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New classes of bi-axially symmetric solutions to four-dimensional Vasiliev higher spin gravity

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Abstract: We present new infinite-dimensional spaces of bi-axially symmetric asymptotically anti-de Sitter solutions to four-dimensional Vasiliev higher spin gravity, obtained by modifications of the Ansatz used in arXiv:1107.1217, which gave rise to a Type-D solution space. The current Ansatz is based on internal semigroup algebras (without identity) generated by exponentials formed out of the bi-axial symmetry generators. After having switched on the vacuum gauge function, the resulting generalized Weyl tensor is given by a sum of generalized Petrov type-D tensors that are Kerr-like or 2-brane-like in the asymptotic AdS_4 region, and the twistor space connection is smooth in twistor space over finite regions of spacetime. We provide evidence for that the linearized twistor space connection can be brought to Vasiliev gauge.

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Contents

1 Introduction

Vasiliev's equations [1] (for a recent review, see [2]) provide a fully nonlinear description of higher spin gauge fields in four dimensions coupled to gravity and matter fields. The basic feature of Vasiliev's theory is that the full field configurations are captured by master fields that live on an extension of spacetime by a noncommutative twistor space. The equations admit an exact solution given by the direct product of anti-de Sitter spacetime and an undeformed twistor space. In a specific gauge, certain linearized perturbations of the non-commutative twistor space structure give rise to Fronsdal fields. This suggests a holographic relationship to three-dimensional conformal field theories [3–5]; see also [6–8]. In [9, 10] this relation was examined under the assumption that the Gubser-Klebanov-Polyakov-Witten (GKPW) prescription [11, 12] for on-shell computations of Witten diagrams can be applied to classical field configurations obtained from Vasiliev's equations.

However, the Fronsdal fields embedded into Vasiliev's master fields have non-local interactions [13–15][1] that belong to a functional class widely separated [17] from that of the quasi-local Fronsdal theory [18], which is built by applying the canonical Noether approach to Fronsdal fields in anti-de Sitter spacetime. The GKPW prescription applies to the quasi-local theory by construction, as its action has self-adjoint kinetic terms, and the resulting holographic correlation functions indeed correspond to free three-dimensional conformal field theories.[2] Recent work [22] shows that there exists an explicit field redefinition that maps Vasiliev's theory to a quasi-local theory on-shell, obtained by carefully fine-tuning the perturbative expansion on the Vasiliev side, though it remains to be seen whether it coincides with that of [18]. Moreover, as later shown in [23] the required field redefinition is large, and hence it is unclear to what extent the method can be used to actually compute any holographic correlation functions. Thus, to our best understanding, the issue of whether holographic amplitudes can be extracted by applying the GKPW prescription to the Fronsdal fields embedded into Vasiliev's master fields remains an open problem.

An alternative approach, pursued in [24], is to seek a weaker relation between the two theories, namely at the level of two distinct effective actions, derived in their own rights following different principles, and then evaluated subject to suitable dual boundary conditions. To this end, one starts from Hamilton's principle applied to a covariant Hamiltonian action formulated using Weyl order on a noncommutative manifold whose boundary is given by the direct product of spacetime and twistor space [24, 25]; the Weyl order is required for the noncommutative version of the Stokes' theorem to hold and for the imposition of boundary conditions. The resulting variational principle yields Vasiliev's equations in Weyl order, that can be mapped back to Vasiliev's normal order for special classes of initial data in twistor space following the perturbative scheme set up in [26, 27]. The resulting form of the higher spin amplitudes [28–30] is closely related to first-quantized topological open

[1] For a review, see [16].

[2] The functional class encountered in the quasi-local Fronsdal theory in [18] (within the AdS/CFT context) has not yet been identified completely; for a discussion, see [17, 19] and section 7 of [20]. At the cubic order, the separation between the functional classes of this theory and the Vasiliev theory has been spelled out in [21].

string amplitudes [31], but nonetheless reproduce exactly the same correlation functions as the Witten diagrams computed in the quasi-local theory. We would like to stress the fact that the Hamiltonian form of the action implies that the dependence of the classical Vasiliev master fields on classical sources are of a different type than for fields obeying equations of motion following from an action with self-adjoint kinetic terms. Indeed, instead of applying the GKPW prescription, the higher spin amplitudes are obtained from functionals given by topological boundary terms added to the Hamiltonian action [24, 29, 32], whose on-shell values are given by higher spin invariants, as we shall comment on further below.

In this paper, we shall construct new perturbatively defined solution spaces to Vasiliev's equations in Weyl order, by taking into account classes of functions that resemble closely those used in [26]. We shall then demonstrate explicitly that they can be mapped to Vasiliev's normal order, at least at the linearized level, thus providing further evidence in favour of the covariant Hamiltonian approach outlined above.

To this end, we recall that at the linearized level, the fluctuations in the master fields that are asymptotic to anti-de Sitter spacetime form various representation spaces of the anti-de Sitter isometry algebra, including lowest-weight spaces as well as spaces associated to linearized solitons [33] and generalized Petrov type-D solutions [26, 33, 34]. Nonlinear completions of various Type-D solution spaces were constructed in [26, 33, 34]; for a review, see [35]. Of direct relevance for the work in this paper is the subspace that contains the the black-hole-like solutions,[3] including spherically symmetric solutions. In these solutions, each individual Fronsdal field has a point-like source at the origin, showing up as a divergence in its Weyl tensor. However, upon packing all curvatures into a master zero-form, one obtains the symbol of a quantum-mechanical operator that approaches a delta function distribution at the origin [26], which defines a smooth state as seen via classical observables given by zero-form charges [29, 30, 37]. In this sense, the black-hole-like Type-D solutions to Vasiliev's theory are source free at the origin.[4] Furthermore, it is possible to dress these solutions with lowest-weight space modes [27] at the fully nonlinear level; in doing so, the latter modes induce Type-D modes already at the second order of classical perturbation theory.[5]

Clearly, the full extent of the moduli space of the theory yet remains to be determined. In this paper, we shall present a new infinite-dimensional class of bi-axially symmetric exact solutions that are asymptotic to anti-de Sitter spacetime and singularity free at the level of zero-form charges. We shall furthermore propose a super-selection mechanism based on requiring that the solutions can be brought to Vasiliev gauge (where the asymptotic linearized fluctuations are in terms of Fronsdal fields).

Our construction method follows closely the one devised in [26] using gauge functions and separation of twistor space variables, which is in effect equivalent to starting from an Ansatz in Weyl order. The key difference is that we shall expand the master fields over a

[3]This subspace is related to the massless spectrum by means of a $\mathbf{Z}_2$-operation [26], reminiscent of a U-duality transformation [36].

[4]It remains to be examined whether additional topological two-forms describing Dirac strings need to be activated in the dynamical two-form [24, 38].

[5]This phenomena resembles some of the scattering processes in U-duality covariant field theory [36].

new set of elements in the associative fiber algebra, thus adding a branch to the existing moduli space. In a generic gauge, the expansion coefficients are functions on the base manifold. However, in the holomorphic gauge of [26] the Weyl zero-form is a constant while the twistor space one-form is given by a universal set of functions, related to Wigner's deformed oscillators, originally derived within the context of three-dimensional matter coupled higher spin gravity [39]. The resulting solution space is then mapped to Vasiliev gauge in which the spacetime one-form consists of nonlinear Fronsdal tensors (after a suitable field redefinition in order to reinstate manifest Lorentz covariance). This map is achieved by means of two consecutive (large) gauge transformations: first, one uses a vacuum gauge function in SO(2,3)/SO(1,3).[6] Provided that the resulting twistor space connection is smooth at the origin of the base of the twistor space, Vasiliev gauge can be reached by means of a second perturbatively defined gauge transformation. As we shall see, the real-analyticity requirement constrains the initial data in the Weyl zero-form already at the linearized level.[7]

More specifically, the new sector of the fiber algebra is isomorphic to the group algebra $\mathbb{C}[\mathbb{Z} \times \mathbb{Z}]$ where $\mathbb{Z} \times \mathbb{Z}$ is generated by two elements in $Sp(4;\mathbb{C})$ given by exponentials of a pair of Cartan generators of $sp(4;\mathbb{R})$. These correspond to linear symmetries of the two-dimensional harmonic oscillator, and generate the Killing symmetries of the solutions (including higher spin symmetries). As we shall see, the aforementioned super-selection rule amounts to restricting the master fields to a subalgebra of the group algebra not containing the unity.

The paper is organized as follows: in section 2 we review parts of Vasiliev's bosonic higher spin gravity model that we shall use in constructing and interpreting the exact solutions. Solution spaces based on (semi)group algebras are constructed in section 3 using the aforementioned method; the singular nature of the contribution from the identity is pointed out in section 3.4. In section 4, we show that the Weyl tensor is given by a sum of Petrov type-D tensors that are Kerr-like or 2-brane-like in the asymptotic AdS_4 region, and we compute higher spin curvature invariants. In section 5, we show in special cases that the twistor space one-form is real-analytic in twistor space over finite regions of spacetime, and that its linearized part can be brought to Vasiliev gauge. We conclude in section 6.

2 Bosonic Vasiliev model

In this section, we describe the non-minimal bosonic higher spin gravity model of Vasiliev type [1],[8] for which we shall present exact solutions in the next section. The model is characterized by the fact that it admits a linearization consisting of real Fronsdal fields in four-dimensional anti-de Sitter spacetime of spins $s = 0, 1, 2, \ldots$ with each spin occurring once; for further details, we refer to [13, 26] and the review [2].

[6]Whether a more general vacuum gauge function can introduce additional classical moduli remains an open problem.

[7]An optional criterion is that the fiber algebra is a unitarizable representation of the higher spin algebra and hence the anti-de Sitter isometry algebra; we expect this property to arise at higher orders of classical perturbation theory by requiring positivity of a suitable free energy functional.

[8]For recent reformulations containing the original Vasiliev system as consistent truncations, see [24, 38].

We first provide the formal definition in terms of master fields on the direct product of a commuting space and a noncommutative twistor space. We then spell out the component form of the equations, including their reformulation in terms of deformed oscillators. Finally, we remark on choices of bases for the internal algebra, and the Lorentz covariant weak field expansion scheme leading to Fronsdal fields, stressing the role of Vasiliev gauge and smoothness in twistor space.

2.1 Master field equations

Vasiliev's original formulation of higher spin gravity is given in terms of two master fields Φ and A of degrees 0 and 1, respectively, and two closed and twisted-central elements I and $\overline{I}$ of degree 2, all of which are elements of a differential graded associative algebra $\Omega(\mathcal{M})$ of forms on a non-commutative manifold $\mathcal{M}$, valued in an internal associative algebra $\mathcal{A}$. Letting $\star$ denote the associative product of $\Omega(\mathcal{M})\otimes\mathcal{A}$, which is assumed to be compatible with d, the fully nonlinear master field equations read

$$F+\mathcal{B}\star\Phi\star I-\overline{\mathcal{B}}\star\Phi\star\overline{I}=0\,, \tag{2.1}$$

$$D\Phi=0\,, \tag{2.2}$$

where

$$F:=dA+A\star A\,,\qquad D\Phi:=d\Phi+A\star\Phi-\Phi\star\pi\left(A\right)\,, \tag{2.3}$$

and π denotes an automorphism of the differential graded associative algebra. The two-forms are characterised by the subsidiary constraints

$$dI=0\,,\qquad I\star f\;=\;\pi(f)\star I\,, \tag{2.4}$$

for any $f\in\Omega(\mathcal{M})\otimes\mathcal{A}$, idem $\overline{I}$. Finally, the star functions

$$\mathcal{B}:=\sum_{n=0}^{\infty}b_n(\Phi\star\pi(\Phi))^{\star n}\,,\qquad\overline{\mathcal{B}}:=\sum_{n=0}^{\infty}\bar{b}_n(\Phi\star\pi(\Phi))^{\star n}\,, \tag{2.5}$$

where $b_n,\bar{b}_n\in\mathbb{C}$. It follows that $\Phi\star\pi(\Phi)$ and hence $\mathcal{B}$ is covariantly constant, viz.

$$d\mathcal{B}+A\star\mathcal{B}-\mathcal{B}\star A=0\,, \tag{2.6}$$

idem $\overline{\mathcal{B}}$. As $\Phi\star I$ and $\Phi\star\overline{I}$ are covariantly constant as well, it follows that the constraint on F is compatible with its Bianchi identity. The integrability of the constraint on $D\Phi$, on the other hand, requires $F\star\Phi-\Phi\star\pi(F)$ to vanish, which is indeed a consequence of the constraint on F. The resulting Cartan integrability, i.e. consistency with $d^2\equiv0$, holds for any dimension of $\mathcal{M}$ and any star functions $\mathcal{B}$ and $\overline{\mathcal{B}}$, which are hence not fixed uniquely by the requirement of higher spin symmetry alone.

In the context of higher spin gravity, it is usually assumed that

$$\mathcal{M}=\mathcal{X}_4\times\mathcal{Z}_4\,, \tag{2.7}$$

where $\mathcal{X}_4$ is a four-dimensional real commuting manifold, with coordinates x^μ, and $\mathcal{Z}_4$ is a four-dimensional real non-commutative symplectic manifold, with canonical coordinates

$Z^{\underline{\alpha}}$. The compatibility between the star product and the differential amounts to the Leibniz' rule

$$d(f \star g) = df \star g + (-1)^{\deg(f)} f \star dg \,. \tag{2.8}$$

The differential star product algebra is assumed to be trivial in strictly positive degrees, in the sense that $d\Xi^M := (dx^\mu, dz^\alpha, d\bar{z}^{\dot{\alpha}})$ are taken to be graded anti-commuting elements obeying

$$d\Xi^M \star f = d\Xi^M \wedge f \,, \qquad f \star d\Xi^M = f \wedge d\Xi^M \,, \tag{2.9}$$

which are consistent with associativity. The algebra $\Omega(\mathcal{M}) \otimes \mathcal{A}$ is also assumed to be equipped with an anti-linear anti-automorphism $\dagger$, for which we use the convention

$$(f_1 \star f_2)^\dagger = (-1)^{\deg(f_1)\deg(f_2)} f_2^\dagger \star f_1^\dagger \,, \qquad (df)^\dagger = d(f^\dagger) \,. \tag{2.10}$$

In case of the basic bosonic models, without internal Yang-Mills symmetries, the internal algebra $\mathcal{A}$ consists of classes of functions on yet one more four-dimensional real non-commutative symplectic manifold, that we shall denote by $\mathcal{Y}_4$, with canonical coordinates $Y^{\underline{\alpha}}$. We shall refer to $\mathcal{Y}_4 \times \mathcal{Z}_4$ as the full twistor space, and $\mathcal{Y}_4$ and $\mathcal{Z}_4$, respectively, as the internal and external twistor spaces.[9] The Sp(4;$\mathbb{R}$) quartets are split into SL(2;$\mathbb{C}$) doublets, viz.[10]

$$Y^{\underline{\alpha}} = (y^\alpha, \bar{y}^{\dot{\alpha}}) \,, \qquad Z^{\underline{\alpha}} = (z^\alpha, \bar{z}^{\dot{\alpha}}) \,, \tag{2.11}$$

obeying

$$\bar{y}^{\dot{\alpha}} = (y^\alpha)^\dagger \,, \qquad \bar{z}^{\dot{\alpha}} = -\,(z^\alpha)^\dagger \,, \tag{2.12}$$

The automorphism π and its hermitian conjugate $\bar{\pi}$ are defined by

$$\pi\left(x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}}\right) = \left(x^\mu; -y^\alpha, \bar{y}^{\dot{\alpha}}; -z^\alpha, \bar{z}^{\dot{\alpha}}\right) \,, \tag{2.13}$$
$$\bar{\pi}\left(x^\mu; y^\alpha, \bar{y}^{\dot{\alpha}}; z^\alpha, \bar{z}^{\dot{\alpha}}\right) = \left(x^\mu; y^\alpha, -\bar{y}^{\dot{\alpha}}; z^\alpha, -\bar{z}^{\dot{\alpha}}\right) \,, \tag{2.14}$$

and $\pi \circ d = d \circ \pi$ idem $\bar{\pi}$. Imposing

$$\Phi^\dagger = \pi(\Phi) \,, \qquad A^\dagger = -A \,, \qquad I^\dagger = \bar{I} \,, \tag{2.15}$$

and

$$\mathcal{B}^\dagger = \overline{\mathcal{B}} \,, \tag{2.16}$$

that is, $(b_n)^\dagger = \bar{b}_n$, and

$$\pi\bar{\pi}(\Phi) = \Phi \,, \qquad \pi\bar{\pi}(A) = A \,, \qquad \pi\bar{\pi}(I) = I \,, \qquad \pi\bar{\pi}\left(\bar{I}\right) = \bar{I} \tag{2.17}$$

yields a model with a perturbative expansion around four-dimensional anti-de Sitter spacetime in terms of Fronsdal fields of all integer spins.

