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SEQUENCES WITH CHANGING DEPENDENCIES*

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Abstract. Consider words over an alphabet with n letters. Fisher [*Amer. Math. Monthly*, 96 (1989), pp. 610–614] calculated the number of distinct words of length ℓ assuming certain pairs of letters commute. In this paper we are interested in a more general setting where the pairs of letters that commute at a certain position of a word depend on the initial segment of the word. In particular, we show that if for each word at each position any letter fails to commute with at most a constant number of other letters, then the number of distinct words of length ℓ is at most $C^{n+\ell}$ for some constant C. We use this result to obtain a lower bound on the number of diagonal flips required in the worst case to transform one n-vertex labeled triangulated planar graph into some other one. This has previously been proved in [D. D. Sleator, R. E. Tarjan, and W. P. Thurston, *SIAM J. Discrete Math.*, 5 (1992), pp. 428–450] by different methods.

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1. Introduction.

Let $\mathcal{A} = \{a_1, \ldots, a_m\}$ be a finite alphabet consisting of letters $a_1, \ldots, a_m$. We denote the set of all words of finite length with letters from $\mathcal{A}$ by $\mathcal{A}^*$. Given a word $w \in \mathcal{A}^*$, we write $[w]_t$ for the prefix of w of length t (that is, $[w]_t$ consists of the first t letters of w). For each word w in A^* we fix a set $C_w \subseteq \mathcal{A}$. We call the collection $\mathcal{C} = (C_w : w \in \mathcal{A}^*)$ the *commuting family.* Let $\sim_{\mathcal{C}}$ be the smallest equivalence relation such that

$$a_1 \ldots a_{t-1} a_t a_{t+1} \ldots a_\ell \sim_{\mathcal{C}} a_1 \ldots a_{t-1} a_{t+1} a_t \ldots a_\ell$$

whenever $a_{t+1} \in C_{a_1 \ldots a_t}$. By enlarging the sets C_w, if necessary, we may assume that for all $a, b \in \mathcal{A}$ and $w \in \mathcal{A}^*$, we have $a \in C_{wb} \Leftrightarrow b \in C_{wa}$. Note that this does not change the equivalence relation. We call two words w and w' equivalent (with respect to $\mathcal{C}$) if $w \sim_{\mathcal{C}} w'$.

Fisher [4] considered the number T_ℓ of equivalent classes of words of length ℓ, when C_w depends only on the last letter of w (that is, we specify the pairs of letters that commute). In this case one can construct a dependency graph $G = (V, E)$ where the vertices correspond to the letters of the alphabet and two vertices are adjacent if the corresponding letters commute. Fisher showed that for $\ell > 0$, $T_\ell = \sum_{k=1}^{\ell} (-1)^{k+1} c_k T_{\ell-k}$ with $T_0 = 1$, where c_k is the number of complete subgraphs of size k in G. It follows that the generating function $t(x) = \sum_{k=0}^{\infty} T_k x^k$ is closely related to the well-known independence polynomial $Z_G(x) = \sum_{k=0}^{|V|} s_k x^k$, where s_k is the number of independent sets with k vertices. More precisely, $t(x) = 1/Z_{\overline{G}}(-x)$,

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where $\overline{G}$ is the complement of G. The notation we use here, Z_G, is the one used in statistical physics, where (a generalization of) the polynomial goes under the name of the "lattice-gas partition function" associated with the graph G. For a review of various combinatorial results concerning the independence polynomial, see Levit and Mandrescu [7].

Clearly, $s_0 = 1$, $s_1 = |V|$ for every graph $G = (V, E)$, and $s_n = 0$ unless G has no edges, in which case it is 1. In particular, Z_G is nonzero in some complex disc

$$D_R = \{z \in \mathbb{C} : |z| \leq R\},$$

where $R > 0$.

Extending a result of Kotecký and Preiss [6], Dobrushin [1] (see also [2]) proved the following fundamental result in statistical physics, widely known as *Dobrushin's theorem*.

