

**Electro- and magnetostatics of topological insulators as modeled by planar, spherical, and cylindrical  $\theta$  boundaries: Green's function approach**A. Martín-Ruiz,<sup>1,\*</sup> M. Cambiaso,<sup>2</sup> and L. F. Urrutia<sup>1</sup><sup>1</sup>*Instituto de Ciencias Nucleares, Universidad Nacional Autónoma de México,  
04510 México, Distrito Federal, Mexico*<sup>2</sup>*Universidad Andres Bello, Departamento de Ciencias Físicas, Facultad de Ciencias Exactas,  
Avenida Republica 220, Santiago, Chile*

(Received 30 November 2015; published 17 February 2016)

The Green's function method is used to analyze the boundary effects produced by a Chern-Simons extension to electrodynamics. We consider the electromagnetic field coupled to a  $\theta$  term that is piecewise constant in different regions of space, separated by a common interface  $\Sigma$ , the  $\theta$  boundary, model which we will refer to as  $\theta$  electrodynamics. This model provides a correct low-energy effective action for describing topological insulators. Features arising due to the presence of the boundary, such as magnetoelectric effects, are already known in Chern-Simons extended electrodynamics, and solutions for some experimental setups have been found with a specific configuration of sources. In this work we construct the static Green's function in  $\theta$  electrodynamics for different geometrical configurations of the  $\theta$  boundary, namely, planar, spherical and cylindrical  $\theta$ -interfaces. Also, we adapt the standard Green's theorem to include the effects of the  $\theta$  boundary. These are the most important results of our work, since they allow one to obtain the corresponding static electric and magnetic fields for arbitrary sources and arbitrary boundary conditions in the given geometries. Also, the method provides a well-defined starting point for either analytical or numerical approximations in the cases where the exact analytical calculations are not possible. Explicit solutions for simple cases in each of the aforementioned geometries for  $\theta$  boundaries are provided. On the one hand, the adapted Green's theorem is illustrated by studying the problem of a pointlike electric charge interacting with a planar topological insulator with prescribed boundary conditions. On the other hand, we calculate the electric and magnetic static fields produced by the following sources: (i) a pointlike electric charge near a spherical  $\theta$  boundary, (ii) an infinitely straight current-carrying wire near a cylindrical  $\theta$  boundary and (iii) an infinitely straight uniformly charged wire near a cylindrical  $\theta$  boundary. Our generalization, when particularized to specific cases, is successfully compared with previously reported results, most of which have been obtained by using the method of images.

DOI: [10.1103/PhysRevD.93.045022](https://doi.org/10.1103/PhysRevD.93.045022)**I. INTRODUCTION**

It is not seldom that seemingly abstract mathematical models find widespread application in several fields of theoretical as well as applied physics. A paramount example of this is Einstein's theory of general relativity, because in order to achieve the accuracy that render GPSs useful, relativistic effects must be taken into account. Filling the gap from theory to applied engineering is only a matter of time. Fortunately, the time span from conception to application gets shorter and shorter. So is the case with the study of Chern-Simons (CS) forms [1] to its applications in topological insulators, spintronics and topological quantum computer science [2,3].

The relevance of CS forms also is apparent in theoretical physics. In quantum field theories, they play a prominent role in regard to anomalies. The existence of anomalies can jeopardize the consistence of the theory as they would be indicative of gauge symmetry violation, hence the need for

anomaly cancellation mechanisms. Furthermore, anomalies are needed to account for certain experimental observables, e.g., the proper neutral pion decay rate into two photons as predicted by the nonconservation of the axial-vector current generated by the Adler-Bell-Jackiw anomaly [4,5]. The latter can be expressed as the one-loop contribution to the divergence of a pseudovector or axial current  $\partial^\mu j_{\mu A} \propto *FF$ . In general relativity a similar argument holds. In  $3 + 1$  spacetime dimensions, general relativity can be thought of as a non-Abelian gauge theory for the  $SO(3, 1)$  gauge group of local Lorentz transformations, for which case a gravitational anomaly exists  $D^\mu J_{\mu\ell} \propto \mathcal{P}$ , where  $J_{\mu\ell}$  is the lepton-number current and  $\mathcal{P} = *RR$  is the Pontryagin class in  $3 + 1$  dimensions. Here one can cancel the anomaly by adding the appropriate counterterm to the Einstein-Hilbert action, which turns out to be a CS extension of general relativity. Further, this CS extended general relativity when applied to the cosmological context was posed as a possible explanation for matter-antimatter (baryon) asymmetry in the Universe [6]. Actually it is a lepton-number asymmetry; however, particular weak interaction processes in the SM

\*alberto.martin@nucleares.unam.mx

mediated by  $SU(2)$  instantons (sphalerons) can transmute leptons into baryons under certain conditions that would have been met during the out-of-equilibrium inflationary epoch of the Universe [7,8].

In a very simple guise, CS terms were used by Peccei and Quinn by the introduction of the axion in order to solve the strong  $CP$  problem of QCD [9]. Soon after 't Hooft realized that the unobserved  $U(1)$  symmetry of QCD can be understood due to the dynamics of instantons [10]. Wilczek explicitly predicted applications of axion-electrodynamics to the field of material science [11]. Further studies involve its uses in topological quantum field theory [12], topological string theory [13] and as a quantum gravity candidate [14].

In this work we will be concerned with a simple case of CS theories, akin to axion-electrodynamics, introduced by Wilczek as mentioned above in the context of particle physics. We will refer to this particular model as  $\theta$ -electrodynamics or simply  $\theta$  ED, and it amounts to extending Maxwell electromagnetism by a gauge invariant term of the form

$$\Delta\mathcal{L}_\theta = \theta(\alpha/4\pi^2)\mathbf{E} \cdot \mathbf{B}. \quad (1)$$

Here though,  $\theta$  is no longer a dynamical field, but rather we take it as a constant, a genuine Lorentz scalar, and thus Eq. (1) is a pseudoscalar. Written in a manifestly covariant way

$$\Delta\mathcal{L}_\theta = -\frac{\theta}{4}(\alpha/4\pi^2)F_{\mu\nu}\tilde{F}^{\mu\nu}, \quad (2)$$

the identification with the Pontryagin invariant associated with the  $U(1)$  gauge connection  $A = A_\mu dx^\mu$  is immediate. In the latter we introduced the Hodge dual field strength electromagnetic tensor  $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta}$ , and  $\epsilon^{\mu\nu\alpha\beta}$  is the Levi-Civita symbol.

The idea of  $\theta$  ED has been studied in several situations and also extrapolated to study other systems. In other contexts,  $\theta$  ED under consideration here has been studied as a  $3 + 1$  particular kind of Maxwell-Chern-Simons electrodynamics, where it has received considerable attention [15–31]. And also it has been studied as a restricted subset of the Standard Model extension [32,33], where several results have been achieved, too [34–40].

Further, the topological nature of the  $\theta$  term of Eq. (2) can be seen by the fact that this CS extension is a total derivative. Therefore, it produces no contribution to the field equations of motion when usual boundary conditions are met; the contribution is a boundary term that vanishes whenever one imposes the vanishing of the fields at the boundary (or at infinity in the case the theory is defined over the whole space). Should  $\theta$  cease to be a constant in the manifold where the theory is defined, the CS term would fail to be a topological invariant, and therefore the

corresponding modifications to the field equations would need to be taken into consideration.

In this paper, we will be concerned with the simplest nontrivial case in this context; namely, we will study the modifications to Maxwell's theory defined on a manifold in which either (a) there are two domains defined by their different constant values of  $\theta$ , i.e.,  $\theta$  is space dependent  $\theta(x)$ , or (b) there is a nonvanishing  $\theta$  value and the manifold has a boundary where the fields or their derivatives are not vanishing, with either Dirichlet or Neumann boundary conditions. A constant  $\theta$  can be thought of as an effective parameter characterizing properties of a novel electromagnetic vacuum, possibly arising from a more fundamental theory, where discontinuities in the value of  $\theta$  has interesting properties. This approach has been taken in the context of classical  $\theta$  ED [41–44] and in the context of quantum vacuum [45]. A similar avenue has been taken in the context of Janus field theories [46–51]. These were motivated from the gravitational sector of the AdS/CFT correspondence, by an exact and nonsingular solution for the dilatonic field in type IIB supergravity [52]. The similarity is due to the fact that the coupling constant of the ensuing four-dimensional  $\mathcal{N} = 4$  super Yang-Mills theory living in the boundary exhibits a spacetime dependent character. For further insight on the similarities and differences between Janus field theories and  $\theta$  ED as studied in this work, see Ref. [53] and references therein. On the other hand, as applied to material media,  $\theta$  can be regarded as an effective macroscopic parameter to describe new degrees of freedom of quantum matter. This approach has been thoroughly used in the context of topological insulators (TIs) as will be explained below.

First, recall that in general CS forms are amenable for capturing topological features of the physical system that they describe. Formally, this can be seen from the action principle. Given a symmetry group and an odd-dimensional differentiable manifold where fields and functions are defined, a gauge connection one-form can be defined of which the associated curvature two-form can be used to build  $2k$ -forms that (i) are gauge invariant under the symmetry group, (ii) are closed and therefore expressible in terms of a  $(2k - 1)$ -form and (iii) the integral of which is a topological invariant. This last point is crucial revealing the importance of boundaries. Its many uses in gravitation and the former description are clearly reviewed in Ref. [54].

The latter description is tailor made for the understanding of what came to be known as topological phases. The discovery of the quantum Hall (QH) state made manifest the existence of new states of matter that do not fall into Landau-Ginzburg's effective field theory paradigm. In it, the quantum mechanical states of matter that determine the different phases are characterized by the spontaneous breaking of a global symmetry of the quantum mechanical system. Von Klitzing's discovery of the astonishing precision with which the Hall conductance of a sample is

quantized [55], despite the varying irregularities of the sample, turned out to have a topological origin. The ensuing electric current along the edge of a 2D electron gas at very low temperature, due to an external magnetic field applied perpendicular to the sample, is in fact insensitive to the sample's geometric details. The reason for this lies in the band structure of the sample. For the QH state, the system is insulating in the bulk and conducting in the boundary. The Hamiltonian of the many-particle system in the bulk exhibits an energy gap separating the ground state from the excited states. On the contrary, for the edge states, the band structure is gapless. Furthermore, one can define a smooth deformation in the Hamiltonian parameter space (the symmetry transformation referred to in the previous paragraph, the transformation taking the coffee mug to a torus) that does not close the bulk gap. To finally understand the connection with topology, one can recall the Gauss-Bonnet formula and Berry's phase, the generalization of the concept of curvature to quantum mechanical systems. The former allows one to express the genus  $g$  of a surface  $S$  in terms of an integral over the local curvature of the surface. The latter is a measure of the phase accumulated by a wave function as it evolves under a slow and closed variation in the parameter space of the Hamiltonian. Back in the QH scenario, the Hall conductance can be expressed as an invariant integral over the frequency momentum space, more precisely as an integral of the Berry curvature over the Brillouin zone [56]. This quantity plays the role of a topological order parameter uniquely determining the nature of the quantum state inasmuch as the order parameter in Landau-Ginzburg effective field theory determines the usual phases of quantum matter.

As mentioned previously, CS terms lend themselves for the description of topological features of a given physical system. Concretely, in material science systems, the low-energy limit of the electrodynamics of topological insulators can be described by extending Maxwell electrodynamics precisely by the  $\theta$  term of Eq. (2), originally formulated in  $4 + 1$  dimensions but appropriately adapted to lower dimensions by dimensional reduction [57]. Thus,  $\theta$  ED as a topological field theory serves as model for many theoretical [58–60] and experimental realizations for studying detailed properties of topological states of quantum matter [2,41,61,62]. See also the review papers [63,64] and references therein.

The general scope of this work is to introduce Green's function (GF) methods in  $\theta$  ED, which are well suited to deal with the calculation of electric and magnetic fields arising from arbitrary sources, as well as to solve problems with given Dirichlet or Neumann boundary conditions on arbitrary surfaces. This approach is more general than the method of images which, to our knowledge, has been systematically employed in most of the previous works on the related literature. On the other hand, the GF method provides a precise starting point from where either

analytical or numerical approximations can be performed. In Ref. [53], we have already presented the first steps in this direction by constructing the GF for a  $\theta$  boundary with planar geometry. The method was applied to the calculation of some specific examples. The GF for a stack of layered time-reversal-symmetry-broken TIs has been constructed in Ref. [65] and applied to describe novel field patterns arising from the magnetoelectric effect due to a dipole close to the surface of the TI.

The paper is organized as follows. In Sec. II, we review the basics of Chern-Simons electrodynamics defined on a four-dimensional spacetime characterized by a piecewise constant value of  $\theta$  in different regions of space separated by a common boundary  $\Sigma$ . To isolate the effects of the  $\theta$  term on the GFs and of the stress-energy tensor, we will take the media on either side of the  $\Sigma$  boundary with no dielectric nor magnetic properties, i.e.,  $\epsilon = 1$  and  $\mu = 1$  across  $\Sigma$ . Including electric and magnetic susceptibility properties proceeds accordingly. We will restrict our analysis to the case of electro- and magnetostatics. As in usual Maxwell electrodynamics, a robust static theory is necessary and important to understand the dynamical case as well as the ensuing quantum electrodynamics. In this scenario, the field equations remain the standard Maxwell equations in the bulk, but the discontinuity of  $\theta$  modifies the behavior of the fields at the interface  $\Sigma$ . The most striking feature of this theory is that even in the static limit, electric and magnetic fields are intertwined. Aspects about a possibility to circumvent Earnshaw's theorem and how to obtain the modified conservation laws are revised; in particular, a detailed construction of the stress-energy tensor is presented.

In Sec. III, we start by adapting Green's theorem to incorporate the contributions of the  $\theta$  boundary. Also, we classify the different boundary conditions that can be imposed there. Restricting ourselves to the case where we do not impose boundary conditions at the  $\theta$  interface (i.e., boundary conditions only at infinity), we consider the simplest geometries for the  $\Sigma$ : planar, cylindrical and spherical. We present a brief review of the planar case and construct the static GF matrix for the remaining geometries, thus providing the general solution to the modified field equations for the case of arbitrary configuration of sources. The method can be extended to the case of other geometries, but we focus on these configurations given the fact that those are the ones that have attracted the most of the experimental efforts. This is an important part of our paper. In it, not only do we aim to provide practical solutions for a given boundary-value problem with arbitrary external sources but also to provide a means to improve the understanding of the theory.

In the appendices, we present the detailed calculations of the Green's matrix elements for the cases of cylindrical and spherical geometries for the  $\theta$  boundary. In Sec. IV, we present a simple application of the adapted Green's theorem

when Dirichlet boundary conditions are imposed on a planar  $\Sigma$ . Also, we discuss other applications, where we make contact between the results obtained with our method and others in the existing literature, a comparison that endorses our approach, for instance, the problem of a pointlike charge near a spherical  $\theta$  boundary together with the cases of a current-carrying wire and a uniformly charged wire near a cylindrical  $\theta$  boundary. Finally, in Sec. V, we summarize our results and give further concluding remarks. Throughout the paper, Lorentz-Heaviside units are assumed ( $\hbar = c = 1$ ), the metric signature will be taken as  $(+, -, -, -)$ , and the convention  $\epsilon^{0123} = +1$  is adopted.

## II. $\theta$ ELECTRODYNAMICS IN A BOUNDED REGION

### A. Field equations and boundary conditions

Let us consider the additional coupling of electrodynamics with the electromagnetic Pontryagin invariant

$$\mathcal{P} = F_{\mu\nu} \tilde{F}^{\mu\nu}, \quad (3)$$

via a scalar field  $\theta$  defined by the action

$$S = \int_{\mathcal{M}} d^4x \left[ -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{4} \frac{\theta(x)}{2\pi} \frac{\alpha}{2\pi} F_{\mu\nu} \tilde{F}^{\mu\nu} - j^\mu A_\mu \right], \quad (4)$$

where  $\alpha = e^2/\hbar c$  is the fine structure constant,  $\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta}$  is the dual of the field strength tensor and  $j^\mu$  is a conserved external current. The coupling constant for the  $\theta$  term,  $\alpha/4\pi^2$ , is chosen in such a way that the Dirac quantization condition for the magnetic charge be an integer multiple of  $e/2\alpha$ , as discussed in Ref. [11]. The topological nature of the Pontryagin density makes the equations of motions arising from the action (4) invariant under the change  $\theta(x) \rightarrow \theta(x) + C$ , with  $C$  any constant. Quantum mechanical arguments impose further conditions on  $C$ , which will be discussed in the following.