The equations given so far provide a formal definition of the basic bosonic model.

[9] Taking the master fields to be smooth functions of $\mathcal{Y}_4$ yields an anti-de Sitter analog of the Penrose-Newman transformation; to our best understanding, the precise relation between $\mathcal{Y}_4 \times \mathcal{Z}_4$ and the original (commuting) twistor space of Penrose remains to be spelled out in detail.

[10] The doublet indices are raised and lowered using $f^\alpha = \varepsilon^{\alpha\beta} f_\beta$ and $f_\beta = f^\alpha \varepsilon_{\alpha\beta}$ idem $f^{\dot{\alpha}}$.

2.2 Star product, twisted central element and traces

In what follows, we shall use Vasiliev's original realization of the $\star$-product given by

$$\begin{aligned} & f_1(y,\bar y,z,\bar z)\star f_2(y,\bar y,z,\bar z) \\ &= \int \frac{d^2u d^2\bar u d^2 v d^2\bar v}{(2\pi)^4} e^{iv^\alpha u_\alpha + i\bar v^{\dot\alpha}\bar u_{\dot\alpha}} f_1(y+u,\bar y+\bar u; z+u,\bar z-\bar u)\, f_2(y+v,\bar y+\bar v; z-v,\bar z+\bar v)\,. \end{aligned} \tag{2.18}$$

We shall encounter $\star$-product compositions leading to Gaussian integrals involving indefinite bilinear forms. To define these we use the fact that the auxiliary integration is a formal representation of the original Moyal-like contraction formula, which means that the integration must be performed by means of analytical continuations of the eigenvalues of the bilinear forms.

Symbol calculus. The star product rule implies that

$$[f_1(y,\bar y), f_2(z,\bar z)]_\star = 0\,, \tag{2.19}$$

that is, the variables $Y^{\underline{\alpha}}$ and $Z^{\underline{\alpha}}$ are mutually commuting. Moreover, from

$$y_\alpha\star y_\beta = y_\alpha y_\beta + i\varepsilon_{\alpha\beta}\,,\ \ y_\alpha\star z_\beta = y_\alpha z_\beta - i\varepsilon_{\alpha\beta}\,,\ \ z_\alpha\star y_\beta = z_\alpha y_\beta + i\varepsilon_{\alpha\beta}\,,\ \ z_\alpha\star z_\beta = z_\alpha z_\beta - i\varepsilon_{\alpha\beta}\,, \tag{2.20}$$

it follows that

$$a^\pm_\alpha := \frac{1}{2}(y_\alpha \pm z_\alpha)\ , \tag{2.21}$$

obey

$$\left[a^-_\alpha, a^+_\beta\right]_\star = \left[a^+_\alpha, a^-_\beta\right]_\star = i\varepsilon_{\alpha\beta}\,,\ \ \left[a^+_\alpha, a^+_\beta\right]_\star = \left[a^-_\alpha, a^-_\beta\right]_\star = 0\,. \tag{2.22}$$

Letting $\mathcal{O}_{\text{Weyl}}$ and $\mathcal{O}_{\text{Normal}}$ denote the Wigner maps that send a classical function f to the operator with symbol f in the Weyl and normal order, respectively, where an operator is said to be in normal order if all $\mathcal{O}_{\text{Normal}}(a^+_\alpha)$ stand to the left of all $\mathcal{O}_{\text{Normal}}(a^-_\alpha)$. As a result, one has

$$\mathcal{O}_{\text{Normal}}(f_1(y,z)\star f_2(y,z)) = \mathcal{O}_{\text{Normal}}(f_1(y,z))\mathcal{O}_{\text{Normal}}(f_2(y,z))\,. \tag{2.23}$$

One also has

$$\mathcal{O}_{\text{Weyl}}(f(y)) = \mathcal{O}_{\text{Normal}}(f(y))\,,\qquad \mathcal{O}_{\text{Weyl}}(f(z)) = \mathcal{O}_{\text{Normal}}(f(z))\,, \tag{2.24}$$

resulting in that

$$\mathcal{O}_{\text{Weyl}}(f_1(y)\star f_2(y)) = \mathcal{O}_{\text{Weyl}}(f_1(y))\mathcal{O}_{\text{Weyl}}(f_2(y))\,, \tag{2.25}$$

$$\mathcal{O}_{\text{Weyl}}(f_1(z)\star f_2(z)) = \mathcal{O}_{\text{Weyl}}(f_1(z))\mathcal{O}_{\text{Weyl}}(f_2(z))\,, \tag{2.26}$$

and also

$$\mathcal{O}_{\text{Normal}}(f_1(y)\star f_2(z)) = \mathcal{O}_{\text{Weyl}}(f_1(y)\, f_2(z)) = \mathcal{O}_{\text{Weyl}}(f_1(y))\mathcal{O}_{\text{Weyl}}(f_2(z))\,. \tag{2.27}$$

Twisted central element. The condition (2.4) can be solved by

$$I = j_z \star \kappa_y \,, \qquad j_z = \frac{i}{4} dz^\alpha \wedge dz^\beta \varepsilon_{\alpha\beta} \kappa_z \,, \qquad \kappa_y = 2\pi \delta^2(y) \,, \qquad \kappa_z = 2\pi\delta^2(z) \,, \tag{2.28}$$

where κ_y is an inner Klein operator obeying

$$\kappa_y \star f(y) \star \kappa_y = f(-y) \,, \qquad \kappa_y \star \kappa_y \;=\; 1 \,, \tag{2.29}$$

idem κ_z. Thus, one may write

$$I = \frac{i}{4} dz^\alpha \wedge dz^\beta \varepsilon_{\alpha\beta} \kappa \,, \qquad \kappa := \kappa_y \star \kappa_z = \exp(i y^\alpha z_\alpha) \,, \tag{2.30}$$

where thus

$$\kappa \star f(y,z) = \kappa f(z,y) \,, \qquad f(y,z) \star \kappa = \kappa f(-z,-y) \,, \tag{2.31}$$

$$\kappa \star f(y,z) \star \kappa = \pi(f(y,z)) \,, \qquad \kappa \star \kappa = 1 \,. \tag{2.32}$$

By hermitian conjugation one obtains

$$\overline{I} = -I^\dagger = \frac{i}{4} d\bar{z}^{\dot\alpha} \wedge d\bar{z}^{\dot\beta} \varepsilon_{\dot\alpha\dot\beta} \bar\kappa \,. \tag{2.33}$$

The two-forms j_z and $\overline{j}_{\bar z}$ can be extended to globally defined forms on a non-commutative space $\mathcal{Z}_4$ having the topology of a direct product of two complexified two-spheres [24, 26], with nontrivial flux

$$\int_{\mathcal{Z}_4} j_z \star \overline{j}_{\bar z} = -\frac{1}{4} \,. \tag{2.34}$$

In this topology, it is furthermore assumed that Φ belongs to a section that is bounded at infinity, while the twistor-space one-form is a connection whose curvature two-form falls off at infinity.

We note that the form of I given in eq. (2.30) is useful in deriving the perturbative expansion in terms of Fronsdal fields in Vasiliev gauge, while the factorized form in eq. (2.28) is useful in finding exact solutions.

Trace operations. The detailed form of the symbol of an operator depends on the basis with respect to which it is defined. Its trace, on the other hand, is basis independent, and in addition gauge invariant. The star product algebra admits two natural trace operations. The basic operation is given by the integral over phase space using the symplectic measure, viz.

$$\mathrm{Tr} f := \int_{\mathcal{Z}_4 \times \mathcal{Y}_4} j_y \star \overline{j}_{\bar y} \star \kappa_y \star \bar\kappa_{\bar y} \star f \,, \qquad f \in \Omega(\mathcal{Z}_4) \otimes \mathcal{A} \,, \tag{2.35}$$

where j_y is given by replacing z^α by y^α in j_z defined in eq. (2.28). An alternative trace operation, of relevance to higher spin gauge theory, can be defined if $\mathcal{A}$ admits the decomposition

$$\mathcal{A} = \bigoplus_{n,\bar n = 0,1} \mathcal{A}_{n,\bar n} \star (\kappa_y)^n \star (\bar\kappa_{\bar y})^{\bar n} \,, \tag{2.36}$$

where $\mathcal{A}_{n,\bar{n}}$ consist of operators whose symbols in Weyl order are regular at the origin of $\mathcal{Y}_4$. One may then define the trace operation

$$\mathrm{Tr}'f := \int_{\mathcal{Y}_4} j_y \star \bar{j}_{\bar{y}} \star f_{1,\bar{1}} = -\frac{1}{4} f_{1,\bar{1}}|_{y=0=\bar{y}}\,, \tag{2.37}$$

using the decomposition (2.36), with the convention that

$$\kappa_y \star \bar{\kappa}_{\bar{y}} \star f = \pm f \quad \Rightarrow \quad \mathrm{Tr}'f = \mp\frac{1}{8} f|_{y=0=\bar{y}}\,. \tag{2.38}$$

One may view Tr' as a regularized version of Tr in the sense that if f admits a decomposition of the form (2.36) then

$$\mathrm{Tr}f = \sum_{n,\bar{n}=0,1} \mathrm{Tr}f_{n,\bar{n}} \star (\kappa_y)^n \star (\bar{\kappa}_{\bar{y}})^{\bar{n}} \tag{2.39}$$

$$= \mathrm{Tr}'f + \mathrm{Tr}(f_{0,\bar{0}} + f_{1,\bar{0}} \star \kappa_y + f_{0,\bar{1}} \star \bar{\kappa}_{\bar{y}})\,, \tag{2.40}$$

that is,

$$\mathrm{Tr}'f = \mathrm{Tr}f - \mathrm{Tr}(f_{0,\bar{0}} + f_{1,\bar{0}} \star \kappa_y + f_{0,\bar{1}} \star \bar{\kappa}_{\bar{y}})\,. \tag{2.41}$$

Indeed, in several applications it turns out that $\mathrm{Tr}f$ is ill-defined while $\mathrm{Tr}'f$ is well-defined, as for example in the case that f is a polynomial on $\mathcal{Y}_4$.

2.3 Equations in components and deformed oscillators

We decompose the master one-form into locally defined components as follows:

$$A = U_\mu dx^\mu + V_\alpha dz^a + V_{\dot{\alpha}} d\bar{z}^{\dot{\alpha}}\,, \tag{2.42}$$

The reality condition (2.15) and the bosonic projection (2.17) imply

$$U_\mu^\dagger = -U_\mu\,, \qquad V_\alpha^\dagger = \bar{V}_{\dot{\alpha}}\,, \tag{2.43}$$

$$\pi\bar{\pi}\,(U_\mu) = U_\mu\,, \qquad \pi\bar{\pi}\,(V_\alpha) = -V_\alpha\,. \tag{2.44}$$

Decomposing master equations into components using inner derivatives $\imath_{\partial_\mu}$, $\imath_{\partial_\alpha}$ and $\imath_{\partial_{\dot{\alpha}}}$, where $\partial_\alpha \equiv \partial/\partial z^\alpha$ idem $\partial_{\dot{\alpha}}$, one has

$$\partial_{[\mu} U_{\nu]} + U_{[\mu} \star U_{\nu]} = 0\,, \tag{2.45}$$

$$\partial_\mu \Phi + U_\mu \star \Phi - \Phi \star \pi\,(U_\mu) = 0\,, \tag{2.46}$$

the mixed components

$$\partial_\mu V_\alpha - \partial_\alpha U_\mu + [U_\mu, V_\alpha]_\star = 0\,, \qquad \partial_\mu \bar{V}_{\dot{\alpha}} - \partial_{\dot{\alpha}} U_\mu + \left[U_\mu, \bar{V}_{\dot{\alpha}}\right]_\star = 0\,, \tag{2.47}$$

which are related by hermitian conjugation, and

$$\partial_{[\alpha} V_{\beta]} + V_{[\alpha} \star V_{\beta]} + \frac{i}{4}\varepsilon_{\alpha\beta}\mathcal{B} \star \Phi \star \kappa = 0\,, \quad \partial_{[\dot{\alpha}} \bar{V}_{\dot{\beta}]} + \bar{V}_{[\dot{\alpha}} \star \bar{V}_{\dot{\beta}]} + \frac{i}{4}\varepsilon_{\dot{\alpha}\dot{\beta}}\overline{\mathcal{B}} \star \Phi \star \bar{\kappa} = 0\,, \tag{2.48}$$

$$\partial_\alpha \Phi + V_\alpha \star \Phi - \Phi \star \bar{\pi}\,(V_\alpha) = 0\,, \qquad \partial_{\dot{\alpha}} \Phi + \bar{V}_{\dot{\alpha}} \star \Phi - \Phi \star \pi\left(\bar{V}_{\dot{\alpha}}\right) = 0\,, \tag{2.49}$$

$$\partial_\alpha \bar{V}_{\dot{\alpha}} - \partial_{\dot{\alpha}} V_\alpha + \left[V_\alpha, \bar{V}_{\dot{\alpha}}\right]_\star = 0\,, \tag{2.50}$$

where the two equations in eq. (2.48) are related by hermitian conjugation idem eq. (2.49).