THEOREM 1.1. *Let Z_G be the independence polynomial of a graph $G = (V, E)$. Let r_v, $v \in V$, be constants such that $0 \leq r_v < 1$, and*

$$R \leq r_v \prod_{u\colon uv \in E} (1 - r_u)$$

for every $v \in V$. Then Z_G is nonzero in the disc $|z| \leq R$.

In fact, if $|z| \leq R$, then

$$|Z_G(z)| \geq Z_G(-R) \geq \prod_{v \in V} (1 - r_v). \tag{1}$$

Twenty years earlier, Lovász had proved a celebrated result in combinatorics with analogous conditions (see [3]). This result, the *Lovász local lemma*, gives necessary conditions for the probability of the intersection of a family of events to be strictly positive, when these events are not far from being independent. Recently, Scott and Sokal [9], [8] showed that these two fundamental results, the Lovász local lemma and Dobrushin's theorem, are, in fact, equivalent. The proof of this equivalence is not too easy: it is based on the result of Shearer [10] that the conditions in the Lovász local lemma are best possible.

As a consequence of Dobrushin's theorem, if the degree of every vertex in $G = (V, E)$ is bounded by k, then one can take $R = R(k) > 0$ independently of the size of G. Since $t(x) = 1/Z_{\overline{G}}(-x)$, this implies, for example, that there are at most $C(k)^{\ell+|V|}$ different equivalence classes of words of length ℓ if there are at most k letters that do not commute with any given letter (in other words the dependency graph G has minimum degree at least $|V| - k - 1$).

Sleater, Tarjan, and Thurston [11] used a different setup to count the number of different outcomes of sequences of transformations of graphs. Each transformation is obtained by replacing a subgraph isomorphic to one of a fixed family of graphs by another graph. Whether or not two transformations commute depends on the previous transformations applied. In this sense it is very similar to our approach, but ours appears to be more general as well as simpler.

As an example where the sets C_w depend on the entire word and not only the last letter, we consider words in connection with the class of labeled triangulations on n vertices, that is, maximal planar graphs on the vertex set $\{1, \ldots, n\}$. Consider a fixed triangulation that is embedded in the plane. Due to Whitney [14], this embedding is essentially unique. If one removes an edge $\{i, j\}$ of a triangulation, then there is a face $\{i, k, j, l\}$ of size four, and if the edge $\{k, l\}$ does not already exist, then one can add $\{k, l\}$ instead of $\{i, j\}$ to obtain a new triangulation. Such an operation is called a *flip*. It is well known [13] that each triangulation can be reached from any other triangulation by a finite number of flips. We consider such *flip sequences* from a fixed given embedded triangulation and also *extended flip sequences* where we allow multiple edges but insist that each of the multiple edges has its own face, namely, the face of size four that is obtained by deleting the edge before the flip.

Let $\mathcal{A} = \{a_0, \ldots, a_{3n-6}\}$ be the alphabet where for $i = 1, \ldots, 3n - 6$ the flip of edge i is represented by the letter a_i and a nonflip is represented by a_0. After a flip of edge i, the new edge created will still be numbered i. Some flip sequences may lead to the same triangulation. To capture this we define for an extended flip sequence $w = a_i \ldots a_j$, the set C_w as the nonflip a_0, and the set of all edges except for the four edges having a common face with the last edge a_j after the flips of the sequence w have taken place.

