

[Home](#) [Search](#) [Collections](#) [Journals](#) [About](#) [Contact us](#) [My IOPscience](#)

## A note on a gauge–gravity relation and functional determinants

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2010 J. Phys. A: Math. Theor. 43 295401

(<http://iopscience.iop.org/1751-8121/43/29/295401>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 200.27.72.254

The article was downloaded on 30/08/2012 at 23:27

Please note that [terms and conditions apply](#).

## A note on a gauge–gravity relation and functional determinants

**R Aros, D E Díaz and A Montecinos**

Departamento de Ciencias Físicas, Universidad Andrés Bello, República 220, Santiago, Chile

E-mail: [raros@unab.cl](mailto:raros@unab.cl), [danielodiaz@unab.cl](mailto:danielodiaz@unab.cl) and [alejandramontecinos@unab.cl](mailto:alejandramontecinos@unab.cl)

Received 15 April 2010, in final form 19 May 2010

Published 17 June 2010

Online at [stacks.iop.org/JPhysA/43/295401](http://stacks.iop.org/JPhysA/43/295401)

### Abstract

We present a refinement of a recently found gauge–gravity relation between one-loop effective actions: on the gauge side, for a massive charged scalar in  $2d$  dimensions in a constant maximally symmetric electromagnetic field; on the gravity side, for a massive spinor in  $d$ -dimensional (Euclidean) anti-de Sitter space. The inclusion of the dimensionally regularized volume of AdS leads to a complete mapping within dimensional regularization. In even-dimensional AdS, we acquire a small correction to the original proposal, whereas in odd-dimensional AdS, the mapping is totally new and subtle, with the ‘holographic trace anomaly’ playing a crucial role.

PACS numbers: 11.10.Kk, 04.62.+v, 11.15.Kc

Functional determinants of geometric differential operators, such as the Laplacian and Dirac’s operator, are ubiquitous in mathematical and theoretical physics (cf [1, 2]). In particular, in quantum field theory, the one-loop effective action is given by the determinant of the ‘propagating’ differential operator. Exact results for these determinants in gauge theory date back at least to the works of Heisenberg and Euler and of Weisskopf, for constant electromagnetic backgrounds. In special gravitational backgrounds, as in anti-de Sitter, the high degree of symmetry also allows for exact expressions.

Even though such one-loop effective actions have been known for quite some time, it was only very recently that a remarkable gauge–gravity relation was identified [3]. On the gauge side, one considers the one-loop effective action for a massive charged scalar in  $2d$  dimensions in a constant maximally symmetric electromagnetic field, and on the gravity side, the one-loop effective action for a massive spinor in  $d$ -dimensional (Euclidean) anti-de Sitter space. The observation was inspired by factorization of graviton amplitudes in terms of gauge amplitudes and the prospects for the inclusion of one-loop effects, while the similarity was revealed by the presence of Barnes’ multiple gamma function [6, 7]. Possible directions, in which the observation might be extended, include relaxing the maximal symmetry of the electromagnetic field or its homogeneity, and considering higher loops. In addition, one might also contemplate including finite-temperature effects.

Our concern here, however, is with a seemingly overlooked detail, namely that the mapping is only established for even-dimensional AdS. On the gauge side, as long as the number of dimensions is twice larger than that of AdS, the effective action will be given in terms of Barnes' multiple gamma function. In odd-dimensional AdS, in turn, the renormalized effective Lagrangian is polynomial in the mass parameter and little hope is left to meet Barnes' multiple gamma in this case. In this paper we show how to amend this apparent mismatch by taking into account the dimensionally regularized volume of AdS space.