The twistor space equations (2.48)–(2.50) can be rewritten by introducing Vasiliev's deformed oscillators [1]

$$S_\alpha = z_\alpha - 2iV_\alpha\,, \;\; \bar{S}_{\dot\alpha} = \bar{z}_{\dot\alpha} - 2i\bar{V}_{\dot\alpha}\,, \tag{2.51}$$

for which the reality condition and the bosonic projection take the form:

$$(S_\alpha)^\dagger = -\bar{S}_{\dot\alpha}\,, \tag{2.52}$$

$$\pi\bar\pi\,(S_\alpha) = -S_\alpha\,. \tag{2.53}$$

In terms of the new fields, the aforementioned equations read

$$[S_\alpha, S_\beta]_\star = -2i\varepsilon_{\alpha\beta}\,(1 - \mathcal{B}\star\Phi\star\kappa) \quad \text{and h.c.}\,, \tag{2.54}$$

$$S_\alpha\star\Phi + \Phi\star\pi\,(S_\alpha) = 0 \quad \text{and h.c.}\,, \tag{2.55}$$

$$\left[S_\alpha, \bar{S}_{\dot\beta}\right]_\star = 0\,, \tag{2.56}$$

as can be seen using

$$[z_\alpha, f]_\star = -2i\partial_\alpha f\,, \qquad [\bar{z}_{\dot\alpha}, f]_\star = -2i\partial_{\dot\alpha} f\,, \tag{2.57}$$

$$[z_\alpha, z_\beta]_\star = -2i\varepsilon_{\alpha\beta}\,, \qquad \left[\bar{z}_{\dot\alpha}, \bar{z}_{\dot\beta}\right]_\star = -2i\varepsilon_{\dot\alpha\dot\beta}\,, \tag{2.58}$$

$$[z_\alpha, \bar{z}_{\dot\alpha}]_\star = 0\,. \tag{2.59}$$

As we shall see below, the deformed oscillators are useful in defining the field redefinition to Lorentz covariant basis. They also provide a useful basis for finding exact solutions as they convert the differential equations on $\mathcal{Z}_4$ into algebraic equations that can be solved using Laplace transformation methods [39]; for related details, see [26].

2.4 Lorentz covariance, Fronsdal fields and Weyl tensors

To arrive at a perturbative formulation in terms of Fronsdal fields on $\mathcal{X}_4$, one first solves eqs. (2.47)–(2.50) subject to an initial datum for Φ and U_μ at $Z^{\underline{\alpha}} = 0$ in a perturbative expansion in the zero-form initial data in Vasiliev gauge[11]

$$z^\alpha V_\alpha = 0\,. \tag{2.60}$$

In this gauge, initial data for the zero-form given by generic smooth symbols on $\mathcal{Y}_4$ yields twistor space configurations that are smooth functions on $\mathcal{Y}_4\times\mathcal{Z}_4$. Letting $\omega_\mu^{\alpha\beta}$ denote the canonical Lorentz connection, one can show that [42] Φ, V_α and[12]

$$W_\mu := U_\mu - \frac{1}{4i}\left(\omega_\mu^{\alpha\beta} M_{\alpha\beta} + \bar\omega_\mu^{\dot\alpha\dot\beta} M_{\dot\alpha\dot\beta}\right)\,, \tag{2.61}$$

[11] At the linearized level, this gauge yields the canonical basis for unfolded linearized Fronsdal fields [1]; for further details, see [13] and the review [2]. Beyond the linearized approximation, it has been used in amplitude computations [9, 10, 28, 29] and related recent works [38, 40]. Most exact solutions found so far, however, have been given in other gauges argued to be equivalent to Vasiliev gauge; for example, see [26, 37, 41].

[12] The resulting manifestly Lorentz covariant form of the master field equations can be found in [26, 28].

where

$$M_{\alpha\beta} := y_\alpha y_\beta - z_\alpha z_\beta + S_\alpha \star S_\beta \,, \tag{2.62}$$

have Taylor expansions in $(Y^{\underline{\alpha}}, Z^{\underline{\alpha}})$ around $Y^{\underline{\alpha}} = Z^{\underline{\alpha}} = 0$ in terms of Lorentz tensors. The redefinition induces a shift symmetry that can be used to set the coefficient of $y_\alpha y_\beta$ in W_μ to zero, such that

$$W_\mu|_{Z=0} = e_\mu + W'_\mu \,, \qquad e_\mu = \frac{1}{2i} e_\mu^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} \,, \tag{2.63}$$

where W'_μ consists of a spin-one field and a tower of higher spin gauge fields with $s = 3, 4, \ldots$. Proceeding by assuming that $e_\mu^{\alpha\dot{\alpha}}$ defines a vierbein, and taking $\Phi|_{Z=0}$ and W'_μ to be weak fields in which the couplings in eqs. (2.45)–(2.46) can be expanded perturbatively, one can show that the resulting algebraically independent fields are given by the Lorentz scalar

$$\varphi := \Phi|_{Y=Z=0} \,, \tag{2.64}$$

the metric

$$g_{\mu\nu} := e_\mu^a e_{\nu,a} \,, \tag{2.65}$$

and the tower of doubly traceless tensor gauge fields

$$\varphi_{a_1 \ldots a_s} := (e^{-1})_{(a_1}{}^\mu W'_{\mu, a_2 \ldots a_s)} \,, \qquad s = 1, 3, 4, \ldots \,, \tag{2.66}$$

where $W'_{\mu, a_1 \ldots a_n}$ is the coefficient in W'_μ of $(\sigma^{a_1})_{\alpha\dot{\alpha}} y^\alpha \bar{y}^{\dot{\alpha}} \cdots (\sigma^{a_n})_{\alpha\dot{\alpha}} y^\alpha \bar{y}^{\dot{\alpha}}$. These fields obey equations of motion on the Lorentzian manifold $(\mathcal{X}_4, g_{\mu\nu})$ with second-order kinetic terms, critical masses and dynamical metric.[13]

The virtue of Vasiliev gauge is that the metric and the gauge fields (2.66) are identical to the Fronsdal tensors that can be obtained at the linearized level by integrating the generalized Weyl tensor

$$C_{\alpha_1 \ldots \alpha_{2s}} = \left(\frac{\partial^{2s}}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_{2s}}} \Phi \right) \Bigg|_{Y=Z=0} \,, \qquad s = 1, 2, 3, \ldots \,, \tag{2.67}$$

using the generalized Poincare lemma (for example, see [43–45]). In other words, an asymptotic observer who sources the bulk using a linearized spin-s Fronsdal field will activate the corresponding component field given above, whose boundary value can thus be identified with a dual conformal field theory source coupled to a conserved spin-s current.

The higher order couplings depend on the choice of gauge as well as the initial data for Φ and W_μ; as proposed by Vasiliev [22], these initial data can be fine-tuned at higher orders in order to obtain quasi-local equations of motion in the gauge (2.60).

An alternative approach, which we shall follow here, is to restrict the initial data for the zero-form to specific classes of functions on $\mathcal{Y}_4$, corresponding to associative subalgebras of $\mathcal{A}$ leading to well-defined field configurations obeying physical boundary conditions on $\mathcal{M}$.

[13] Whether the resulting system admit any consistent truncation to a pure higher-derivative gravity theory remains an open problem.

2.5 Internal star product algebras and solution spaces

A parameterised set $(\Phi(\nu, G), U(\nu, G), V(\nu, G), \bar{V}(\nu, L))$, where ν belongs to a parameter space and G is a gauge function, obeying the master field equations form an admissible solution space if they generate a free differential algebra together with I and $\overline{I}$ (for each fixed value of ν). To construct such spaces we use associative star product algebras[14]

$$\mathcal{A}_{\mathcal{S}} = \bigoplus_{\lambda \in \mathcal{S}} T_\lambda \otimes \mathbb{C}\,, \tag{2.68}$$

that are closed under the actions of π, $\bar{\pi}$, $\dagger$ and star multiplication by κ_y and $\bar{\kappa}_{\bar{y}}$, and whose basis elements T_λ, labeled by λ in a discrete set $\mathcal{S}$, have finite traces. We say that $\mathcal{A}_{\mathcal{S}}$ is contained in $\mathcal{A}_{\mathcal{S}'}$ if there exists a monomorphism $\rho : \mathcal{A}_{\mathcal{S}'} \to \mathcal{A}_{\mathcal{S}}$ such that $\mathrm{Tr}' \circ \rho = \mathrm{Tr}'$ i.e. if the elements in $\mathcal{A}_{\mathcal{S}}$ can be expanded in terms of the elements in $\mathcal{A}_{\mathcal{S}'}$ in a way compatible with the trace operation.

Expanding the master fields over $\mathcal{A}_{\mathcal{S}}$ yields a set of modes on $\mathcal{X}_4$ and $\mathcal{Z}_4$ that forms a free differential algebra together with j_z and its hermitian conjugate. Using Cartan integration methods, the modes can be expressed locally in terms of zero-form integration constants, which define the ν parameters, and gauge functions. These data can then be adapted to boundary conditions, which may require a change of basis from $\mathcal{A}_{\mathcal{S}}$ to a basis $\mathcal{A}_{\mathcal{S}'}$ containing $\mathcal{A}_{\mathcal{S}}$; for example, in asymptotically anti-de Sitter spacetimes, it makes sense to impose boundary conditions in a Lorentz covariant basis adapted to a dual conformal field theory. We shall say that a subalgebra $\mathcal{A}_{\mathcal{S}}$ yields a higher spin gravity solution space if the resulting Lorentz covariant master fields in Vasiliev gauge have symbols defined in normal order that can be expanded over finite regions of $\mathcal{X}_4$ in terms of the set of monomials on $\mathcal{Y}_4 \times \mathcal{Z}_4$ that vanish at the origin of $\mathcal{Y}_4 \times \mathcal{Z}_4$, i.e. they are real-analytic at this point.

The resulting moduli spaces can be coordinatized by higher spin invariant functionals, playing the role of classical higher spin observables [28, 32, 38, 40]. By choosing a structure group [32] and fixing a topology for the base manifold, one may extend the locally defined solutions to globally defined higher spin geometries supporting various types of topologically nontrivial observables. Working locally on $\mathcal{X}_4$, the accessible observables are on-shell closed zero-forms on $\mathcal{X}_4$ given by combined integrals over $\mathcal{Z}_4$ and traces over $\mathcal{Y}_4$ of adjoint constructs built from $(\Phi, V_\alpha, \bar{V}_{\dot{\alpha}}; I, \overline{I}; \kappa, \bar{\kappa})$, referred to as zero-form charges. Evaluated on solutions that are asymptotical to anti-de Sitter spacetime, these observables have been shown to have a physical interpretation as generating functionals for correlation functions of holographically dual conformal field theories.

We remark that various subalgebras of $\mathcal{A}$ can be obtained from different quantum mechanical systems in four-dimensional phase space. It is an interesting problem to examine which of these are admissible in the above sense, and to furthermore distinguish between these systems using higher spin invariant observables.

[14]The multiplication table of $\mathcal{A}_{\mathcal{S}}$ may involve fusion rules [26, 46], which stipulate which pairs of basis elements that have nontrivial star products and which basis elements that are to be used to expand the result.

3 New class of biaxially symmetric solutions

In this section, we construct a new class of exact solutions to Vasiliev's equations on a direct product manifold of the form (2.7) using a gauge function and expansions in terms of exponentials of two Cartan generators of $sp(4;\mathbb{C})$, which leads to biaxial symmetry.

3.1 Gauge function

From eq. (2.45) and the fact that $\mathcal{X}_4$ is commuting, it follows that U_μ can be expressed in terms of a gauge function G defined locally on $\mathcal{X}_4 \times \mathcal{Z}_4$. Thus, setting[15]

$$U_\mu^{(G)} = G^{-1} \star \partial_\mu G\,, \tag{3.1}$$
$$\Phi^{(G)} = G^{-1} \star \Phi' \star \pi\,(G) \tag{3.2}$$
$$V_\alpha^{(G)} = G^{-1} \star \partial_\alpha G + G^{-1} \star V'_\alpha \star G\,, \qquad \bar{V}_{\dot\alpha}^{(G)} = G^{-1} \star \partial_{\dot\alpha} G + G^{-1} \star \bar{V}'_{\dot\alpha} \star G\,, \tag{3.3}$$

eqs. (2.46) and (2.47) reduce to

$$\partial_\mu \Phi' = 0\,, \qquad \partial_\mu V'_\alpha = 0\,, \qquad \partial_\mu \bar{V}'_{\dot\alpha} = 0\,, \tag{3.4}$$

i.e. the primed fields are constant on $\mathcal{X}_4$, and eqs. (2.48)–(2.50) take the form

$$\partial_{[\alpha} V'_{\beta]} + V'_{[\alpha} \star V'_{\beta]} + \frac{i}{4}\varepsilon_{\alpha\beta}\mathcal{B}' \star \Phi' \star \kappa = 0 \quad \text{and h.c.}\,, \tag{3.5}$$
$$\partial_\alpha \Phi' + V'_\alpha \star \Phi' - \Phi' \star \bar\pi\left(V'_\alpha\right) = 0 \quad \text{and h.c.}\,, \tag{3.6}$$
$$\partial_\alpha \bar{V}'_{\dot\alpha} - \partial_{\dot\alpha} V'_\alpha + \left[V'_\alpha, \bar{V}'_{\dot\alpha}\right]_\star = 0\,, \tag{3.7}$$

where $\mathcal{B}' := \sum_{n=0}^\infty b_n (\Phi' \star \pi(\Phi'))^{\star n}$.

In order to obtain solutions that are asymptotic to AdS_4, we choose[16]

$$G = L \star H\,, \tag{3.8}$$

where L, which we shall refer to as the vacuum gauge function, is a locally defined map from $\mathcal{X}_4$ to SO(2,3)/SO(1,3) that is constant on $\mathcal{Z}_4$, i.e.

$$\partial_\alpha L = \partial_{\dot\alpha} L = 0\,, \tag{3.9}$$

and H is determined by imposing the Vasiliev gauge condition (2.60), viz.

$$z^\alpha V_\alpha^{(G)} = 0\,, \qquad \bar{z}^{\dot\alpha} \bar{V}_{\dot\alpha}^{(G)} = 0\,, \tag{3.10}$$

in a perturbative expansion

$$H = 1 + \sum_{n=1}^\infty H^{(n)}\,, \tag{3.11}$$

[15] We denote the star product inverse of G by G^{-1}, that is, $G \star G^{-1} = 1$.

[16] The reality condition and bosonic projection of a gauge function G takes the form $G^\dagger = G^{-1}$ and $\pi\bar\pi\,(G) = G$.

where the superscript (n) denotes an n-linear function of Φ'. Thus, the master fields in Vasiliev gauge are given by perturbative corrections of

$$U_\mu^{(L)} = L^{-1} \star \partial_\mu L\,, \tag{3.12}$$

$$\Phi^{(L)} = L^{-1} \star \Phi' \star \pi\,(L) \tag{3.13}$$

$$V_\alpha^{(L)} = L^{-1} \star \partial_\alpha L + L^{-1} \star V'_\alpha \star L\,, \qquad \bar{V}_{\dot{\alpha}}^{(L)} = L^{-1} \star \partial_{\dot{\alpha}} L + L^{-1} \star \bar{V}'_\alpha \star L\,, \tag{3.14}$$

where the Maurer-Cartan form $U_\mu^{(L)}$ consists of the frame field and Lorentz connection on the anti-de Sitter background spacetime, for which we shall use the explicit form in stereographic coordinates given in appendix C. As for H, its existence requires that $V_{\underline{\alpha}}^{(L)}$ admits a power series expansion on $\mathcal{Z}_4$ around $Z^{\underline{\alpha}} = 0$, to be examined in more detail in section 5.

Thus, the dependence on $\mathcal{X}_4$ arises via the gauge function, leaving $\mathcal{X}_4$-independent equations (3.5)–(3.7), to which we turn next.