Consider the commuting family $\mathcal{C} = (C_w : w \in \mathcal{A}^*)$ and the set $\mathcal{A}^\ell$ of all flip sequences of length ℓ starting with a given embedded triangulation. Note that if two flip sequences in $\mathcal{A}^\ell$ are equivalent, then the two sequences yield the same triangulation. This can be seen by observing that it does not matter at which point an edge is flipped as long as the edges incident to a common face have not been changed, and $\mathcal{C}$ is chosen in such a way that the order of edges incident to a common face can never be changed. Hence the number of equivalence classes is at least the number of triangulations that can be reached with ℓ flips from the given triangulation. We want to find a lower bound on the number of flips ℓ needed to reach all triangulations from a fixed given triangulation. Since Tutte [12] has shown that the number of triangulations is at least $C^n n!$ for any $C < 256/27$ and sufficiently large n, it suffices to find an upper bound on the number of equivalence classes of $\mathcal{A}^\ell$, as we need more equivalence classes than triangulations to reach all triangulations from a given one. We find such an upper bound in Theorem 2.2, where we show that the number of equivalence classes is at most $\widetilde{C}^{|\mathcal{A}|+\ell}$ for some constant $\tilde{C}$. Thus to reach all triangulations from a fixed triangulation we have to ensure that the number of flips ℓ satisfies $\widetilde{C}^{3n-5+\ell} \geq C^n n!$. It follows that $\Omega(n \log n)$ flips are needed. This lower bound matches asymptotically the upper bound, as it is known (see [11] and [5]) that $O(n \log n)$ flips suffice. The lower bound has also been shown in [11] with a slightly longer, more complicated proof. In particular, in [11] one needs to consider the dual of a triangulation before one can apply their method.

2. Main result.

To prove our main result we first show that the number of words over the alphabet $\{1, \ldots, m\}$ that never decrease much can be bounded from above.

LEMMA 2.1. *Let $k \in \mathbb{N}$ be a constant. The number of words $a_1 \ldots a_\ell$ of length ℓ over the alphabet $\mathcal{A} = \{1, \ldots, m\}$ such that $a_{j+1} - a_j \geq -k$ for all $1 \leq j \leq \ell - 1$ is at most*

$$\binom{m + \ell k + \ell}{\ell} \leq ((k + 1)e)^\ell e^{m/(k+1)},$$

where e denotes the base of the natural logarithm.

Proof. We can view the sequence $a_1, \ldots, a_\ell$ as the following walk on the integers starting at 0. We consider $a_1, \ldots, a_\ell$ in turn. When considering a_i, $b_i = a_i - a_{i-1} + k$ steps of size 1 are taken up (where we set $a_0 = k$) and then one step of size k is taken down; see Figure 1. Note that $b_i \geq 0$ as $a_{j+1} - a_j \geq -k$ for all $1 \leq j \leq \ell - 1$. Given

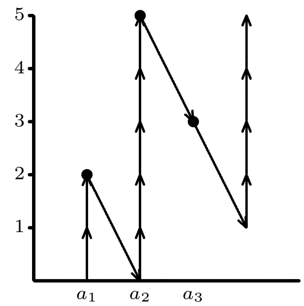

FIG. 1. *Sequence UUDUUUUUDDUUUU corresponds to* $(a_1, a_2, a_3) = (2, 5, 3)$, *where* $k = 2$, $m = 5$.

such a walk we can easily reconstruct the sequence $a_1 \dots a_\ell$, as a_i is just the height of the walk before stepping down for the ith time. Hence it remains to count the number of such walks.

Note that when considering the word $a_1, \dots, a_\ell$ we can step up a total of at most $m + \ell k$ times. By adding additional steps up after the last step down, we may assume without loss of generality that there are exactly $m + \ell k$ steps up and ℓ steps down. Therefore the number of such walks is at most the number of ways we can choose the positions of the ℓ steps down out of the $m + \ell k + \ell$ steps and is thus at most

$$\binom{m+\ell k+\ell}{\ell} \le \left(\frac{e(m+\ell(k+1))}{\ell}\right)^\ell$$

$$\le \left((k+1)e\left(\frac{m}{(k+1)\ell}+1\right)\right)^\ell \le ((k+1)e)^\ell e^{m/(k+1)}.$$

(Here we used the well-known facts that $\binom{a}{b} \le (ea/b)^b$ and $(1 + x) \le e^x$.) ∎

THEOREM 2.2. *Let $k \in \mathbb{N}$ be a constant, let $\mathcal{A}$ be a finite alphabet with m letters, and let $\mathcal{A}^\ell$ be a subset of all words in $\mathcal{A}^*$ of length ℓ. If every member C_w of a commuting family $\mathcal{C}$ has at least $m - k$ elements, then there exists a set $\mathcal{S}$ of size at most $((k+1)e)^\ell e^{m/(k+1)}$ such that for each word $w \in \mathcal{A}^\ell$ there exists a word $w' \in \mathcal{S}$ with $w \sim_{\mathcal{C}} w'$.*

Proof. We shall define an injection from the set of equivalence classes to the set of sequences $b_1, \dots, b_\ell$ of length ℓ with $b_j \in \{1, \dots, m\}$ and the property that $b_{j+1} - b_j \ge -k$. The result then follows by Lemma 2.1.

