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# Properties and Applications of Extended Hypergeometric Functions

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## Abstract

In this article, we study several properties of extended Gauss hypergeometric and extended confluent hypergeometric functions. We derive several integrals, inequalities and establish relationship between these and other special functions. We also show that these functions occur naturally in statistical distribution theory.

**Key words:** Beta distribution; extended beta function; extended confluent hypergeometric function; extended Gauss hypergeometric function; gamma distribution; Gauss hypergeometric function.

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## Propiedades y aplicaciones de Funciones Hipergeométricas Extendida

**Resumen**
En este artículo estudiamos varias propiedades de las funciones hipergeométrica de Gauss extendida e hipergeométrica confluente extendida. Derivamos varias integrales, desigualdades y establecemos relaciones entre estas y otras funciones especiales. También mostramos que estas funciones ocurren naturalmente en la teoría de distribuciones estadísticas.

**Palabras clave:** Distribución beta; función beta extendida; función hipergeométrica confluente extendida; función hipergeométrica de Gauss extendida; distribución gamma; función hipergeométrica de Gauss.

## 1 Introduction

The classical beta function, denoted by $B(a,b)$, is defined (see Luke [1]) by the Euler's integral

$$\begin{aligned} B(a,b) &= \int_0^1 t^{a-1}(1-t)^{b-1}\,\mathrm{d}t, \\ &= \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \quad \mathrm{Re}(a)>0, \quad \mathrm{Re}(b)>0. \end{aligned} \tag{1}$$

Based on the beta function, the Gauss hypergeometric function, denoted by $F(a,b;c;z)$, and the confluent hypergeometric function, denoted by $\Phi(b;c;z)$, for $\mathrm{Re}(c)>\mathrm{Re}(b)>0$, are defined as (see Luke [1]),

$$F(a,b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a}\,\mathrm{d}t, \quad |\arg(1-z)|<\pi, \tag{2}$$

and

$$\Phi(b;c;z) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\exp(zt)\,\mathrm{d}t. \tag{3}$$

Using the series expansions of $(1-zt)^{-a}$ and $\exp(zt)$ in (2) and (3), respectively, the series representations of $F(a,b;c;z)$ and $\Phi(b;c;z)$, for $\text{Re}(c) > \text{Re}(b) > 0$, are obtained as

$$F(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \quad |z| < 1, \tag{4}$$

and

$$\Phi(b; c; z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}, \tag{5}$$

respectively.

In 1997, Chaudhry et al. [2] extended the classical beta function to the whole complex plane by introducing in the integrand of (1) the exponential factor $\exp\left[-\sigma/t(1-t)\right]$, with $\text{Re}(\sigma) > 0$. Thus, the extended beta function is defined as

$$B(a, b; \sigma) = \int_0^1 t^{a-1}(1-t)^{b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \quad \text{Re}(\sigma) > 0. \tag{6}$$

If we take $\sigma = 0$ in (6), then for $\text{Re}(a) > 0$ and $\text{Re}(b) > 0$ we have $B(a, b; 0) = B(a, b)$. Further, replacing $t$ by $1-t$ in (6), one can see that $B(a, b; \sigma) = B(b, a; \sigma)$. The rationale and justification for introducing this function are given in Chaudhry et al. [2] where several properties and a statistical application have also been studied. Miller [3] further studied this function and has given several additional results.

In 2004, Chaudhry et al. [4] gave definitions of the extended Gauss hypergeometric function and the extended confluent hypergeometric function, denoted by $F_\sigma(a, b; c; z)$ and $\Phi_\sigma(b; c; z)$, respectively. These definitions were developed by considering the extended beta function (6) instead of beta function (1) that appear in the general term of the series (4) and (5). Thus, for $\text{Re}(c) > \text{Re}(b) > 0$, $F_\sigma(a, b; c; z)$ and $\Phi_\sigma(b; c; z)$ are defined by

$$F_\sigma(a, b; c; z) = \sum_{n=0}^{\infty} \frac{(a)_n B(b+n, c-b; \sigma)}{B(b, c-b)} \frac{z^n}{n!}, \quad \sigma \geq 0, \quad |z| < 1, \tag{7}$$

and

$$\Phi_\sigma(b;c;z) = \sum_{n=0}^{\infty} \frac{B(b+n, c-b;\sigma)}{B(b, c-b)} \frac{z^n}{n!}, \quad \sigma \geq 0, \tag{8}$$

respectively. Further, using the integral representation of the extended beta function (6) in (7) and (8), Chaudhry et al. [4] obtained integral representations, for $\sigma \geq 0$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, of the extended Gauss hypergeometric function (EGHF) and the extended confluent hypergeometric function (ECHF) as

$$F_\sigma(a,b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-zt)^a} \exp\left[-\frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \\ |\arg(1-z)| < \pi, \tag{9}$$

and

$$\Phi_\sigma(b;c;z) = \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[zt - \frac{\sigma}{t(1-t)}\right] \mathrm{d}t, \tag{10}$$

respectively.