**Gauge side**

On the gauge side, one considers a massive ( $M$ ) charged scalar field  $\phi$  in flat Euclidean  $2d$ -dimensional space in the constant background of a maximally symmetric electromagnetic field<sup>1</sup>. The field strength can be written in block form:

$$F_{\mu\nu} = f \operatorname{diag}_d \left\{ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right\}. \tag{1}$$

The *dramatis persona* on the gauge side is the functional determinant arising in the one-loop effective action

$$S_{\text{gauge}}^{2d} = \ln \det\{-D^2 + M^2\}. \tag{2}$$

Standard manipulations lead to an expression in terms of the Green's function  $G$ , schematically  $(-D^2 + M^2)G = -1$ ,

$$S_{\text{gauge}}^{2d} = - \int^{M^2} \operatorname{tr} G^{2d}, \tag{3}$$

where the trace  $\operatorname{tr}$  means volume integral (large box of length  $L$ ) of the diagonal entry  $G^{2d}(x, x)$ . The heat kernel gives a useful representation of this diagonal entry, and a starting point to deal with the inherent short-distance divergence:

$$- \int^{M^2} \operatorname{tr} G^{2d} = L^{2d} \int_0^\infty \frac{ds}{s} \frac{1}{(4\pi s)^d} \left( \frac{fs}{\operatorname{sh}(fs)} \right)^d e^{-sM^2}. \tag{4}$$

The divergence at the lower integration value of the proper-time integral could be cured if the dimension  $d$  were negative enough. This observation is the key to dimensional regularization where the dimension is  $D \equiv d - \epsilon$  and one can work out the value of the effective action in closed form:

$$S_{\text{gauge}}^{2D} = - \left( \frac{fL^2}{2\pi} \right)^D \Gamma(1 - D) \int^{\frac{M^2}{f}} d\mu \frac{\Gamma(\frac{D}{2} + \mu)}{\Gamma(1 - \frac{D}{2} + \mu)}. \tag{5}$$

The divergence as  $D \rightarrow d$  comes entirely from the  $\Gamma(1 - D)$  factor for any integer value of  $d$ , even or odd.

Now, the crucial observation in [3] is related to the presence of Barnes' multiple gamma function in the renormalized value of this expression. A quick way to understand the Barnes' multiple gamma function is to go back to the heat kernel representation and perform a 'Weierstrass regularization', that is, subtract successive terms of the Taylor expansion of the integrand to achieve convergence in the lower integration limit. We stick to the conventions in [6, 7] for Barnes' multiple gamma function (A.1), modulo renormalization scheme-dependent

<sup>1</sup> We refer to [3] for conventions and further references.

terms which are polynomial and logarithmic in  $M$  (cf [3]), and obtain the finite renormalized value

$$S_{\text{gauge}}^{2d} = \left(\frac{fL^2}{2\pi}\right)^d \cdot \ln \Gamma_d \left(\frac{d}{2} + \frac{M^2}{2f}\right). \quad (6)$$

Alternatively, expanding in  $\epsilon$  and cancelling the pole from  $\Gamma(1 - D)$  with the linear-in- $\epsilon$  term in the expansion of the integrand in equation (5), one obtains

$$\begin{aligned} S_{\text{gauge}}^{2d} &= \left(\frac{fL^2}{2\pi}\right)^d \frac{(-1)^{d-1}}{2(d-1)!} \int^{\frac{M^2}{2f}} d\mu \frac{\Gamma\left(\frac{d}{2} + \mu\right)}{\Gamma\left(1 - \frac{d}{2} + \mu\right)} \left(\psi\left(\frac{d}{2} + \mu\right) + \psi\left(1 - \frac{d}{2} + \mu\right)\right) + \dots \\ &= \left(\frac{fL^2}{2\pi}\right)^d \frac{(-1)^{d-1}}{(d-1)!} \int^{\frac{M^2}{2f}} d\mu \left\{ \prod_{j=0}^{d-2} \left(\frac{d}{2} + \mu - 1 - j\right) \right\} \psi\left(\frac{d}{2} + \mu\right) + \dots \\ &= \left(\frac{fL^2}{2\pi}\right)^d \cdot \ln \Gamma_d \left(\frac{d}{2} + \frac{M^2}{2f}\right) + \dots \end{aligned} \quad (7)$$

Any subtraction or renormalization prescription will eventually lead to Barnes' multiple gamma, modulo the aforementioned logarithmic and polynomial terms related to a renormalization scheme ambiguity.

