Noether–Wald energy in Critical Gravity

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A B S T R A C T

Criticality represents a specific point in the parameter space of a higher-derivative gravity theory, where the linearized field equations become degenerate. In 4D Critical Gravity, the Lagrangian contains a Weyl-squared term, which does not modify the asymptotic form of the curvature. The Weyl^2 coupling is chosen such that it eliminates the massive scalar mode and it renders the massive spin-2 mode massless. In doing so, the theory turns consistent around the critical point.

Here, we employ the Noether–Wald method to derive the conserved quantities for the action of Critical Gravity. It is manifest from this energy definition that, at the critical point, the mass is identically zero for Einstein spacetimes, what is a defining property of the theory. As the entropy is obtained from the Noether–Wald charges at the horizon, it is evident that it also vanishes for any Einstein black hole.

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1. Introduction

General Relativity (GR) is a successful theory of gravity at a classical level but it lacks of consistency in a quantum regime because it is not renormalizable. On the other hand, in the low energy limit of String Theory, which should be finite to all orders, there appear contributions that are quadratic in the curvature. As a consequence, higher curvature extensions of Einstein gravity are expected to give rise to a gravity theory with a better ultraviolet behavior. Early work on the subject has suggested that this class of theories should be renormalizable [1].

Lower-dimensional examples have been extensively studied in recent literature. They are regarded as insightful toy models which capture essential features of 4D gravity. One of them is New Massive Gravity (NMG) [2], a parity-even three-dimensional theory which describes two propagating massive spin-2 modes, in contrast to 3D Einstein gravity which is topological. Picking up the conventional sign of the Einstein–Hilbert action, the energy of the massive excitations is negative (ghost modes), while the mass of the Banados–Teitelboim–Zanelli (BTZ) black hole is positive. Clearly, this inconsistency persists even if one reverses the sign of the kinetic term. A physically reasonable theory arises at a specific point of parametric space, where the massive spin-2 field turns massless [3]. At this particular point, both the energy of the graviton and the mass of the BTZ black hole vanish identically [4].

Furthermore, both central charges turn into zero, what leads to a vanishing entropy [4]. Another feature of the theory is the presence of new modes with logarithmic behavior at the critical point [5]. These modes are eliminated when standard Brown–Henneaux boundary conditions are considered. Relaxing the asymptotic conditions to include log terms switches on new holographic sources at the boundary [6].

Another theory in three dimensions sharing similar features with NMG is Topologically Massive Gravity (TMG) [7]. The corresponding critical point defines the concept of Chiral Gravity. However, in this case, the central charges are different from each other due to a parity-violating term in the action. As a consequence, neither mass nor entropy vanish for BTZ black holes at the chiral point.

The generalization of the concept of criticality, present in these models, to four dimensions is given by theories which include quadratic terms in the curvature with particular couplings on top of the Einstein–Hilbert action. The most general form of a gravity action with quadratic-curvature corrections in 4D is given by

$$I = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left(R - 2\Lambda + \alpha R_{\mu\nu} R^{\mu\nu} + \beta R^2 \right), \qquad (1)$$

where α and β are arbitrary couplings, and $\Lambda = -3/\ell^2$ is the cosmological constant in terms of the AdS radius ℓ. The Riemann-squared term is not present, as it can be always traded off by the Gauss–Bonnet (GB) invariant plus the curvature-squared terms present in the action (1). The GB term does not affect the field equations in the bulk but it does modify the boundary dynamics.

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This class of theories leads to equations of motion (EOM) with up to four derivatives in the metric. Generically, they describe modes that represent a massless spin-2 graviton, a massive spin-2 field and a massive scalar. For a quadratic-curvature gravity theory with arbitrary coupling constants, perturbations around a given background would give rise to ghosts. The problem with the sign of the energy of these modes can be circumvented by a sign flip of the constant in front of Einstein kinetic term. On the other hand, Einstein black holes are solutions to the theory defined by Eq. (1). Therefore, the change in the sign mentioned above would lead to a negative mass for Schwarzschild-AdS black hole. Needless to say, this picture is clearly unphysical as the energy of the perturbations around a background and the mass of a black hole carry opposite signs.