Let $\mathcal{A} = \{a_1, \dots, a_m\}$. We start with an arbitrary ordering of the letters, say, $a_1 < a_2 < \cdots < a_m$. This ordering might change at each step of the following procedure. We assign the following sequence $b_1, \dots, b_\ell$ to a word $w = a_{i_1}, \dots, a_{i_\ell} \in \mathcal{A}^\ell$: At each step j, $b_j \in \{1, \dots, m\}$ is equal to the position of a_{i_j} in the current ordering. Thus $b_1 = i_1$. Given the ordering when we have considered k letters of w, the next ordering is obtained by moving the letters not in $C_{[w]_k}$ that are below a_{i_k} directly below a_{i_k} so that the relative order is conserved; see Table 1. Observe that letters that are above a_{i_k} in the current ordering are not moved.

Note first that given the set $\mathcal{C}$ and the initial ordering $a_1 < a_2 < \cdots < a_m$, a sequence $b_1, \dots, b_\ell$ with $b_i \in \{1, \dots, m\}$ for all $1 \le i \le \ell$ represents a unique word in $\mathcal{A}^*$. Thus it remains to show that we can choose a word in each equivalence class in

Table 1

The change of the ordering with the word decaf *which is mapped to* $(4,5,4,3,6)$. *We start with the ordering* a,b,c,d,e,f *and consider the letters of the word* decaf *in turn. After considering* d *we have to move the letters not in* C_{d} *that are below* d *directly behind* d. C_{d} *consists of two elements* b, f. *Thus we have to move* a,c *directly behind* d *to obtain* b,a,c,d,e,f. *After considering* de *we have to move all the elements below* e *that are not in* $C_{\mathrm{de}} = \{\mathrm{a}, \mathrm{d}\}$ *directly behind* e. *Thus we have to move* b *and* c *to obtain* a,d,b,c,e,f. *We continue and obtain the word* $(4,5,4,3,6)$.

Commuting sets: $C_{\mathrm{d}} = \{\mathrm{b},\mathrm{f}\}$, $C_{\mathrm{de}} = \{\mathrm{a},\mathrm{d}\}$, $C_{\mathrm{dec}} = \{\mathrm{b},\mathrm{d}\}$, $C_{\mathrm{deca}} = \{\mathrm{e},\mathrm{f}\}$					
Order	Initially	After: d	de	dec	deca
6	f	f	f	f	f
5	e	e	e	e	e
4	d	d	c	c	c
3	c	c	b	a	a
2	b	a	d	b	b
1	a	b	a	d	d

such a way that each word is mapped to a sequence $b_1, \ldots, b_\ell$ with $b_{j+1} - b_j \geq -k$ for all $j \in \{1, \ldots, \ell-1\}$. For each equivalence class we choose the word that is assigned the lexicographically smallest sequence, that is, if $w = a_{i_1}, \ldots, a_{i_\ell}$ is chosen and is mapped to $b_1, \ldots, b_\ell$, then every word that is equivalent to w is assigned to a sequence that is bigger than $b_1, \ldots, b_\ell$ in the lexicographic order.