### Gravity side

Let us now turn to the gravity side and consider a massive ( $m$ ) Dirac spinor  $\psi$  minimally coupled to gravity in the Euclidean  $\text{AdS}_d$  background, i.e. hyperbolic space  $H^d$ . The *dramatis persona* on the gravity side is the functional determinant in the one-loop effective action

$$S_{\text{gravity}}^d = -\ln \det\{\not{\nabla} + m\}. \quad (8)$$

Again, standard manipulations bring in the Green's function,  $(\not{\nabla} + m)\mathcal{D} = -\mathbb{I}$ ,

$$S_{\text{gravity}}^d = \int^m \text{tr } \mathcal{D}^d, \quad (9)$$

where now the spinor indices are also traced out.

The spinor propagator in  $\text{AdS}_d$  is a well-known quantity [4], involving Gauss' hypergeometric function, and its diagonal entry can be worked out in dimensional regularization

$$\mathcal{D}^D(x, x) = -\frac{\mathbb{I}}{\sqrt{\Lambda}} \left(\frac{\Lambda}{4\pi}\right)^{D/2} \Gamma\left(1 - \frac{D}{2}\right) \frac{\Gamma\left(\frac{D}{2} + \frac{m}{\sqrt{\Lambda}}\right)}{\Gamma\left(1 - \frac{D}{2} + \frac{m}{\sqrt{\Lambda}}\right)}, \quad (10)$$

with  $\mathbb{I}$  denoting the identity matrix in spinor indices, whose dimensionality is  $2^{[D/2]}$ . The analogy with the gauge result is already apparent, but there is a seemingly small mismatch in the gamma factor:  $\Gamma\left(1 - \frac{D}{2}\right)$  instead of  $\Gamma(1 - D)$ . This makes a tiny difference when  $d = \text{even}$ ; both expressions have a pole and the finite piece involves Barnes' multiple gamma. However, when  $d = \text{odd}$ , there is a huge difference and there is no mapping of gauge and gravity quantities. On the gauge side there is still a pole, but the gravity side is finite in the limit  $D \rightarrow d$ . Our contribution in this paper is to amend this discrepancy by taking into account the dimensionally regularized volume of Euclidean AdS which is finite for  $d = \text{even}$ , whereas for  $d = \text{odd}$  one finds the missing pole in the above discussion.

To compute the dimensionally regularized volume of  $H^D$ , start with the metric

$$ds^2 = \frac{R^2}{r^2} \left[ dr^2 + \frac{(1-r^2)^2}{4} d\Omega_{D-1}^2 \right], \quad (11)$$

where  $R = 1/\sqrt{\Lambda}$  is the radius of AdS. The volume is then [5]

$$\begin{aligned} \text{vol}(H^D) &= R^D 2^{1-D} \text{vol}(S^{D-1}) \int_0^1 dr r^{-D} (1-r^2)^{D-1} \\ &= \pi^{\frac{D}{2}-\frac{1}{2}} R^D \Gamma\left(\frac{1}{2} - \frac{D}{2}\right). \end{aligned} \quad (12)$$

Note that the volume has a pole for  $D \rightarrow$  odd and is finite for  $D \rightarrow$  even, so that when combined with the effective Lagrangian there is always a divergence as  $D$  goes to the physical dimension, even or odd, as in the gauge side. Using Legendre’s duplication formula for the gamma functions we end up with

$$S_{\text{gravity}}^D = -2^{[D/2]} \Gamma(1-D) \int_{\sqrt{\Lambda}}^{\frac{m}{\sqrt{\Lambda}}} d\mu \frac{\Gamma\left(\frac{D}{2} + \mu\right)}{\Gamma\left(1 - \frac{D}{2} + \mu\right)}. \quad (13)$$

Now one can appreciate the mapping to the gauge one-loop effective action (5).