3.2 Exact solutions in holomorphic gauge from abelian group algebras

One class of solution spaces arise from star product algebras

$$\mathcal{A}_\Lambda = \bigoplus_{n,\bar{n}=0,1} \bigoplus_{\vec{\lambda}\in\Lambda} (T_{\vec{\lambda}} \star \kappa_y{}^n \star \bar{\kappa}_{\bar{y}}{}^{\bar{n}}) \otimes \mathbb{C}\,, \tag{3.15}$$

where $\vec{\lambda} = (\lambda_1, \ldots, \lambda_N)$ belongs to an N-dimensional lattice Λ and

$$T_{\vec{\lambda}} \star T_{\vec{\lambda}'} = T_{\vec{\lambda}+\vec{\lambda}'}\,, \qquad [T_{\vec{\lambda}}, \kappa_y \star \bar{\kappa}_{\bar{y}}]_\star = 0\,, \qquad (T_{\vec{\lambda}})^\dagger = T_{c(\vec{\lambda})}\,, \qquad \pi(T_{\vec{\lambda}}) = T_{\pi(\vec{\lambda})}\,, \tag{3.16}$$

for $c, \pi : \Lambda \to \Lambda$. The second relation, which is equivalent to the bosonic projection $\pi\bar{\pi}\left(T_{\vec{\lambda}}\right) = T_{\vec{\lambda}}$, makes it possible to decompose under

$$\Pi_\sigma := \frac{1}{2}\left(1 + \sigma \kappa_y \star \bar{\kappa}_{\bar{y}}\right) = \frac{1}{2}\left(1 + \sigma \kappa_y \bar{\kappa}_{\bar{y}}\right)\,, \tag{3.17}$$

by expanding

$$\Phi' = \sum_{\sigma;\vec{\lambda}} T_{\vec{\lambda}} \star \Pi_\sigma \star (\nu_{\sigma;\vec{\lambda}} \kappa_y + \check{\nu}_{\sigma;\vec{\lambda}})\,, \qquad \nu_{\sigma;\vec{\lambda}}\,,\ \check{\nu}_{\sigma;\vec{\lambda}} \in \mathbb{C}\,, \tag{3.18}$$

$$V'_\alpha = \sum_{\sigma;\vec{\lambda}} T_{\vec{\lambda}} \star \Pi_\sigma \star \left(a_{\sigma;\vec{\lambda};\alpha} + \check{a}_{\sigma;\vec{\lambda};\alpha} \star \kappa_y\right)\,, \tag{3.19}$$

where $a_{\sigma;\vec{\lambda};\alpha}$ and $\check{a}_{\sigma;\vec{\lambda};\alpha}$ are holomorphic functions on $\mathcal{Z}_4$ and are constant over $\mathcal{Y}_4$, which may be viewed as a gauge choice (for given zero-form initial data). Expanding[17]

$$\mathcal{B}' \star \Phi' \star \kappa_y = \sum_{\sigma;\vec{\lambda}} T_{\vec{\lambda}} \star \Pi_\sigma \star (\mu_{\sigma;\vec{\lambda}} + \check{\mu}_{\sigma;\vec{\lambda}}\, \kappa_y)\,, \qquad \mu_{\sigma;\vec{\lambda}}\,,\ \check{\mu}_{\sigma;\vec{\lambda}} \in \mathbb{C}\,, \tag{3.20}$$

and introducing

$$\mathring{\mu}_\sigma(\vec{\zeta}) := \sum_{\vec{\lambda}} \mu_{\sigma;\vec{\lambda}} (\vec{\zeta})^{\vec{\lambda}}\,, \qquad \mathring{\check{\mu}}_\sigma(\vec{\zeta}) := \sum_{\vec{\lambda}} \check{\mu}_{\sigma;\vec{\lambda}} (\vec{\zeta})^{\vec{\lambda}}\,, \tag{3.21}$$

$$\mathring{a}_\sigma(\vec{\zeta}) := \sum_{\vec{\lambda}} dz^\alpha a_{\sigma;\vec{\lambda};\alpha} (\vec{\zeta})^{\vec{\lambda}}\,, \qquad \mathring{\check{a}}_\sigma(\vec{\zeta}) := \sum_{\vec{\lambda}} dz^\alpha \check{a}_{\sigma;\vec{\lambda};\alpha} (\vec{\zeta})^{\vec{\lambda}}\,, \tag{3.22}$$

[17] We use a convention such that if $\mathcal{B}' = b_0$ then $\mu_{\sigma;\vec{\lambda}} = b_0 \nu_{\sigma;\vec{\lambda}}$ and $\check{\mu}_{\sigma;\vec{\lambda}} = b_0 \check{\nu}_{\sigma;\vec{\lambda}}$.

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where $\vec{\zeta} := (\zeta_1, \ldots, \zeta_N) \in \mathbb{C}^N$ and $(\vec{\zeta})^{\vec{\lambda}} := (\zeta_1)^{\lambda_1} \cdots (\zeta_N)^{\lambda_N}$, the remaining equations on $\mathcal{Z}_4$ take the form

$$d\mathring{a}_\sigma + \mathring{a}_\sigma \star \mathring{a}_\sigma + \check{\mathring{a}}_\sigma \star \gamma \star \check{\mathring{a}}_\sigma \star \gamma + j_z \mathring{\mu}_\sigma = 0\,, \tag{3.23}$$

$$d\check{\mathring{a}}_\sigma \star \gamma + \mathring{a}_\sigma \star \check{\mathring{a}}_\sigma \star \gamma + \check{\mathring{a}}_\sigma \star \gamma \star \mathring{a}_\sigma + j_z \check{\mathring{\mu}}_\sigma \star \gamma = 0\,, \tag{3.24}$$

where the element γ obeys

$$\gamma \star (\vec{\zeta})^{\vec{\lambda}} \star \gamma = (\vec{\zeta})^{\pi(\vec{\lambda})}\,, \qquad [\gamma, z^\alpha]_\star = 0\,. \tag{3.25}$$

Defining

$$\mathring{\mu}^{\pm}_\sigma = \mathring{\mu}_\sigma \pm \check{\mathring{\mu}}_\sigma \star \gamma\,, \qquad \mathring{a}^{\pm}_\sigma = \mathring{a}_\sigma \pm \check{\mathring{a}}_\sigma \star \gamma\,, \tag{3.26}$$

we obtain two decoupled systems of the form

$$d\mathring{a}^{\pm}_\sigma + \mathring{a}^{\pm}_\sigma \star \mathring{a}^{\pm}_\sigma + j_z \mathring{\mu}^{\pm}_\sigma = 0\,, \tag{3.27}$$

that can be solved using the method of [26] (see also [35]), drawn from the original method devised in [39]. Omitting discrete moduli which arise via projector algebras on $\mathcal{Z}_4$, two particular solutions that we label by $\varsigma = \pm 1$, are given by

$$\left(\mathring{a}^{\pm}_{\sigma;\varsigma}\right)_\alpha = 2iz_\alpha \int_{-1}^{1} \frac{d\tau}{(\tau+1)^2} j_\sigma\left(\varsigma \mathring{\mu}^{\pm}_\sigma\,;\,\tau\right) \exp\left[\varsigma c(\tau)\, U^{\beta\gamma} z_\beta z_\gamma\right]\,, \tag{3.28}$$

where

$$j_\sigma\left(\varsigma\mathring{\mu}^{\pm}_\sigma;\tau\right) := -\frac{\varsigma\mathring{\mu}^{\pm}_\sigma}{4}\, {}_1F_1\left[\frac{1}{2};2;\frac{\varsigma\mathring{\mu}^{\pm}_\sigma}{2}\log\tau^2\right]\,, \qquad c(\tau) := i\frac{\tau-1}{\tau+1}\,, \tag{3.29}$$

and

$$U^{\beta\gamma} := \left(u^+\right)^{(\beta}\left(u^-\right)^{\gamma)}\,, \tag{3.30}$$

where u^+ and u^- are a set of spinor basis vectors obeying

$$\left(u^+\right)^\alpha\left(u^-\right)_\alpha = 1\,, \quad \left(u^+\right)^\alpha\left(u^+\right)_\alpha = \left(u^-\right)^\alpha\left(u^-\right)_\alpha = 0\,. \tag{3.31}$$

Using (B.4), we can choose

$$\left(u^+\right)^\alpha = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\,, \quad \left(u^-\right)^\alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\,. \tag{3.32}$$

The original twistor space connection can thus be obtained by expanding the confluent hypergeometric function in a power series, followed by identifying powers of $\vec{\zeta}$ and γ, though in what follows we shall mainly work directly with the generating functions.

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3.3 Twistor space connection in Weyl order in holomorphic gauge

The twistor space connection V'_α is given in the holomorphic gauge by (3.19). From eq. (2.27), it follows that

$$\begin{aligned}&\mathcal{O}_{\text{Normal}}\left(\sum_{\sigma;\vec{\lambda}} T_{\vec{\lambda}}\star\Pi_\sigma\star\left(a_{\sigma;\vec{\lambda};\alpha}+\check{a}_{\sigma;\vec{\lambda};\alpha}\star\kappa_y\right)\right)\\&\qquad=\mathcal{O}_{\text{Weyl}}\left(\sum_{\sigma;\vec{\lambda}}\left((T_{\vec{\lambda}}\star\Pi_\sigma)a_{\sigma;\vec{\lambda};\alpha}+(T_{\vec{\lambda}}\star\Pi_\sigma\star\kappa_y)\check{a}_{\sigma;\vec{\lambda};\alpha}\right)\right)\,,\end{aligned}\tag{3.33}$$

that is, the symbol in Weyl order of V'_α is given by the argument of the Wigner map on the right-hand side. This quantity contains singular distributions on $\mathcal{Y}_4$, which we shall examine in more detail later, and on $\mathcal{Z}_4$, which we shall examine in what follows. To this end, we observe that the integrand in (3.28) has potential divergences at $\tau=0$, where $\log(\tau^2)$ goes to infinity, and at $\tau=-1$, where denominators vanish.

As for the potential divergence at $\tau=0$, it does not lead to any non-real-analyticity in $\mathcal{Z}_4$ to any finite order in perturbation theory as follows from the fact that[18]

$$\left|{}_1F_1\left[\frac{1}{2};2;\frac{\varsigma\mathring{\mu}^\pm_\sigma}{2}\log\left(\tau^2\right)\right]\right|\leq\left|\tau^{\varsigma\mathring{\mu}^\pm_\sigma}\right|\,,\qquad\text{Re}(\varsigma\mathring{\mu}^\pm_\sigma)<0\,,\tag{3.34}$$

for $\tau\in[-1,1]$, while the same quantity is bounded for $\tau\in[-1,1]$ if $\text{Re}(\varsigma\mathring{\mu}^\pm_\sigma)\geqslant 0$. Thus, at $\tau=0$ there is no singularity as long as $\mathring{\mu}^\pm_\sigma$ lies inside the unit disc; indeed, for $\mathring{\mu}^\pm_\sigma$ sufficiently close to zero, the power series expansion of the confluent hypergeometric function yields a basis of functions of τ that can be used to convert the integral equation, obtained by inserting eq. (3.28) into the deformed oscillator equation, into an algebraic equation for symbols (for details, see [26, 39]). Thus, in order for (3.28) to provide a solution, there has to exist an annulus of convergence in the $\vec{\zeta}$-space for the Laurent series defining $\mathring{\mu}^\pm_\sigma$ where its modulus is less than one, which can be achieved by tuning the overall strength of the ν- and b_n-parameters. In other words, the contribution to (3.28) from the region around $\tau=0$ is real-analytic on $\mathcal{Z}_4$ to any finite order in perturbation theory.

Turning to the divergence at $\tau=-1$, it induces a simple pole $\mathring{a}^\pm_\alpha|_{\text{pole}}$ in $\mathring{a}^\pm_\alpha$ at $z^\alpha=0$, which can be extracted using the formula

$$\int_{-1}^{1}\frac{d\tau}{(\tau+1)^2}e^{\frac{\tau-1}{\tau+1}p}=\frac{1}{2p}\,,\qquad\text{Re}\,p>0\,,\tag{3.35}$$

[18] The confluent hypergeometric function ${}_1F_1(a;b;x):=\sum_{n=0}^{\infty}\frac{(a)_n x^n}{(b)_n n!}$ obeys $0<{}_1F_1(a;b;x)<e^x$ for $b>a>0$ and $x>0$. Its asymptotic form for large $|x|$ is given by ${}_1F_1(a;b;x)\sim\frac{\Gamma(b)}{\Gamma(a)}x^{a-b}e^x+\frac{\Gamma(b)}{\Gamma(b-a)}(-x)^{-a}$.

and analytical continuation of $U^{\beta\gamma}z_\beta z_\gamma$. It follows that

$$\begin{aligned}(\mathring{a}^{\pm}_{\sigma;\varsigma})_\alpha|_{\rm pole} &= 2iz_\alpha\int_{-1}^{1}\frac{d\tau}{(\tau+1)^2}j_\sigma\left(\varsigma\mathring{\mu}^{\pm}_\sigma;\tau\right)\exp\left[\frac{\varsigma i(\tau-1)}{\tau+1}U^{\beta\gamma}z_\beta z_\gamma\right]\Big|_{\rm pole}\\ &= -iz_\alpha\frac{\varsigma\mathring{\mu}^{\pm}_\sigma}{2}\int_{-1}^{1}\frac{d\tau}{(\tau+1)^2}\exp\left[\frac{\varsigma i(\tau-1)}{\tau+1}U^{\beta\gamma}z_\beta z_\gamma\right]\Big|_{\rm pole}\\ &= -\frac{\mathring{\mu}^{\pm}_\sigma z_\alpha}{4U^{\beta\gamma}z_\beta z_\gamma}\quad \text{for}\quad \mathrm{Re}\left(2\varsigma iU^{\beta\gamma}z_\beta z_\gamma\right)>0\,.\end{aligned}\tag{3.36}$$

Indeed, taking the exterior derivative of the right-hand side one obtains a delta function on the holomorphic slice of $\mathcal{Z}_4$ that cancels the linear source term in eq. (3.27). As for the higher order corrections to $\mathring{a}_\alpha$ in the ν-expansion, they are finite but not analytic at $z^\alpha=0$, given by combinations of positive powers and logarithms of z^α.

As we shall see in section 5, the nature of the twistor space connection as a distribution on $\mathcal{Y}_4\times\mathcal{Z}_4$, changes drastically once the vacuum gauge function is switched on and the connection is given in normal order.

3.4 Singularities in L-gauge from $T_{\vec{0}}$

We note that if the unity $T_{\vec{0}}$ of the star product algebra $\mathcal{A}_\Lambda$ in (3.15) is represented by the constant symbol on $\mathcal{Y}_4$, then its contributions to both $V^{(L)}_\alpha$ and $\Phi^{(L)}$ that are not real-analytic at the origin of $\mathcal{Y}_4\times\mathcal{Z}_4$ for generic points in $\mathcal{X}_4$. More precisely, the singular contributions to $V^{(L)}_\alpha$ are given by $\Pi_\sigma\star a_{\sigma;\vec{0};\alpha}$, where $a_{\sigma;\vec{0};\alpha}$ is given by $\vec{\zeta}$-independent contribution to (3.28); and those to $\Phi^{(L)}$ are given by $\nu_{\sigma;\vec{0}}\Pi_\sigma\star\kappa_y$. They are hence singular at $Z^{\underline{\alpha}}=0$ and $Y^{\underline{\alpha}}=0$, respectively. Thus, in order for a star product algebra to give rise to proper higher spin gravity configurations, it cannot contain the constant symbol on $\mathcal{Y}_4$; in the case of a group algebra this can be achieved by a truncation to a proper semigroup algebra (without the unity), as we shall analyse in more detail in sections 4 and 5.