For $\sigma = 0$ in (9), we have $F_0(a,b;c;z) = F(a,b;c;z)$, that is, the classical Gauss hypergeometric function is a special case of the extended Gauss hypergeometric function. Likewise, taking $\sigma = 0$ in (10) yields $\Phi_0(b;c;z) = \Phi(b;c;z)$, which means that the classical confluent hypergeometric function is a special case of the extended confluent hypergeometric function. Chaudhry et al. [4] and Miller [3] found that extended forms of beta and hypergeometric functions are related to the beta, Bessel and Whittaker functions, and also gave several alternative integral representations.

In this article, we give several interesting results on extended beta, extended Gauss hypergeometric and extended confluent hypergeometric functions and show that they occur in a natural way in statistical distribution theory.

This paper is divided into five sections. Section 2 deals with some well known definitions and results on special functions . In Section 3, several properties of the extended beta, the extended Gauss hypergeometric and the extended confluent hypergeometric functions have been studied. Section 4 deals with the integrals involving EGHF and ECHF. Finally, applications of the extended Gauss hypergeometric and the extended confluent hypergeometric functions are demonstrated in Section 5.

## 2 Some Known Definitions and Results

An integral representation of the type 2 modified Bessel function (Gradshteyn and Ryzhik [5, Eq. 3.471.9]) is given by

$$K_\nu(2\sqrt{ab}) = \frac{1}{2}\left(\frac{a}{b}\right)^{\nu/2} \int_0^\infty t^{\nu-1} \exp\left[-\left(at + \frac{b}{t}\right)\right] \mathrm{d}t, \tag{11}$$

where $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$.

If we make the transformation $t = (1+u)^{-1}u$ in (2) and (3) with the Jacobian $J(t \to u) = (1+u)^{-2}$, we obtain alternative integral representations for $F(a, b; c; z)$ and $\Phi(b; c; z)$ as

$$F(a, b; c; z) = \frac{1}{B(b, c-b)} \int_0^\infty \frac{u^{b-1}(1+u)^{a-c}}{[1+(1-z)u]^a} \,\mathrm{d}u, \tag{12}$$

and

$$\Phi(b; c; z) = \frac{1}{B(b, c-b)} \int_0^\infty \frac{u^{b-1}\exp[z(1+u)^{-1}u]}{(1+u)^c} \,\mathrm{d}u, \tag{13}$$

respectively.

Putting $z = 1$ in (2) and evaluating the resulting integral using (1), one obtains

$$F(a, b; c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \quad \mathrm{Re}(c-a-b) > 0. \tag{14}$$

In the remainder of this section we give several properties of extended beta, extended Gauss hypergeometric, and extended confluent hypergeometric functions, most of them have been derived by Chaudhry et al. [2],[4].

Using the transformation $t = (1+u)^{-1}u$ in (6), with the Jacobian $J(t \to u) = (1+u)^{-2}$, we arrive at

$$B(a,b;\sigma) = \exp(-2\sigma)\int_0^\infty \frac{u^{a-1}\exp[-\sigma(u+u^{-1})]}{(1+u)^{a+b}}\,\mathrm{d}u. \tag{15}$$

For $\sigma = 0$ with $\mathrm{Re}(a) > 0$ and $\mathrm{Re}(b) > 0$, the above expression gives the well-known integral representation of $B(a,b)$ as

$$B(a,b) = \int_0^\infty \frac{u^{a-1}}{(1+u)^{a+b}}\,\mathrm{d}u. \tag{16}$$

If we take $b = -a$ in (15) and compare the resulting expression with (11) we obtain an interesting relationship between the extended beta function and the type 2 modified Bessel function as

$$B(a,-a;\sigma) = 2\exp(-2\sigma)K_a(2\sigma). \tag{17}$$

If we consider $z = 1$ in (9) and compare the resulting expression with the representation (6), we find that the extended beta function and EGHF are related by the expression

$$F_\sigma(a,b;c;1) = \frac{B(b,c-b-a;\sigma)}{B(b,c-b)}, \quad \mathrm{Re}(c) > \mathrm{Re}(b) > 0. \tag{18}$$

Further, substituting $c = a$ in (18) and using (17), we obtain, for $\sigma > 0$,

$$F_\sigma(a,b;a;1) = \frac{B(b,-b;\sigma)}{B(b,a-b)} = \frac{2\exp(-2\sigma)}{B(b,a-b)}K_b(2\sigma), \tag{19}$$

where $\mathrm{Re}(a) > \mathrm{Re}(b) > 0$.

Note that (19) can also be obtained by taking $z = 1$ and $a = c$ in (20), and then using the integral representation (11).

In the integral representation of EGHF and ECHF given in (9) and (10), respectively, substituting $t = (1+u)^{-1}u$, with the Jacobian

$J(t \to u) = (1+u)^{-2}$, alternative integral representations are obtained as

$$F_\sigma(a,b;c;z) = \frac{\exp(-2\sigma)}{B(b,c-b)} \int_0^\infty \frac{u^{b-1}\exp[-\sigma(u+u^{-1})]}{(1+u)^{c-a}[1+(1-z)u]^a}\, du \tag{20}$$

and

$$\Phi_\sigma(b;c;z) = \frac{\exp(-2\sigma)}{B(b,c-b)} \int_0^\infty \frac{u^{b-1}}{(1+u)^c} \exp\left[\frac{zu}{1+u} - \sigma\left(u+\frac{1}{u}\right)\right] du. \tag{21}$$

If we take $\sigma = 0$ in (20) and (21), we arrive at the representations (12) and (13) of the classical Gauss hypergeometric function and the classical confluent hypergeometric function, respectively.