*Even dimensions:  $d = 2n$*

Let us now go to the physical dimension  $d = 2n$ , as first considered in [3]. The volume asymptotics,  $D = 2n - \epsilon$  as  $\epsilon \rightarrow 0$ , is finite:

$$\text{vol}(H^D) = \mathcal{V}_{2n} + o(1), \quad (14)$$

with  $\mathcal{V}_{2n} = \pi^{n-1/2} R^{2n} \Gamma\left(\frac{1}{2} - n\right) = (-1)^n \frac{\pi^{n+1/2}}{\Gamma(n+\frac{1}{2})} R^{2n}$ .

Note that our renormalized volume differs by a factor  $(-1)^n$  from the volume element used in [3]. We welcome this extra factor for it makes more precise the gauge–gravity mapping.

The important cancelation occurs in the expansion of the effective Lagrangian, very much as in the gauge side,

$$S_{\text{gravity}}^{2n} = \mathcal{V}_{2n} \left(\frac{\Lambda}{4\pi}\right)^n \frac{(-2)^{n+1}}{(n-1)!} \int_{\sqrt{\Lambda}}^{\frac{m}{\sqrt{\Lambda}}} d\mu \left\{ \prod_{j=0}^{2n-2} (n + \mu - 1 - j) \right\} \psi(n + \mu) + \dots \quad (15)$$

After including the volume factor and using again the duplication formula for the gamma function, we finally obtain [3]

$$S_{\text{gravity}}^{2n} = 2^n \cdot \ln \Gamma_{2n} \left( n + \frac{m}{\sqrt{\Lambda}} \right), \quad (16)$$

which ought to be compared with the gauge result (6).

*Odd dimensions:  $d = 2n + 1$*

This case is qualitatively different from the above; now the volume gets mixed with the subleading term in the  $\epsilon$ –expansion of the effective Lagrangian. The volume asymptotics behaves as

$$\text{vol}(H^D) = \frac{1}{\epsilon} \mathcal{L}_{2n+1} + \mathcal{V}_{2n+1} + o(1). \quad (17)$$

The precise value of the renormalized volume  $\mathcal{V}_{2n+1}$  will not be relevant for it only contributes a polynomial in the mass. The important term is the ‘integrated holographic trace anomaly’ [8–10] which is unambiguously given by  $\mathcal{L}_{2n+1} = (-1)^n \frac{2\pi^n}{\Gamma(n+1)} R^{2n+1}$ .

The effective Lagrangian is now finite as  $D \rightarrow 2n + 1$ , and polynomial in  $m$ . The nontrivial part, which eventually leads to Barnes’ multiple gamma function, comes from the

cancelation of the pole in the volume against the linear-in- $\epsilon$  term in the expansion of the effective Lagrangian<sup>2</sup>

$$\begin{aligned}
 S_{\text{gravity}}^{2n+1} &= \frac{\Gamma\left(\frac{1}{2} - n\right)}{(-2)^n \pi^{\frac{1}{2}} \Gamma(1+n)} \int^{\frac{m}{\sqrt{\Lambda}}} d\mu \frac{\Gamma\left(n + \frac{1}{2} + \mu\right)}{\Gamma\left(\frac{1}{2} - n + \mu\right)} \\
 &\quad \times \left( \psi\left(n + \frac{1}{2} + \mu\right) + \psi\left(\frac{1}{2} - n + \mu\right) \right) + \dots \\
 &= \frac{2^{1-n} \pi^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2} + n\right) \Gamma(1+n)} \int^{\frac{m}{\sqrt{\Lambda}}} d\mu \left\{ \prod_{j=0}^{2n-1} \left(n - \frac{1}{2} + \mu - j\right) \right\} \psi\left(n + \frac{1}{2} + \mu\right) + \dots
 \end{aligned} \tag{18}$$

A little algebraic manipulation yields

$$S_{\text{gravity}}^{2n+1} = 2^n \cdot \ln \Gamma_{2n+1} \left( n + \frac{1}{2} + \frac{m}{\sqrt{\Lambda}} \right). \tag{19}$$

In all, on the gravity side the one-loop effective action after inclusion of the volume term in  $d$  dimensions, even and odd, reads

$$S_{\text{gravity}}^d = 2^{\lfloor d/2 \rfloor} \cdot \ln \Gamma_d \left( \frac{d}{2} + \frac{m}{\sqrt{\Lambda}} \right) \tag{20}$$

in remarkable correspondence with the gauge one-loop effective action (6) and with  $d = \text{odd}$  and  $d = \text{even}$  on equal footing.