The  $(3+1)$ -dimensional spacetime is  $\mathcal{M} = \mathcal{U} \times \mathbb{R}$ , where  $\mathcal{U}$  is a three-dimensional manifold and  $\mathbb{R}$  corresponds to the temporal axis. We make a partition of space in two regions,  $\mathcal{U}_1$  and  $\mathcal{U}_2$ , in such a way that manifolds  $\mathcal{U}_1$  and  $\mathcal{U}_2$  intersect along a common two-dimensional boundary  $\Sigma$ , to be called the  $\theta$  boundary, so that  $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_2$  and  $\Sigma = \mathcal{U}_1 \cap \mathcal{U}_2$ , as shown in Fig. 1. We also assume that the field  $\theta$  is piecewise constant in such way that it takes the constant value  $\theta = \theta_1$  in region  $\mathcal{U}_1$  and the constant value  $\theta = \theta_2$  in region  $\mathcal{U}_2$ . This situation is expressed in the characteristic function

$$\theta(\mathbf{x}) = \begin{cases} \theta_1, & \mathbf{x} \in \mathcal{U}_1 \\ \theta_2, & \mathbf{x} \in \mathcal{U}_2 \end{cases}. \quad (5)$$

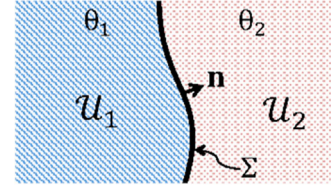


FIG. 1. Region over which the electromagnetic field theory is defined.

In this scenario, the  $\theta$  term in the action fails to be a global topological invariant because it is defined over a region with the boundary  $\Sigma$ . Varying the action gives rise to a set of Maxwell's equations with an effective additional current with support at the boundary

$$\partial_\mu F^{\mu\nu} = \tilde{\theta} \delta(\Sigma) n_\mu \tilde{F}^{\mu\nu} + 4\pi j^\nu, \quad (6)$$

where  $n_\mu = (0, \mathbf{n})$ ,  $\mathbf{n}$  is the outward unit normal to  $\Sigma$  and

$$\tilde{\theta} = \frac{\alpha}{\pi} (\theta_1 - \theta_2). \quad (7)$$

Current conservation can be verified directly by taking the divergence at both sides of Eq. (6),

$$\partial_\nu \partial_\mu F^{\mu\nu} = \tilde{\theta}'(\Sigma) n_\mu n_\nu \tilde{F}^{\mu\nu} + \tilde{\theta} \delta(\Sigma) n_\mu \partial_\nu \tilde{F}^{\mu\nu} + 4\pi \partial_\nu j^\nu, \quad (8)$$

where the left-hand side and the first two terms in the right-hand side vanish due to symmetry properties.

Together with the Bianchi identity  $\partial_\mu \tilde{F}^{\mu\nu} = 0$ , the set of Eqs. (6) for  $\theta$  ED can be written in coordinates adapted to the surface as

$$\nabla \cdot \mathbf{E} = \tilde{\theta} \delta(\Sigma) \mathbf{B} \cdot \mathbf{n} + 4\pi \rho, \quad (9)$$

$$\nabla \times \mathbf{B} - \frac{\partial \mathbf{E}}{\partial t} = \tilde{\theta} \delta(\Sigma) \mathbf{E} \times \mathbf{n} + 4\pi \mathbf{J}, \quad (10)$$

$$\nabla \cdot \mathbf{B} = 0, \quad (11)$$

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \quad (12)$$

where  $\mathbf{n}$  is the unit normal to  $\Sigma$  shown in Fig. 1. In this work, we consider the simplest geometries corresponding to the cases where the surface  $\Sigma$  is taken as (i) the plane  $z = a$ , (ii) a sphere with center at the origin and radius  $a$  and (iii) an infinitely straight cylinder of radius  $a$  with axis parallel to the  $z$  direction. In these cases, the choice of adapted coordinates has an obvious meaning: (i) Cartesian coordinates, (ii) spherical coordinates and (iii) cylindrical coordinates, respectively.

As we see from Eqs. (9)–(12) the behavior of  $\theta$  ED in the bulk regions  $\mathcal{U}_1$  and  $\mathcal{U}_2$  is the same as in standard

electrodynamics. The  $\theta$  term modifies Maxwell's equations only at the surface  $\Sigma$ . Here,  $F^{i0} = E^i$ ,  $F^{ij} = -\epsilon^{ijk} B^k$  and  $\tilde{F}^{i0} = B^i$ ,  $\tilde{F}^{ij} = \epsilon^{ijk} E^k$ . Equations (9)–(12) also suggest that the electromagnetic response of a system in the presence of a  $\theta$  term can be described in terms of Maxwell equations in matter

$$\nabla \cdot \mathbf{D} = 4\pi\rho, \quad \nabla \times \mathbf{H} = 4\pi\mathbf{J}, \quad (13)$$

with constitutive relations

$$\mathbf{D} = \mathbf{E} + \frac{\alpha}{\pi}\theta(\mathbf{x})\mathbf{B}, \quad \mathbf{H} = \mathbf{B} - \frac{\alpha}{\pi}\theta(\mathbf{x})\mathbf{E}, \quad (14)$$

where  $\theta(\mathbf{x})$  is given in Eq. (5). It is clear that if  $\theta(\mathbf{x})$  is globally constant in  $\mathcal{M}$ , there is no contribution to Maxwell's equations from the  $\theta$  term in the action, even though  $\theta(\mathbf{x})$  still is present in the constitutive relations. In fact, the additional contributions of a globally constant  $\theta(\mathbf{x})$  to each of the equations (13) cancel due to the homogeneous equations (11) and (12).

So far we have considered  $\theta$  as an external parameter that can take arbitrary values, albeit respecting the rescaling symmetry  $\theta \rightarrow \theta + C$  that we already mentioned. Naively, this would allow one to set  $\theta$  to zero at the classical level. However, quantum mechanically, given that for properly quantized electric and magnetic fluxes  $S_\theta/\hbar$  is an integer multiple of  $\theta$ , then the only allowed values of  $C$  are  $C = 2\pi n$  for integer  $n$ ; otherwise, nontrivial contributions to the path integral would result. The above argument together with additional considerations about the behavior of the system under time reversal (TR) can further constrain  $\theta$ . In fact, in the context of TIs, it was originally assumed that only systems with broken TR symmetry are of relevance; however, it was soon realized that systems preserving TR symmetry are as important; see Ref. [66]. Given the pseudoscalar nature under TR of the CS coupling,  $T(\mathbf{E} \cdot \mathbf{B}) = -\mathbf{E} \cdot \mathbf{B}$ , where  $T$  is the time-reversal operator  $T:t \rightarrow -t$ , it seems that breaking TR symmetry is inescapable. However, TR symmetry can be preserved as long as  $\theta$  takes the values 0 or  $\pi \pmod{2\pi}$ . For  $\theta = 0$ , the system is trivially TR invariant, whereas for  $\theta = \pi$ , it can be restored by suitably rescaling  $\theta$ ; i.e.,  $T(\pi\mathbf{E} \cdot \mathbf{B}) = -\pi\mathbf{E} \cdot \mathbf{B}$  can be made TR invariant by  $\theta \rightarrow \theta' = \theta + 2\pi$ . For the sake of generality, in the sequel, we will not restrict to systems with a definite TR symmetry property, and therefore the only relevant constraint for  $\theta$  that we will consider is that its observable effects satisfy the  $\theta \rightarrow \theta + 2\pi n$  symmetry, [63]. This property is made self-evident because all our observable results will depend only on the combination  $\tilde{\theta}$  given in Eq. (7).

Assuming that the time derivatives of the fields are finite, in the vicinity of the surface  $\Sigma$ , the field equations imply that the normal component of  $\mathbf{E}$ , and the tangential components of  $\mathbf{B}$ , acquire discontinuities additional to those produced by superficial free charges and currents,

while the normal component of  $\mathbf{B}$ , and the tangential components of  $\mathbf{E}$ , are continuous. For vanishing external sources on  $\Sigma$ , the boundary conditions read

$$\Delta\mathbf{E}_n|_\Sigma = \tilde{\theta}\mathbf{B}_n|_\Sigma, \quad (15)$$

$$\Delta\mathbf{B}_\parallel|_\Sigma = -\tilde{\theta}\mathbf{E}_\parallel|_\Sigma, \quad (16)$$

$$\Delta\mathbf{B}_n|_\Sigma = 0, \quad (17)$$

$$\Delta\mathbf{E}_\parallel|_\Sigma = 0. \quad (18)$$

The notation is  $\mathbf{V}_i|_{z=a^-}^{z=a^+} = \mathbf{V}_i(z)|_{z=a^-}^{z=a^+} = \lim_{\epsilon \rightarrow 0} (\mathbf{V}_i(z=a+\epsilon) - \mathbf{V}_i(z=a-\epsilon))$ ,  $\epsilon > 0$  and  $\mathbf{V}_i|_{z=a} = \mathbf{V}_i(z=a)$ , for any vector  $\mathbf{V}$ .

The continuity conditions, Eqs. (17) and (18), imply that the right-hand sides of Eqs. (15) and (16) are well defined and they represent surface charge and current densities, respectively. An immediate consequence of the boundary conditions is that the presence of a magnetic field crossing the surface  $\Sigma$  is sufficient to generate an electric field, even in the absence of free electric charges. For a  $\theta$  boundary characterizing different  $\theta$  values of the electromagnetic vacuum, these issues together with some of its consequences over the propagation of electromagnetic waves across such interfaces were studied in Refs. [43,44]. It is worth mentioning that with the modified boundary conditions, several properties of conductors in static fields still hold as far as the conductor does not lie in the  $\Sigma$  boundary. In particular, conductors are equipotential surfaces, and the electric field just outside the conductor is normal to its surface.

## B. Possibility to circumvent Earnshaw's theorem

It is a well-established fact in ordinary electrostatics that, due to electrostatic forces alone, a charge cannot be in stable equilibrium (Earnshaw's theorem). More generally, no static object comprised of electric charges, magnets and masses can be held in static equilibrium by any combination of electric, magnetic or gravitational forces.

It is natural to ask whether in the case of  $\theta$  boundaries a charge can or cannot be held in stable equilibrium due to electrostatic forces alone. This question is certainly relevant from an experimental point of view but also theoretically. As we will show below, the case of  $\theta$  ED is an interesting one as it opens a window to circumvent Earnshaw's theorem. In usual electromagnetism, the lack of the existence of stable equilibrium points is proven by contradiction, owing to the fact that, away from the sources, the electrostatic potential satisfies Laplace's equation. We will show that in  $\theta$  electromagnetism, there exist points where such a contradiction no longer takes place. From Eq. (9) and  $\mathbf{E} = -\nabla\phi$ , we have

$$-\nabla^2\phi = 4\pi\rho + \tilde{\theta}\delta(\Sigma)\mathbf{B} \cdot \mathbf{n}. \quad (19)$$

We inquire whether the potential has an extremum at a point  $P$  where a test charge is to be held in equilibrium. The argument proceeds in the usual manner. Suppose that there exists a point  $P$  at which  $\phi$  acquires a minimum. Then, at every point over the surface of any small closed surface enclosing  $P$ , one must have  $\partial_n\phi > 0$ , where  $n$  denotes the outward normal direction to the surface and thus one must have

$$\oint_{\partial\mathcal{V}} \partial_n\phi ds > 0, \quad (20)$$

where the volume  $\mathcal{V}$  encloses  $P$ . Due to  $\partial_n\phi = \nabla\phi \cdot \mathbf{n}$  and to the divergence theorem, this latter integral can be cast into a volume integral of the Laplacian of the potential. Again, to isolate the contribution due to the  $\theta$  term, we set  $\rho = 0$  to obtain

$$\begin{aligned} \oint_{\partial\mathcal{V}} \partial_n\phi ds &= \int_{\mathcal{V}} \nabla^2\phi dv \\ &= -\tilde{\theta} \int_{\mathcal{V}} \delta(\Sigma)\mathbf{B} \cdot \mathbf{n} dv \\ &= -\tilde{\theta} \int_{\Sigma} \mathbf{B}_n|_{\Sigma} d^2x. \end{aligned} \quad (21)$$

In ordinary Maxwell electrostatics ( $\tilde{\theta} = 0$ ), the right-hand side of Eq. (21) vanishes, leading to a contradiction with Eq. (20) and thus proving that  $P$  cannot be a local minimum. In the case of  $\theta$  ED, however, as far as  $\tilde{\theta} \neq 0$  and a nonvanishing normal magnetic field at  $\Sigma$  exists, the aforementioned contradiction no longer occurs, for points  $P$  at  $\Sigma$ , thus removing the immediate obstruction of ordinary electromagnetism for the existence of local minima. We emphasize that by no means are we claiming to have proven the existence of points of stable equilibrium in  $\theta$  electrostatics, which can be dealt with elsewhere. To this end, a more thorough analysis is required along the lines of Ref. [67] in the context of ordinary electromagnetism. We consider this an important issue. Charged or magnetized matter can be held in static equilibrium in a given experimental setup. However, if this were possible only with the aid of external devices such as tweezers or dynamical mechanisms or EM traps, e.g., Penning traps [68], these could compromise the precision of the measurements or interact with the EM fields under study.

### C. Stress-energy tensor

For the case of  $\theta$  ED, we still expect the electromagnetic field to be an energy-momentum transmitting entity. In this way, for example, we should be able to compute the net force over a given region in 3-space in terms of the values of the electromagnetic fields on the bounding surface of

that region. This requires establishing conservation laws, as in ordinary Maxwell electrodynamics. In Sec. II A, we showed that the  $\theta$  term modifies the behavior of the fields at the surface  $\Sigma$  only. This suggests that, for each region in the bulk, the energy-momentum tensor for  $\theta$  ED has the same form as that in standard electrodynamics, where the  $\tilde{\theta}$  dependence appears through the contribution to the fields arising from the additional sources present on the boundary  $\Sigma$ . In fact, the identification of the stress-energy tensor proceeds along the standard lines of electrodynamics in a medium (see, for example, Ref. [69]), where we read the rate at which the electric field does work on the free charges

$$\mathbf{J} \cdot \mathbf{E} = -\nabla \cdot \left( \frac{1}{4\pi} \mathbf{E} \times \mathbf{H} \right) - \frac{1}{4\pi} \left( \mathbf{E} \cdot \frac{\partial \mathbf{D}}{\partial t} + \mathbf{H} \cdot \frac{\partial \mathbf{B}}{\partial t} \right) \quad (22)$$

and the rate at which momentum is transferred to the charges

$$\begin{aligned} \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} &= -\frac{1}{4\pi} \frac{\partial}{\partial t} (\mathbf{D} \times \mathbf{B}) - \frac{1}{4\pi} [D_i \nabla E_i - \nabla \cdot (\mathbf{D}\mathbf{E})] \\ &\quad - \frac{1}{4\pi} [B_i \nabla H_i - \nabla \cdot (\mathbf{B}\mathbf{H})]. \end{aligned} \quad (23)$$

Here, the notation is  $A_i \nabla B_i = (A_i \partial_k B_i) \hat{e}_k$  and  $\nabla \cdot (\mathbf{A}\mathbf{B}) = (\partial_i A_i) B_k \hat{e}_k$  for any vectors  $\mathbf{A}$  and  $\mathbf{B}$ . Using the constitutive relations in Eq. (14), we recognize from Eq. (22) the energy flux  $\mathbf{S}$  and the energy density  $U$  as

$$\mathbf{S} = \frac{1}{4\pi} \mathbf{E} \times \mathbf{B}, \quad U = \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2), \quad (24)$$

while from Eq. (23), we obtain the momentum density  $\mathbf{G}$ , and we identify the stress tensor  $T_{ij}$  as

$$\begin{aligned} \mathbf{G} &= \frac{1}{4\pi} \mathbf{E} \times \mathbf{B}, \\ T_{ij} &= \frac{1}{8\pi} (\mathbf{E}^2 + \mathbf{B}^2) \delta_{ij} - \frac{1}{4\pi} (E_i E_j + B_i B_j). \end{aligned} \quad (25)$$

Outside the free sources, the conservation equations read

$$\nabla \cdot \mathbf{S} + \frac{\partial U}{\partial t} = 0, \quad \frac{\partial G_k}{\partial t} + \partial_i T_{ik} = \frac{\alpha}{\pi} (\mathbf{E} \cdot \mathbf{B}) \partial_k \theta(\mathbf{x}). \quad (26)$$

In other words, the stress-energy tensor has in fact the same form as in vacuum, but, as expected, it is not conserved on the  $\theta$  boundary because of the self-induced charge and current densities arising there.