For $|\arg(1-z)| < 1$, the transformation formula is given by

$$F_\sigma(a,b;c;z) = (1-z)^{-a} F_\sigma\left(a, c-b; c; -\frac{z}{1-z}\right). \tag{22}$$

It is noteworthy that $\sigma = 0$ in (22) gives the well-known transformation formula

$$F(a,b;c;z) = (1-z)^{-a} F\left(a, c-b; c; -\frac{z}{1-z}\right).$$

Also, putting $c = b$ in the above expression, one obtains

$$F(a,b;b;z) = (1-z)^{-a}.$$

In the integral representation of the ECHF (10) consider the substitution $1 - u = t$, whose Jacobian is given by $J(t \to u) = 1$, to obtain

$$\Phi_\sigma(b;c;z) = \frac{\exp(z)}{B(b,c-b)} \int_0^1 (1-u)^{b-1} u^{c-b-1} \exp\left[-zu - \frac{\sigma}{u(1-u)}\right] du. \tag{23}$$

By evaluating the integral in (23) using (10), Kummer's relation for extended confluent hypergeometric function is derived as

$$\Phi_\sigma(b;c;z) = \exp(z)\Phi_\sigma(c-b;c;-z). \tag{24}$$

For $\sigma = 0$, the expression (24) reduces to the well known Kummer's first formula for the classical confluent hypergeometric function.

## 3 Properties of the EGHF and ECHF

This section gives several properties of the the EGHF and ECHF. Writing $F_\sigma(a, b; c; z/a)$ in terms of integral representation using (9) and taking $a \to \infty$, we obtain

$$\lim_{a\to\infty} F_\sigma\left(a, b; c; \frac{z}{a}\right) = \Phi_\sigma(b; c; z).$$

Replacing $\exp(-\sigma/t)$ and $\exp[-\sigma/(1-t)]$ by their respective series expansions involving Laguerre polynomials $L_n(\sigma) \equiv L_n^{(0)}(\sigma)$ $(n = 0, 1, 2, \ldots)$ given in Miller [3, Eq. 3.4a, 3.4b], namely,

$$\exp\left(-\frac{\sigma}{t}\right) = \exp(-\sigma)t\sum_{n=0}^{\infty} L_n(\sigma)(1-t)^n, \quad |t| < 1,$$

and

$$\exp\left(-\frac{\sigma}{1-t}\right) = \exp(-\sigma)(1-t)\sum_{m=0}^{\infty} L_m(\sigma)t^m, \quad |t| < 1,$$

in (9) and (10), and integrating with respect to $t$ using (2) and (3), EGHF and ECHF can also be expressed as

$$F_\sigma(a, b; c; z) = \frac{\exp(-2\sigma)}{B(b, c-b)}\sum_{m,n=0}^{\infty} B(b+m+1, c+n+1-b) \times L_m(\sigma)L_n(\sigma)F(a, b+m+1; c+m+n+2; z)$$

and

$$\Phi_\sigma(b; c; z) = \frac{\exp(-2\sigma)}{B(b, c-b)}\sum_{m,n=0}^{\infty} B(b+m+1, c+n+1-b) \times L_m(\sigma)L_n(\sigma)\Phi(b+m+1; c+m+n+2; z),$$

respectively.

**Theorem 3.1.** If $z$ is such that $z < 1$, $\sigma > 0$ and $c > b > 0$, then

$$|F_\sigma(a, b; c; z)| \leq \exp(-4\sigma) F(a, b; c; z) \leq \frac{\exp(-1)}{4\sigma} F(a, b; c; z). \quad (25)$$

*Proof.* It follows that for $u > 0$ and $\sigma > 0$, $\sigma(u + u^{-1} - 2) \geq 0$ implies that $\sigma(u + u^{-1}) \geq 2\sigma$ and $\exp[-\sigma(u + u^{-1})] \leq \exp(-2\sigma)$. Now, using this inequality in the representation given in (20), we get

$$\begin{aligned} |F_\sigma(a, b; c; z)| &\leq \frac{\exp(-4\sigma)}{B(b, c-b)} \int_0^\infty \frac{u^{b-1}(1+u)^{a-c}}{[1 + (1-z)u]^a} \, du \\ &= \exp(-4\sigma) F(a, b; c; z), \end{aligned}$$

where the last line has been obtained by using (12). Further, the inequality $\ln v \leq v - 1$, $v > 0$, for $v = 4\sigma$, yields

$$\exp(-4\sigma) \leq \frac{\exp(-1)}{4\sigma},$$

which gives the second part of the inequality. □

Using special cases of the Gauss hypergeometric function in (25), several inequalities for EGHF can be obtained. For example, application of