In view of this general obstruction to obtain a four-dimensional gravity theory which is free of the inconsistencies discussed above, it was quite surprising when the authors of Ref. [8] pointed out the fact that, for the particular couplings $\alpha = -3\beta$ and $\beta = -1/2\Lambda$, the massive scalar is eliminated and the massive spin-2 mode turns massless. This choice renders the theory physically sensible around the critical point. This fact is confirmed by using the Ostrogradsky method for Lagrangians with derivatives of higher order: the energy for the massive mode vanishes for the critical value of the couplings. From the point of view of the energy of the black holes of the theory, one can use the Abbott–Deser–Tekin (ADT) formula [9,10] to evaluate the mass of Schwarzschild-AdS solution, what results in

$$M = m\left(1 + 2\Lambda(\alpha + 4\beta)\right),\tag{2}$$

where m is the mass parameter in the solution.

The general formula Eq. (2), makes evident that, for the critical condition mentioned above, the mass for Schwarzschild-AdS black hole vanishes.

In the present work, as an alternative to Deser–Tekin procedure, we employ Noether–Wald method [11,12] to compute the charges in Critical Gravity. This full (non-linearized) expression derived in this way has a remarkable property: the energy of any Einstein space is identically zero, as long anticipated in Ref. [13].

2. Deser–Tekin energy in 4D quadratic-curvature gravity

As mentioned in the previous section, in Refs. [9,10], the authors provide a generic definition of energy for an arbitrary curvature-squared gravity theory. That definition of the energy is obtained as an extension of the Abbott–Deser method [14].

In order to obtain the ADT mass for a general asymptotically AdS (AAdS) solution, we need to write down the metric of the spacetime in the form of $g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}$, where $\bar{g}_{\mu\nu}$ is the metric of the background and $h_{\mu\nu}$ is the perturbation tensor. Such construction leaves the first-order variation of field equations as

$$\delta\left(G_{\mu\nu} + E_{\mu\nu}\right) = [1 + 2\Lambda(\alpha + 4\beta)]\,G^L_{\mu\nu}$$
$$+ \alpha\left[\left(\bar{\Box} - \frac{2\Lambda}{3}\right)G^L_{\mu\nu} - \frac{2\Lambda}{3}R^L\bar{g}_{\mu\nu}\right] +$$
$$+ (\alpha + 2\beta)\left[-\bar{\nabla}_\mu\bar{\nabla}_\nu + \bar{g}_{\mu\nu}\bar{\Box} + \Lambda\bar{g}_{\mu\nu}\right]R^L,\tag{3}$$

where $G^L_{\mu\nu}$ and R^L are the linearized expression of Einstein tensor and Ricci scalar, respectively. The tensor $E_{\mu\nu}$ is the contribution of fourth order in the derivatives to the field equations. The equation (3) has to be equal to an effective energy–momentum tensor $T_{\mu\nu}$, which is covariantly conserved. One can write a conserved current, for a set of Killing fields $\{\bar{\xi}^\mu\}$ that represents the isometries of the background

$$J^\mu_{ADT} = 8\pi\,GT^{\mu\nu}\bar{\xi}_\nu.\tag{4}$$

In order to evaluate the mass of a gravitational object, the Killing vector needs to be timelike, at least, at infinity.

Whenever there is a current which is conserved, one is able to write down J^μ as the divergence of a 2-form prepotential, i.e.,

$$J^\mu_{ADT} = \nabla_\nu\mathcal{F}^{\mu\nu}.\tag{5}$$

One can consider a spacetime foliated by a normal (radial) direction z

$$ds^2 = N^2(z)dz^2 + h_{ij}(z,x)dx^i dx^j,\tag{6}$$

where $h_{ij}(z,x)$ is the induced metric on ∂M, and its radial evolution is defined by the unit vector $n_\nu = N(z)\delta^z_\nu$.