In the remainder of this section, however, we shall proceed with the construction of solution spaces in the holomorphic gauge without truncating the underlying group algebras.

3.5 Abelian group algebra from Cartan subalgebra of sp(4; $\mathbb{R}$)

In what follows, we shall give an explicit example of a solution space of the type introduced above in the case when the lattice is two-dimensional, i.e. $\vec{\lambda}=(m,\tilde{m})$ with $m,\tilde{m}\in\mathbb{Z}$. The underlying group algebra $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$ is realized as

$$\mathcal{A}_{E,J}:=\bigoplus_{\sigma=\pm}\mathcal{A}_{E,J;\sigma}\,,\qquad \mathcal{A}_{E,J;\sigma}:=\bigoplus_{m,\tilde{m}\in\mathbb{Z}}(T_{m,\tilde{m}}\star\Pi_\sigma)\otimes\mathbb{C}\,,\tag{3.37}$$

in terms of group elements

$$T_{m,\tilde{m}}:=e_\star^{-4m\theta E}\star e_\star^{-4\tilde{m}\tilde{\theta}J}\,,\tag{3.38}$$

generated by the anti-de Sitter energy and spin operators[19]

$$E = \frac{1}{8} E_{\underline{\alpha\beta}} Y^{\underline{\alpha}} \star Y^{\underline{\beta}} = \frac{1}{8} E_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}} \ , \tag{3.39}$$

$$J = \frac{1}{8} J_{\underline{\alpha\beta}} Y^{\underline{\alpha}} \star Y^{\underline{\beta}} = \frac{1}{8} J_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}} \ , \tag{3.40}$$

respectively, using $\mathrm{sp}(4;\mathbb{R})$ valued matrices obeying[20]

$$(E^2)_{\underline{\alpha}}{}^{\underline{\beta}} = (J^2)_{\underline{\alpha}}{}^{\underline{\beta}} = -\delta_{\underline{\alpha}}{}^{\underline{\beta}} \,, \tag{3.41}$$

$$(EJ)_{\underline{\alpha}}{}^{\underline{\beta}} = (JE)_{\underline{\alpha}}{}^{\underline{\beta}} \,, \qquad (EJ)_{\underline{\alpha\beta}} + (JE)_{\underline{\beta\alpha}} = 0 \,, \qquad (EJ)_{\underline{\alpha}}{}^{\underline{\alpha}} = 0 \,, \tag{3.42}$$

from which it follows that

$$\det \left(\delta_{\underline{\alpha}}{}^{\underline{\beta}} + a E_{\underline{\alpha}}{}^{\underline{\gamma}} J_{\underline{\gamma}}{}^{\underline{\beta}} \right) = \left(1 - a^2\right)^2 \ . \tag{3.43}$$

As for the parameters, we take

$$\theta \in \mathbb{R} \cup i\mathbb{R} \,, \qquad i\mathbb{Z}\theta \cap \left(\frac{\pi}{2} + \mathbb{Z}\pi \right) = \emptyset \,, \tag{3.44}$$

idem $\tilde{\theta}$. The basis elements obey (3.16), viz.

$$T_{m,\tilde{m}} \star T_{n,\tilde{n}} = T_{m+n,\tilde{m}+\tilde{n}} \,, \qquad [T_{m,\tilde{m}}, \Pi_\sigma]_\star = 0 \,, \qquad \pi(T_{m,\tilde{m}}) = T_{-m,\tilde{m}} \,. \tag{3.45}$$

To compute the symbol of $T_{m,\tilde{m}}$ in Weyl order, we first use (A.25) with $N = 4$ to compute

$$e_\star^{-4m\theta E} = \mathbf{S}^2 \, e^{-4\mathbf{T}E} \ , \quad e_\star^{-4\tilde{m}\tilde{\theta}J} = \widetilde{\mathbf{S}}^2 \, e^{-4\widetilde{\mathbf{T}}J} \,, \tag{3.46}$$

where

$$\mathbf{S} := \mathrm{sech}\,(m\theta) \ , \quad \mathbf{T} := \tanh\,(m\theta) \ , \quad \widetilde{\mathbf{S}} := \mathrm{sech}\left(\tilde{m}\tilde{\theta}\right) \ , \quad \widetilde{\mathbf{T}} := \tanh\left(\tilde{m}\tilde{\theta}\right) \ . \tag{3.47}$$

In what follows, we make the convention that all boldfaced quantities depend on $m\theta$ and $\tilde{m}\tilde{\theta}$. The symbol of $T_{m,\tilde{m}}$ is thus given by

$$\begin{aligned} T_{m,\tilde{m}} &= \left[\mathbf{S}^2 e^{-4\mathbf{T}E}\right] \star \left[\widetilde{\mathbf{S}}^2 e^{-4\widetilde{\mathbf{T}}J}\right] \\ &= \left(\mathbf{S}\widetilde{\mathbf{S}}\right)^2 \int \frac{d^4U d^4V}{(2\pi)^4} \exp\left\{ i \left(V^{\underline{\alpha}} - Y^{\underline{\alpha}}\right) \left(U_{\underline{\alpha}} - Y_{\underline{\alpha}}\right) \right\} \\ &\qquad \times \exp\left\{ -\frac{1}{2} \left[\mathbf{T} E_{\underline{\alpha\beta}} U^{\underline{\alpha}} U^{\underline{\beta}} + \widetilde{\mathbf{T}} J_{\underline{\alpha\beta}} V^{\underline{\alpha}} V^{\underline{\beta}} \right] \right\} \ . \end{aligned} \tag{3.48}$$

By performing the Gaussian integrals, we obtain

$$T_{m,\tilde{m}} = \mathbf{A} \exp\left\{ -\frac{1}{2} \mathbf{K}_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}} \right\} \ , \qquad \mathbf{K}_{\underline{\alpha\beta}} := \mathbf{B} E_{\underline{\alpha\beta}} + \mathbf{C} J_{\underline{\alpha\beta}} \,, \tag{3.49}$$

[19] Inequivalent exact solution spaces can be obtained by replacing E and J by Cartan subalgebra generators in $\mathrm{sp}(4;\mathbb{C})$, which we leave for future work.

[20] We have suppressed the dummy indices, which are contracted using the north-west to south-east convention.

where

$$\mathbf{A} := \frac{\left(\mathbf{S}\widetilde{\mathbf{S}}\right)^2}{1-\left(\mathbf{T}\widetilde{\mathbf{T}}\right)^2}\,, \quad \mathbf{B} := \frac{\mathbf{T}\left(1-\widetilde{\mathbf{T}}^2\right)}{1-\left(\mathbf{T}\widetilde{\mathbf{T}}\right)^2}\,, \quad \mathbf{C} := \frac{\widetilde{\mathbf{T}}\left(1-\mathbf{T}^2\right)}{1-\left(\mathbf{T}\widetilde{\mathbf{T}}\right)^2}\,. \tag{3.50}$$

We also need the symbol of $T_{m,\tilde{m}} \star \kappa_y \bar{\kappa}_{\bar{y}}$, which is given by

$$\begin{aligned}
T_{m,\tilde{m}} \star \kappa_y \bar{\kappa}_{\bar{y}} &= (2\pi)^2\, \mathbf{A} \exp\left\{-\frac{1}{2}\mathbf{K}_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}}\right\} \star \delta^4(Y) \\
&= \mathbf{A} \int \frac{d^4U d^4V}{(2\pi)^2} e^{iV^{\underline{\alpha}} U_{\underline{\alpha}}} e^{-\frac{1}{2}\mathbf{K}_{\underline{\alpha\beta}}(Y^{\underline{\alpha}}+U^{\underline{\alpha}})(Y^{\underline{\beta}}+U^{\underline{\beta}})} \delta^4(Y+V) \\
&= \mathbf{A} \int \frac{d^4U d^4V}{(2\pi)^2} e^{-iY^{\underline{\alpha}} U_{\underline{\alpha}}} e^{-\frac{1}{2}\mathbf{K}_{\underline{\alpha\beta}} U^{\underline{\alpha}} U^{\underline{\beta}}} \\
&= \frac{\mathbf{A}}{\sqrt{\det(\mathbf{K})}} \exp\left\{-\frac{1}{2}\left(\mathbf{K}^{-1}\right)^{\underline{\alpha\beta}} Y_{\underline{\alpha}} Y_{\underline{\beta}}\right\}\,,
\end{aligned} \tag{3.51}$$

where

$$\mathbf{K}_{\underline{\alpha\beta}} \left(\mathbf{K}^{-1}\right)^{\underline{\beta\gamma}} := \delta_{\underline{\alpha}}{}^{\underline{\gamma}}\,. \tag{3.52}$$

3.6 New exact biaxially symmetric solutions in holomorphic gauge

The above construction of $\mathcal{A}_{E,J}$ thus allows us to solve equations (3.5)–(3.7) using the Ansatz (3.18)–(3.19). In order to keep matters simple, we shall assume that $\check{\nu} = \check{\alpha} = 0$, and work with the following reduced version:

$$\Phi' = \sum_{\sigma;m,\tilde{m}} \nu_{\sigma;m,\tilde{m}} T_{m,\tilde{m}} \star \Pi_\sigma \star \kappa_y\,, \tag{3.53}$$

$$V'_\alpha = \sum_{\sigma;m,\tilde{m}} T_{m,\tilde{m}} \star \Pi_\sigma \star (a_{\sigma;m,\tilde{m}}(z))_\alpha\,, \tag{3.54}$$

$$\bar{V}'_{\dot{\alpha}} = (V'_\alpha)^\dagger = \sum_{\sigma;m,\tilde{m}} T^\dagger_{m,\tilde{m}} \star \Pi_\sigma \star (\bar{a}_{\sigma;m,\tilde{m}}(\bar{z}))_{\dot{\alpha}}\,, \tag{3.55}$$

where thus $\nu_{\sigma;m,\tilde{m}} \in \mathbb{C}$ and we recall that the twistor space connection is (anti-)holomorphic, as indicated above. From

	$\theta \in \mathbb{R}$	$\theta \in i\mathbb{R}$
$\tilde{\theta} \in \mathbb{R}$	$T^\dagger_{m,\tilde{m}} = T_{m,\tilde{m}}$	$T^\dagger_{m,\tilde{m}} = T_{-m,\tilde{m}}$
$\tilde{\theta} \in i\mathbb{R}$	$T^\dagger_{m,\tilde{m}} = T_{m,-\tilde{m}}$	$T^\dagger_{m,\tilde{m}} = T_{-m,-\tilde{m}}$

(3.56)

it follows that the reality condition $(\Phi')^\dagger = \pi(\Phi')$ implies that

	$\theta \in \mathbb{R}$	$\theta \in i\mathbb{R}$
$\tilde{\theta} \in \mathbb{R}$	$\nu^*_{\sigma;m,\tilde{m}} = \sigma\nu_{\sigma;m,\tilde{m}}$	$\nu^*_{\sigma;m,\tilde{m}} = \sigma\nu_{\sigma;-m,\tilde{m}}$
$\tilde{\theta} \in i\mathbb{R}$	$\nu^*_{\sigma;m,\tilde{m}} = \sigma\nu_{\sigma;m,-\tilde{m}}$	$\nu^*_{\sigma;m,\tilde{m}} = \sigma\nu_{\sigma;-m,-\tilde{m}}$

. (3.57)

We note that the Ansatz (3.53)–(3.55) identically obeys (3.6) and (3.7) since

$$\left[(a_{\sigma;m,\tilde{m}})_\alpha\,, (\bar{a}_{\sigma;m,\tilde{m}})_{\dot{\alpha}}\right]_\star = 0\,, \tag{3.58}$$

while (3.5) reduces to

$$\partial_{[\alpha}\left(a_{\sigma;m,\tilde{m}}\right)_{\beta]}+\sum_{n,\tilde{n}}\left(a_{\sigma;n,\tilde{n}}\right)_{[\alpha}\star\left(a_{\sigma;m-n,\tilde{m}-\tilde{n}}\right)_{\beta]}+\frac{i}{4}\varepsilon_{\alpha\beta}\mu_{\sigma;m,\tilde{m}}\kappa_z=0\quad\text{and h.c.}\,,\tag{3.59}$$

where $\mu_{\sigma;m,\tilde{m}}$ are defined as in (3.20). Finally, multiplying (3.59) with $\zeta^m\tilde{\zeta}^{\tilde{m}}$, where $\zeta,\tilde{\zeta}\in\mathbb{C}$, and summing over m and $\tilde{m}$, yields the equivalent equation

$$\partial_{[\alpha}\left(\mathring{a}_\sigma\right)_{\beta]}+\left(\mathring{a}_\sigma\right)_{[\alpha}\star\left(\mathring{a}_\sigma\right)_{\beta]}+\frac{i}{4}\varepsilon_{\alpha\beta}\mathring{\mu}_\sigma\kappa_z=0\quad\text{and h.c.}\,,\tag{3.60}$$

where the generating functions

$$\left(\mathring{a}_\sigma\right)_\alpha\left(\zeta,\tilde{\zeta}\right):=\sum_{m,\tilde{m}}\left(a_{\sigma,m,\tilde{m}}\right)_\alpha\zeta^m\tilde{\zeta}^{\tilde{m}}\;\;,\quad\mathring{\mu}_\sigma\left(\zeta,\tilde{\zeta}\right):=\sum_{m,\tilde{m}}\mu_{\sigma;m,\tilde{m}}\zeta^m\tilde{\zeta}^{\tilde{m}}\,,\tag{3.61}$$

for which we shall use the particular solutions in (3.28) with $\check{\nu}=\check{a}=0$.

By definition, the symmetries of the solution are generated by generalized Killing gauge parameters $\epsilon^{(G)}$ leaving $(\Phi^{(G)},U^{(G)}_\mu,V^{(G)}_\alpha)$ invariant. Locally, the space of such parameters is given by

$$\epsilon^{(G)}=G^{-1}\star\epsilon'\star G\,,\qquad\epsilon'=\epsilon'(E,J)\,,\tag{3.62}$$

where the parameters are arbitrary star polynomials in E and J; and globally, a Killing parameter belongs to an adjoint section obeying suitable boundary conditions, and we shall assume that $\epsilon^{(G)}$ is real-analytic on $\mathcal{Y}_4\times\mathcal{Z}_4$ and falls off at infinity of $\mathcal{X}_4$, such that they leave the background spacetime invariant. This implies that the solutions have time-translational and rotational symmetries generated by E and J, respectively. Furthermore, if the Ansatz is expanded over only $T_{m,0}$ or $T_{0,\tilde{m}}$, respectively, then the symmetry is further enhanced to the enveloping algebras of $\mathrm{so}(2)_E\oplus\mathrm{so}(3)$ or $\mathrm{so}(1,2)\oplus\mathrm{so}(2)_J$, where $\mathrm{so}(3)$ is the subalgebras of $\mathrm{sp}(4;\mathbb{R})$ commuting to E idem $\mathrm{so}(1,2)$ and J. Acting on the solutions with the full higher spin algebra leads to an orbit that forms a higher spin representation space. The trace operation Tr' equips this space with an indefinite sesqui-linear form, as we shall comment on below in the context of higher spin invariant functionals.