$$F\left(a, a + \frac{1}{2}; 2a + 1; z\right) = \left[\frac{2}{1 + \sqrt{1-z}}\right]^{2a}$$

and

$$F\left(a, a + \frac{1}{2}; \frac{1}{2}; z\right) = \frac{1}{2}\left[(1 + \sqrt{z})^{-2a} + (1 - \sqrt{z})^{-2a}\right]$$

yield

$$\left|F_\sigma\left(a, a + \frac{1}{2}; 2a + 1; z\right)\right| \leq \exp(-4\sigma)\left[\frac{2}{1 + \sqrt{1-z}}\right]^{2a}$$

and

$$\left|F_\sigma\left(a, a + \frac{1}{2}; \frac{1}{2}; z\right)\right| \leq \frac{\exp(-4\sigma)}{2}\left[(1 + \sqrt{z})^{-2a} + (1 - \sqrt{z})^{-2a}\right].$$

Further, using the Clausen's identity

$$\left[F\left(a,b;a+b+\frac{1}{2};z\right)\right]^2 = {}_3F_2\left(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};z\right)$$

in (25), one gets

$$\left[F_\sigma\left(a,b;a+b+\frac{1}{2};z\right)\right]^2 \\ \leq \exp(-8\sigma)\,{}_3F_2\left(2a,2b,a+b;2a+2b,a+b+\frac{1}{2};z\right).$$

If we put $z=1$ in (25), and then use (18) and (14) in the resulting expression, we obtain

$$|B(b,d;\sigma)| \leq \exp(-4\sigma)B(b,d) \leq \frac{\exp(-1)}{4\sigma}B(b,d),$$

where $d=c-a-b>0$. If we replace $z=0$ in (10) and compare the resulting expression with (6), we see that the ECHF and the extended beta function have the relationship

$$\Phi_\sigma(b;c;0) = \frac{B(b,c-b;\sigma)}{B(b,c-b)}.$$

**Theorem 3.2.** If $\alpha$ and $\beta$ are two scalars such that $\beta-\alpha>0$, then

$$\Phi_\sigma(b;c;z) = \frac{(\beta-\alpha)^{-c+1}}{B(b,c-b)}\int_\alpha^\beta (u-\alpha)^{b-1}(\beta-u)^{c-b-1}\exp\left[\frac{z(u-\alpha)}{\beta-\alpha}\right] \\ \times\exp\left[-\frac{\sigma(\beta-\alpha)^2}{(u-\alpha)(\beta-u)}\right]\,\mathrm{d}u. \tag{26}$$

*Proof.* Using the transformation $t=(u-\alpha)/(\beta-\alpha)$ with the Jacobian $(\beta-\alpha)^{-1}$ in the representation (10), we obtain the result. □

If we consider $\beta=1$ and $\alpha=-1$ in (26), we have another integral representation of extended confluent hypergeometric function as

$$\Phi_\sigma(b;c;z) = \frac{2^{-c+1}\exp(z/2)}{B(b,c-b)}$$

$$\times \int_{-1}^{1} (1+u)^{b-1}(1-u)^{c-b-1} \exp\left(\frac{zu}{2} - \frac{4\sigma}{1-u^2}\right) \mathrm{d}u.$$

**Theorem 3.3.** If $\sigma > 0$ and $c > b > 0$, then

$$|\Phi_\sigma(b; c; z)| \leq \exp(-4\sigma)\Phi(b; c; z) \leq \frac{\exp(-1)}{4\sigma}\Phi(b; c; z).$$

*Proof.* Similar to the proof of Theorem 3.1. □

## 4 Integrals involving EGHF and ECHF

In this section we evaluate some integrals that are related to EGHF and ECHF.

**Theorem 4.1.** If $\sigma \geq 0$, $\alpha > \beta > 0$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(a) > 0$, then

$$\int_0^\infty \exp(-\alpha x) x^{a-1} \Phi_\sigma(b; c; \beta x)\, \mathrm{d}x = \Gamma(a)\alpha^{-a} F_\sigma(a, b; c; \beta\alpha^{-1}). \quad (27)$$

*Proof.* Using the integral representation (10) and changing the order of integration, we have

$$\int_0^\infty x^{a-1} \exp(-\alpha x) \Phi_\sigma(b; c; \beta x)\, \mathrm{d}x$$
$$= \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right]$$
$$\times \int_0^\infty x^{a-1} \exp[-(\alpha - \beta t)x]\, \mathrm{d}x\, \mathrm{d}t.$$

Now, integrating with respect to $x$ using Euler's gamma integral and then $t$ using the representation (9), we get the desired result. □

**Corollary 4.1.** If $\sigma > 0$, $\alpha > 0$, $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $\mathrm{Re}(a) > 0$, then

$$\int_0^\infty x^{a-1} \Phi_\sigma(b; c; -\alpha x)\, \mathrm{d}x = \frac{\Gamma(a) B(b-a, c-b; \sigma)}{B(b, c-b)} \alpha^{-a} \quad (28)$$

and

$$\int_0^\infty \exp(-x) x^{a-1} \Phi_\sigma(b; c; x)\,\mathrm{d}x = \frac{\Gamma(a) B(b, c-b-a; \sigma)}{B(b, c-b)}. \qquad (29)$$