In this coordinate frame, the conserved charge can be expressed as an integral on the co-dimension two surface Σ

$$Q^\mu_{ADT}[\bar{\xi}] = \int_\Sigma dS_\nu\mathcal{F}^{\mu\nu}.\tag{7}$$

Here, $dS_\nu = d^2x\sqrt{-h}\,n_\nu$ is a surface normal vector that defines the integration for a fixed time and radius. For the case of curvature-squared gravity in four dimensions, the conserved quantity adopts the form[1]

$$8\pi\,G\,Q^\mu_{ADT}[\bar{\xi}] = [1 + 2\Lambda(\alpha + 4\beta)]\int_{\partial M} d^3x\,G^{\mu\lambda}_L\bar{\xi}_\lambda$$
$$+ (\alpha + 2\beta)\int_\Sigma dS_\nu\left(2\bar{\xi}^{[\mu}\bar{\nabla}^{\nu]}R^L + R^L\bar{\nabla}^\mu\bar{\xi}^\nu\right)$$
$$- \alpha\int_\Sigma dS_\nu\left(2\bar{\xi}_\lambda\bar{\nabla}^{[\mu}G^{\nu]\lambda}_L + 2G^{\lambda[\mu}_L\bar{\nabla}^{\nu]}\bar{\xi}_\lambda\right).\tag{8}$$

3. Critical Gravity

In Ref. [8], the energy of the graviton modes in quadratic-curvature gravity was studied. These excitations come from the linearized EOM (3). The choice $\alpha = -3\beta$ leads to a traceless perturbation ($h = 0$) which eliminates the massive scalar mode. Consequently, the equation for the propagating mode takes the form

$$\left(\bar{\Box} - \frac{2\Lambda}{3}\right)\left(\bar{\Box} - \frac{2\Lambda}{3} - \frac{2\Lambda\beta + 1}{3\beta}\right)h_{\mu\nu} = 0.\tag{9}$$

The first factor of the equation describes the propagation of a massless graviton in an AdS background while the second one represents a massive spin-2 field.

It is clear that the latter becomes massless by imposing the critical value $\beta = -1/2\Lambda$. This particular coupling produces the fourth order equation

$$\left(\bar{\Box} - \frac{2\Lambda}{3}\right)^2 h_{\mu\nu} = 0,\tag{10}$$

which reflects the appearance of both massless and logarithmic modes [8].

In order to obtain the energy of the excitations, the authors in Ref. [16] followed a Hamiltonian approach. For an unrestricted value of β, the action up to quadratic order in $h^{\mu\nu}$ is

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[1] For a generalized ADT procedure see, e.g., Ref. [15].

$$I = -\frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left[\frac{1}{2} \left(1 + 6\beta\Lambda \right) \bar{\nabla}^\lambda h^{\mu\nu} \bar{\nabla}_\lambda h_{\mu\nu} \right.$$
$$\left. + \frac{3}{2}\beta\bar{\Box}h^{\mu\nu}\bar{\Box}h_{\mu\nu} + \frac{\Lambda}{3}\left(1+4\beta\Lambda\right)h^{\mu\nu}h_{\mu\nu} \right]. \tag{11}$$

Using the Ostrogradsky method for higher-derivative Lagrangians, one obtains the following conjugate momenta

$$\pi^{\mu\nu}_{(1)} = \frac{1}{16\pi G} \sqrt{-g}\bar{\nabla}^0 \left[(1 + 6\beta\Lambda) h^{\mu\nu} - 3\beta\bar{\Box}h^{\mu\nu} \right], \tag{12}$$

$$\pi^{\mu\nu}_{(2)} = \frac{3\beta}{16\pi G} \sqrt{-g}\bar{g}^{00} \bar{\Box}h^{\mu\nu}. \tag{13}$$