4 Weyl zero-form and Weyl tensors

In this section we compute the Weyl zero-form, Weyl tensors and higher spin invariants formed out of them.

4.1 The Weyl zero-form in L-gauge

The Weyl tensors in the L-gauge are contained in the zero-form master field. From (3.13) and (3.53) it follows that

$$\begin{aligned}\Phi^{(L)}&=\frac{1}{2}\sum_{\sigma,m,\tilde{m}}\nu_{\sigma,m,\tilde{m}}L^{-1}\star T_{m,\tilde{m}}\star\left(\kappa_y+\sigma\bar{\kappa}_{\bar{y}}\right)\star\pi\left(L\right)\,,\\&=\frac{1}{2}\sum_{\sigma,m,\tilde{m}}\nu_{\sigma,m,\tilde{m}}\left(L^{-1}\star T_{m,\tilde{m}}\star L\right)\star\left(\kappa_y+\sigma\bar{\kappa}_{\bar{y}}\right)\\&=\sum_{m,\tilde{m}}\left(\nu_{1,m,\tilde{m}}T^L_{m,\tilde{m}}\star\kappa_y+\nu_{2,m,\tilde{m}}T^L_{m,\tilde{m}}\star\bar{\kappa}_{\bar{y}}\right)\,,\end{aligned}\tag{4.1}$$

where

$$T^L_{m,\tilde{m}} := L^{-1} \star T_{m,\tilde{m}} \star L \,, \tag{4.2}$$

and the parameters

$$\nu_{1,m,\tilde{m}} := \frac{1}{2}\left(\nu_{+,m,\tilde{m}} + \nu_{-,m,\tilde{m}}\right) \ , \ \nu_{2,m,\tilde{m}} := \frac{1}{2}\left(\nu_{+,m,\tilde{m}} - \nu_{-,m,\tilde{m}}\right) \,, \tag{4.3}$$

obey the reality conditions

	$\theta \in \mathbb{R}$	$\theta \in i\mathbb{R}$
$\tilde{\theta} \in \mathbb{R}$	$\nu^*_{1,m,\tilde{m}} = \nu_{2,m,\tilde{m}}$	$\nu^*_{1,m,\tilde{m}} = \nu_{2,-m,\tilde{m}}$
$\tilde{\theta} \in i\mathbb{R}$	$\nu^*_{1,m,\tilde{m}} = \nu_{2,m,-\tilde{m}}$	$\nu^*_{1,m,\tilde{m}} = \nu_{2,-m,-\tilde{m}}$

(4.4)

To compute $T^L_{m,\tilde{m}}$ we use the lemma

$$L^{-1} \star f\left(Y_{\underline{\alpha}}\right) \star L = f\left(L_{\underline{\alpha}}{}^{\underline{\beta}} Y_{\underline{\beta}}\right) \,, \tag{4.5}$$

where $L_{\underline{\alpha}}{}^{\underline{\beta}}$ is a matrix that depends on the spacetime coordinates (see appendix C for an explicit expression). It follows from (3.49) that

$$T^L_{m,\tilde{m}} = \mathbf{A}\exp\left\{\left(-\frac{1}{2}\right)\mathbf{K}^L_{\underline{\alpha\beta}} Y^{\underline{\alpha}} Y^{\underline{\beta}}\right\} \,, \tag{4.6}$$

where

$$\mathbf{K}^L_{\underline{\alpha\beta}} := \mathbf{B}E^L_{\underline{\alpha\beta}} + \mathbf{C}J^L_{\underline{\alpha\beta}} \,, \qquad E^L_{\underline{\alpha\beta}} := E_{\underline{\gamma\delta}} L^{\underline{\gamma}}{}_{\underline{\alpha}} L^{\underline{\delta}}{}_{\underline{\beta}} \,, \qquad J^L_{\underline{\alpha\beta}} := J_{\underline{\gamma\delta}} L^{\underline{\gamma}}{}_{\underline{\alpha}} L^{\underline{\delta}}{}_{\underline{\beta}} \,. \tag{4.7}$$

Under $Y^{\underline{\alpha}} = \left\{y^\alpha, \bar{y}^{\dot{\alpha}}\right\}$, the above matrices decompose into

$$E^L_{\underline{\alpha\beta}} =: \begin{pmatrix} (\kappa^L_E)_{\alpha\beta} & (v^L_E)_{\alpha\dot{\beta}} \\ (\bar{v}^L_E)_{\dot{\alpha}\beta} & (\bar{\kappa}^L_E)_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \,, \tag{4.8}$$

idem $J^L_{\underline{\alpha\beta}}$, whose components obey[21]

$$v^L_{\alpha\dot{\beta}} = \bar{v}^L_{\dot{\beta}\alpha} \,, \qquad \left(v^L\right)_{\alpha\dot{\beta}} \left(\bar{v}^L\right)^{\dot{\beta}\gamma} = \left(v^L\right)^2 \delta_\alpha{}^\gamma \,, \qquad \left(\bar{v}^L\right)_{\dot{\alpha}\beta} \left(v^L\right)^{\beta\dot{\gamma}} = \left(v^L\right)^2 \delta_{\dot{\alpha}}{}^{\dot{\gamma}} \,, \tag{4.9}$$

where $\left(v^L\right)^2 := \frac{1}{2}\left(v^L\right)_{\alpha\dot{\beta}}\left(v^L\right)^{\alpha\dot{\beta}}$, and

$$\left(\kappa^L\right)^2 := \frac{1}{2}\left(\kappa^L\right)_{\alpha\beta}\left(\kappa^L\right)^{\alpha\beta} = \det\left(\kappa^L\right) \quad , \quad \left(\kappa^L\right)_{\alpha\beta}\left(\kappa^L\right)^{\beta\gamma} = \left(\kappa^L\right)^2 \delta_\alpha{}^\gamma \ , \tag{4.10}$$

idem $\bar{\kappa}^L$, which are derived from general properties of any 2×2 symmetric matrix. Furthermore, from (3.41) it follows that

$$\left(\kappa^L\right)_{\alpha\beta}\left(v^L\right)^{\beta\dot{\gamma}} + \left(v^L\right)_{\alpha\dot{\beta}}\left(\bar{\kappa}^L\right)^{\dot{\beta}\dot{\gamma}} = 0 \,, \qquad \left(\kappa^L\right)^2 + \left(v^L\right)^2 = \left(\bar{\kappa}^L\right)^2 + \left(v^L\right)^2 = 1 \,, \tag{4.11}$$

[21] Eqs. (4.9)–(4.12) hold, if we label all components with either "E" or "J" (not a mixture of both).

which in its turn implies

$$\left(\bar{\kappa}^L\right)^2\left(\kappa^L\right)_{\alpha\beta}-\left(\bar{\kappa}^L\right)^{\dot{\alpha}\dot{\beta}}\left(v^L\right)_{\alpha\dot{\alpha}}\left(v^L\right)_{\beta\dot{\beta}}=\left(\kappa^L\right)_{\alpha\beta}\ , \tag{4.12}$$

which will be useful later when we determine the Petrov type.

Returning to (4.6), we thus have

$$\mathbf{K}^L_{\underline{\alpha\beta}}=\begin{pmatrix}\mathbf{F}_{\alpha\beta} & \mathbf{G}_{\alpha\dot{\beta}}\\ \mathbf{G}_{\dot{\alpha}\beta} & \mathbf{H}_{\dot{\alpha}\dot{\beta}}\end{pmatrix}:=\begin{pmatrix}\mathbf{B}\left(\kappa^L_E\right)_{\alpha\beta}+\mathbf{C}\left(\kappa^L_J\right)_{\alpha\beta} & \mathbf{B}\left(v^L_E\right)_{\alpha\dot{\beta}}+\mathbf{C}\left(v^L_J\right)_{\alpha\dot{\beta}}\\ \mathbf{B}\left(\bar{v}^L_E\right)_{\dot{\alpha}\beta}+\mathbf{C}\left(\bar{v}^L_J\right)_{\dot{\alpha}\beta} & \mathbf{B}\left(\bar{\kappa}^L_E\right)_{\dot{\alpha}\dot{\beta}}+\mathbf{C}\left(\bar{\kappa}^L_J\right)_{\dot{\alpha}\dot{\beta}}\end{pmatrix}\ , \tag{4.13}$$

where $\mathbf{G}_{\dot{\alpha}\beta}=\mathbf{G}_{\beta\dot{\alpha}}$, and correspondingly

$$T^L_{m,\tilde{m}}=\mathbf{A}\exp\left\{\left(-\frac{1}{2}\right)\left[y^\alpha\mathbf{F}_{\alpha\beta}y^\beta+\bar{y}^{\dot{\alpha}}\mathbf{H}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\beta}}+2y^\alpha\mathbf{G}_{\alpha\dot{\beta}}\bar{y}^{\dot{\beta}}\right]\right\}\ . \tag{4.14}$$

Finally, for $(m,\tilde{m})\neq(0,0)$, by performing Gaussian integrals we obtain[22]

$$\begin{aligned}&T^L_{m,\tilde{m}}\star\kappa_y \\ &=\frac{\mathbf{A}}{\sqrt{\mathbf{F}^2}}\exp\left\{\frac{1}{2\mathbf{F}^2}\left[\left(\mathbf{F}^{\alpha\beta}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}-\mathbf{F}^2\mathbf{H}_{\dot{\alpha}\dot{\beta}}\right)\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}-\mathbf{F}_{\alpha\beta}y^\alpha y^\beta+2i\mathbf{F}_\alpha{}^\beta\mathbf{G}_{\beta\dot{\beta}}y^\alpha\bar{y}^{\dot{\beta}}\right]\right\}\ ,\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&T^L_{m,\tilde{m}}\star\bar{\kappa}_{\bar{y}} \\ &=\frac{\mathbf{A}}{\sqrt{\mathbf{H}^2}}\exp\left\{\frac{1}{2\mathbf{H}^2}\left[\left(\mathbf{H}^{\alpha\beta}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}-\mathbf{H}^2\mathbf{F}_{\dot{\alpha}\dot{\beta}}\right)y^\alpha y^\beta-\mathbf{H}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}+2i\mathbf{H}_{\dot{\alpha}}{}^{\dot{\beta}}\mathbf{G}_{\beta\dot{\beta}}\bar{y}^{\dot{\alpha}}y^\beta\right]\right\}\ ,\end{aligned} \tag{4.16}$$

while $T^L_{0,\tilde{0}}\star\kappa_y=\kappa_y$ and $T^L_{0,\tilde{0}}\star\bar{\kappa}_{\bar{y}}=\bar{\kappa}_{\bar{y}}$.

Substituting the above formulae into (4.1), we obtain

$$\begin{aligned}&\Phi^{(L)} \\ &=\nu_{1,0,0}\kappa_y\ +\ \nu_{2,0,0}\bar{\kappa}_{\bar{y}}\ + \\ &+\sum_{(m,\tilde{m})\neq(0,0)}\mathbf{A}\left(\frac{\nu_{1,m,\tilde{m}}}{\sqrt{\mathbf{F}^2}}\exp\left\{\frac{1}{2\mathbf{F}^2}\left[\left(\mathbf{F}^{\alpha\beta}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}-\mathbf{F}^2\mathbf{H}_{\dot{\alpha}\dot{\beta}}\right)\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}-\mathbf{F}_{\alpha\beta}y^\alpha y^\beta+2i\mathbf{F}_\alpha{}^\beta\mathbf{G}_{\beta\dot{\beta}}y^\alpha\bar{y}^{\dot{\beta}}\right]\right\}\right. \\ &\quad\left.+\frac{\nu_{2,m,\tilde{m}}}{\sqrt{\mathbf{H}^2}}\exp\left\{\frac{1}{2\mathbf{H}^2}\left[\left(\mathbf{H}^{\dot{\alpha}\dot{\beta}}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}-\mathbf{H}^2\mathbf{F}_{\alpha\beta}\right)y^\alpha y^\beta-\mathbf{H}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\alpha}}\bar{y}^{\dot{\beta}}+2i\mathbf{H}_{\dot{\alpha}}{}^{\dot{\beta}}\mathbf{G}_{\beta\dot{\beta}}\bar{y}^{\dot{\alpha}}y^\beta\right]\right\}\right)\ .\end{aligned} \tag{4.17}$$

The expression $\mathbf{H}^2\mathbf{F}_{\alpha\beta}-\mathbf{H}^{\dot{\alpha}\dot{\beta}}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}$ can be factorized as

$$\mathbf{H}^2\mathbf{F}_{\alpha\beta}-\mathbf{H}^{\dot{\alpha}\dot{\beta}}\mathbf{G}_{\alpha\dot{\alpha}}\mathbf{G}_{\beta\dot{\beta}}\ =\ \left(\mathbf{B}^2-\mathbf{C}^2\right)\breve{\mathbf{F}}_{\alpha\beta}\,, \tag{4.18}$$

where $\breve{\mathbf{F}}_{\alpha\beta}$ satisfies

$$\breve{\mathbf{F}}^2\ =\ \mathbf{H}^2\,. \tag{4.19}$$

Then, assuming that

$$\nu_{1,0,0}=0=\nu_{2,0,0}\,, \tag{4.20}$$

[22]We note the useful relations $\mathbf{F}^2:=\frac{1}{2}\mathbf{F}_{\alpha\beta}(\mathbf{F})^{\alpha\beta}=\det(\mathbf{F})$ and $\mathbf{F}_{\alpha\beta}\mathbf{F}^{\beta\gamma}=\mathbf{F}^2\delta_\alpha{}^\gamma$, idem $\mathbf{H}$.

the resulting generalized spin-s Weyl tensor in the L-gauge reads

$$\begin{aligned} C_{\alpha_1\cdots\alpha_{2s}} &:= \left[\frac{\partial}{\partial y^{\alpha_1}}\cdots\frac{\partial}{\partial y^{\alpha_{2s}}}\Phi^{(L)}\right]_{Y=0} \\ &= \frac{(2s)!}{s!}\sum_{(m,\tilde m)\neq(0,0)} \mathbf{A}\Bigg\{\frac{\nu_{1,m,\tilde m}}{\sqrt{\mathbf{F}^2}}\left(\frac{-1}{2\mathbf{F}^2}\right)^s \mathbf{F}_{(\alpha_1\alpha_2}\cdots\mathbf{F}_{\alpha_{2s-1}\alpha_{2s})} \\ &\qquad + \left(\mathbf{B}^2-\mathbf{C}^2\right)^s\frac{\nu_{2,m,\tilde m}}{\sqrt{\breve{\mathbf{F}}^2}}\left(\frac{-1}{2\breve{\mathbf{F}}^2}\right)^s \breve{\mathbf{F}}_{(\alpha_1\alpha_2}\cdots\breve{\mathbf{F}}_{\alpha_{2s-1}\alpha_{2s})}\Bigg\}, \end{aligned} \tag{4.21}$$

where there are two separate generalized Petrov type-D tensors summed for each $(m,\tilde m)$ for positive s.[23]

4.2 Petrov types of the Weyl tensors

In what follows, we analyze in a few special cases whether the Weyl tensor as the sum (4.21) is of Petrov type D.