*Proof.* Application of Kummer's relation (24) yields

$$\int_0^\infty x^{a-1} \Phi_\sigma(b; c; -\alpha x)\,\mathrm{d}x = \int_0^\infty \exp(-\alpha x) x^{a-1} \Phi_\sigma(c-b; c; \alpha x)\,\mathrm{d}x.$$

Evaluating the above integral by applying (27) and then using the relation (18), we get (28). To prove (29) just take $\alpha = \beta = 1$ in (27) and use (18). □

**Corollary 4.2.** If $\sigma > 0$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$\int_0^\infty \exp(-x) x^{c-1} \Phi_\sigma(b; c; x)\,\mathrm{d}x = \frac{2\Gamma(c) \exp(-2\sigma) K_b(2\sigma)}{B(b, c-b)}.$$

*Proof.* Just take $a = c$ in (29), and then use (17). □

**Theorem 4.2.** For $\sigma \geq 0$, $\alpha < 1$, $\mathrm{Re}(a) > \mathrm{Re}(d) > 0$ and $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$, we have

$$\int_0^1 x^{d-1}(1-x)^{a-d-1} F_\sigma(a, b; c; \alpha x)\,\mathrm{d}x = B(d, a-d) F_\sigma(d, b; c; \alpha). \qquad (30)$$

*Proof.* Using (9) and changing the order of integration

$$\begin{aligned}
&\int_0^1 x^{d-1}(1-x)^{a-d-1} F_\sigma(a, b; c; \alpha x)\,\mathrm{d}x \\
&\quad = \frac{1}{B(b, c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \\
&\qquad \times \int_0^1 \frac{x^{d-1}(1-x)^{a-d-1}}{(1-\alpha t x)^a}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad = \frac{B(d, a-d)}{B(b, c-b)} \int_0^1 \frac{t^{b-1}(1-t)^{c-b-1}}{(1-\alpha t)^d} \exp\left[-\frac{\sigma}{t(1-t)}\right]\,\mathrm{d}t,
\end{aligned}$$

where the integral involving $x$ has been evaluated using (2). Finally, using the representation (9), we arrive at the desired result. □

**Corollary 4.3.** For $\sigma > 0$ and $\text{Re}(a) > \text{Re}(c) > \text{Re}(b) > 0$, we have

$$\int_0^1 x^{c-1}(1-x)^{a-c-1} F_\sigma(a, b; c; x)\,\mathrm{d}x = \frac{2B(c, a-c)\exp(-2\sigma)K_b(2\sigma)}{B(b, c-b)}.$$

*Proof.* Just take $\alpha = 1$ and $c = d$ in (30), and then use (19). □

**Theorem 4.3.** For $\sigma > 0$, $\text{Re}(c) > \text{Re}(b) > 0$ and $\text{Re}(a) > \text{Re}(d) > 0$, we have

$$\int_0^1 x^{d-1}(1-x)^{a-d-1} F_\sigma(a, b; c; 1-x)\,\mathrm{d}x = \frac{B(d, a-d)B(b, c+d-a-b; \sigma)}{B(b, c-b)}.$$

*Proof.* Using (30), one gets

$$\int_0^1 (1-x)^{d-1} x^{a-d-1} F_\sigma(a, b; c; 1-x)\,\mathrm{d}x = B(d, a-d) F_\sigma(d, b; c; 1).$$

Now, replacing $d$ and $a-d$ by $a-d$ and $d$, respectively, in the above expression, one gets

$$\int_0^1 (1-x)^{a-d-1} x^{d-1} F_\sigma(a, b; c; 1-x)\,\mathrm{d}x = B(a-d, d) F_\sigma(a-d, b; c; 1).$$

Finally, substituting for $F_\sigma(a-d, b; c; 1)$ from (18), we get the desired result. □

**Theorem 4.4.** If $\sigma > 0$, $\alpha > 0$, $\text{Re}(a) > \text{Re}(d) > 0$ and $\text{Re}(c) > \text{Re}(b) > 0$, then

$$\int_0^\infty x^{d-1} F_\sigma(a, b; c; -\alpha x)\,\mathrm{d}x = \frac{\alpha^{-d} B(a-d, d) B(b-d, c-b; \sigma)}{B(b, c-b)}. \quad (31)$$

*Proof.* Replacing $F_\sigma(a,b;c;-\alpha x)$ by its integral representation (9) and changing the order of integration, we get

$$\int_0^\infty x^{d-1} F_\sigma(a,b;c;-\alpha x)\,\mathrm{d}x$$
$$= \frac{1}{B(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1} \exp\left[-\frac{\sigma}{t(1-t)}\right] \int_0^\infty \frac{x^{d-1}}{(1+\alpha t x)^a}\,\mathrm{d}x\,\mathrm{d}t.$$

