this direction. In particular, Theorem 3 of [11] guarantees that for large enough k there exist such graphs k vertices and

$$\geqslant \left(\frac{k}{2}\right)^{1+\frac{2}{3\ell-2}} \tag{3}$$

edges. An explicit deterministic construction of such bipartite graphs can be found in [8] and [9] (also see [7]). We can now use this result to prove our corollary.

Proof. For sufficiently large n, let S_n be the set of $n+2$ vectors with $\{0, \pm 1\}$ coordinates obtained in Section 4, hence S_n cannot be covered by $(n+2)-n+1=3$ parallel hyperplanes. Let Γ be a bipartite graph on the $n+2$ vertices corresponding to the vectors of S_n with the number of edges satisfying (3). Let D be the set of difference vectors corresponding to the edges of Γ, then for every subset D' of D consisting of ℓ vectors $c(D') > \ell$. Therefore by Theorem 14, $A(D)$ is a sensing matrix for ℓ-sparse vectors, and we have $|A(D)| = 2$. Furthermore, $A(D)$ is an $n \times d$ integer matrix where

$$d \geqslant \left(\frac{n+2}{2}\right)^{1+\frac{2}{3\ell-2}},$$

by (3). Notice that if $\ell \leqslant (\log n)^{\varepsilon}$ for any $\varepsilon \in (0,1)$, then $d/n \to \infty$ as $n \to \infty$, meaning that d is greater than linear in n. □

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