We now claim that these chosen words are assigned to sequences $b_1, \ldots, b_\ell$ with $b_{j+1} - b_j \geq -k$ for all $1 \leq i \leq \ell$. Assume for contradiction that a chosen word $w = a_{i_1} \ldots a_{i_\ell}$ is mapped to a sequence $b_1, \ldots, b_\ell$ such that $b_{t+1} - b_t < -k$ for some index $t \in \{1, \ldots, \ell-1\}$. Then by the assumption that $|C_{[w]_t}| \geq m-k$ and the choice of our ordering, $a_{i_{t+1}} \in C_{[w]_t}$. Thus w is equivalent to $w' = a_{i_1} \ldots a_{i_{t-1}} a_{i_{t+1}} a_{i_t} a_{i_{t+2}} \ldots a_{i_\ell}$. But it is clear that $a_{i_1} \ldots a_{i_{t-1}}$ is mapped to the same subsequence in both words. Moreover, after we have considered $a_{i_1} \ldots a_{i_{t-1}}$, in the current ordering we have $a_{i_{t+1}} < a_{i_t}$, since $a_{i_{t+1}} \geq a_{i_t}$ at this point would imply that $a_{i_{t+1}} \geq a_{i_t}$ in the ordering after $[w]_t$ has been considered. But this contradicts our assumption. Hence w' is mapped to a sequence that is smaller with respect to the lexicographic ordering, a contradiction to the choice of w. ◻

As we have discussed in the introduction one can consider the set $\mathcal{A}^\ell$ of all flip sequences of length ℓ from a given triangulation T, and construct sets C_w, $w \in \mathcal{A}^*$ with $|C_w| \leq (3n-5) - 4$ for all $C_w \in \mathcal{C}$ such that every triangulation T' that can be reached from T with at most ℓ flips is represented by at least one equivalence class. For any $C < 256/27$, there are at least $C^n n!$ triangulations of size n, and by Theorem 2.2 we can reach at most $t(\ell) = (5e)^\ell e^{(3n-5)/5}$ triangulations from T with ℓ flips. Thus we have to choose ℓ such that $t(\ell) \geq C^n n!$ in order to achieve that the number of triangulations that can be reached from T with at most ℓ flips is bigger than the number of all triangulations. But $t(\ell) \geq C^n n!$ is equivalent to $\ell \geq \alpha n \log n$ for some constant $\alpha > 0$. Hence we obtain the following result, a more complicated proof of which can be found in [11].

Corollary 2.3. *There are labeled triangulations* T *and* T' *on* n *vertices such that* $\Theta(n \log n)$ *flips are needed to transform* T *into* T'.

The alert reader might have observed that we did not need all the properties of the equivalence relation $\sim_{\mathcal{C}}$. Thus we can slightly generalize the definition of a commuting family. We consider sets $\mathcal{B} \subseteq \mathcal{A}^*$ where some of the words in $\mathcal{B}$ are *dominated* by others. More precisely, we are given a set $(C'_w \subseteq \mathcal{A} : w \in \mathcal{A}^*)$ that we call a *generalized* *commuting family*, and a word $w_1 \in \mathcal{B}$ dominates a word $w_2 \in \mathcal{B}$ if and only if there exists a sequence of words $w'_1, w'_2, \ldots, w'_l \in \mathcal{A}^*$ with $w_1 = w'_1$ and $w_2 = w'_l$ such that for each $j = 1, \ldots, l-1$, w'_{j+1} can be obtained from w'_j by exchanging two consecutive letters at positions t_j and $t_j + 1$ and the $t_j + 1$st letter of w'_{j+1} is contained in $C'_{[w'_j]_{t_j}}$. For example if $\mathcal{B}$ consists of all words of length ℓ and $C'_w = \mathcal{A}$ for all $w \in \mathcal{A}^*$, then every word dominates every other word. If $C'_w = \emptyset$ for all $w \in \mathcal{A}^*$, then no word dominates any other word. Finally, for a set $\mathcal{B}$ with generalized commuting family $\mathcal{C}$, a *basic set* is a subset of $\mathcal{A}^*$ such that each word of $\mathcal{B}$ dominates at least one word of the basic set. Trivial basic sets of $\mathcal{B}$ with commuting family $\mathcal{C}$ are supersets of $\mathcal{B}$. The size of a smallest basic set $\mathcal{B}$ with commuting family $\mathcal{C}$ is denoted by $|\mathcal{B}, \mathcal{C}|$. Now, as in Theorem 2.2 one can prove that $|\mathcal{B}, \mathcal{C}| \leq C(k)^{m+\ell}$ if every element of the generalized commuting family contains at least $m - k$ elements.

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