Now, we integrate $x$ using (16) and then $t$ using (6) to obtain the result. □

**Corollary 4.4.** If $\sigma > 0$, $\alpha > 0$ and $\mathrm{Re}(a) > \mathrm{Re}(c) > \mathrm{Re}(b) > 0$, then

$$\int_0^\infty x^{c-1} F_\sigma(a,b;c;-\alpha x)\,\mathrm{d}x = \frac{2B(a-c,c)\alpha^{-c}\exp(-2\sigma)K_{b-c}(2\sigma)}{B(b,c-b)}.$$

*Proof.* Just take $c = d$ in (31) and use the relation (17). □

## 5 Statistical Distributions

In this section, we define the extended Gauss hypergeometric function and the extended confluent hypergeometric function distributions. We study several properties of these new distributions and their relationships with other known distributions. We also show that these distributions occur naturally as the distribution of the quotient $U/V$, where $U$ and $V$ are independent, $U$ has a gamma or beta type 2 distribution and the random variable $V$ has an extended beta type 1 distribution. In the end, we derive results on products and quotients of independent random variables.

First, we define the gamma, beta type 1 and beta type 2 distributions. These definitions can be found in Johnson, Kotz and Balakrishnan [6], and Gupta and Nagar [7].

A random variable $X$ is said to have a gamma distribution with parameters $\theta\,(> 0)$, $\kappa\,(> 0)$, denoted by $X \sim \mathrm{Ga}(\kappa,\theta)$, if its probability density function (pdf) is given by

$$\{\theta^{\kappa}\Gamma(\kappa)\}^{-1}\, x^{\kappa-1}\exp\left(-\frac{x}{\theta}\right), \quad x>0. \tag{32}$$

Note that for $\theta = 1$, the above distribution reduces to a standard gamma distribution and in this case we write $X \sim \mathrm{Ga}(\kappa)$.

A random variable $X$ is said to have a beta type 1 distribution with parameters $(a, b)$, $a > 0$, $b > 0$, denoted as $X \sim \mathrm{B1}(a, b)$, if its pdf is given by

$$\{B(a,b)\}^{-1}x^{a-1}(1-x)^{b-1}, \quad 0<x<1,$$

where $B(a, b)$ is the beta function.

A random variable $X$ is said to have a beta type 2 distribution with parameters $(a, b)$, denoted as $X \sim \mathrm{B2}(a, b)$, $a > 0$, $b > 0$, if its pdf is given by

$$\{B(a,b)\}^{-1}x^{a-1}(1+x)^{-(a+b)}, \quad x>0. \tag{33}$$

A random variable $X$ is said to have an extended beta (type 1) distribution with parameters $\alpha$, $\beta$ and $\lambda$, denoted by $X \sim \mathrm{EB1}(\alpha, \beta; \lambda)$, if its pdf is given by (Chaudhry et al. [2]),

$$\{B(\alpha,\beta;\lambda)\}^{-1}x^{\alpha-1}(1-x)^{\beta-1}\exp\left[-\frac{\lambda}{x(1-x)}\right], \quad 0<x<1, \tag{34}$$

where $B(\alpha, \beta; \lambda)$ is the extended beta function defined by (6), $\lambda > 0$, and $-\infty < \alpha, \beta < \infty$.

For $\lambda = 0$ with $\alpha > 0$ and $\beta > 0$, the density (34) reduces to a beta type 1 density.

**Definition 5.1.** A random variable $X$ is said to have an extended Gauss hypergeometric function distribution with parameters $\nu, \alpha, \beta$, $\gamma$ and $\sigma$, denoted by $X \sim \mathrm{EGH}(\nu, \alpha, \beta, \gamma; \sigma)$, if its pdf is given by

$$\frac{B(\beta,\gamma-\beta)}{B(\nu,\alpha-\nu)B(\beta-\nu,\gamma-\beta;\sigma)}x^{\nu-1}F_{\sigma}(\alpha,\beta;\gamma;-x), \quad x>0,$$

where $\alpha > \nu > 0$, $\gamma > \beta > 0$ if $\sigma > 0$ and $\alpha > \nu > 0$, $\gamma > \beta > \nu > 0$ if $\sigma = 0$.

The following theorem derives the extended Gauss hypergeometric function distribution as the distribution of the ratio of two independent random variables distributed as beta type 2 and extended beta type 1.

**Theorem 5.1.** Suppose that the random variables $U$ and $V$ are independent, $U \sim \mathrm{B2}(\nu, \gamma)$ and $V \sim \mathrm{EB1}(\alpha, \beta; \sigma)$. Then $U/V \sim \mathrm{EGH}(\nu, \nu+\gamma, \nu+\alpha, \nu+\alpha+\beta; \sigma)$.