In the standard case, the knowledge of both the stress-energy tensor together with the GF of the system is also relevant in the quantum situation. For example, when dealing with the calculation of Casimir energies, the

vacuum expectation value of the energy momentum can be obtained with the aid of the corresponding GF for Maxwell electrodynamics [70–72]. Using this method, insights regarding corrections to the Casimir energy in the context of  $\theta$  ED have been developed in Refs. [45,73], as well as Ref. [39] for a related perspective. This approach serves as an additional motivation for the following Sec. III.

### III. GREEN'S MATRIX METHOD

In this section, we use the GF method to solve boundary-value problems in  $\theta$  ED in terms of the electromagnetic potential  $A^\mu$ . Certainly, one could solve for the electric and magnetic fields from the modified Maxwell equations (9) through (12) together with the boundary conditions in Eqs. (15)–(18); however, just as in ordinary electrodynamics, there might be occasions where, besides the knowledge of the external sources, we are provided with requirements on the fields at the given boundaries. In these cases, the GF method provides the general solution to a given boundary-value problem (Dirichlet or Neumann) for arbitrary sources. The importance of such a general solution is evident from an experimental point of view. For example, it would allow one, at least in principle, to predict the electromagnetic response of topological insulators with planar, spherical or cylindrical boundaries in the presence of more intricate configuration of sources. Moreover, as already mentioned in Sec. II C, the GF method is useful for computing the vacuum expectation value of the energy-momentum tensor in the context of Casimir forces. Furthermore, the GF method should be also useful for the solution of dynamical problems in  $\theta$  ED.

In the following, we concentrate only on the static case. Since the homogeneous Maxwell equations that express the relationship between potentials and fields are not modified in  $\theta$  ED, the electrostatic and magnetostatic fields can be written in terms of the 4-potential  $A^\mu = (\phi, \mathbf{A})$  according to  $\mathbf{E} = -\nabla\phi$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ , as usual. In the Coulomb gauge  $\nabla \cdot \mathbf{A} = 0$ , the 4-potential satisfies the equations of motion

$$[-\eta^\mu{}_\nu \nabla^2 - \tilde{\theta}\delta(\Sigma)n_\alpha \epsilon^{\alpha\mu\beta}{}_\nu \partial_\beta]A^\nu = 4\pi j^\mu, \quad (27)$$

together with the boundary conditions (BCs)

$$\begin{aligned} \Delta A^\mu|_\Sigma &= 0, \\ \Delta(n^\alpha \partial_\alpha A^\mu)|_\Sigma &= -\tilde{\theta}n_\alpha \epsilon^{\alpha\mu\beta}{}_\nu \partial_\beta A^\nu|_\Sigma. \end{aligned} \quad (28)$$

Here,  $n_\alpha$  is the unit normal to  $\Sigma$ , which depends on the geometry of the  $\theta$  boundary. One can further verify that the BCs of Eq. (28) yield those obtained in Eqs. (16)–(18), starting from the modified Maxwell equations.

To obtain a general solution for the potentials  $\phi$  and  $\mathbf{A}$  in the presence of arbitrary external sources  $j^\mu(\mathbf{x})$ , we

introduce the GF matrix  $G^\mu{}_\nu(\mathbf{x}, \mathbf{x}')$ , solving Eq. (27) for a pointlike source,

$$[-\eta^\mu{}_\nu \nabla^2 - \tilde{\theta}\delta(\Sigma)n_\alpha \epsilon^{\alpha\mu\beta}{}_\nu \partial_\beta]G^\nu{}_\sigma(\mathbf{x}, \mathbf{x}') = 4\pi\eta^\mu{}_\sigma \delta^3(\mathbf{x} - \mathbf{x}'), \quad (29)$$

together with the BCs of Eq. (28). In the following, we discuss the general solution to Eq. (29). To this end, we require an appropriate adaptation of the standard Green's theorem, from which the solution of Eq. (29) can be constructed using well-known methods.

#### A. Green's theorem and boundary conditions on $\Sigma$

We begin this section by introducing the differential operator

$$\mathcal{O}^\mu{}_\nu{}^i = \eta^\mu{}_\nu \partial^i + \tilde{\theta}\delta(\Sigma)n_j \epsilon^{j\mu}{}_\nu{}^i, \quad (30)$$

from which the relations  $\mathcal{O}^\mu{}_\nu{}^i \partial_i A^\nu = 4\pi j^\mu$  and  $\mathcal{O}^\mu{}_\nu{}^i \partial_i G^\nu{}_\sigma = 4\pi\eta^\mu{}_\sigma \delta^3(\mathbf{x} - \mathbf{x}')$  correctly yield the differential equations for the 4-potential in Eq. (27) and the GF in Eq. (29). Here, the indexes  $i, j$  range from 1 to 3, while  $\mu, \nu$  range from 0 to 3. These choices reflect that we are working with static fields and that the normal to the  $\theta$  boundary is always spacelike.

The relevant Green's theorem can be given in terms of two arbitrary fields,  $X_\mu$  and  $Z^\mu{}_\nu$ . Defining the tensor  $T^i{}_\sigma = X_\mu \mathcal{O}^\mu{}_\nu{}^i Z^\nu{}_\sigma$ , using the divergence theorem for  $\partial_i T^i{}_\sigma$  and subtracting the equation arising from the interchange  $X \leftrightarrow Z$ , we find

$$\begin{aligned} \oint_S dS \mathbf{n}_i (X_\mu \mathcal{O}^\mu{}_\nu{}^i Z^\nu{}_\sigma - Z^\nu{}_\sigma \mathcal{O}^\mu{}_\nu{}^i X_\mu) \\ = \int_V d^3x [X_\mu \mathcal{O}^\mu{}_\nu{}^i (\partial_i Z^\nu{}_\sigma) - Z^\nu{}_\sigma \mathcal{O}^\mu{}_\nu{}^i (\partial_i X_\mu)] \\ - \int_V d^3x [(\partial_i X_\mu) \mathcal{O}^\mu{}_\nu{}^i Z^\nu{}_\sigma - (\partial_i Z^\nu{}_\sigma) \mathcal{O}^\mu{}_\nu{}^i X_\mu], \end{aligned} \quad (31)$$

where  $\mathbf{n}$  is the outward normal to the surface  $S$  bounding the volume  $V$ . In deriving Eq. (31), we use the result  $\partial_i \mathcal{O}^\mu{}_\nu{}^i = \mathcal{O}^\mu{}_\nu{}^i \partial_i$ , which follows directly from Eq. (30).

Substituting Eqs. (27) and (29) in Eq. (31), we find that the general solution for the 4-potential in the Coulomb gauge is

$$\begin{aligned} A^\mu(\mathbf{x}) &= \int_V d^3x' G^\mu{}_\nu(\mathbf{x}, \mathbf{x}') j^\nu(\mathbf{x}') \\ &+ \frac{1}{4\pi} \oint_S dS' \mathbf{n}_i [A_\nu(\mathbf{x}') \partial^i G^{\mu\nu}(\mathbf{x}, \mathbf{x}') \\ &- G^{\mu\nu}(\mathbf{x}, \mathbf{x}') \partial^i A_\nu(\mathbf{x}')] \\ &+ \frac{\tilde{\theta}}{4\pi} \int_\Sigma d^2\mathbf{x}'_\Sigma n_j \epsilon^{j\alpha}{}_\nu{}^i [A_\alpha(\mathbf{x}') \partial_i G^{\mu\nu}(\mathbf{x}, \mathbf{x}') \\ &- G^{\mu\nu}(\mathbf{x}, \mathbf{x}') \partial_i A_\alpha(\mathbf{x}')]|_\Sigma, \end{aligned} \quad (32)$$

where  $n_j$  is the normal to the  $\theta$  boundary. This result yields the standard interpretation where the first term is the contribution from the sources inside the volume  $V$ , the second term represents the effects of the bounding surface  $S$ , while the remaining term replaces the contributions from the surface  $S$  by those of the  $\theta$  boundary.

We next consider the issue of the appropriate BCs for the fields at the  $\theta$  interface, when we take  $S$  as a surface at infinity where the usual BCs are imposed. Inspection of Eq. (32) reveals that there are four classes of BCs on  $\Sigma$  that specify a solution.

The class BC-I is defined by fixing, on  $\Sigma$ , the scalar potential  $A^0$  and the vector potential parallel to the  $\theta$  boundary  $\mathbf{n} \times \mathbf{A}$ , together with the requirement  $n_j \epsilon^{j\alpha} \nu^i G^{\mu\nu}|_{\Sigma} = 0$  on  $\Sigma$ . This class describes the case which corresponds to the nearest analogy with the standard Dirichlet BCs. Besides, it is the only class for which the explicit GF is independent of the area of the  $\theta$  boundary. In Sec. IV, we solve the problem of a pointlike charge near a planar  $\theta$  boundary at fixed potential using class BC-I.

The class BC-II is specified by fixing on  $\Sigma$  the normal component of the magnetic field  $\mathbf{B}_n$  and the parallel component of the electric field  $\mathbf{n} \times \mathbf{E}$ , plus the condition  $n_j \epsilon^{j\alpha} \nu^i \partial_i G^{\mu\nu}|_{\Sigma} = 1/A_\theta$ , where  $A_\theta = \int d^2\mathbf{x}_\Sigma$  is the surface area of the  $\theta$  boundary. This class corresponds to the Neumann BCs in standard electrostatics, which incorporates a factor of the inverse surface area, generating a term in the solution involving the average contribution of the potential.

The class BC-III fixes the scalar potential  $A^0$  and the normal component of the magnetic field  $\mathbf{B}_n$  on  $\Sigma$ , together with the conditions  $n_j \epsilon^{j0} \nu^i G^{\mu k}|_{\Sigma} = 0$  and  $n_j \epsilon^{jk} \nu^i \partial_i G^{\mu 0}|_{\Sigma} = 1/A_\theta$ .

The class BC-IV requires specifying  $\mathbf{n} \times \mathbf{A}$  and  $\mathbf{n} \times \mathbf{E}$  on  $\Sigma$ . For this class, we must also demand  $n_j \epsilon^{jk} \nu^i G^{\mu 0}|_{\Sigma} = 0$  and  $n_j \epsilon^{j0} \nu^i \partial_i G^{\mu k}|_{\Sigma} = 1/A_\theta$ .

In the next subsections, we deal with the problem of constructing the GFs for different geometrical configurations of the  $\theta$  boundary, considering those which could be more relevant in experimental works, namely, planar, spherical and cylindrical  $\theta$  interfaces. Moreover, having set the surface  $S$  at infinity, with standard BCs there, we restrict ourselves to the case where we do not specify any additional condition for the fields at the  $\theta$  boundary. In this way,  $A^\mu$  is given only by the first term of the right-hand side in Eq. (32).

## B. Planar $\theta$ boundary

For the case of planar symmetry, we choose the simplest setup in which the value of  $\theta$  jumps across the plane  $\Sigma$  and remains constant at either side of  $\Sigma$ , defined by  $z = a$  and indicated in Fig. 2. In this way, the adapted coordinates to this system are the Cartesian ones. The GF we consider has translational invariance in the directions parallel to  $\Sigma$ , that is

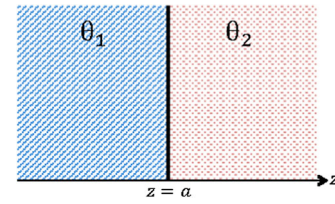


FIG. 2. Geometry of the semi-infinite planar  $\theta$ -region.

in the transverse directions  $x$  and  $y$ , but this invariance is broken in the direction  $z$ . Exploiting this symmetry, we further introduce the Fourier transform in the direction parallel to the plane  $\Sigma$ , taking the coordinate dependence to be  $(\mathbf{x} - \mathbf{x}')_{\parallel} = (x - x', y - y')$ , and define

$$G^\mu{}_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \int \frac{d^2\mathbf{p}}{(2\pi)^2} e^{i\mathbf{p} \cdot (\mathbf{x} - \mathbf{x}')_{\parallel}} g^\mu{}_\nu(z, z', \mathbf{p}), \quad (33)$$

where  $\mathbf{p} = (p_x, p_y)$  is the momentum parallel to the plane  $\Sigma$ . In the following, we suppress the dependence on  $\mathbf{p}$  of the reduced GF  $g^\mu{}_\nu$ . In this case, the reduced GF satisfies

$$[\partial^2 \eta^\mu{}_\nu + i\tilde{\theta} \delta(z - a) \epsilon^{3\mu\alpha} \nu p_\alpha] g^\mu{}_\nu(z, z') = \eta^\mu{}_\sigma \delta(z - z'), \quad (34)$$

where  $\partial^2 = \mathbf{p}^2 - \partial_z^2$ ,  $p^\alpha = (0, \mathbf{p})$  and  $\mathbf{p}^2 = -p^\alpha p_\alpha = p_x^2 + p_y^2$ . The solution of Eq. (34) is a simple, but not straight-forward, task. For solving it, we employ a method similar to that used for obtaining the GF for the one-dimensional  $\delta$ -function potential in quantum mechanics, where the free-particle GF is used for integrating the GF equation with the  $\delta$ -interaction. A main simplification arises in this case because what normally results in an integral equation reduces to an algebraic equation in virtue of the fact that the integration over the  $\delta$ -potential can be performed. To proceed in an analogous way, we consider the reduced free GF having the form  $\mathcal{G}^\mu{}_\nu(z, z') = \mathbf{g}(z, z') \eta^\mu{}_\nu$ , which solves the equation

$$\partial^2 \mathcal{G}^\mu{}_\nu(z, z') = \eta^\mu{}_\nu \delta(z - z'). \quad (35)$$

The details of the calculation for solving Eq. (34) were presented in Ref. [53]. For completeness, we remind the reader of the general solution

$$\begin{aligned} g^\mu{}_\nu(z, z') &= \eta^\mu{}_\nu \mathbf{g}(z, z') - \tilde{\theta} \frac{\mathbf{g}(z, a) \mathbf{g}(a, z')}{1 + p^2 \tilde{\theta}^2 \mathbf{g}^2(a, a)} \\ &\times \{ \tilde{\theta} \mathbf{g}(a, a) [p^\mu p_\nu + (\eta^\mu{}_\nu + n^\mu n_\nu) p^2] \\ &+ i \epsilon^\mu{}_\nu{}^{\alpha 3} p_\alpha \}, \end{aligned} \quad (36)$$

where  $n_\mu = (0, 0, 0, 1)$  is the normal to  $\Sigma$ .

The reciprocity between the position of the unit charge and the position at which the GF is evaluated



$G_{\mu\nu}(\mathbf{x}, \mathbf{x}') = G_{\nu\mu}(\mathbf{x}', \mathbf{x})$  is one of its most remarkable properties. From Eq. (33), this condition requires

$$g_{\mu\nu}(z, z', \mathbf{p}) = g_{\nu\mu}(z', z, -\mathbf{p}), \quad (37)$$

which we verify directly from Eq. (36). The symmetry  $g_{\mu\nu}(z, z') = g_{\nu\mu}^*(z, z') = g_{\mu\nu}^\dagger(z, z')$  is also manifest. The various components of the static GF matrix in coordinate representation were obtained by Fourier transforming the reduced GF, as defined in Eq. (33). In vacuum, with no additional boundaries, the reduced GF is  $\mathbf{g}(z, z') = e^{-p|z-z'|}/2p$ , and the corresponding GF matrix in coordinate representation is (see Ref. [53])

$$G^0_0(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{1}{\sqrt{R^2 + Z^2}}, \quad (38)$$

$$G^0_i(\mathbf{x}, \mathbf{x}') = -\frac{2\tilde{\theta}}{4 + \tilde{\theta}^2} \frac{\epsilon_{0ij3} R^j}{R^2} \left(1 - \frac{Z}{\sqrt{R^2 + Z^2}}\right), \quad (39)$$

$$G^i_j(\mathbf{x}, \mathbf{x}') = \eta^i_j G^0_0(\mathbf{x}, \mathbf{x}') - \frac{i}{2} \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \partial^i K_j(\mathbf{x}, \mathbf{x}'), \quad (40)$$

where  $Z = |z - a| + |z' - a|$ ,  $R^j = (\mathbf{x} - \mathbf{x}')_{\parallel}^j = (x - x', y - y')$ ,  $R = |(\mathbf{x} - \mathbf{x}')_{\parallel}|$  and

$$K^j(\mathbf{x}, \mathbf{x}') = 2i \frac{\sqrt{R^2 + Z^2} - Z}{R^2} R^j. \quad (41)$$

Finally, we observe that Eqs. (38)–(40) contain all the required elements of the GF matrix, according to the choices of  $z$  and  $z'$  in the function  $Z$ .