The case $\theta\tilde\theta = 0$. If $\theta \neq 0$ and $\tilde\theta = 0$, then $\widetilde{\mathbf{S}} = 1$, $\widetilde{\mathbf{T}} = 0$ and

$$\mathbf{A} = \mathbf{S}^2\,,\quad \mathbf{B} = \mathbf{T}\,,\quad \mathbf{C} = 0\,. \tag{4.22}$$

It follows that

$$\begin{pmatrix} \mathbf{F}_{\alpha\beta} & \mathbf{G}_{\alpha\dot\beta} \\ \mathbf{G}_{\dot\alpha\beta} & \mathbf{H}_{\dot\alpha\dot\beta}\end{pmatrix} = \mathbf{T}\begin{pmatrix} \left(\kappa_E^L\right)_{\alpha\beta} & \left(v_E^L\right)_{\alpha\dot\beta} \\ \left(\bar v_E^L\right)_{\dot\alpha\beta} & \left(\bar\kappa_E^L\right)_{\dot\alpha\dot\beta}\end{pmatrix}. \tag{4.23}$$

Furthermore, using (4.12) we obtain

$$\breve{\mathbf{F}}_{\alpha\beta} = \mathbf{T}\left(\kappa_E^L\right)_{\alpha\beta}\,. \tag{4.24}$$

The resulting spin-s Weyl tensor reads[24]

$$\begin{aligned} &C_{\alpha_1\cdots\alpha_{2s}}\big|_{\tilde\theta=0} \\ &= \frac{(2s)!}{s!}\sum_{m\neq 0}\frac{\mathbf{S}^2}{\sqrt{\mathbf{T}^2\left(\kappa_E^L\right)^2}}\frac{1}{\left[-2\left(\kappa_E^L\right)^2\right]^s}\left(\nu_{1,m}\mathbf{T}^{-s}+\nu_{2,m}\mathbf{T}^s\right)\left(\kappa_E^L\right)_{(\alpha_1\alpha_2}\cdots\left(\kappa_E^L\right)_{\alpha_{2s-1}\alpha_{2s})}\,, \end{aligned} \tag{4.25}$$

where $\nu_{1,m} = \sum_{\tilde m}\nu_{1,m,\tilde m}$ and $\nu_{2,m} = \sum_{\tilde m}\nu_{2,m,\tilde m}$, which we assume to be finite and vanishing for $m = 0$.

If instead $\theta = 0$ and $\tilde\theta \neq 0$, then $\mathbf{S} = 1$, $\mathbf{T} = 0$, and

$$\mathbf{A} = \widetilde{\mathbf{S}}^2\,,\quad \mathbf{B} = 0\,,\quad \mathbf{C} = \widetilde{\mathbf{T}}\,. \tag{4.26}$$

[23]A generalized spin-s Petrov type-D tensor is defined as a symmetric rank-$2s$ tensor with spinor indices that can be decomposed into the products of two spinors, each of which has the power s [26].

[24]One can show that $\left(\kappa_E^L\right)^2 = -\lambda^2 r^2$. The analytical continuation involves the choice of sign in front of the square roots. These must be correlated to analogous choices in the expression for the twistor space connection.

Then we have

$$\begin{pmatrix} \mathbf{F}_{\alpha\beta} & \mathbf{G}_{\alpha\dot{\beta}} \\ \mathbf{G}^T_{\dot{\alpha}\beta} & \mathbf{H}_{\dot{\alpha}\dot{\beta}} \end{pmatrix} = \widetilde{\mathbf{T}} \begin{pmatrix} (\kappa^L_J)_{\alpha\beta} & (v^L_J)_{\alpha\dot{\beta}} \\ (\bar{v}^L_J)_{\dot{\alpha}\beta} & (\bar{\kappa}^L_J)_{\dot{\alpha}\dot{\beta}} \end{pmatrix} , \tag{4.27}$$

and hence using (4.12) it follows that

$$\check{\mathbf{F}}_{\alpha\beta} = -\widetilde{\mathbf{T}} \, (\kappa^L_J)_{\alpha\beta} , \tag{4.28}$$

and the spin-s Weyl tensor becomes

$$\begin{aligned} & C_{\alpha_1\cdots\alpha_{2s}}|_{\theta=0} \\ & = \frac{(2s)!}{s!} \sum_{\tilde{m}\neq 0} \frac{\widetilde{\mathbf{S}}^2}{\sqrt{\widetilde{\mathbf{T}}^2 \left(\kappa^L_J\right)^2}} \frac{1}{\left[-2\left(\kappa^L_J\right)^2\right]^s} \left(\nu_{1,\tilde{m}} \widetilde{\mathbf{T}}^{-s} + \nu_{2,\tilde{m}} \widetilde{\mathbf{T}}^s\right) (\kappa^L_J)_{(\alpha_1\alpha_2} \cdots (\kappa^L_J)_{\alpha_{2s-1}\alpha_{2s})} , \end{aligned} \tag{4.29}$$

where $\nu_{1,\tilde{m}} = \sum_m \nu_{1,m,\tilde{m}}$ and $\nu_{2,\tilde{m}} = \sum_m \nu_{2,m,\tilde{m}}$, which we assume to be finite and vanishing for $\tilde{m} = 0$.

Thus, to summarize, if the Weyl zero-form depends on either E or J, but not both, in the holomorphic gauge, then Weyl tensors in L-gauge become proportional to direct products of κ^L's, which means they are of generalized Petrov type D.

The case $\theta\tilde{\theta} \neq 0$. If both θ and $\tilde{\theta}$ are non-zero, i.e. if both E and J are present in the Weyl zero-form in the holomorphic gauge, then we can simplify the analysis by substituting the explicit expressions provided in appendices B and C into the spin-s Weyl tensor (4.21) in L-gauge.

If $\mathbf{B}^2 - \mathbf{C}^2 = 0$ i.e. $m\theta = \pm\tilde{m}\tilde{\theta}$, then the second set of terms in (4.21) vanishes, and in the first set of terms $\mathbf{F}_{\alpha\beta} = \mathbf{B}\left[(\kappa^L_E)_{\alpha\beta} \pm (\kappa^L_J)_{\alpha\beta}\right]$. This means that if $\theta/\tilde{\theta}$ is a rational number, and if furthermore we turn on only the terms with $m\theta = \pm\tilde{m}\tilde{\theta}$, then the Weyl tensors become proportional to direct products of $(\kappa^L_E \pm \kappa^L_J)$'s, i.e. they are of generalized Petrov type D.[25] However, for generic values of $\theta/\tilde{\theta}$, (4.21) is not of type D,[26] though it is a sum of type-D tensors.

4.3 Asymptotic behaviour of the Weyl tensors

By using the gamma matrix realization in appendix B and the global coordinates in appendix C, we can investigate the asymptotic behaviour of the Weyl tensors. When $r \to \infty$, we have

$$\mathbf{F}^2|_{r\to\infty} = \check{\mathbf{F}}^2|_{r\to\infty} = \lambda^2 r^2 \left[-\mathbf{B}^2 + \mathbf{C}^2 \sin^2(\vartheta)\right] , \tag{4.30}$$

Then the terms in (4.21) of spin-s Weyl tensor at large radius, by a simple power counting, scale as

$$\left\{\lambda^2 r^2 \left[-\mathbf{B}^2 + \mathbf{C}^2 \sin^2(\vartheta)\right]\right\}^{-\frac{1}{2}(s+1)} , \tag{4.31}$$

and hence each term is either Kerr-like (when $\mathbf{B}^2 \neq \mathbf{C}^2$) or 2-brane-like (when $\mathbf{B}^2 = \mathbf{C}^2$) in the asymptotic region. The Weyl tensor as the sum of these terms falls off as $\frac{1}{r^{s+1}}$, which is the regular boundary condition of asymptotically AdS_4 solutions.

[25] Note, however, the matrix $\mathbf{K}_{\underline{\alpha\beta}} \equiv \left[\mathbf{B}E_{\underline{\alpha\beta}} + \mathbf{C}J_{\underline{\alpha\beta}}\right]$ in this special case has determinant $\left(\mathbf{B}^2 - \mathbf{C}^2\right)^2 = 0$, which has consequences for the twistor space connection; see eq. (5.7).

[26] See appendix D for details on spin-2.

4.4 Zero-form charges

Although the separate spin-s Weyl tensors blow up at the origin of spacetime, the limit of the full Weyl zero-form remains well-defined as the symbol of an operator. From this operator, it is possible to obtain higher spin gauge invariant quantities given by

$$\mathcal{I}_{2p} := \int_{\mathcal{Z}_4} \mathrm{Tr}' \left\{ I \star \bar{I} \star [\Phi \star \pi(\Phi)]^{\star p} \right\} , \tag{4.32}$$

which are referred to as zero-form charges [37] and that are related to higher spin amplitudes [28–30]. On our exact solutions, i.e. by substituting (3.13) and (3.53), these charges are given by

$$\mathcal{I}_{2p}|_{\text{on-solution}} := \frac{1}{32} \sum_{\substack{\sigma,\\ m_1,m_2,\cdots,m_{2p},\\ \tilde{m}_1,\tilde{m}_2,\cdots,\tilde{m}_{2p}}} \mathbf{A}_{\sum_{j=1}^{2p}(-1)^{j+1}m_j,\sum_{j=1}^{2p}\tilde{m}_j} \prod_{j=1}^{2p} \nu_{\sigma,m_j,\tilde{m}_j} , \tag{4.33}$$

where

$$\mathbf{A}_{m,\tilde{m}} := \frac{\left[\mathrm{sech}(m\theta)\ \mathrm{sech}(\tilde{m}\tilde{\theta})\right]^2}{1-\left[\tanh(m\theta)\ \tanh(\tilde{m}\tilde{\theta})\right]^2} . \tag{4.34}$$

The simplest case is $p = 1$:

$$\mathcal{I}_2|_{\text{on-solution}} = \frac{1}{32} \sum_{\sigma,m,\tilde{m},n,\tilde{n}} \mathbf{A}_{m-n,\tilde{m}+\tilde{n}} \nu_{\sigma,m,\tilde{m}} \nu_{\sigma,n,\tilde{n}} . \tag{4.35}$$

In [24], this zero-form charge has been proposed to be one of the contributions to the effective action for higher spin gravity in asymptotically anti-de Sitter spacetimes. As noted at the end of section 3.6, the resulting contribution to the free energy functional is not positive definite.

5 Twistor space connection

In this section, we first compute the twistor space connection $V^{(L)}_{\underline{\alpha}}$, and show in special cases that it admits a regular power series expansion on $\mathcal{Z}_4$ around $Z^{\underline{\alpha}} = 0$ over finite regions of spacetime provided that the group algebra $\mathbb{C}[\mathbb{Z}\times\mathbb{Z}]$ is truncated down to a non-unital subalgebra. We then demonstrate the existence of the linearized gauge function $H^{(1)}$ taking the linearized twistor space connection to Vasiliev gauge in a special case.

5.1 Generating function for twistor space connection in L-gauge

In order to facilitate the analysis, we write

$$\begin{aligned} V^{(L)}_{\alpha} &= L^{-1} \star V'_{\alpha} \star L \\ &= \sum_{\sigma,m,\tilde{m}} L^{-1} \star T_{m,\tilde{m}} \star \Pi_\sigma \star L \star (a_{\sigma,m,\tilde{m}})_\alpha \\ &= \sum_{\sigma,m,\tilde{m}} \oint_0 \frac{d\zeta}{2\pi i \zeta^{m+1}} \oint_0 \frac{d\tilde{\zeta}}{2\pi i \tilde{\zeta}^{\tilde{m}+1}} \frac{1}{2} \left(\mathring{V}^{(L)}_{0;\sigma,m,\tilde{m}} + \sigma \mathring{V}^{(L)}_{1;\sigma,m,\tilde{m}} \right)_\alpha , \end{aligned} \tag{5.1}$$

in terms of the generating functions ($n = 0, 1$)

$$\left(\mathring{V}^{(L)}_{n;\sigma,m,\tilde{m}}\right)_\alpha := 2i\frac{\partial}{\partial\rho^\alpha}\int_{-1}^{1}\frac{d\tau j_\sigma(\tau)}{(\tau+1)^2}T^L_{m,\tilde{m}}\star(\kappa_y\bar{\kappa}_{\bar{y}})^n\star\left\{\exp\left[\varsigma c(\tau)\,U^{\beta\gamma}z_\beta z_\gamma+\rho^\beta z_\beta\right]\right\}_{\rho=0}, \tag{5.2}$$

where ρ^α is an auxiliary commuting spinor, and we denote $j_\sigma(\tau) \equiv j_\sigma(\varsigma\mathring{\mu}_\sigma;\tau)$. Thus, if these two integrals are finite for bounded $\mathring{\mu}_\sigma$ and finite ρ^α, then V_α is real-analytic in $\mathcal{Y}_4 \times \mathcal{Z}_4$.

5.2 Singular twistor space connection in L-gauge from $T_{0,0}$

From the discussion in section 3.3 and the fact that $T^L_{0,0} = 1$, it follows that (5.1) contains a term given by $(a_{\sigma,0,0})_\alpha$, which is not real-analytic in $\mathcal{Z}_4$. Thus, real-analyticity of $V^{(L)}_\alpha$ in $\mathcal{Z}_4$ requires

$$(a_{\sigma,0,0})_\alpha = 0\,. \tag{5.3}$$

This can be achieved by a consistent truncation of the Ansatz (3.53)–(3.55) by taking $\mathcal{A}_{E,J}$ to be a semigroup without identity.[27]

If $\theta\tilde{\theta} \neq 0$ this can be achieved by taking

$$\nu_{\sigma,m,\tilde{m}} = (a_{\sigma,m,\tilde{m}})_\alpha = 0 \quad \text{for} \quad m \leqslant 0 \;\; \text{and/or} \;\; \tilde{m} \leqslant 0\,. \tag{5.4}$$

In other words, in the original Ansatz we sum over $m, \tilde{m} \in \mathbb{Z}$, but due to the requirement of real-analyticity, we instead sum over only positive m and/or positive $\tilde{m}$. Furthermore, as can be seen from the table (3.57), for compatibility with the reality condition, along with the truncation we must set θ and/or $\tilde{\theta}$ to be real.

If $\theta = 0$ (or $\tilde{\theta} = 0$) then we need to restrict $\tilde{m} \in \mathbb{Z}^+$, $\tilde{\theta} \in \mathbb{R}$ (or $m \in \mathbb{Z}^+$, $\theta \in \mathbb{R}$).

θ and $\tilde{\theta}$ cannot be both zero.

To summarize, in the following table, we give notations to the consistent truncations, and "×" means that the situation either includes the unity or is inconsistent with the reality condition.