Due to the fact that the Lagrangian is time independent, the Hamiltonian can be written as its time average, that is

$$H = \frac{1}{16\pi GT} \int_M d^4x \sqrt{-g} \left[(1 + 6\beta\Lambda) \bar{\nabla}^0 h^{\mu\nu} \dot{h}_{\mu\nu} \right.$$
$$\left. - 6\beta \left(\frac{\partial}{\partial t} \left(\bar{\Box}h^{\mu\nu} \right) \right) \bar{\nabla}^0 h_{\mu\nu} \right] - \frac{1}{T}I. \tag{14}$$

Evaluating for the case of massless and massive propagating modes, one obtains the following expressions for the corresponding on-shell energies

$$E_{(m)} = -\frac{1}{16\pi GT} \left(1 + 2\beta\Lambda \right) \int_M d^4x \bar{\nabla}^0 h^{\mu\nu}_m \dot{h}^m_{\mu\nu}, \tag{15}$$

$$E_{(M)} = \frac{1}{16\pi GT} \left(1 + 2\beta\Lambda \right) \int_M d^4x \bar{\nabla}^0 h^{\mu\nu}_M \dot{h}^M_{\mu\nu}, \tag{16}$$

where the subscripts m and M stand for massless graviton and massive spin-2 field, respectively.

In a gravity theory with quadratic terms in the curvature, where the couplings are related as $\alpha = -3\beta$, there is only a specific value of β that kills the negative energy states. More specifically, from Eqs. (15), (16) it is shown that for $\beta = -1/2\Lambda$, the energy of both the massless and the massive modes is zero. Hence, all the ghosts disappear leading to a consistent theory of gravity.

Therefore, the action of Critical Gravity reads

$$I_{critical} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left[\left(R + \frac{6}{\ell^2} \right) \right.$$
$$\left. - \frac{\ell^2}{2} \left(R_{\mu\nu} R^{\mu\nu} - \frac{1}{3} R^2 \right) \right]. \tag{17}$$

On the other hand, the generic expression for the energy of the black holes in this gravity theory is given by Eq. (8). For any static black hole, the only nonvanishing contribution comes from the first term on the right hand side of Eq. (8). In particular, for a Schwarzschild-AdS black hole, the ADT charge leads to the result in Eq. (2). It is easy to notice that, for the particular value of the couplings which define Critical Gravity ($\alpha = -3\beta$, $\beta = -1/2\Lambda$), the mass of the black hole vanish.

In what follows, we provide an alternative formula of conserved charges in Critical Gravity, which makes manifest the fact that the energy for Einstein black holes is identically zero.

4. Noether–Wald charges in Critical Gravity

A general prescription to define conserved charges in an arbitrary theory of gravity was given in Refs. [11,12,17]. For the purpose of the discussion below, we will restrict ourselves to the case

where Lagrangian density is a functional only of the metric and the curvature, $\mathcal{L}(g_{\mu\nu}, R_{\mu\nu\alpha\beta})$. For a given set of Killing vectors $\{\xi^\mu\}$, the Noether current is written down as

$$\sqrt{-g}J^\mu = \Theta^\mu \left(\delta_\xi g \right) + \Theta^\mu \left(\delta_\xi \Gamma \right) + \sqrt{-g}\mathcal{L}\xi^\mu. \tag{18}$$

For simplicity, we assume that the surface term Θ^μ is separable into a part that contains variations of the Christoffel symbol and another part that contains variations of the metric. As we are interested in diffeomorphic charges for gravity, all the variations are replaced by a Lie derivative along the vector $\{\xi^\mu\}$.