### C. Spherical $\theta$ boundary

In the preceding section, the problem of an arbitrary charge and current distributions in the presence of a plane  $\theta$  boundary was discussed by the method of GF matrix. In this section, following the same procedure as in the planar situation, we discuss the spherical case, in which the value of  $\theta$  has a discontinuity across the surface  $r = a$ . In the adapted spherical coordinates  $r, \vartheta, \varphi$ , it proves convenient to introduce explicitly the angular momentum operator  $\hat{\mathbf{L}} = \frac{1}{i} \mathbf{x} \times \nabla$ . In fact, the GF equation (29) can be written as

$$\left[ -\eta^\mu_\alpha \nabla^2 - i \frac{\tilde{\theta}}{a} \delta(r - a) (\eta^\mu_0 \eta^k_\alpha - \eta^{\mu k} \eta^0_\alpha) \hat{\mathbf{L}}_k \right] G^\alpha_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \eta^\mu_\nu \delta^3(\mathbf{x} - \mathbf{x}'), \quad (42)$$

with  $k = 1, 2, 3$ . Since the square of the angular momentum  $\hat{\mathbf{L}}^2$  commutes with the operator appearing in the left-hand side of Eq. (42), its solution has the form

$$G^\mu_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{m'=-l}^{+l} g_{lmm'\nu}^\mu(r, r') Y_{lm}(\vartheta, \varphi) Y_{l m'}^*(\vartheta', \varphi'), \quad (43)$$

with the reduced GF  $g_{lmm'\nu}^\mu(r, r')$  satisfying the equation

$$\hat{\mathcal{O}}_r g_{lmm'\nu}^\mu(r, r') = \eta^\mu_\nu \frac{\delta(r - r')}{r^2} \delta_{mm'} + i \frac{\tilde{\theta}}{a} \delta(r - a) (\eta^\mu_0 \eta^k_\alpha - \eta^{\mu k} \eta^0_\alpha) \times \sum_{m''=-l}^{+l} \langle lm | \hat{\mathbf{L}}_k | lm'' \rangle g_{l m'' m' \nu}^\alpha(r, r'), \quad (44)$$

where  $\hat{\mathcal{O}}_r = l(l+1)r^{-2} - r^{-2}\partial_r(r^2\partial_r)$ . This equation can be integrated in the same way as for the planar symmetry. The detailed calculation is presented in Appendix A. The solution is

$$g_{lmm'\nu}^\mu(r, r') = \eta^\mu_\nu \mathbf{g}_l(r, r') \delta_{mm'} - a^2 \tilde{\theta}^2 l(l+1) \mathbf{g}_l(a, a) \times S_l(r, r') \langle lm | \hat{\mathbf{L}}_\mu \hat{\mathbf{L}}_\nu | lm' \rangle + ia \tilde{\theta} S_l(r, r') \times \langle lm | \hat{\mathbf{L}}_\alpha | lm' \rangle (\eta^\mu_0 \Gamma^\alpha_\nu + \Gamma^{\mu\alpha} \eta^0_\nu), \quad (45)$$

where  $\hat{\mathbf{L}}_0$  is the identity operator, the operator  $\Gamma^{\mu\nu} = \eta^{\mu\nu} - \eta^\mu_0 \eta^\nu_0$  projects a 4-vector into the 3-space, and

$$S_l(r, r') = \frac{\mathbf{g}_l(r, a) \mathbf{g}_l(a, r')}{1 + a^2 \tilde{\theta}^2 l(l+1) \mathbf{g}_l^2(a, a)}. \quad (46)$$

Here,  $\mathbf{g}_l(r, r')$  is the solution of the free reduced GF equation in the absence of the  $\theta$  boundary which satisfies Eq. (A5).

### D. Cylindrical $\theta$ boundary

In this section, we discuss the problem of an arbitrary charge and current distributions in the presence of a cylindrical  $\theta$  boundary. Let us consider an infinite cylinder of which the axis lies along the  $z$  direction, such that the value of  $\theta$  has a discontinuity across its surface  $\rho = a$ , as shown in Fig. 4. In this way, the adapted coordinates are the cylindrical ones,  $\rho, \varphi, z$ , and the GF equation is

$$\left[ -\eta^\mu_\nu \nabla^2 - \tilde{\theta} \delta(\rho - a) n_\alpha \epsilon^{\alpha\mu\beta} \partial_\beta \right] G^\nu_\sigma(\mathbf{x}, \mathbf{x}') = 4\pi \eta^\mu_\sigma \delta(\mathbf{x} - \mathbf{x}'), \quad (47)$$

where  $n_\alpha = (0, \cos \varphi, \sin \varphi, 0)$  is the normal to the  $\theta$  interface. The GF must be invariant under translations in the  $z$  direction. In accordance with this symmetry, we start by writing the solution of Eq. (47) as

$$G^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}') = 4\pi \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \times \sum_{m'=-\infty}^{+\infty} g^{\mu}_{mm',\nu}(\rho, \rho'; k) e^{i(m\varphi - m'\varphi')} \quad (48)$$

with the reduced GF  $g^{\mu}_{mm',\nu}(\rho, \rho'; k)$  satisfying the equation

$$\hat{O}_{\rho}^{(m)} g^{\mu}_{mm',\sigma} - i\tilde{\theta}\delta(\rho - a) \left[ k \sum_{m''=-\infty}^{+\infty} A^{\mu}_{m''m,\nu} g^{\nu}_{m''m',\sigma} + \epsilon^{1\mu 2}_{\nu} \frac{m}{\rho} \delta_{mm'} g^{\nu}_{mm',\sigma} \right] = \eta^{\mu}_{\sigma} \frac{\delta(\rho - \rho')}{\rho} \delta_{mm'}, \quad (49)$$

where

$$\hat{O}_{\rho}^{(m)} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{m^2}{\rho^2} + k^2, \quad (50)$$

$$A^{\mu}_{mm'',\nu} = \frac{1}{2} [\delta_{m,m''-1} (\tilde{\epsilon}^{\mu}_{\nu})^* + \delta_{m,m''+1} \tilde{\epsilon}^{\mu}_{\nu}],$$

with  $\tilde{\epsilon}^{\mu}_{\nu} = \epsilon^{1\mu 3}_{\nu} + i\epsilon^{2\mu 3}_{\nu}$ . This equation can be integrated in the same way as for the planar and spherical cases. Detailed calculations are presented in Appendix B.

The solution is

$$g^0_{mm',\sigma}(\rho, \rho') = \eta^0_{\sigma} \delta_{mm'} [\mathbf{g}_m(\rho, \rho') - \tilde{\theta}^2 \mathbf{f}_m(k) C_{mm}(\rho, \rho')] + i\tilde{\theta} (m\delta_{mm'} \eta^3_{\sigma} + ka A^0_{m'm,\sigma}) C_{mm'}(\rho, \rho'), \quad (51)$$

$$g^3_{mm',\sigma}(\rho, \rho') = \eta^3_{\sigma} \delta_{mm'} [\mathbf{g}_m(\rho, \rho') - m^2 \tilde{\theta}^2 \mathbf{g}_m(a, a) C_{mm}(\rho, \rho')] + im\tilde{\theta} (\eta^0_{\sigma} + ika\tilde{\theta} A^0_{mm',\sigma}) C_{mm'}(\rho, \rho'), \quad (52)$$

$$g^j_{mm',j}(\rho, \rho') = \eta^j_{j} \delta_{mm'} \mathbf{g}_m(\rho, \rho') - \tilde{\theta}^2 k^2 a^2 \mathbf{g}_m(\rho, a) \times \sum_{m''=-\infty}^{+\infty} A^j_{m'm,0} A^0_{m''m',j} C_{m''m'}(a, \rho'), \quad (53)$$

where  $\mathbf{f}_m(k) = m^2 \mathbf{g}_m(a, a) + \frac{k^2 a^2}{2} [\mathbf{g}_{m+1}(a, a) + \mathbf{g}_{m-1}(a, a)]$  and

$$C_{mm'}(\rho, \rho') = \frac{\mathbf{g}_m(\rho, a) \mathbf{g}_{m'}(a, \rho')}{1 + \tilde{\theta}^2 \mathbf{f}_m(k) \mathbf{g}_m(a, a)}. \quad (54)$$

Here,  $\mathbf{g}_m(\rho, \rho')$  is the solution to the reduced GF equation (49) in the absence of the  $\theta$  boundary. Note that the remaining  $\mu\nu$ -components of the GF matrix can be obtained from the symmetry property

$$g_{mm',\mu\nu}(\rho, \rho'; k) = g_{m'm,\nu\mu}(\rho', \rho; -k). \quad (55)$$

## IV. APPLICATIONS

### A. Pointlike charge near a planar $\theta$ boundary at fixed potentials

Now, we deal with the problem of one or more point charges in the presence of a planar  $\theta$  boundary surface at fixed potential. This case falls under the BC-I class of boundary conditions at the  $\theta$  interface. In the planar case, these BCs reduce to demanding the full BC-I Green function  $(G_1)^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}')$  to be zero. As an example, let us consider a pointlike electric charge in front of an infinite planar  $\theta$  boundary at zero potentials  $(A^0, \mathbf{A}_{\parallel}) = (0, \mathbf{0})$  located at  $z = 0$ . No additional bounding surfaces are considered. As discussed in Ref. [53], the problem of a pointlike charge in front of an infinite planar  $\theta$  boundary is equivalent to the problem of the original charge, together with an electric charge and a magnetic monopole located at the mirror-image point behind the plane but with the  $\theta$  interface removed, i.e.,  $\tilde{\theta} = 0$ . These electric and magnetic images reproduce the boundary conditions in Refs. [15–18], induced by the nontrivial jump of the  $\theta$ -value.

The problem at hand can also be solved in terms of images. Following similar steps as in the standard case, the corresponding GF,  $(G_1)^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}')$ , can be constructed as

$$(G_1)^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}') = G^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}') + F^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}'), \quad (56)$$

with the addition of a matrix  $F^{\mu}_{\nu}$  which satisfies the homogeneous equation

$$[-\eta^{\mu}_{\nu} \nabla^2 - \tilde{\theta} \delta(z) \epsilon^{3\mu\alpha}_{\nu} \partial_{\alpha}] F^{\nu}_{\sigma}(\mathbf{x}, \mathbf{x}') = 0. \quad (57)$$

This freedom in the definition of the GF allow us to choose appropriately the matrix  $F^{\mu}_{\nu}$  in such a way that  $(G_1)^{\mu}_{\nu}(\mathbf{x}, \mathbf{x}') = 0$  for  $\mathbf{x}'$  on  $\Sigma$ . Let us consider a pointlike electric charge located at  $z' > 0$  and assume that  $z > 0$ . The required components of the GF matrix for solving this problem are

$$G^0_0(\mathbf{x}, \mathbf{x}') = \frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{1}{|\mathbf{x} - \mathbf{x}''|}, \quad (58)$$

$$G^0_i(\mathbf{x}, \mathbf{x}') = -\frac{2\tilde{\theta}}{4 + \tilde{\theta}^2} \frac{\epsilon_{0ij3}(\mathbf{x} - \mathbf{x}')^j}{R^2} \left( 1 - \frac{z + z'}{|\mathbf{x} - \mathbf{x}''|} \right), \quad (59)$$

where  $\mathbf{x}' = (x', y', z')$  denotes the position of the charge and  $\mathbf{x}'' = (x', y', -z')$  indicates the position of the images.

Regarding the problem of the BC-I Green function, we find that

$$\begin{aligned}
(G_1^0)^0(\mathbf{x}, \mathbf{x}') &= G^0_0(\mathbf{x}, \mathbf{x}') - G^0_0(\mathbf{x}, \mathbf{x}'') \\
&= \left(1 + \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2}\right) \left(\frac{1}{|\mathbf{x} - \mathbf{x}'|} - \frac{1}{|\mathbf{x} - \mathbf{x}''|}\right), \quad (60)
\end{aligned}$$

$$\begin{aligned}
(G_1^0)^i(\mathbf{x}, \mathbf{x}') &= G^0_i(\mathbf{x}, \mathbf{x}') - G^0_i(\mathbf{x}, \mathbf{x}'') \\
&= \frac{2\tilde{\theta}}{4 + \tilde{\theta}^2} \frac{\epsilon_{0ij3}(\mathbf{x} - \mathbf{x}')^j}{R^2} \left(\frac{z + z'}{|\mathbf{x} - \mathbf{x}''|} - \frac{z - z'}{|\mathbf{x} - \mathbf{x}'|}\right), \quad (61)
\end{aligned}$$

which effectively vanish at  $z' = 0$ . We interpret the fields as follows. The electric field in the region  $z > 0$  is generated by four electric charges: (i) the original charge of unit strength at  $z'$ , (ii) an equal and opposite charge located at the mirror-image point, (iii) an electric charge of strength  $-\tilde{\theta}^2/(4 + \tilde{\theta}^2)$  located at the mirror-image point and (iv) an electric charge of strength  $+\tilde{\theta}^2/(4 + \tilde{\theta}^2)$  located at  $z'$ . The magnetic field can be interpreted as that produced by two monopoles, one of strength  $2\tilde{\theta}/(4 + \tilde{\theta}^2)$  located at the mirror-image point, induced by the  $\theta$  boundary, and the other of strength  $-2\tilde{\theta}/(4 + \tilde{\theta}^2)$  at  $z'$ , arising from the BC  $\mathbf{A}_{\parallel}|_{\Sigma} = 0$ .

In a similar fashion, one can further check that the electric field in the region  $z < 0$  is due to an electric charge of strength  $-4/(4 + \tilde{\theta}^2)$  located at  $z'$  plus an electric charge of strength  $+4/(4 + \tilde{\theta}^2)$  located at  $-z'$ . The magnetic field can be interpreted as being generated by two magnetic monopoles, one of strength  $-2\tilde{\theta}/(4 + \tilde{\theta}^2)$  located at  $z'$  together with its image of strength  $+2\tilde{\theta}/(4 + \tilde{\theta}^2)$  located at  $-z'$ .

## B. Pointlike charge near a spherical $\theta$ boundary

The problem we shall discuss is that of a pointlike charge in vacuum ( $\theta_2 = 0$ ) near a spherical topological medium of radius  $a$  and  $\theta_1 \neq 0$ . For simplicity, we choose the line connecting the center of the sphere and the point charge as the  $z$ -axis. Thus, the current density can be written as  $j^\mu(\mathbf{x}') = \frac{q}{b^2} \eta^\mu_0 \delta(r' - b) \delta(\cos \vartheta' - 1) \delta(\varphi')$ , with  $b > a$ .