*Proof.* As $U$ and $V$ are independent, by (33) and (34), the joint density of $U$ and $V$ is given by

$$\{B(\nu, \gamma)B(\alpha, \beta; \sigma)\}^{-1} \frac{u^{\nu-1} v^{\alpha-1}(1-v)^{\beta-1}}{(1+u)^{\nu+\gamma}} \exp\left[-\frac{\sigma}{v(1-v)}\right],$$

where $u > 0$ and $0 < v < 1$. Using the transformation $X = U/V$, with the Jacobian $J(u \to x) = v$, we obtain the joint density of $V$ and $X$ as

$$\{B(\nu, \gamma)B(\alpha, \beta; \sigma)\}^{-1} \frac{x^{\nu-1} v^{\nu+\alpha-1}(1-v)^{\beta-1}}{(1+xv)^{\nu+\gamma}} \exp\left[-\frac{\sigma}{v(1-v)}\right],$$

where $0 < v < 1$ and $x > 0$. Now, integration of the above expression with respect to $v$ using (9) yields the desired result. □

If $X \sim \mathrm{EGH}(\nu, \alpha, \beta, \gamma; \sigma)$, then

$$\mathrm{E}(X^h) = \frac{B(\beta, \gamma-\beta)}{B(\nu, \alpha-\nu)B(\beta-\nu, \gamma-\beta; \sigma)} \int_0^\infty x^{\nu+h-1} F_\sigma(\alpha, \beta; \gamma; -x)\,\mathrm{d}x.$$

Now, evaluation of the above integral by using (31) yields

$$\mathrm{E}(X^h) = \frac{B(\nu+h, \alpha-\nu-h)B(\beta-\nu-h, \gamma-\beta; \sigma)}{B(\nu, \alpha-\nu)B(\beta-\nu, \gamma-\beta; \sigma)}.$$

where $-\nu < \mathrm{Re}(h) < \alpha-\nu$ if $\sigma > 0$, and $-\nu < \mathrm{Re}(h) < \alpha-\nu$ and $\mathrm{Re}(h) < \beta-\nu$ if $\sigma = 0$.

Next, we define and study the extended confluent hypergeometric function distribution.

**Definition 5.2.** A random variable $X$ is said to have an extended confluent hypergeometric function distribution with parameters $(\nu, \alpha, \beta, \sigma)$, denoted by $X \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$, if its pdf is given by

$$\frac{B(\alpha, \beta - \alpha)x^{\nu-1}\Phi_\sigma(\alpha; \beta; -x)}{\Gamma(\nu)B(\alpha - \nu, \beta - \alpha; \sigma)}, \quad x > 0, \tag{35}$$

where $\nu > 0$, $\beta > \alpha > 0$ if $\sigma > 0$ and $\beta > \alpha > \nu > 0$ if $\sigma = 0$.

The extended confluent hypergeometric function distribution can be derived as the distribution of the quotient of independent gamma and extended beta type 1 variables as given in the following theorem.

**Theorem 5.2.** If $U \sim \text{Ga}(a)$ and $V \sim \text{EB1}(b, c; \sigma)$ are independent, then $X = U/V \sim \text{ECH}(a, a+b, a+b+c; \sigma)$.

*Proof.* As $U$ and $V$ are independent, from (32) and (34), the joint density of $U$ and $V$ is given by

$$\frac{u^{a-1}v^{b-1}(1-v)^{c-1}\exp[-u - \sigma/v(1-v)]}{\Gamma(a)B(b, c; \sigma)}, \quad u > 0, \quad 0 < v < 1.$$

Making the transformation $X = U/V$, with the Jacobian $J(u \to x) = v$, we find the joint density of $V$ and $X$ as

$$\frac{x^{a-1}v^{a+b-1}(1-v)^{c-1}\exp[-vx - \sigma/v(1-v)]}{\Gamma(a)B(b, c; \sigma)}, \quad 0 < v < 1, \quad x > 0.$$

Now, the density of $X$ is obtained by integrating the above expression with respect to $v$ using the integral representation (10). □

By using (28), the expected value of $X^h$, when $X \sim \text{ECH}(\nu, \alpha, \beta; \sigma)$, is derived as

$$\text{E}(X^h) = \frac{\Gamma(\nu + h)B(\alpha - \nu - h, \beta - \alpha; \sigma)}{\Gamma(\nu)B(\alpha - \nu, \beta - \alpha; \sigma)},$$

where $\text{Re}(\nu + h) > 0$ if $\sigma > 0$ and $\beta > \alpha > \text{Re}(\nu + h) > 0$ if $\sigma = 0$.

In the remainder of this section we derive results on products and quotients of independent random variables. The derivation and final result in each case involves extended forms of beta, confluent hypergeometric, Gauss hypergeometric or generalized hypergeometric functions showing ample applications of these functions and further advancing statistical distribution theory.

**Theorem 5.3.** Suppose that the random variables $X$ and $Y$ are independent, $X \sim \mathrm{Ga}(\lambda)$ and $Y \sim \mathrm{ECH}(\nu, \alpha, \beta; \sigma)$. Then, the pdf of $R = Y/(Y+X)$ is given by

$$\frac{B(\alpha, \beta-\alpha)}{B(\nu, \lambda) B(\alpha-\nu, \beta-\alpha; \sigma)} r^{\nu-1}(1-r)^{\lambda-1} F_\sigma(\nu+\lambda, \beta-\alpha; \beta; r),$$

where $0 < r < 1$.