Using the Killing equation, $\delta_\xi g_{\mu\nu} = \nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu = 0$, one can notice that first term in Eq. (18) vanishes. The same relation, this time for the Lie derivative of the Christoffel connection, would produce a combination of double covariant derivatives and curvatures. This casts the current, for a generic gravity theory, in the form

$$J^\mu = 2E^{\mu\nu}_{\alpha\beta} \left(\nabla_\nu \nabla^\alpha \xi^\beta + R^{\alpha\beta}_{\nu\sigma} \xi^\sigma \right) + \xi^\mu \mathcal{L}. \tag{19}$$

Here, the tensor $E^{\mu\nu}_{\alpha\beta}$ is the functional derivative of $\mathcal{L}$ with respect to the spacetime Riemann tensor $R^{\mu\nu}_{\alpha\beta}$, that is,

$$E^{\mu\nu}_{\alpha\beta} = \frac{\delta\mathcal{L}}{\delta R^{\alpha\beta}_{\mu\nu}}. \tag{20}$$

It can be shown, by means of the general form of the field equations for these class of gravity theories, that the last two terms on the right hand side of (19) form the EOM contracted with the Killing field.

Thus, on-shell, the first term on the right side of (19) is the only nonvanishing part.

As the tensor $E^{\mu\nu}_{\alpha\beta}$ satisfies Bianchi identity, the conserved current turns into a total derivative

$$J^\mu = 2\nabla_\nu \left(E^{\mu\nu}_{\alpha\beta} \nabla^\alpha \xi^\beta \right). \tag{21}$$

As the Noether current J^μ can be written as $J^\mu = \nabla_\nu q^{\mu\nu}$, the conserved charge is expressed as an integral on the co-dimension two surface Σ

$$Q^\mu[\xi] = \int_\Sigma dS_\nu q^{\mu\nu} \tag{22}$$

as mentioned previously in Section 2. Finally the conserved charge is written as

$$Q^\mu[\xi] = 2 \int_\Sigma dS_\nu E^{\mu\nu}_{\alpha\beta} \nabla^\alpha \xi^\beta. \tag{23}$$

An alternative form for the action of Critical Gravity considers the difference between Weyl^2 and the GB term $\mathcal{E}_4$, as the GB invariant term does not alter the bulk dynamics [18]

$$I_{critical} = \frac{1}{16\pi G} \int_M d^4x \sqrt{-g} \left[\left(R + \frac{6}{\ell^2} \right) \right.$$
$$\left. + \frac{\ell^2}{4} \left(\mathcal{E}_4 - W^{\mu\nu\alpha\beta} W_{\alpha\beta\mu\nu} \right) \right]. \tag{24}$$

We can split the action in two parts: the first one is the MacDowell–Mansouri action, I_{MM}, which is given by the Einstein–Hilbert plus GB terms, the latter with a fixed coupling [19]. In Einstein gravity, this corresponds to a built-in renormalized AdS action [20]. The second part is minus the action of Conformal Gravity I_{CG}.

Using the Noether–Wald formula for the current (21) for the first part, the functional derivative with respect to the Riemann tensor of the Lagrangian in I_{MM} produces

$$E_{\alpha\beta}^{\mu\nu} = \frac{\ell^2}{128\pi G} \delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\sigma\lambda]} \left(R_{\sigma\lambda}^{\gamma\delta} + \frac{1}{\ell^2}\delta_{[\gamma\lambda]}^{[\rho\delta]} \right), \tag{25}$$

whereas, for the Conformal Gravity part I_{CG}, we get

$$\tilde{E}_{\alpha\beta}^{\mu\nu} = -\frac{\ell^2}{128\pi G} \delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\sigma\lambda]} W_{\sigma\lambda}^{\gamma\delta} \tag{26}$$

Using the Noether–Wald formula (23), the total charge for the theory

$$Q^{\mu}[\xi] = \frac{\ell^2}{64\pi G} \int_{\Sigma} dS_{\nu} \delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\sigma\lambda]} \nabla^{\alpha}\xi^{\beta}$$

$$\times \left[\left(R_{\sigma\lambda}^{\gamma\delta} + \frac{1}{\ell^2}\delta_{[\sigma\lambda]}^{[\gamma\delta]} \right) - W_{\sigma\lambda}^{\gamma\delta} \right]. \tag{27}$$