The solution for this problem is then

$$\phi(\mathbf{x}) = \int G^0_\mu(\mathbf{x}, \mathbf{x}') j^\mu(\mathbf{x}') d^3 \mathbf{x}' = q G^0_0(\mathbf{x}, \mathbf{b}), \quad (62)$$

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= \sum_{k=1}^3 \int G^k_\mu(\mathbf{x}, \mathbf{x}') j^\mu(\mathbf{x}') \hat{\mathbf{e}}_k d^3 \mathbf{x}' \\
&= \sum_{k=1}^3 q G^k_0(\mathbf{x}, \mathbf{b}) \hat{\mathbf{e}}_k, \quad (63)
\end{aligned}$$

where  $\mathbf{b} = b \hat{\mathbf{e}}_z$ . With the use of the corresponding components of the GF matrix, Eq. (45), the scalar and vector potentials are

$$\begin{aligned}
\phi(\mathbf{x}) &= 4\pi q \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{m'=-l}^{+l} [\mathbf{g}_l(r, b) - a^2 \tilde{\theta}^2 l(l+1) \\
&\quad \times \mathbf{g}_l(a, a) S_l(r, b)] \delta_{mm'} Y_{lm}(\vartheta, \varphi) Y_{lm'}^*(0, \varphi'), \quad (64)
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= 4\pi q \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \sum_{m'=-l}^{+l} ia \tilde{\theta} S_l(r, b) Y_{lm}(\vartheta, \varphi) \\
&\quad \times Y_{lm'}^*(0, \varphi') \langle lm | \hat{\mathbf{L}} | lm' \rangle. \quad (65)
\end{aligned}$$

From the relations  $Y_{lm}(0, \varphi) = \delta_{m0} \sqrt{\frac{2l+1}{4\pi}}$  and  $Y_{l0}(\vartheta, \varphi) = \sqrt{\frac{2l+1}{4\pi}} P_l(\cos \vartheta)$ , we find

$$\begin{aligned}
\phi(\mathbf{x}) &= q \sum_{l=0}^{\infty} [\mathbf{g}_l(r, b) - a^2 \tilde{\theta}^2 l(l+1) \mathbf{g}_l(a, a) S_l(r, b)] \\
&\quad \times (2l+1) P_l(\cos \vartheta), \quad (66)
\end{aligned}$$

$$\begin{aligned}
\mathbf{A}(\mathbf{x}) &= q \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} ia \tilde{\theta} \sqrt{4\pi(2l+1)} S_l(r, b) Y_{lm}(\vartheta, \varphi) \\
&\quad \times \langle lm | \hat{\mathbf{L}} | l0 \rangle, \quad (67)
\end{aligned}$$

which immediately yields  $A_z = 0$ . The remaining components of the vector potential can be calculated by introducing the combinations  $A_{\pm} = A_x \pm iA_y$ . The result is

$$A_{\pm}(\mathbf{x}) = q \sum_{l=0}^{\infty} ia \tilde{\theta} \sqrt{4\pi(2l+1)l(l+1)} S_l(r, b) Y_{l\pm 1}(\vartheta, \varphi), \quad (68)$$

where the expected symmetry  $A_+ = A_-^*$  follows from the relation  $Y_{l1}^* = -Y_{l-1}$ . Recalling that

$$\begin{aligned}
A_\theta &= -\sin \vartheta A_z + \frac{1}{2} \cos \vartheta (A_+ e^{-i\varphi} + A_- e^{i\varphi}), \\
A_\varphi &= \frac{1}{2i} (A_+ e^{-i\varphi} - A_- e^{i\varphi}), \quad (69)
\end{aligned}$$

we obtain

$$A_\theta = 0, \quad A_\varphi = q \sum_{l=0}^{\infty} a \tilde{\theta} (2l+1) S_l(r, b) \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta}, \quad (70)$$

in spherical coordinates.

Now, we analyze the field strengths for the regions 1,  $r > b > a$ , and 2,  $b > a > r$ . Using the free GF,  $\mathbf{g}_l(r, r') = \frac{1}{2l+1} \frac{r'_l}{r^{l+1}}$ , the scalar potential and the (nonzero component of the) vector potential for the region 1 take the form

$$\begin{aligned} \phi_{(1)}(\mathbf{x}) &= \frac{q}{|\mathbf{x} - \mathbf{b}|} - q \sum_{l=1}^{\infty} \frac{\tilde{\theta}^2 l(l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \\ &\quad \times \frac{a^{2l+1}}{r^{l+1} b^{l+1}} P_l(\cos \gamma), \end{aligned} \quad (71)$$

$$A_{(1)\varphi}(\mathbf{x}) = q \sum_{l=0}^{\infty} \frac{\tilde{\theta}(2l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \frac{a^{2l+1}}{r^{l+1} b^{l+1}} \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta}, \quad (72)$$

respectively. The corresponding electric and magnetic field strengths can be calculated directly from  $\mathbf{E} = -\nabla\phi$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ , respectively. The result is

$$\begin{aligned} \mathbf{E}_{(1)}(\mathbf{x}) &= q \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^3} - q \sum_{l=1}^{\infty} \frac{\tilde{\theta}^2 l(l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \\ &\quad \times \frac{a^{2l+1}}{r^{l+2} b^{l+1}} \left[ -(l+1) P_l(\cos \vartheta) \hat{\mathbf{r}} + \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta} \hat{\vartheta} \right], \end{aligned} \quad (73)$$

$$\begin{aligned} \mathbf{B}_{(1)}(\mathbf{x}) &= q \sum_{l=1}^{\infty} \frac{\tilde{\theta} l(2l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \frac{a^{2l+1}}{r^{l+2} b^{l+1}} \\ &\quad \times \left[ -(l+1) P_l(\cos \vartheta) \hat{\mathbf{r}} + \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta} \hat{\vartheta} \right]. \end{aligned} \quad (74)$$

We now ask what is the behavior of these field strengths when the separation between the point charge and the sphere is large compared to the radius of the sphere,  $b \gg a$ . Since the  $l$ th term in the sum behaves as  $(a/b)^{l+1}$ , only small values of  $l$  contribute. The leading contribution arises from  $l = 1$ , and we obtain

$$\mathbf{E}_{(1)}(\mathbf{x}) \sim q \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^3} + \frac{p}{r^3} (2 \cos \vartheta \hat{\mathbf{r}} + \sin \vartheta \hat{\vartheta}), \quad (75)$$

$$\mathbf{B}_{(1)}(\mathbf{x}) \sim \frac{m}{r^3} (2 \cos \vartheta \hat{\mathbf{r}} + \sin \vartheta \hat{\vartheta}), \quad (76)$$

which corresponds to the electric and magnetic fields generated by an electric dipole  $\mathbf{p}$  and a magnetic dipole  $\mathbf{m}$  lying at the origin and pointing in the  $z$  direction,

$$\mathbf{p} = \frac{2q\tilde{\theta}}{9 + 2\tilde{\theta}^2} \frac{a^3}{b^2} \hat{\mathbf{e}}_z = -\frac{2}{3} \mathbf{m}.$$

Next, we consider the field strengths in region 2:  $b > a > r$ . The scalar potential and the  $\varphi$ -component of the vector potential become

$$\begin{aligned} \phi_{(2)}(\mathbf{x}) &= \frac{q}{|\mathbf{x} - \mathbf{b}|} - q \sum_{l=1}^{\infty} \frac{\tilde{\theta}^2 l(l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \\ &\quad \times \frac{r^l}{b^{l+1}} P_l(\cos \gamma), \end{aligned} \quad (77)$$

$$A_{(2)\varphi}(\mathbf{x}) = q \sum_{l=0}^{\infty} \frac{\tilde{\theta}(2l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \frac{r^l}{b^{l+1}} \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta}, \quad (78)$$

respectively. The corresponding field strengths are

$$\begin{aligned} \mathbf{E}_{(2)}(\mathbf{x}) &= q \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^3} + q \sum_{l=1}^{\infty} \frac{\tilde{\theta}^2 l(l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \\ &\quad \times \frac{r^{l-1}}{b^{l+1}} \left[ l P_l(\cos \vartheta) \hat{\mathbf{r}} + \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta} \hat{\vartheta} \right], \end{aligned} \quad (79)$$

$$\begin{aligned} \mathbf{B}_{(2)}(\mathbf{x}) &= -q \sum_{l=1}^{\infty} \frac{\tilde{\theta} l(2l+1)(l+1)}{(2l+1)^2 + \tilde{\theta}^2 l(l+1)} \frac{r^{l-1}}{b^{l+1}} \\ &\quad \times \left[ l P_l(\cos \vartheta) \hat{\mathbf{r}} + \frac{\partial P_l(\cos \vartheta)}{\partial \vartheta} \hat{\vartheta} \right]. \end{aligned} \quad (80)$$

When the separation between the point charge and the sphere is large compared to the radius of the sphere,  $b \gg a$ , the field strengths in the region  $r < a$  become

$$\mathbf{E}_{(2)}(\mathbf{x}) \sim q \frac{\mathbf{x} - \mathbf{b}}{|\mathbf{x} - \mathbf{b}|^3} - \frac{1}{3} \mathbf{P}, \quad (81)$$

$$\mathbf{B}_{(2)}(\mathbf{x}) \sim \frac{2}{3} \mathbf{M}, \quad (82)$$

where

$$\mathbf{P} = \frac{6q\tilde{\theta}}{9 + 2\tilde{\theta}^2} \frac{1}{b^2} \hat{\mathbf{e}}_z = -\frac{2}{3} \mathbf{M}. \quad (83)$$

An interesting feature to note is the form of the field strengths in such region. The electric field behaves as the field produced by a uniformly polarized sphere with polarization  $\mathbf{P}$ , while the magnetic field resembles the one produced by a uniformly magnetized sphere with magnetization  $\mathbf{M}$ .

### C. Infinitely straight current-carrying wire near a cylindrical $\theta$ boundary

Let us consider an infinite straight wire parallel to the  $z$ -axis carrying a current  $I$  in the  $+z$  direction. The wire is located in vacuum ( $\theta_2 = 0$ ) at a distance  $b$  from the  $z$ -axis. Also, we assume a cylindrical nontrivial topological insulator of radius  $a < b$ , with its axis parallel to the wire and passing through the origin. For simplicity, we choose the

coordinates such that  $\varphi' = 0$ . Therefore, the current density is  $j^\mu(\mathbf{x}') = \frac{I}{b} \eta^\mu_3 \delta(\varphi') \delta(\rho' - b)$ . The solution for this problem is

$$\begin{aligned} A^\mu(\mathbf{x}) &= \int G^{\mu_3}(\mathbf{x}, \mathbf{x}') j^3(\mathbf{x}') d\mathbf{x}' \\ &= 2I \lim_{k \rightarrow 0} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} g_{mm',3}^\mu(\rho, b, k) e^{im\varphi}, \end{aligned} \quad (84)$$

where the various components of the reduced GF in cylindrical coordinates are given by Eqs. (51)–(53). With the use of the corresponding components of the GF matrix, the scalar and the (nonzero component of the) vector potential are

$$A^0(\mathbf{x}) = -4I\tilde{\theta} \lim_{k \rightarrow 0} \sum_{m=1}^{+\infty} m C_{mm}(\rho, b) \sin m\varphi, \quad (85)$$

$$\begin{aligned} A^3(\mathbf{x}) &= 2I \lim_{k \rightarrow 0} \left\{ g_0(\rho, b) + 2 \sum_{m=1}^{+\infty} [g_m(\rho, b) \right. \\ &\quad \left. - \tilde{\theta}^2 m^2 g_m(a, a) C_{mm}(\rho, b)] \cos m\varphi \right\}. \end{aligned} \quad (86)$$

The reduced GF in vacuum is  $g_m(\rho, \rho'; k) = I_m(k\rho_<) K_m(k\rho_>)$ , where  $I_m$  and  $K_m$  are the modified Bessel functions of the first and second kinds, respectively. Using the limiting form of the modified Bessel functions for small arguments [74], the limit  $k \rightarrow 0$  of the reduced GF becomes

$$\lim_{k \rightarrow 0} g_m(\rho, \rho'; k) = \frac{1}{2m} \left( \frac{\rho_<}{\rho_>} \right)^m, \quad (87)$$

and then  $\lim_{k \rightarrow 0} \mathbf{f}_m(k) = m/2$ . The potentials now become

$$A^0(\mathbf{x}) = -\frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \sum_{m=1}^{+\infty} \frac{1}{m} \left( \frac{a_<}{a_>} \right)^m \sin m\varphi, \quad (88)$$

$$\begin{aligned} A^3(\mathbf{x}) &= 2I \left\{ -\log \rho_> + \sum_{m=1}^{+\infty} \left[ \frac{1}{m} \left( \frac{b_<}{\rho_>} \right)^m \right. \right. \\ &\quad \left. \left. - \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{1}{m} \left( \frac{a_<}{\rho_>} \right)^m \right] \cos m\varphi \right\}, \end{aligned} \quad (89)$$

where the symbols  $>$  and  $<$  denote the greater and lesser in the ratio  $a/\rho$ . The summations can be performed analytically, with the result

$$A^0(\mathbf{x}) = \frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \arctan \left( \frac{\sin \varphi}{\cos \varphi - \frac{\rho_> b}{a_<}} \right), \quad (90)$$

$$\begin{aligned} A^3(\mathbf{x}) &= I \left\{ -\log [\rho^2 - 2b\rho \cos \varphi + b^2] \right. \\ &\quad \left. + \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \log \left[ 1 - 2 \frac{a_<}{\rho_>} \frac{a}{b} \cos \varphi + \left( \frac{a_<}{\rho_>} \frac{a}{b} \right)^2 \right] \right\}. \end{aligned} \quad (91)$$

Next, we analyze the field strengths for regions 1,  $\rho > b > a$ , and 2,  $b > a > \rho$ .

In region 1, the potentials take the form

$$A^0_{(1)}(\mathbf{x}) = \frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \arctan \left( \frac{\sin \varphi}{\cos \varphi - \frac{a}{d}} \right). \quad (92)$$

$$\begin{aligned} A^3_{(1)}(\mathbf{x}) &= I \left[ -\log (\rho^2 + b^2 - 2b\rho \cos \varphi) \right. \\ &\quad \left. + \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \log \left( 1 - 2 \frac{d}{\rho} \cos \varphi + \frac{d^2}{\rho^2} \right) \right], \end{aligned} \quad (93)$$

where  $d = a^2/b$ . The corresponding electric and magnetic field can be calculated directly as  $\mathbf{E} = -\nabla A^0$  and  $\mathbf{B} = \nabla \times \mathbf{A}$ , respectively. The result is

$$\begin{aligned} \mathbf{E}_{(1)}(\mathbf{x}) &= -\frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \left[ \frac{d \sin \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \hat{\rho} \right. \\ &\quad \left. + \left( \frac{1}{\rho} - \frac{\rho - d \cos \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right) \hat{\varphi} \right], \\ \mathbf{B}_{(1)}(\mathbf{x}) &= \hat{\rho} \left[ \frac{-2Ib \sin \varphi}{\rho^2 + b^2 - 2b\rho \cos \varphi} \right. \\ &\quad \left. + \frac{2I\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{d \sin \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right] \\ &\quad + \hat{\varphi} \left[ \frac{2I(\rho - b \cos \varphi)}{\rho^2 + b^2 - 2b\rho \cos \varphi} \right. \\ &\quad \left. + \frac{2I\tilde{\theta}^2}{4 + \tilde{\theta}^2} \left( \frac{1}{\rho} - \frac{\rho - d \cos \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right) \right]. \end{aligned} \quad (94)$$

The fields can be interpreted as follows. The magnetic field corresponds to that generated by the wire with current  $I$ , plus two image currents: one of strength  $I\tilde{\theta}^2/(4 + \tilde{\theta}^2)$  located at the origin and the other of strength  $-I\tilde{\theta}^2/(4 + \tilde{\theta}^2)$  located at  $d = a^2/b$ .

In region 2, the potentials are

$$A^0_{(2)}(\mathbf{x}) = \frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \arctan \left( \frac{\sin \varphi}{\cos \varphi - \frac{b}{\rho}} \right), \quad (95)$$

$$A^3_{(2)}(\mathbf{x}) = \frac{4I}{4 + \tilde{\theta}^2} \log (\rho^2 + b^2 - 2b\rho \cos \varphi). \quad (96)$$

The fields can be calculated directly. The result is

$$\mathbf{E}_{(2)}(\mathbf{x}) = \frac{4I\tilde{\theta}}{4 + \tilde{\theta}^2} \left[ \frac{b \sin \varphi}{\rho^2 + b^2 - 2b\rho \cos \varphi} \hat{\rho} - \frac{\rho - b \cos \varphi}{\rho^2 + b^2 - 2b\rho \cos \varphi} \hat{\varphi} \right], \quad (97)$$

$$\mathbf{B}_{(2)}(\mathbf{x}) = \frac{4I}{4 + \tilde{\theta}^2} \left[ -\frac{2b \sin \varphi}{\rho^2 + b^2 - 2b\rho \cos \varphi} \hat{\rho} + \frac{2(\rho - b \cos \varphi)}{\rho^2 + b^2 - 2b\rho \cos \varphi} \hat{\varphi} \right]. \quad (98)$$

The magnetic field in this region corresponds to the one produced by a current  $4I/(4 + \tilde{\theta}^2)$  located at  $b$ .