*Proof.* Since $X$ and $Y$ are independent, from (32) and (35), we write the joint density of $X$ and $Y$ as

$$\frac{B(\alpha, \beta-\alpha)}{\Gamma(\nu)\Gamma(\lambda) B(\alpha-\nu, \beta-\alpha; \sigma)} x^{\lambda-1} y^{\nu-1} \exp(-x) \Phi_\sigma(\alpha; \beta; -y),$$

where $x > 0$ and $y > 0$. Now, making the transformation $S = Y + X$ and $R = Y/(Y+X)$ with the Jacobian $J(x, y \to r, s) = s$ and using (24), we obtain the joint density of $S$ and $R$ as

$$\frac{B(\alpha, \beta-\alpha)}{\Gamma(\nu)\Gamma(\lambda) B(\alpha-\nu, \beta-\alpha; \sigma)} r^{\nu-1}(1-r)^{\lambda-1}$$
$$\times s^{\nu+\lambda-1} \exp(-s) \Phi_\sigma(\beta-\alpha; \beta; rs), \quad s > 0, \quad 0 < r < 1.$$

Clearly, $R$ and $S$ are not independent. Integrating the previous expression with respect to $s$ by using (27) the density of $R$ is obtained. □

**Corollary 5.1.** The density of $W = X/Y$ is given by

$$\frac{B(\alpha, \beta-\alpha)}{B(\nu, \lambda) B(\alpha-\nu, \beta-\alpha; \sigma)} \frac{w^{\lambda-1}}{(1+w)^{\nu+\lambda}} F_\sigma\left(\nu+\lambda, \beta-\alpha; \beta; \frac{1}{1+w}\right),$$

where $w > 0$.

**Theorem 5.4.** Suppose that the random variables $U$ and $V$ are independent, $U \sim \mathrm{B2}(\nu,\gamma)$ and $V \sim \mathrm{EB1}(\alpha,\beta;\sigma)$. Then $Y = UV$ has the density

$$\frac{B(\alpha+\gamma,\beta)}{B(\nu,\gamma)B(\alpha,\beta;\sigma)}\frac{y^{\nu-1}}{(1+y)^{\nu+\gamma}}F_\sigma\left(\nu+\gamma,\beta;\alpha+\beta+\gamma;\frac{1}{1+y}\right), \quad y>0.$$

*Proof.* As $U$ and $V$ are independent, from (33) and (34), the joint density of $U$ and $V$ is given by

$$\{B(\nu,\gamma)B(\alpha,\beta;\sigma)\}^{-1}\frac{u^{\nu-1}v^{\alpha-1}(1-v)^{\beta-1}}{(1+u)^{\nu+\gamma}}\exp\left[-\frac{\sigma}{v(1-v)}\right],$$

where $u>0$ and $0<v<1$. Using the transformation $Y = UV$, with the Jacobian $J(u \to y) = 1/v$, we obtain the joint density of $V$ and $Y$ as

$$\{B(\nu,\gamma)B(\alpha,\beta;\sigma)\}^{-1}\frac{y^{\nu-1}}{(1+y)^{\nu+\gamma}}\frac{v^{\alpha+\gamma-1}(1-v)^{\beta-1}\exp\left[-\sigma/v(1-v)\right]}{[1-(1-v)/(1+y)]^{\nu+\gamma}},$$

where $0<v<1$ and $y>0$. The marginal density of $Y$ is obtained by integrating the above expression with respect to $v$ using (9). □

**Corollary 5.2.** Suppose that the random variables $U$ and $V$ are independent, $U \sim \mathrm{B2}(\nu,\gamma)$ and $V \sim \mathrm{B1}(\alpha,\beta)$. Then $Y = UV$ has the density

$$\frac{B(\alpha+\gamma,\beta)}{B(\nu,\gamma)B(\alpha,\beta)}\frac{y^{\nu-1}}{(1+y)^{\nu+\gamma}}F\left(\nu+\gamma,\beta;\alpha+\beta+\gamma;\frac{1}{1+y}\right), \quad y>0.$$

The above corollary has also been derived in Nagar and Zarrazola [8] and Morán-Vásquez and Nagar [9].

## 6 Conclusion

We have given several interesting properties of extended beta, extended Gauss hypergeometric and extended confluent hypergeometric functions. We have also evaluated a number of integrals involving these function. Finally, we have shown that these functions occur in a natural way in statistical distribution theory.

In a series of papers Castillo-Pérez and his co-authors [10],[11],[12],[13] have studied a generalization of the Gauss hypergeometric function defined by

$$R(a,b;c;\tau;z)=\sum_{n=0}^{\infty}\frac{B(b+\tau\, n,c-b)}{B(b,c-b)}\frac{z^n}{n!},\quad \mathrm{Re}(c)>\mathrm{Re}(b)>0.$$

Replacing $B(b+\tau\, n,c-b)$ by $B(b+\tau\, n,c-b;\sigma)$, an extended form of the above function can be defined as

$$R_\sigma(a,b;c;\tau;z)=\sum_{n=0}^{\infty}\frac{B(b+\tau\, n,c-b;\sigma)}{B(b,c-b)}\frac{z^n}{n!},\quad \mathrm{Re}(c)>\mathrm{Re}(b)>0.$$

The function defined above is a generalization of the extended Gauss hypergeometric function and will be considered for further research.

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