By definition, the Weyl tensor is

$$W_{\sigma\lambda}^{\gamma\delta} = R_{\sigma\lambda}^{\gamma\delta} - \frac{1}{2}\left(R_{\sigma}^{\gamma}\delta_{\lambda}^{\delta} - R_{\sigma}^{\delta}\delta_{\lambda}^{\gamma} - R_{\lambda}^{\gamma}\delta_{\sigma}^{\delta} + R_{\lambda}^{\delta}\delta_{\sigma}^{\gamma} \right) + \frac{1}{6}R\delta_{[\sigma\lambda]}^{[\gamma\delta]}. \tag{28}$$

For Einstein spaces, $R_{\mu\nu} = -(3/\ell^2)g_{\mu\nu}$, the Weyl tensor adopts the particular form

$$W_{(E)\sigma\lambda}^{\gamma\delta} = R_{\sigma\lambda}^{\gamma\delta} + \frac{1}{\ell^2}\delta_{[\sigma\lambda]}^{[\gamma\delta]}, \tag{29}$$

where the right hand side, is referred to as AdS curvature[2] Using the above fact, the conserved quantity in Critical Gravity is identically zero for Einstein spaces.

5. Electric part of the Weyl tensor and Einstein modes in Conformal Gravity

CG in four dimensions is invariant under local Weyl rescalings of the metric ($g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\omega}g_{\mu\nu}$). Solutions to CG are Bach-flat geometries, which include Einstein spacetimes.

From a holographic viewpoint, asymptotically AdS space in CG are endowed with new sources at the conformal boundary. Indeed, we can set any AAdS spacetime in Fefferman–Graham (FG) form of the metric

$$ds^2 = \frac{\ell^2}{z^2}dz^2 + \frac{1}{z^2}g_{ij}(z,x)dx^i dx^j, \tag{30}$$

where the metric $g_{ij}(z,x)$ is expanded as a power series around the boundary $z = 0$, i.e.,

$$g_{ij}(z,x) = g_{(0)ij} + z g_{(1)ij} + z^2 g_{(2)ij} + z^3 g_{(3)ij} + \dots. \tag{31}$$

Here, the ellipsis denotes higher-order terms which do not enter into the holographic description of $4D$ AAdS spaces.

The presence of the term $z g_{(1)ij}$ reflects the fact the space contains a non-Einstein part. By demanding the vanishing of the linear term on z, one recovers the Einstein branch, with only even powers of z in the expansion. This is achieved by imposing a Neumann boundary condition on the metric, $\partial_z g|_{z=0} = 0$ [21].

On the other hand, the Noether–Wald charge for Conformal Gravity is proportional to the Weyl tensor, as shown by Eq. (26).

However, it is not obvious whether, for Einstein spaces, the holographic modes of CG at the boundary are contained in the electric part of the Weyl tensor

$$E_j^i = W_{j\nu}^{i\mu} n_{\mu}n^{\nu} = W_{jz}^{iz}, \tag{32}$$

as it is the case in Einstein gravity.

As Einstein spaces are solutions of the EOM of CG in the bulk, we restrict the discussion to the surface term in the variation of I_{CG}, that is,

$$\delta I_{CG} = \frac{\ell^2}{64\pi G} \int_{\partial M} d^3x \sqrt{-h} \delta_{[\nu_1\nu_2\nu_3\nu_4]}^{[\mu_1\mu_2\mu_3\mu_4]} \left[n_{\mu_1}\delta\Gamma_{\beta\mu_2}^{\nu_1} g^{\nu_2\beta} W_{(E)\mu_3\mu_4}^{\nu_3\nu_4} \right.$$

$$\left. + n^{\nu_1}\nabla_{\mu_1}W_{(E)\mu_2\mu_3}^{\nu_2\nu_3} \left(g^{-1}\delta g \right)_{\mu_4}^{\nu_4} \right], \tag{33}$$

where $W_{(E)}$ is the Einstein part of the Weyl tensor (29).