#### D. Infinitely uniformly charged wire near a cylindrical $\theta$ boundary

Let us consider an infinite straight wire which carries the uniform charge per unit length  $\lambda$ . The wire is placed parallel to the  $z$ -axis and is located in vacuum ( $\theta_2 = 0$ ) at a distance  $b$  from the  $z$ -axis. Also, we assume a cylindrical nontrivial topological insulator of radius  $a < b$ . Again, we choose the coordinates such that  $\varphi' = 0$ . Therefore, the current density is  $j^\mu(\mathbf{x}') = \frac{\lambda}{b} \eta^\mu_0 \delta(\varphi') \delta(\rho' - b)$ . The solution for this problem is

$$\begin{aligned} A^\mu(\mathbf{x}) &= \int G^\mu_0(\mathbf{x}, \mathbf{x}') j^0(\mathbf{x}') d\mathbf{x}' \\ &= 2\lambda \lim_{k \rightarrow 0} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} g_{mm',0}^\mu(\rho, b, k) e^{im\varphi}. \end{aligned} \quad (99)$$

The nonzero components can be calculated in the same way as in the previous example. The final result is

$$\begin{aligned} A^0(\mathbf{x}) &= \lambda \left\{ -\log[\rho^2 - 2b\rho \cos \varphi + b^2] \right. \\ &\quad \left. + \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \log \left[ 1 - 2 \frac{a_{<} a}{\rho_{>} b} \cos \varphi + \left( \frac{a_{<} a}{\rho_{>} b} \right)^2 \right] \right\}, \end{aligned} \quad (100)$$

$$A^3(\mathbf{x}) = \frac{4\lambda\tilde{\theta}}{4 + \tilde{\theta}^2} \arctan \left( \frac{\sin \varphi}{\cos \varphi - \frac{\rho_{>} b}{a_{<} a}} \right). \quad (101)$$

Now, we analyze the field strengths for regions 1,  $\rho > b > a$ , and 2,  $b > a > \rho$ . In region 1, the potentials take the form

$$\begin{aligned} A^0_{(1)}(\mathbf{x}) &= \lambda \left[ -\log[\rho^2 - 2b\rho \cos \varphi + b^2] \right. \\ &\quad \left. + \frac{\tilde{\theta}^2}{4 + \tilde{\theta}^2} \log \left( 1 - 2 \frac{d}{\rho} \cos \varphi + \frac{d^2}{\rho^2} \right) \right], \end{aligned} \quad (102)$$

$$A^3_{(1)}(\mathbf{x}) = \frac{4\lambda\tilde{\theta}}{4 + \tilde{\theta}^2} \arctan \left( \frac{\sin \varphi}{\cos \varphi - \frac{\rho}{d}} \right), \quad (103)$$

where  $d = a^2/b$ . The corresponding electric and magnetic fields can be calculated as usual. In region 1, the result is

$$\begin{aligned} \mathbf{E}_{(1)}(\mathbf{x}) &= \left[ \frac{2\lambda(\rho - b \cos \varphi)}{\rho^2 + b^2 - 2\rho b \cos \varphi} \right. \\ &\quad \left. + \frac{2\lambda\tilde{\theta}^2}{4 + \tilde{\theta}^2} \left( \frac{1}{\rho} - \frac{\rho - d \cos \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right) \right] \hat{\rho} \\ &\quad + \left( \frac{2\lambda b \sin \varphi}{\rho^2 + b^2 - 2\rho b \cos \varphi} \right. \\ &\quad \left. - \frac{2\lambda\tilde{\theta}^2}{4 + \tilde{\theta}^2} \frac{d \sin \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right) \hat{\varphi}, \end{aligned} \quad (104)$$

$$\begin{aligned} \mathbf{B}_{(1)}(\mathbf{x}) &= \frac{4\lambda\tilde{\theta}}{4 + \tilde{\theta}^2} \left[ \left( \frac{1}{\rho} - \frac{\rho - d \cos \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \right) \hat{\rho} \right. \\ &\quad \left. - \frac{d \sin \varphi}{\rho^2 + d^2 - 2d\rho \cos \varphi} \hat{\varphi} \right]. \end{aligned} \quad (105)$$

In region 2, the result is

$$\begin{aligned} \mathbf{E}_{(2)}(\mathbf{x}) &= \frac{4\lambda}{4 + \tilde{\theta}^2} \left[ \frac{2(\rho - b \cos \varphi)}{\rho^2 + b^2 - 2\rho b \cos \varphi} \hat{\rho} \right. \\ &\quad \left. + \frac{2b \sin \varphi}{\rho^2 + b^2 - 2\rho b \cos \varphi} \hat{\varphi} \right], \end{aligned} \quad (106)$$

$$\begin{aligned} \mathbf{B}_{(2)}(\mathbf{x}) &= \frac{4\lambda\tilde{\theta}}{4 + \tilde{\theta}^2} \left( \hat{\rho} \frac{\rho - b \cos \varphi}{\rho^2 + b^2 - 2\rho b \cos \varphi} \right. \\ &\quad \left. + \hat{\varphi} \frac{b \sin \varphi}{\rho^2 + b^2 - 2\rho b \cos \varphi} \right). \end{aligned} \quad (107)$$

#### V. SUMMARY

In this paper, we have considered an appealing topological extension of Maxwell electrodynamics which constitutes a low-energy effective theory to study the response of topological insulators. The model is defined by supplementing the Lagrange density of classical electrodynamics in 3 + 1 spacetime dimensions with the  $U(1)$  Pontryagin invariant coupled to a scalar field  $\theta$ . We take the field  $\theta$  as an external prescribed quantity that is a function of space. This coupling violates Lorentz, parity and time-reversal symmetries, while preserving gauge invariance. We have restricted to the case where  $\theta$  is piecewise constant in two different regions of space separated by a common interface  $\Sigma$ , denoted also as the  $\theta$  boundary. Nevertheless, our methods can be directly generalized to include additional spatial interfaces. The related problem of considering

timelike interfaces is interesting on its own, but it lies outside the scope of this work and can be dealt with elsewhere. This would model a system with  $\theta = \theta(t)$  rather than  $\theta(\mathbf{x})$ . One can anticipate that to tackle this problem correctly, a fully dynamical theory would be necessary. The  $\theta$ -value can be thought of as an effective parameter characterizing the properties of a novel electromagnetic media, possibly arising from a more fundamental theory of matter, which encodes the effect of novel quantum degrees of freedom. We have referred to this model as  $\theta$ -electrodynamics ( $\theta$  ED). In this scenario, the field equations in the bulk remain the standard Maxwell equations, but the discontinuity of  $\theta$  at the surface  $\Sigma$  alters the behavior of the fields, as shown in Eq. (6), giving rise to magnetoelectric effects such as a nontrivial Faraday- and Kerr-like rotation of the plane of polarization of electromagnetic waves traversing the interface  $\Sigma$  as analyzed in Refs. [42–44]. In a preceding work [53], we introduced the Green’s function method in static  $\theta$  ED, and we provided the calculation of the Green’s function for a planar  $\theta$  boundary, which is summarized here in Eqs. (38)–(40), for completeness. As a first application, we tackled the problem of a pointlike electric charge located near the planar  $\theta$  interface, and we recovered the results of Ref. [41] which were obtained with the use of the method of images. However, the Green’s function approach is far more general given that it is well suited to deal with the calculation of electric and magnetic fields arising from arbitrary sources. The force between the charge and the  $\theta$  boundary was also computed by two different methods: (i) we used the Green’s function to calculate the interaction energy between a charge-current distribution and the  $\theta$  boundary, with the vacuum energy removed; alternatively, (ii) we arrived at the same result by considering the momentum flux perpendicular to the interface in terms of the stress-energy tensor. The problem of an infinitely straight current-carrying wire near a planar  $\theta$  boundary was also discussed in detail.

In this work we extend our method to compute the static Green’s function for a  $\theta$  boundary with spherical and cylindrical geometries, given the fact that those geometries seems to be relevant for a large number of experimental settings. The results are presented in Eq. (45) for the spherical case and Eqs. (51)–(53) for the cylindrical case. Prior to this, we have dealt with some important structural aspects of classical static electromagnetic theory, namely the issue of the possibility of having stable equilibrium due to electromagnetic forces only and the correct construction of the stress-energy tensor to further analyze the status of conservation laws. Regarding the former, in Eq. (21), we have shown that for the case of  $\theta$  ED, points of stable equilibrium are not *a priori* forbidden, at least not

in a trivial way as in ordinary electromagnetism. Thus, TIs as modeled by  $\theta$  interfaces have a chance to circumvent Earnshaw’s theorem. With respect to the stress-energy tensor, we found that the energy density, energy flux, momentum density and the stress tensor are defined in the usual way; however, the ensuing conservation laws reveal a nonconservation of the stress-energy tensor on the  $\theta$  boundary, which in retrospect is not unexpected since the mere existence of the boundary breaks translational symmetry along the direction perpendicular to it. Also, as shown in Eq. (32), we extend the Green’s theorem to  $\theta$  ED, and we classify the boundary conditions that can be imposed on  $\Sigma$  in four different classes. Class I makes contact with the Dirichlet boundary-value problem in standard electrostatics, while the remaining classes yield boundary conditions depending on the surface area of the  $\theta$  boundary. These are the most important results of our work, since they allow one to obtain the corresponding static electric and magnetic fields for arbitrary sources and arbitrary boundary conditions in the given geometries. Also, the method provides a well-defined starting point for either analytical or numerical approximations in the cases where the exact analytical calculations are not possible. As an illustration of the extended Green’s theorem, we analyze the problem of a pointlike electric charge near a grounded planar  $\theta$  boundary, i.e., having zero scalar potential, together with zero parallel components of the vector potential. In this case, the boundary conditions imply that the Green’s function is zero at the  $\theta$  boundary. In close analogy with the standard case, this GF is subsequently constructed starting from the original plane symmetric GF given in Eqs. (38)–(40), by adding a homogeneous solution of Eq. (29) in order to fulfill the boundary conditions. In this simple situation, the method of images allows one to readily identify these solutions, and the final configuration can be interpreted in terms of suitable image charges and induced magnetic monopoles. Regarding the interpretation of the solution as due to image charges and image magnetic monopoles, let us recall that these images are just artifacts. In fact, the physical situation under study is mimicked by a hypothetical one with the same physical sources with the  $\theta$  boundary removed plus the fictitious sources (image charges and monopoles) to ensure that the boundary conditions of the fields are met at the location of the boundary. The appearance of magnetic monopoles in this solution seems to violate the Maxwell law  $\nabla \cdot \mathbf{B} = 0$ , which remained unaltered in the case of  $\theta$  ED. However, this is not the case. In fact, given  $(\mathbf{x} \pm \mathbf{r})/|\mathbf{x} \pm \mathbf{r}|^3 \sim \nabla_x(1/|\mathbf{x} \pm \mathbf{r}|)$ , we have  $\nabla \cdot \mathbf{B} \sim \nabla_x^2(1/|\mathbf{x} \pm \mathbf{r}|) \sim \delta(\mathbf{x} \pm \mathbf{r})$  in a region where  $\mathbf{x} \neq \pm \mathbf{r}$ . Physically, the magnetic field is induced by a surface current density  $\mathbf{J} = \tilde{\theta}\delta(z)\mathbf{E} \times \mathbf{n}$  that is circulating around the origin.

Additional applications include the use of the spherical Green's function to analyze the problem of a pointlike charge near a spherical  $\theta$  boundary, while the cylindrical Green's function allows for the calculation of the fields produced by an infinitely current-carrying wire and by a uniformly charged wire, both near a cylindrical  $\theta$  boundary and parallel to its axis.

The Green's function method should also be useful for the extension to the dynamical case. In this respect, to our knowledge, little efforts have been done in the context of topological insulators. Furthermore, Green's functions are also relevant for the computation of other effects, such as the Casimir effect.

The method here expanded and initiated in Ref. [53], when applied to specific configurations representing given experimental setups, predicts results that coincide with those in the previously existing literature. Our method, however, enjoys a certain generality in the sense that can be applied to a more intricate configuration of sources, in which case, for example, the method of images can be more cumbersome.

### ACKNOWLEDGMENTS

L. F. U. acknowledges J. Zanelli for introducing him to the  $\theta$ -theories. M. C. has been supported by the project FONDECYT (Chile) Initiation into Research Grant No. 11121633 and also wants to thank Instituto de Ciencias Nucleares, UNAM, for the kind hospitality. L. F. U. has been supported in part by Project No. IN104815 from Dirección General Asuntos del Personal Académico (Universidad Nacional Autónoma de México) and CONACyT (México), Project No. 237503. L. F. U. and A. M. R. thank Universidad Andres Bello for the warm hospitality.

### APPENDIX A: GF FOR A SPHERICAL $\theta$ BOUNDARY

In this section, we construct the GF in spherical coordinates for the configuration shown in Fig. 3 where the  $\theta$  boundary is the surface of a sphere of radius  $a$  with center at the origin. Here, the adapted coordinate system is provided by spherical coordinates. The various components of the GF are the solution of

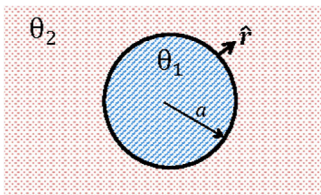


FIG. 3. Spherical region.

$$\left[ -\eta^\mu_\alpha \nabla^2 - i \frac{\tilde{\theta}}{a} \delta(r-a) (\eta^\mu_0 \eta^k_\alpha - \eta^{\mu k} \eta^0_\alpha) \hat{\mathbf{L}}_k \right] G^\alpha_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \eta^\mu_\nu \delta(\mathbf{x} - \mathbf{x}'), \quad (\text{A1})$$

with  $k = 1, 2, 3$ . Here,  $\hat{\mathbf{L}}_k$  are the components of the angular momentum operator. Since the completeness relation for the spherical harmonics is

$$\delta(\cos \vartheta - \cos \vartheta') \delta(\varphi - \varphi') = \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\vartheta, \varphi) Y_{lm}^*(\vartheta', \varphi'), \quad (\text{A2})$$

we look for a solution of the form

$$G^\mu_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \sum_{l=0}^{\infty} \sum_{l'=0}^{\infty} \sum_{m=-l}^{+l} \sum_{m'=-l'}^{+l'} g_{ll'mm',\nu}^\mu(r, r') \times Y_{lm}(\vartheta, \varphi) Y_{l'm'}^*(\vartheta', \varphi'), \quad (\text{A3})$$

where  $g_{ll'mm',\nu}^\mu(r, r')$  is the reduced GF analogous to  $g^\mu_\nu(z, z')$  in the case of planar symmetry. The operator in the left-hand side of Eq. (A1) commutes with  $\hat{\mathbf{L}}^2$  in such a way that

$$g_{ll'mm',\nu}^\mu(r, r') = \delta_{ll'} g_{lmm',\nu}^\mu(r, r'). \quad (\text{A4})$$

In the limiting case  $\tilde{\theta} \rightarrow 0$ , the matrix elements take the simple form  $g_{lm,\nu}^\mu(r, r') = \eta^\mu_\nu g_l(r, r')$ , where  $g_l(r, r')$  solves the equation

$$\hat{\mathcal{O}}_r g_l(r, r') = \frac{\delta(r - r')}{r^2}, \quad (\text{A5})$$

with the radial operator being

$$\hat{\mathcal{O}}_r = \frac{l(l+1)}{r^2} - \frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial}{\partial r} \right). \quad (\text{A6})$$

The solution to Eq. (A5) for different configurations is well known (see for example Ref. [69]). In free space, with boundary conditions at infinity, the solution is

$$g_l(r, r') = \frac{r^l_{<} - 1}{r^{l+1}_{>} - 1}, \quad (\text{A7})$$

where  $r_>$  ( $r_<$ ) is the greater (lesser) of  $r$  and  $r'$ . The substitution of Eq. (A7) into Eq. (A3) correctly reproduces the well-known result  $|\mathbf{x} - \mathbf{x}'|^{-1}$ .

In the following, we focus on determining the various components of the GF matrix in Eq. (A1). The method we shall employ is similar to that used for solving the planar case, but the required mathematical techniques are more



subtle because of the dependence upon the angular momentum operator.