**Acknowledgments**

We kindly acknowledge E Friedman and S Ruijsenaars for a useful conversation on Barnes’ zeta and gamma functions. DED wishes to thank H Dorn for valuable discussions. This work was partially funded through Fondecyt-Chile 3090012, UNAB AR-01-09 and UNAB AR-04-09.

**Appendix. Useful identities**

- Integral representation of Barnes’ multiple gamma, which effectively performs the Weierstrass regularization:

$$\begin{aligned}
 \ln \Gamma_N(w) &= \int_0^\infty \frac{dt}{t} \left( e^{-wt} \prod_{j=1}^N \frac{1}{1 - e^{-a_j t}} \right. \\
 &\quad \left. - t^N \sum_{k=0}^{N-1} \frac{(-t)^k}{k!} B_{N,k}(w) - \frac{(-1)^N}{N!} e^{-t} B_{N,N}(w) \right),
 \end{aligned} \tag{A.1}$$

with  $B_{N,k}(w)$  being the Bernoulli-type polynomials [6, 7].

- Gamma function, duplication formula

$$\Gamma(z) \Gamma\left(z + \frac{1}{2}\right) = 2^{1-2z} \Gamma\left(\frac{1}{2}\right) \Gamma(2z). \tag{A.2}$$

- Gamma function, reflection formula

$$\Gamma\left(\frac{1}{2} + z\right) \Gamma\left(\frac{1}{2} - z\right) = \frac{\pi}{\cos \pi z}. \tag{A.3}$$

<sup>2</sup> An analogous ‘conspiracy’ was also crucial in establishing a holographic formula for functional determinant in AdS [5].

## References

- [1] Branson T 1993 *The Functional Determinant* (Lecture Note Series vol 4) (Seoul: Global Analysis Research Center, Seoul National University)
- [2] Dunne G V 2008 Functional determinants in quantum field theory *J. Phys. A: Math. Theor.* **41** 304006 (arXiv:0711.1178 [hep-th])
- [3] Basar G and Dunne G V 2010 A Gauge-gravity relation in the one-loop effective action *J. Phys. A: Math. Theor.* **43** 072002 (arXiv:0912.1260 [hep-th])
- [4] Camporesi R 1992 The Spinor heat kernel in maximally symmetric spaces *Commun. Math. Phys.* **148** 283
- [5] Diaz D E and Dorn H 2007 Partition functions and double-trace deformations in AdS/CFT *J. High Energy Phys.* **JHEP05(2007)046** (arXiv:hep-th/0702163)
- [6] Ruijsenaars S N M 2000 On Barnes' multiple zeta and gamma functions *Adv. Math.* **156** 107–32
- [7] Friedman E and Ruijsenaars S N M 2004 Shintani–Barnes zeta and gamma functions *Adv. Math.* **187** 362–95
- [8] Henningson M and Skenderis K 1998 The holographic Weyl anomaly *J. High Energy Phys.* **JHEP07(1998)023** (arXiv:hep-th/9806087)  
Henningson M and Skenderis K 2000 Holography and the Weyl anomaly *Fortsch. Phys.* **48** 125 (arXiv:hep-th/9812032)
- [9] Graham C R 2000 Volume and area renormalizations for conformally compact Einstein metrics *Rend. Circ. Mat. Palermo* (2) Suppl. no. 63 31–42 (arXiv:math.DG/9909042)
- [10] Albin P 2005 Renormalizing Curvature Integrals on Poincare–einstein Manifolds arXiv:math.DG/0504161