The second term in the above relation can be eliminated using the Bianchi identity of second kind. A projection of all indices to the boundary can be performed by taking the explicit form of the normal vector n_{μ} in Gaussian coordinates. Then, the surface term takes the form

$$\delta I_{CG} = \frac{\ell^2}{64\pi G} \int_{\partial M} d^3x \sqrt{-h} \delta_{[j_1j_2j_3]}^{[i_1i_2i_3]} N(z) \left[g^{j_1\ell}\delta\Gamma_{i_1\ell}^{z} W_{(E)i_2i_3}^{j_2j_3} \right.$$

$$\left. - g^{zz}\delta\Gamma_{i_1z}^{j_1} W_{(E)i_2i_3}^{j_2j_3} - 2g^{j_2\ell} W_{(E)i_2i_3}^{j_3z}\delta\Gamma_{\ell i_1}^{j_1} \right]. \tag{34}$$

In Gauss normal frame (6), the relevant components of the Christoffel symbol are

$$\Gamma_{ij}^z = \frac{1}{N}K_{ij},$$

$$\Gamma_{zj}^i = -NK_j^i,$$

$$\Gamma_{jk}^i(g) = \Gamma_{jk}^i(h), \tag{35}$$

where $K_{ij} = -\frac{1}{2N}\partial_z h_{ij}$ is the extrinsic curvature at ∂M. Equipped with this result, the variation of the action is written as

$$\delta I_{CG} = \frac{\ell^2}{64\pi G} \int_{\partial M} d^3x \sqrt{-h} \delta_{[j_1j_2j_3]}^{[i_1i_2i_3]} \left[2W_{i_2i_3}^{j_2j_3}\delta K_{i_1}^{j_1} \right.$$

$$\left. + \left(h^{-1}\delta h \right)_{\ell}^{j_1} K_{i_1}^{\ell} W_{i_2i_3}^{j_2j_3} - 2N\delta\Gamma_{i_1\ell}^{j_1} h^{\ell j_2} W_{i_2i_3}^{j_3z} \right], \tag{36}$$

after some algebraic manipulation and index relabeling.

The rest of the proof relies on a power-counting argument in the radial coordinate z. In order to do so, it is required to expand the tensorial quantities which appear at the surface term.

First, we consider the FG expansion for Einstein spacetimes, where $N(z) = \ell/z$ and $h_{ij}(z,x) = g_{ij}(z,x)/z^2$ with the metric at the conformal boundary given by

$$g_{ij}(z,x) = g_{(0)ij} + z^2 g_{(2)ij} + z^3 g_{(3)ij} + \dots. \tag{37}$$

From this form of the metric, the following expressions are straightforwardly derived

$$\sqrt{-h} = \frac{\sqrt{g_{(0)}}}{z^3} + \mathcal{O}\left(z^{-1}\right), \tag{38}$$

$$\left(h^{-1}\delta h \right)_{\ell}^{j} = \left(g_{(0)}^{-1}\delta g_{(0)} \right)_{\ell}^{j} + \mathcal{O}(z^2), \tag{39}$$

$$K_j^i(h) = \frac{1}{\ell}\delta_j^i - \ell z^2 S_j^i(g_{(0)}) + \mathcal{O}(z^3), \tag{40}$$

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[2] The field strength for the AdS group also contains the torsion along the generators of AdS translations in Riemann–Cartan theory. For Riemannian geometry, Eq. (29) is the only nonvanishing part of the curvature of the AdS group.

where S^i_j is the Schouten tensor defined for the boundary metric $g_{(0)}$, i.e.,

$$S^i_j\left(g_{(0)}\right) = \mathcal{R}^i_j\left(g_{(0)}\right) - \frac{1}{4}\delta^i_j \mathcal{R}\left(g_{(0)}\right) . \tag{41}$$