Substituting Eq. (A3) into Eq. (A1) and using  $-\nabla^2 \rightarrow \hat{\mathcal{O}}_r$  gives

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left[ \eta_{\alpha}^{\mu} \hat{\mathcal{O}}_r - i \frac{\tilde{\theta}}{a} \delta(r-a) (\eta_{\alpha}^{\mu} \eta_{\alpha}^k - \eta^{\mu k} \eta_{\alpha}^0) \hat{\mathbf{L}}_k \right] \\ & \times g_{lmm',\nu}^{\alpha}(r, r') Y_{lm}(\vartheta, \varphi) Y_{lm'}^*(\vartheta', \varphi') \\ & = \eta_{\nu}^{\mu} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\vartheta, \varphi) Y_{lm'}^*(\vartheta', \varphi') \frac{\delta(r-r')}{r^2} \delta_{mm'}. \quad (\text{A8}) \end{aligned}$$

The linear independence of the spherical harmonics  $Y_{lm'}^*(\vartheta', \varphi')$  yields

$$\begin{aligned} & \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} \left[ \eta_{\alpha}^{\mu} \hat{\mathcal{O}}_r - i \frac{\tilde{\theta}}{a} \delta(r-a) (\eta_{\alpha}^{\mu} \eta_{\alpha}^k - \eta^{\mu k} \eta_{\alpha}^0) \hat{\mathbf{L}}_k \right] \\ & \times g_{lmm',\nu}^{\alpha}(r, r') Y_{lm}(\vartheta, \varphi) \\ & = \eta_{\nu}^{\mu} \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} Y_{lm}(\vartheta, \varphi) \frac{\delta(r-r')}{r^2} \delta_{mm'}. \quad (\text{A9}) \end{aligned}$$

Next, we multiply Eq. (A9) to the left by  $Y_{l''m''}^*(\vartheta, \varphi)$  and integrate over the solid angle  $d\Omega(\vartheta, \varphi)$ . After using the properties of the spherical harmonics,

$$\begin{aligned} \langle l''m'' | lm \rangle &= \delta_{l''l} \delta_{m''m}, \\ \langle l''m'' | \hat{\mathbf{L}}_k | lm \rangle &= \delta_{l''l} \langle l''m'' | \hat{\mathbf{L}}_k | l'm \rangle, \quad (\text{A10}) \end{aligned}$$

where

$$\langle lm | \hat{\mathbf{L}}_k | lm'' \rangle = \int_{\Omega} Y_{lm}^*(\vartheta, \varphi) \hat{\mathbf{L}}_k Y_{lm''}(\vartheta, \varphi) d\Omega, \quad (\text{A11})$$

we obtain

$$\begin{aligned} & \hat{\mathcal{O}}_r g_{lmm',\nu}^{\mu}(r, r') - \eta_{\nu}^{\mu} \frac{\delta(r-r')}{r^2} \delta_{mm'} \\ & = i \frac{\tilde{\theta}}{a} \delta(r-a) (\eta_{\alpha}^{\mu} \eta_{\alpha}^k - \eta^{\mu k} \eta_{\alpha}^0) \\ & \times \sum_{m''=-l}^{+l} \langle lm | \hat{\mathbf{L}}_k | lm'' \rangle g_{lm''m',\nu}^{\alpha}(r, r'), \quad (\text{A12}) \end{aligned}$$

where we have relabeled  $l'' \rightarrow l$  and  $m'' \leftrightarrow m$ .

The resulting equation can be integrated using the free reduced GF  $\mathbf{g}_l(r, r')$ , satisfying Eq. (A5), with the result

$$\begin{aligned} g_{lmm',\nu}^{\mu}(r, r') &= \eta_{\nu}^{\mu} \mathbf{g}_l(r, r') \delta_{mm'} \\ & + ia \tilde{\theta} (\eta_{\alpha}^{\mu} \eta_{\alpha}^k - \eta^{\mu k} \eta_{\alpha}^0) \mathbf{g}_l(r, a) \\ & \times \sum_{m''=-l}^{+l} g_{lm''m',\nu}^{\alpha}(a, r') \langle lm | \hat{\mathbf{L}}_k | lm'' \rangle. \quad (\text{A13}) \end{aligned}$$

Now, we have to solve for the various components. To this end, we split Eq. (A13) into the components  $\mu = 0$  and  $\mu = k$ ;

$$\begin{aligned} g_{lmm',\nu}^0(r, r') &= \eta_{\nu}^0 \mathbf{g}_l(r, r') \delta_{mm'} \\ & + ia \tilde{\theta} \mathbf{g}_l(r, a) \sum_{k=1}^3 \sum_{m''=-l}^{+l} g_{lm''m',\nu}^k \\ & \times (a, r') \langle lm | \hat{\mathbf{L}}_k | lm'' \rangle, \quad (\text{A14}) \end{aligned}$$

$$\begin{aligned} g_{lmm',\nu}^k(r, r') &= \eta_{\nu}^k \mathbf{g}_l(r, r') \delta_{mm'} + ia \tilde{\theta} \mathbf{g}_l(r, a) \\ & \times \sum_{m''=-l}^{+l} g_{lm''m',\nu}^0(a, r') \langle lm | \hat{\mathbf{L}}_k | lm'' \rangle, \quad (\text{A15}) \end{aligned}$$

where the second term in the right-hand side produces the coupling between the two types of components. Now, we set  $r = a$  in Eq. (A15) and then substitute into Eq. (A14) yielding

$$\begin{aligned} g_{lmm',\nu}^0(r, r') &= \eta_{\nu}^0 \mathbf{g}_l(r, r') \delta_{mm'} \\ & + ia \tilde{\theta} \mathbf{g}_l(r, a) \mathbf{g}_l(a, r') \langle lm | \eta_{\nu}^k \hat{\mathbf{L}}_k | lm' \rangle \\ & - a^2 \tilde{\theta}^2 l(l+1) \mathbf{g}_l(a, a) \mathbf{g}_l(r, a) g_{lmm',\nu}^0(a, r'), \quad (\text{A16}) \end{aligned}$$

where we have used the result

$$\begin{aligned} & \sum_{k=1}^3 \sum_{m''=-l}^{+l} \langle lm | \hat{\mathbf{L}}_k | lm' \rangle \langle lm' | \hat{\mathbf{L}}_k | lm'' \rangle \\ & = \sum_{k=1}^3 \langle lm | \hat{\mathbf{L}}_k^2 | lm'' \rangle = l(l+1) \delta_{mm''}. \quad (\text{A17}) \end{aligned}$$

Now, we set  $r = a$  in Eq. (A16) and solve it for  $g_{lmm',\nu}^0(a, r')$ , obtaining

$$\begin{aligned} g_{lmm',\nu}^0(a, r') &= \frac{\eta_{\nu}^0 \delta_{mm'} + ia \tilde{\theta} \mathbf{g}_l(a, a) \sum_{k=1}^3 \langle lm | \eta_{\nu}^k \hat{\mathbf{L}}_k | lm' \rangle}{1 + a^2 \tilde{\theta}^2 l(l+1) \mathbf{g}_l^2(a, a)} \\ & \times \mathbf{g}_l(a, r'), \quad (\text{A18}) \end{aligned}$$

which we insert back into Eq. (A16) with the final result

$$g_{lmm',\nu}^0(r, r') = \eta^0_\nu \delta_{mm'} [\mathfrak{g}_l(r, r') - a^2 \tilde{\theta}^2 l(l+1) S_l(r, r')] + ia \tilde{\theta} S_l(r, r') \langle lm | \hat{\mathbf{L}}_\nu^k | lm' \rangle, \quad (\text{A19})$$

where the function  $S_l(r, r')$  was defined in Eq. (46).

The remaining components now can be computed directly. The substitution of Eq. (A18) into Eq. (A15) produces

$$g_{lmm',\nu}^k(r, r') = \eta^\mu_\nu \mathfrak{g}_l(r, r') \delta_{mm'} + ia \tilde{\theta} S_l(r, r') \times \eta^0_\nu \langle lm | \hat{\mathbf{L}}_k | lm' \rangle - a^2 \tilde{\theta}^2 l(l+1) \mathfrak{g}_l(a, a) \times S_l(r, r') \langle lm | \eta^r_\nu \hat{\mathbf{L}}_k \hat{\mathbf{L}}_r | lm' \rangle. \quad (\text{A20})$$

One can further check that  $g_{lmm',k}^0(r, r') = g_{lmm',0}^k(r, r')$ . Thus, the general solution can be written in a compact way as

$$g_{lmm',\nu}^\mu(r, r') = \eta^\mu_\nu \mathfrak{g}_l(r, r') \delta_{mm'} - a^2 \tilde{\theta}^2 l(l+1) \times \mathfrak{g}_l(a, a) S_l(r, r') \langle lm | \hat{\mathbf{L}}_\mu \hat{\mathbf{L}}_\nu | lm' \rangle + ia \tilde{\theta} S_l(r, r') (\eta^\mu_0 \Gamma^\alpha_\nu + \Gamma^{\mu\alpha} \eta^0_\nu) \times \langle lm | \hat{\mathbf{L}}_\alpha | lm' \rangle, \quad (\text{A21})$$

where  $\hat{\mathbf{L}}_0$  denotes the identity operator and  $\Gamma^{\mu\nu} = \eta^{\mu\nu} - \eta^\mu_0 \eta^\nu_0$ .

## APPENDIX B: GF FOR A CYLINDRICAL $\theta$ BOUNDARY

Now, we concentrate in constructing the GF in cylindrical coordinates for the configuration shown in Fig. 4 where the  $\theta$  boundary is the surface of a cylinder of radius  $a$  with its axis lying along the  $z$  direction. The various components of the GF are the solution of

$$[-\eta^\mu_\nu \nabla^2 - \tilde{\theta} \delta(\rho - a) n_\alpha \epsilon^{\alpha\mu\beta} \partial_\beta] G^\nu_\sigma(\mathbf{x}, \mathbf{x}') = 4\pi \eta^\mu_\sigma \delta(\mathbf{x} - \mathbf{x}'), \quad (\text{B1})$$

where  $n_\alpha = (0, \cos \varphi, \sin \varphi, 0)$  is the normal to the  $\theta$  interface. Since we have the completeness relation

$$\delta(\varphi - \varphi') \delta(z - z') = \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} \delta_{mm'} e^{i(m\varphi - m'\varphi')}, \quad (\text{B2})$$

we look for a solution of the form

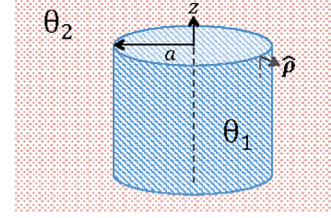


FIG. 4. Cylindrical region.

$$G^\mu_\nu(\mathbf{x}, \mathbf{x}') = 4\pi \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m=-\infty}^{+\infty} \sum_{m'=-\infty}^{+\infty} \times g_{mm',\nu}^\mu(\rho, \rho'; k) e^{i(m\varphi - m'\varphi')}, \quad (\text{B3})$$

where  $g_{mm',\nu}^\mu(\rho, \rho'; k)$  is the reduced GF analogous to  $g^\mu_\nu(z, z')$  in the case of planar symmetry.

In the limiting case  $\tilde{\theta} \rightarrow 0$ , the matrix elements take the simple form  $g_{mm',\nu}^\mu(\rho, \rho'; k) = \eta^\mu_\nu \delta_{mm'} \mathfrak{g}_m(\rho, \rho'; k)$ , where  $\mathfrak{g}_m(\rho, \rho'; k)$  solves

$$\hat{\mathcal{O}}_\rho^{(m)} \mathfrak{g}_m(\rho, \rho'; k) = \frac{\delta(\rho - \rho')}{\rho}, \quad (\text{B4})$$

with the radial operator being

$$\hat{\mathcal{O}}_\rho^{(m)} = -\frac{1}{\rho} \frac{\partial}{\partial \rho} \left( \rho \frac{\partial}{\partial \rho} \right) + \frac{m^2}{\rho^2} + k^2. \quad (\text{B5})$$

The solution to Eq. (B4) in free space, with standard boundary conditions at infinity, is

$$\mathfrak{g}_m(\rho, \rho'; k) = I_m(k\rho_<) K_m(k\rho_>), \quad (\text{B6})$$

where  $\rho_>$  ( $\rho_<$ ) is the greater (lesser) of  $\rho$  and  $\rho'$ . Here,  $I_m$  and  $K_m$  are the modified Bessel functions of the first and second kinds, respectively. The substitution of Eq. (B6) into Eq. (B3) correctly reproduces the well-known result  $|\mathbf{x} - \mathbf{x}'|^{-1}$  for the free case.

In the following, we focus on solving Eq. (B1) for the various components of the GF matrix. To this end, we first observe that the additional differential operator in Eq. (B1) does not involve radial derivatives, but only derivatives with respect to the coordinates  $z$  and  $\varphi$ . That is to say

$$n_\alpha \epsilon^{\alpha\mu\beta} \partial_\beta = (\cos \varphi \epsilon^{1\mu 3} + \sin \varphi \epsilon^{2\mu 3}) \partial_z + \epsilon^{1\mu 2} \frac{1}{\rho} \partial_\varphi. \quad (\text{B7})$$

Substituting Eq. (B3) into Eq. (B1) and using  $\partial_z \rightarrow ik$ ,  $\partial_\varphi \rightarrow im$ ,  $-\nabla^2 \rightarrow \hat{\mathcal{O}}_\rho^{(m)}$  gives

$$\begin{aligned} & \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m,m'=-\infty}^{+\infty} e^{i(m\varphi-m'\varphi')} \left\{ \eta_{\nu}^{\mu} \hat{\mathcal{O}}_{\rho}^{(m)} - i\tilde{\theta}\delta(\rho-a) \left[ k(\cos\varphi\epsilon^{1\mu^3}_{\nu} + \sin\varphi\epsilon^{2\mu^3}_{\nu}) + \epsilon^{1\mu^2}_{\nu} \frac{m}{\rho} \right] \right\} g_{mm',\sigma}^{\nu} \\ &= \eta_{\sigma}^{\mu} \frac{\delta(\rho-\rho')}{\rho} \int_{-\infty}^{+\infty} \frac{dk}{2\pi} e^{ik(z-z')} \frac{1}{2\pi} \sum_{m,m'=-\infty}^{+\infty} \delta_{mm'} e^{i(m\varphi-m'\varphi')}. \end{aligned} \quad (\text{B8})$$

Using the linear independence of  $e^{-im'\varphi'}$  and  $e^{-ikz'}$ , we are left with

$$\sum_{m=-\infty}^{+\infty} e^{im\varphi} \left\{ \eta_{\nu}^{\mu} \hat{\mathcal{O}}_{\rho}^{(m)} - i\tilde{\theta}\delta(\rho-a) \left[ k(\cos\varphi\epsilon^{1\mu^3}_{\nu} + \sin\varphi\epsilon^{2\mu^3}_{\nu}) + \epsilon^{1\mu^2}_{\nu} \frac{m}{\rho} \right] \right\} g_{mm',\sigma}^{\nu} = \eta_{\sigma}^{\mu} \frac{\delta(\rho-\rho')}{\rho} \times \sum_{m=-\infty}^{+\infty} \delta_{mm'} e^{im\varphi}. \quad (\text{B9})$$

In analogy with the spherical case, we next multiply Eq. (B9) to the left by  $e^{-im''\varphi}$  and integrate with respect to  $\varphi$ . After using the relations

$$\delta_{mm'} = \frac{1}{2\pi} \int_0^{2\pi} e^{i(m-m')\varphi} d\varphi, \quad (\text{B10})$$

$$\begin{aligned} A_{mm'',\nu}^{\mu} &= \frac{1}{2\pi} \int_0^{2\pi} d\varphi e^{im\varphi} (\cos\varphi\epsilon^{1\mu^3}_{\nu} + \sin\varphi\epsilon^{2\mu^3}_{\nu}) e^{-im''\varphi} \\ &= \frac{1}{2} [\delta_{m,m''-1} (\tilde{\epsilon}^{\mu}_{\nu})^* + \delta_{m,m''+1} \tilde{\epsilon}^{\mu}_{\nu}], \end{aligned} \quad (\text{B11})$$

where  $\tilde{\epsilon}^{\mu}_{\nu} = \epsilon^{1\mu^3}_{\nu} + \epsilon^{2\mu^3}_{\nu} i$ , Eq. (B9) simplifies to

$$\begin{aligned} & \hat{\mathcal{O}}_{\rho}^{(m)} g_{mm',\sigma}^{\mu} - i\tilde{\theta}\delta(\rho-a) \left[ k \sum_{m''=-\infty}^{+\infty} A_{m''m,\nu}^{\mu} g_{m''m',\sigma}^{\nu} \right. \\ & \left. + \epsilon^{1\mu^2}_{\nu} \frac{m}{\rho} \delta_{mm'} g_{mm',\sigma}^{\nu} \right] = \eta_{\sigma}^{\mu} \frac{\delta(\rho-\rho')}{\rho} \delta_{mm'}, \end{aligned} \quad (\text{B12})$$

where we have relabeled  $m'' \leftrightarrow m$ .