In a similar fashion, one can compute the fall-off of the different components of the spacetime Weyl tensor. Here, we just write down the ones which are of relevance for this holographic discussion

$$W^{iz}_{jk} = \mathcal{O}\left(z^4\right) , \tag{42}$$

$$W^{ik}_{jm} = z^2 \mathcal{W}^{ik}_{jl}\left(g_{(0)}\right) + \frac{3}{2}\frac{z^3}{\ell^2}g^{[i}_{(3)[j}\delta^{k]}_{m]} + \mathcal{O}\left(z^4\right) , \tag{43}$$

where $\mathcal{W}^{ik}_{jm}$ correspond to the boundary Weyl tensor and the indices of $g_{(3)}$ are raised and lowered with the metric $g_{(0)}$.

Replacing all the above quantities in Eq. (36), we realize that the first term and third terms in the integrand are of order $\mathcal{O}\left(z^2\right)$. That implies that these terms do not contribute in the limit $z \to 0$. In turn, the only nonvanishing contribution comes from the second term in Eq. (36) as

$$\delta I_{CG} = \frac{\ell}{16\pi G}\int_{\partial M}d^3x\sqrt{g_{(0)}}\frac{3}{2\ell^2}g^{ij}_{(3)}\delta g_{(0)ij} , \tag{44}$$

expressed in terms of the holographic Einstein modes.

One can take a few steps back in the expansion of the boundary quantities and appropriately covariantize the last result, in order to express it in terms of the subtrace of the spacetime Weyl tensor

$$\delta I_{CG} = \frac{\ell}{16\pi G}\int_{\partial M}d^3x\sqrt{-h}\,W^{j\ell}_{i\ell}\left(h^{-1}\delta h\right)^i_j . \tag{45}$$

Due to the fact that the Weyl tensor is traceless ($W^{j\mu}_{i\mu}$), its subtrace can be traded off by the electric part of the Weyl tensor

$$W^{j\ell}_{i\ell} = -W^{jz}_{iz} . \tag{46}$$

As a consequence, the variation of the Conformal Gravity action is

$$\delta I_{CG} = -\frac{\ell}{16\pi G}\int_{\partial M}d^3x\sqrt{-h}\,E^j_i\left(h^{-1}\delta h\right)^i_j , \tag{47}$$

for the Einstein modes of the theory. At the same time, this means that the definition of conserved quantities for that sector of CG can be mapped to the notion of Conformal Mass in $4D$ [22].

6. Conclusions

In the present work, we have shown that, in Critical Gravity, the energy of any Einstein solution vanishes identically. This proof does not make use of any particular Einstein black hole, nor relies on charge formulas obtained from the linearization of the field equations. In this respect, charge expression (27) provides the explicit realization of a claim originally stated in Ref. [13].

The holographic derivation in Section 5 confirms the fact that the boundary stress tensor for the total action (24) is zero, in a similar way as in Ref. [23].

When one goes beyond Einstein spaces, the expression (27) is able to capture the effects due to the presence of higher-derivative terms in the curvature. Indeed, as it was shown in Ref. [24], only the non-Einstein modes of the Weyl tensor survive in the surface term form the variation of the Critical Gravity action. As a matter of fact, the boundary contributions are expressible in terms of the

Bach tensor, what enormously simplify the computation of holographic correlation functions at the critical point [25].

Noether–Wald charges provides the black hole entropy in a given gravity theory, when evaluated at the horizon $r = r_h$,

$$S = -2\int_{\Sigma_h}dS_\nu E^{0\nu}_{0\alpha}\nabla^\alpha\xi^0 . \tag{48}$$

As the condition in the Weyl tensor (29) holds throughout the spacetime for Einstein solutions, it is evident from the above formula that the entropy vanishes in Critical Gravity. The addition of topological invariants to the four-dimensional AdS gravity action has led to energy definitions which are finite [26,27], but also has provided insight on the problem of holography for asymptotically AdS spaces in Einstein gravity [20]. The result presented here indicates that the Gauss–Bonnet term also plays a role in the holographic description of gravity beyond Einstein theory.

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