The resulting equation can be integrated using the free reduced GF  $\mathfrak{g}_m(\rho, \rho'; k)$ , satisfying Eq. (B4), with the result

$$\begin{aligned} g_{mm',\sigma}^{\mu}(\rho, \rho') &= \eta_{\sigma}^{\mu} \delta_{mm'} \mathfrak{g}_m(\rho, \rho') + i\tilde{\theta} m \epsilon^{1\mu^2}_{\nu} \mathfrak{g}_m(\rho, a) \\ & \times g_{mm',\sigma}^{\nu}(a, \rho') + i\tilde{\theta} k a \mathfrak{g}_m(\rho, a) \\ & \times \sum_{m''=-\infty}^{+\infty} A_{m''m,\nu}^{\mu} g_{m''m',\sigma}^{\nu}(a, \rho'). \end{aligned} \quad (\text{B13})$$

Note that we have suppressed the dependence of the reduced GF on  $k$  for the sake of brevity. Now, we have to solve for the various components. To this end, we observe that the nonzero components of  $A_{mm'',\nu}^{\mu}$  are

$$A_{mm'',0}^1 = A_{mm'',1}^0 = +\frac{i}{2} (\delta_{m,m''+1} - \delta_{m,m''-1}), \quad (\text{B14})$$

$$A_{mm'',0}^2 = A_{mm'',2}^0 = -\frac{1}{2} (\delta_{m,m''+1} + \delta_{m,m''-1}). \quad (\text{B15})$$

This result allows us to split Eq. (B13) into the components  $\mu = 0$ ,  $\mu = 3$  and  $\mu = j = 1, 2$ , obtaining

$$\begin{aligned} g_{mm',\sigma}^0(\rho, \rho') &= \eta_{\sigma}^0 \delta_{mm'} \mathfrak{g}_m(\rho, \rho') + i\tilde{\theta} m \mathfrak{g}_m(\rho, a) g_{mm',\sigma}^3(a, \rho') \\ & + i\tilde{\theta} k a \mathfrak{g}_m(\rho, a) \sum_{m''=-\infty}^{+\infty} A_{m''m,i}^0 g_{m''m',\sigma}^i(a, \rho'), \end{aligned} \quad (\text{B16})$$

$$g_{mm',\sigma}^3(\rho, \rho') = \eta_{\sigma}^3 \delta_{mm'} \mathfrak{g}_m(\rho, \rho') + i\tilde{\theta} m \mathfrak{g}_m(\rho, a) g_{mm',\sigma}^0(a, \rho'), \quad (\text{B17})$$

$$\begin{aligned} g_{mm',\sigma}^j(\rho, \rho') &= \eta_{\sigma}^j \delta_{mm'} \mathfrak{g}_m(\rho, \rho') \\ & + i\tilde{\theta} k a \mathfrak{g}_m(\rho, a) \sum_{m''=-\infty}^{+\infty} A_{m''m,0}^j g_{m''m',\sigma}^0(a, \rho'), \end{aligned} \quad (\text{B18})$$

where the index  $i = 1, 2$ .

Setting  $\rho = a$  in Eqs. (B17) and (B18) and then substituting into Eq. (B16) yields

$$\begin{aligned} g_{mm',\sigma}^0(\rho, \rho') &= \eta_{\sigma}^0 \delta_{mm'} \mathfrak{g}_m(\rho, \rho') + i\tilde{\theta} [m \delta_{mm'} \eta_{\sigma}^3 \\ & + k a A_{m',m,\sigma}^0] \mathfrak{g}_m(\rho, a) \mathfrak{g}_{m'}(a, \rho') \\ & - \tilde{\theta}^2 \mathfrak{g}_m(\rho, a) \mathfrak{f}_m(k) g_{mm',\sigma}^0(a, \rho'), \end{aligned} \quad (\text{B19})$$

where  $\mathfrak{f}_m(k) = m^2 \mathfrak{g}_m(a, a) + \frac{k^2 a^2}{2} [\mathfrak{g}_{m+1}(a, a) + \mathfrak{g}_{m-1}(a, a)]$ . In deriving Eq. (B19), we use the result

$$\sum_{i=1}^2 A_{mm'',i}^0 A_{m''m,0}^i = \frac{1}{2} (\delta_{m,m''+1} \delta_{m',m-1} + \delta_{m,m''-1} \delta_{m',m+1}), \quad (\text{B20})$$

which can be verified directly from Eqs. (B14)–(B15).

Solving for  $g_{mm',\sigma}^0(a, \rho')$  by setting  $\rho = a$  in Eq. (B19) and inserting the result back in this equation, we obtain

$$g_{mm',\sigma}^0(\rho, \rho') = \eta^0_\sigma \delta_{mm'} [\mathfrak{g}_m(\rho, \rho') - \tilde{\theta}^2 \mathfrak{f}_m(k) C_{mm}(\rho, \rho')] + i\tilde{\theta} (m\delta_{mm'} \eta^3_\sigma + kaA_{m'm,\sigma}^0) C_{mm'}(\rho, \rho'), \quad (\text{B21})$$

where the function  $C_{mm'}(\rho, \rho')$  was defined in Eq. (54).

The remaining components now can be computed similarly. The substitution of  $g_{mm',\sigma}^0(a, \rho')$  in Eqs. (B17) and (B18) yields

$$g_{mm',\sigma}^3(\rho, \rho') = \eta^3_\sigma \delta_{mm'} [\mathfrak{g}_m(\rho, \rho') - m^2 \tilde{\theta}^2 \mathfrak{g}_m(a, a) C_{mm}(\rho, \rho')] + im\tilde{\theta} (\eta^0_\sigma + ika\tilde{\theta} A_{m'm,\sigma}^0) C_{mm'}(\rho, \rho'), \quad (\text{B22})$$

$$g_{mm',\sigma}^i(\rho, \rho') = \eta^i_\sigma \delta_{mm'} \mathfrak{g}_m(\rho, \rho') + ika\tilde{\theta} [\eta^0_\sigma + i\eta^3_\sigma m' \mathfrak{g}_{m'}(a, a)] A_{m'm,0}^i C_{m'm}(\rho', \rho) - \tilde{\theta}^2 k^2 a^2 \mathfrak{g}_m(\rho, a) \sum_{m''=-\infty}^{+\infty} A_{m'm,0}^j A_{m'm'',\sigma}^0 C_{m''m'}(a, \rho'). \quad (\text{B23})$$

These results establish Eqs. (51)–(53).

- 
- [1] S. S. Chern and J. Simons, *Ann. Math.* **99**, 48 (1974).  
[2] Y. L. Chen *et al.*, *Science* **325**, 178 (2009).  
[3] A. R. Mellnik *et al.*, *Nature (London)* **511**, 449 (2014).  
[4] J. S. Bell and R. Jackiw, *Nuovo Cimento A* **60**, 47 (1969).  
[5] S. L. Adler, *Phys. Rev.* **177**, 2426 (1969).  
[6] S. H. S. Alexander, M. E. Peskin, and M. M. Sheikh-Jabbari, *Phys. Rev. Lett.* **96**, 081301 (2006).  
[7] V. A. Kuzmin, V. A. Rubakov, and M. E. Shaposhnikov, *Phys. Lett.* **155B**, 36 (1985).  
[8] M. Fukugita and T. Yanagida, *Phys. Lett. B* **174**, 45 (1986).  
[9] R. D. Peccei and H. R. Quinn, *Phys. Rev. Lett.* **38**, 1440 (1977).  
[10] G. 't Hooft, *Phys. Rep.* **142**, 357 (1986).  
[11] F. Wilczek, *Phys. Rev. Lett.* **58**, 1799 (1987).  
[12] E. Witten, *Commun. Math. Phys.* **117**, 353 (1988).  
[13] M. Marino, *Rev. Mod. Phys.* **77**, 675 (2005).  
[14] R. Bonezzi, O. Corradini, and A. Waldron, *Phys. Rev. D* **90**, 084018 (2014).  
[15] S. M. Carroll, G. B. Field, and R. Jackiw, *Phys. Rev. D* **41**, 1231 (1990).  
[16] K. A. Milton and Y. J. Ng, *Phys. Rev. D* **42**, 2875 (1990).  
[17] K. A. Milton and Y. J. Ng, *Phys. Rev. D* **46**, 842 (1992).  
[18] F. P. Devecchi, M. Fleck, H. O. Girotti, M. Gomes, and A. J. da Silva, *Ann. Phys. (N.Y.)* **242**, 275 (1995).  
[19] M. A. Anacleto, A. Ilha, J. R. S. Nascimento, R. F. Ribeiro, and C. Wotzasek, *Phys. Lett. B* **504**, 268 (2001).  
[20] A. Blasi, N. Maggiore, N. Magnoli, and S. Storace, *Classical Quantum Gravity* **27**, 165018 (2010).  
[21] S. Ozonder, *Phys. Rev. C* **81**, 062201 (2010).  
[22] G. Tallarita and S. Thomas, *J. High Energy Phys.* **12** (2010) 090.  
[23] P. Mukherjee and B. Paul, *Phys. Rev. D* **85**, 045028 (2012).  
[24] T. Andrade, J. I. Jottar, and R. G. Leigh, *J. High Energy Phys.* **05** (2012) 071.  
[25] Y. Concha-Sánchez, A. Raya, and M. E. Tejeda-Yeomans, *Phys. Rev. D* **87**, 035001 (2013).  
[26] H. C. Chang, M. Fujita, and M. Kaminski, *J. High Energy Phys.* **10** (2014) 118.  
[27] A. P. Balachandran, L. Chandar, E. Ercolessi, T. R. Govindarajan, and R. Shankar, *Int. J. Mod. Phys. A* **09**, 3417 (1994).  
[28] M. Asorey, A. P. Balachandran, and J. M. Pérez-Pardo, *arXiv:1505.03461*.  
[29] M. Asorey, A. P. Balachandran, and J. M. Pérez-Pardo, *J. High Energy Phys.* **12** (2013) 073.  
[30] V. N. Markov and Y. M. Pis'mak, *J. Phys. A* **39**, 6525 (2006).  
[31] D. Y. Pis'mak, Y. M. Pis'mak, and F. Wegner, *Phys. Rev. E* **92**, 013204 (2015).  
[32] D. Colladay and V. A. Kostelecky, *Phys. Rev. D* **55**, 6760 (1997).  
[33] D. Colladay and V. A. Kostelecky, *Phys. Rev. D* **58**, 116002 (1998).  
[34] H. Belich, Jr., M. M. Ferreira, Jr., J. A. Helayel-Neto, and M. T. D. Orlando, *Phys. Rev. D* **68**, 025005 (2003).  
[35] R. Lehnert and R. Potting, *Phys. Rev. D* **70**, 125010 (2004); R. Lehnert and R. Potting, *Phys. Rev. Lett.* **93**, 110402 (2004).  
[36] J. Alfaro, A. A. Andrianov, M. Cambiaso, P. Giacconi, and R. Soldati, *Phys. Lett. B* **639**, 586 (2006).  
[37] A. J. Hariton and R. Lehnert, *Phys. Lett. A* **367**, 11 (2007).  
[38] C. Kaufhold and F. R. Klinkhamer, *Phys. Rev. D* **76**, 025024 (2007).  
[39] O. G. Kharlanov and V. C. Zhukovsky, *Phys. Rev. D* **81**, 025015 (2010).  
[40] J. Alfaro, A. A. Andrianov, M. Cambiaso, P. Giacconi, and R. Soldati, *Int. J. Mod. Phys. A* **25**, 3271 (2010).  
[41] X. L. Qi, R. Li, J. Zang, S. C. Zhang, *Science* **323**, 1184 (2009).  
[42] Y. N. Obukhov and F. W. Hehl, *Phys. Lett. A* **341**, 357 (2005).  
[43] L. Huerta and J. Zanelli, *Phys. Rev. D* **85**, 085024 (2012).  
[44] L. Huerta, *Phys. Rev. D* **90**, 105026 (2014).

- [45] F. Canfora, L. Rosa, and J. Zanelli, *Phys. Rev. D* **84**, 105008 (2011).
- [46] A. B. Clark, D. Z. Freedmann, A. Karch, and M. Schnabl, *Phys. Rev. D* **71**, 066003 (2005).
- [47] E. D'Hoker, J. Estes, and M. Gutperle, *Nucl. Phys.* **B753**, 16 (2006).
- [48] B. Chen, Z. Xu, and C. Liu, *J. High Energy Phys.* **02** (2009) 036.
- [49] D. Gaiotto and E. Witten, *J. High Energy Phys.* **06** (2010) 097.
- [50] C. Kim, E. Koh, and K. Lee, *J. High Energy Phys.* **06** (2008) 040.
- [51] C. Kim, E. Koh, and K. Lee, *Phys. Rev. D* **79**, 126013 (2009).
- [52] D. Bak, M. Gutperle, and S. Hirano, *J. High Energy Phys.* **05** (2003) 072.
- [53] A. Martín-Ruiz, M. Cambiaso, and L. F. Urrutia, *Phys. Rev. D* **92**, 125015 (2015).
- [54] J. Zanelli, *Classical Quantum Gravity* **29**, 133001 (2012).
- [55] K. von Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980).
- [56] S. Q. Shen, *Topological Insulators, Dirac Equation in Condensed Matter* (Springer, Heidelberg, 2012).
- [57] X. L. Qi, T. Hughes, and S. C. Zhang, *Phys. Rev. B* **78**, 195424 (2008).
- [58] J. Maciejko, X. L. Qi, H. D. Drew, and S. C. Zhang, *Phys. Rev. Lett.* **105**, 166803 (2010).
- [59] J. Maciejko, X. L. Qi, A. Karch, and S. C. Zhang, *Phys. Rev. Lett.* **105**, 246809 (2010).
- [60] Z. Wang, X. L. Qi, and S. C. Zhang, *Phys. Rev. Lett.* **105**, 256803 (2010).
- [61] S. Y. Xu *et al.*, *Science* **349**, 613 (2015).
- [62] L. Ge, T. Zhan, D. Han, X. Liu, and J. Zi, *Sci. Rep.* **5**, 7945 (2015).
- [63] X. L. Qi and S. C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
- [64] M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
- [65] J. A. Crose, S. Fuchs, and S. Y. Buhmann, *Phys. Rev. A* **92**, 063831 (2015).
- [66] B. A. Bernevig and T. L. Hughes, *Topological Insulators and Topological Superconductors* (Princeton University, Princeton, NJ, 2013).
- [67] M. V. Berry and A. K. Geim, *Eur. J. Phys.* **18**, 307 (1997).
- [68] F. M. Penning, *Physica* **3**, 873 (1936).
- [69] J. Schwinger, L. DeRaad, K. Milton, and W. Tsai, *Classical Electrodynamics*, Advanced Book Program (Perseus Books Boulder, CO, 1998).
- [70] L. S. Brown and G. J. Maclay, *Phys. Rev.* **184**, 1272 (1969).
- [71] V. Sopova and L. H. Ford, *Phys. Rev. D* **72**, 033001 (2005).
- [72] V. M. Mostepanenko and N. N. Trunov, *Usp. Fiz. Nauk* **156**, 385 (1988) [*Sov. Phys. Usp.* **31**, 965 (1988)].
- [73] W. J. Nie, R. Zeng, Y. Lang, and S. Zhu, *Phys. Rev. B* **88**, 085421 (2013).
- [74] I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products*, 4th ed., edited by A. Jeffrey and D. Zwillinger (Academic, New York, 1994).