		$\theta \in \mathbb{R}\backslash\{0\}$ $m \in \mathbb{Z}$	$\theta \in \mathbb{R}\backslash\{0\}$ $m \in \mathbb{Z}^+$	$\theta \in i\mathbb{R}\backslash\{0\}$ $m \in \mathbb{Z}$	$\theta \in i\mathbb{R}\backslash\{0\}$ $m \in \mathbb{Z}^+$	$\theta = 0$
$\tilde{\theta} \in \mathbb{R}\backslash\{0\}$	$\tilde{m} \in \mathbb{Z}$	×	$\mathcal{A}_{+,\pm}$	×	×	×
$\tilde{\theta} \in \mathbb{R}\backslash\{0\}$	$\tilde{m} \in \mathbb{Z}^+$	$\mathcal{A}_{\pm,+}$	$\mathcal{A}_{+,+}$	$\mathcal{A}_{\pm i,+}$	×	$\mathcal{A}_{0,+}$
$\tilde{\theta} \in i\mathbb{R}\backslash\{0\}$	$\tilde{m} \in \mathbb{Z}$	×	$\mathcal{A}_{+,\pm i}$	×	×	×
$\tilde{\theta} \in i\mathbb{R}\backslash\{0\}$	$\tilde{m} \in \mathbb{Z}^+$	×	×	×	×	×
$\tilde{\theta} = 0$		×	$\mathcal{A}_{+,0}$	×	×	×

(5.5)

[27]Removing the unity from the presentation of $\mathcal{A}$ also removes a singularity from $\Phi^{(L)}$.

5.3 Regularity of twistor space connection in L-gauge for non-unital $\mathcal{A}_{E,J}$

Under the assumption that $\mathcal{A}_{E,J}$ does not contain the unity, we proceed by investigating (5.2). From (3.51) it follows that

$$T_{m,\tilde{m}} \star \kappa_y \bar{\kappa}_{\bar{y}} = \frac{\mathbf{A}}{\sqrt{\det(\mathbf{K})}} \exp\left\{-\frac{1}{2}\left(\mathbf{K}^{-1}\right)^{\underline{\alpha\beta}} Y_{\underline{\alpha}} Y_{\underline{\beta}}\right\}, \tag{5.6}$$

where

$$\left(\mathbf{K}^{-1}\right)^{\underline{\alpha\beta}} = \frac{1}{\mathbf{B}^2-\mathbf{C}^2}\left[\mathbf{B}E^{\underline{\alpha\beta}} - \mathbf{C}J^{\underline{\alpha\beta}}\right], \qquad \det(\mathbf{K}) = \left(\mathbf{B}^2-\mathbf{C}^2\right)^2. \tag{5.7}$$

Thus the case of $n = 1$ is equivalent to the case of $n = 0$ by replacing $\mathbf{B}$ and $\mathbf{C}$ with $\frac{\mathbf{B}}{\mathbf{B}^2-\mathbf{C}^2}$ and $\frac{-\mathbf{C}}{\mathbf{B}^2-\mathbf{C}^2}$, respectively, and multiplying an overall factor $\frac{1}{\sqrt{(\mathbf{B}^2-\mathbf{C}^2)^2}}$, which is possible provided that $\mathbf{B}^2 \neq \mathbf{C}^2$ i.e. if $m\theta \neq \pm\tilde{m}\tilde{\theta}$ for all allowed values of m and $\tilde{m}$ (in the non-unital case). This can be achieved by a suitable choice of θ and $\tilde{\theta}$.

For $n = 0$ we performing the $\star$-products between $T^L_{m,\tilde{m}}$ and the Z-dependent exponential in (5.2), which yields

$$\begin{aligned}
&\left(\mathring{V}^{(L)}_{0;m,\tilde{m}}\right)_\alpha \\
&= 2i\int_{-1}^{1}\frac{d\tau j_\sigma(\tau)}{(\tau+1)^2}\frac{\mathbf{A}}{\sqrt{\mathbf{F}^2\mathbf{M}^2(\tau)}}\exp\left\{-iz^\beta y_\beta - \frac{1}{2}\bar{y}^{\dot{\alpha}}\mathbf{H}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\beta}} + \frac{\mathbf{F}^{\gamma\delta}}{2\mathbf{F}^2}\left(iz_\gamma + \mathbf{G}_{\gamma\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)\left(iz_\delta + \mathbf{G}_{\delta\dot{\beta}}\bar{y}^{\dot{\beta}}\right)\right\} \\
&\quad\times \frac{\partial}{\partial\rho^\alpha}\left\{\exp\left\{\frac{\mathbf{M}_{\beta\gamma}(\tau)}{2\mathbf{M}^2(\tau)}\left[iy^\beta - \rho^\beta + \frac{i\mathbf{F}^{\beta\delta}}{\mathbf{F}^2}\left(iz_\delta + \mathbf{G}_{\delta\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)\right]\left[iy^\gamma - \rho^\gamma + \frac{i\mathbf{F}^{\gamma\xi}}{\mathbf{F}^2}\left(iz_\xi + \mathbf{G}_{\xi\dot{\beta}}\bar{y}^{\dot{\beta}}\right)\right]\right\}\right\}_{\rho=0} \\
&= 2i\int_{-1}^{1}\frac{d\tau j_\sigma(\tau)}{(\tau+1)^2}\frac{-\mathbf{A}\mathbf{M}_{\alpha\beta}(\tau)}{\sqrt{\mathbf{F}^2\mathbf{M}^2(\tau)}\mathbf{M}^2(\tau)}\left[iy^\beta + \frac{i\mathbf{F}^{\beta\gamma}}{\mathbf{F}^2}\left(iz_\gamma + \mathbf{G}_{\gamma\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)\right] \\
&\qquad\times\exp\left\{-iz^\beta y_\beta - \frac{1}{2}\bar{y}^{\dot{\alpha}}\mathbf{H}_{\dot{\alpha}\dot{\beta}}\bar{y}^{\dot{\beta}} + \frac{\mathbf{F}^{\gamma\delta}}{2\mathbf{F}^2}\left(iz_\gamma + \mathbf{G}_{\gamma\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)\left(iz_\delta + \mathbf{G}_{\delta\dot{\beta}}\bar{y}^{\dot{\beta}}\right)\right. \\
&\qquad\quad\left.+\frac{\mathbf{M}_{\beta\gamma}(\tau)}{2\mathbf{M}^2(\tau)}\left[iy^\beta + \frac{i\mathbf{F}^{\beta\delta}}{\mathbf{F}^2}\left(iz_\delta + \mathbf{G}_{\delta\dot{\alpha}}\bar{y}^{\dot{\alpha}}\right)\right]\left[iy^\gamma + \frac{i\mathbf{F}^{\gamma\xi}}{\mathbf{F}^2}\left(iz_\xi + \mathbf{G}_{\xi\dot{\beta}}\bar{y}^{\dot{\beta}}\right)\right]\right\},
\end{aligned} \tag{5.8}$$

where

$$\mathbf{M}_{\alpha\beta}(\tau) \equiv \frac{\mathbf{F}_{\alpha\beta}}{\mathbf{F}^2} - 2\varsigma c(\tau)\, U_{\alpha\beta}, \tag{5.9}$$

The integrand has potential divergencies at $\tau = 0$, $\tau = -1$ and any value for τ where $\mathbf{F}^2$ or $\mathbf{M}^2(\tau)$ vanishes. As analysed in section 3.3, the potential divergencies in $j_\sigma(\tau)$ at $\tau = 0$ do not spoil the convergence of the integral provided that the ν- and b_n-parameters are sufficiently small. Furthermore, since $\mathbf{M}_{\alpha\beta}(\tau) \sim (\tau+1)^{-1}$ as $(\tau+1) \to 0$, it follows that both the prefactor and the exponent are bounded at $\tau = -1$.

To facilitate the investigation of $\mathbf{F}^2$ and $\mathbf{M}^2(\tau)$, which are thus functions of $m\theta$, $\tilde{m}\tilde{\theta}$, $\mathcal{X}_4$ and τ, we use the gamma matrix realization in appendix B and the coordinates for L in appendix C. We have not succeeded in a complete analysis, but we have been able to cover a few important special cases as follows:

The case $\mathcal{A}_{+,0}$. In this case, we have $\theta \in \mathbb{R}\backslash\{0\}$, $\tilde{\theta} = 0$, $m \in \mathbb{Z}^+$, and hence $\mathbf{A} = \mathrm{sech}^2 m\theta$, $\mathbf{B} = \tanh m\theta$, $\mathbf{C} = 0$. Using the explicit matrices and spherical coordinates defined in the appendices, we obtain

$$\mathbf{F}^2 = -\mathbf{B}^2\lambda^2 r^2\,, \tag{5.10}$$

$$\mathbf{M}^2(\tau) = -\frac{\left[\varsigma c(\tau)\,\mathbf{B}\lambda r + e^{i\vartheta}\right]\left[\varsigma c(\tau)\,\mathbf{B}\lambda r + e^{-i\vartheta}\right]}{\mathbf{B}^2\lambda^2 r^2}\,. \tag{5.11}$$

From $m\theta \neq 0$ it follows that $\mathbf{B} \neq 0$, and hence $\mathbf{F}^2$ does not vanish except at $r = 0$. Moreover, since $c(\tau)$ is purely imaginary, the quantity $\varsigma c(\tau)\,\mathbf{B}\lambda r$ is purely imaginary as well. Thus $\mathbf{M}^2(\tau)$ vanishes iff

$$\vartheta = \frac{\pi}{2}\,, \qquad \tau \in \left\{-\frac{1+\mathbf{B}\lambda r}{1-\mathbf{B}\lambda r}, -\frac{1-\mathbf{B}\lambda r}{1+\mathbf{B}\lambda r}\right\}. \tag{5.12}$$

Thus, in this case the twistor space connection is real-analytic everywhere away from the equatorial plane in the spherical coordinates.[28]

The case $\tilde{\theta} \neq 0$. When $\tilde{\theta} \neq 0$, we resort to case-by-case investigation. We will only show two examples below.

For example, if we consider the region of small r, i.e. a small spatial sphere around the origin point, we have

$$\mathbf{F}^2 = \mathbf{C}^2 - 2i\mathbf{BC}\lambda r\,\cos(\vartheta) + O\left(r^2\right)\,, \tag{5.13}$$

$$\mathbf{M}^2(\tau) = -\mathbf{C}^{-2}\left[\varsigma c(\tau)\,\mathbf{C} - i\right]^2 - 2\mathbf{BC}^{-3}\left[\varsigma c(\tau)\,\mathbf{C} - i\right]\lambda r\,\cos(\vartheta) + O\left(r^2\right)\,. \tag{5.14}$$

A valid choice of the parameters is $\theta \in \mathbb{R}\backslash\{0\}$, $\tilde{\theta} \in i\mathbb{R}\backslash\{0\}$ and $m \in \mathbb{Z}^+$, i.e. the truncation $\mathcal{A}_{+,\pm i}$. With this choice, we have $\mathbf{B} \in \mathbb{R}\backslash\{0\}$, $\mathbf{C} \in i\mathbb{R}$. Then for $\mathbf{C} \neq 0$ i.e. $\tilde{m} \neq 0$, both first leading terms of $\mathbf{F}^2$ and $\mathbf{M}^2(\tau)$ are non-zero. For $\tilde{m} = 0$, the discussion is the same as the above $\tilde{\theta} = 0$ case. To summarize, the one-form field in this case is real-analytic in the small sphere except on the equatorial plane.

For another example, we consider the region of small ϑ, i.e. a narrow cone around the axis of symmetry $\vartheta = 0$, we have

$$\mathbf{F}^2 = \left(\mathbf{C} - i\mathbf{B}\lambda r\right)^2 + O\left(\vartheta^2\right)\,, \tag{5.15}$$

$$\mathbf{M}^2(\tau) = \left[\frac{1}{\mathbf{C} - i\mathbf{B}\lambda r} - i\varsigma c(\tau)\right]^2 + O\left(\vartheta^2\right)\,. \tag{5.16}$$

A valid choice of the parameters is $\theta \in \mathbb{R}\backslash\{0\}$, $\tilde{\theta} \in \mathbb{R}\backslash\{0\}$ and $m \in \mathbb{Z}^+$ i.e. the truncation $\mathcal{A}_{+,\pm}$. With this choice, we have $\mathbf{B} \in \mathbb{R}\backslash\{0\}$, $\mathbf{C} \in \mathbb{R}$. Then for $r \neq 0$ we have $\mathbf{C} - i\mathbf{B}\lambda r \notin \mathbb{R}$, and thus, with $i\varsigma c(\tau) \in \mathbb{R}$, both first leading terms of $\mathbf{F}^2$ and $\mathbf{M}^2(\tau)$ are non-zero. The one-form field in this case is real-analytic in the narrow cone around the axis $\vartheta = 0$ excluding the origin point.

[28] On the equatorial plane, for a certain value of τ between the integration limits, zero-denominators appear in the integrand of (5.8) both on the exponent and in the factor in the front. We leave the consequence of this for future work.

5.4 Linearized twistor space connection in Vasiliev gauge

Finally, let us check in a special case that it is indeed possible to bring the linearized twistor space connection to Vasiliev gauge by means of a linearized gauge transformation, as described in section 3.1, viz.

$$V_\alpha^{(G)(1)} = V_\alpha^{(L)(1)} + \partial_\alpha H^{(1)} , \tag{5.17}$$

where $H^{(1)}$ is formally given by

$$H^{(1)} = H^{(1)}|_{Z=0} - \frac{1}{z^\beta \partial_\beta} \left(z^\alpha V_\alpha^{(L)(1)} \right) . \tag{5.18}$$

Note that, as explained around eq. (5.12), in the case $\tilde{\theta} = 0$ with the truncation $\mathcal{A}_{+,0}$, the regularity of the twistor space connection at $\vartheta = \frac{\pi}{2}$ has not yet been verified in the L-gauge. However, we expect that this problem would not exist in Vasiliev gauge.

To perform the check, we set $\vartheta = \frac{\pi}{2}$, $t = \phi = 0$, $y^\alpha = \bar{y}^{\dot{\alpha}} = 0$ and $\lambda = 1$. The resulting expression of the generating function for the L-gauge twistor space connection reads

$$\left(\mathring{V}^{(L)}_{n=0;\sigma,m} \right)_\alpha \Big|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0} = -2i\mathbf{A} \int_{-1}^{1} \frac{d\tau j_\sigma(\tau)}{(\tau+1)^2} P_{\alpha\beta} z^\beta \exp\left\{ Q_{\alpha\beta} z^\alpha z^\beta \right\} , \tag{5.19}$$

where

$$P_{\alpha\beta} = -\left[1 + \mathbf{B}^2 c^2(\tau) r^2 \right]^{-\frac{3}{2}} \begin{pmatrix} \mathbf{B}\varsigma c(\tau) r & -1 \\ 1 & \mathbf{B}\varsigma c(\tau) r \end{pmatrix} , \tag{5.20}$$

$$Q_{\alpha\beta} = -\frac{1}{2}\varsigma c(\tau) \left[1 + \mathbf{B}^2 c^2(\tau) r^2 \right]^{-1} \begin{pmatrix} -\mathbf{B}\varsigma c(\tau) r & 1 \\ 1 & \mathbf{B}\varsigma c(\tau) r \end{pmatrix} . \tag{5.21}$$

Going to Vasiliev gauge, we obtain

$$\begin{aligned} &\left(\mathring{V}^{(G)(1)}_{n=0;\sigma,m} \right)_\alpha \Big|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0} \\ &= -\frac{i\varsigma b_0 \mathring{\nu}_\sigma}{2} \mathbf{A} z_\alpha \int_{-1}^{1} \frac{d\tau}{(\tau+1)^2} \left\{ P e^{Q(z)} + \frac{e^{Q(z)} - 1 - Q(z) e^{Q(z)}}{Q^2(z)} z^\alpha S_\alpha{}^\beta Q_{\beta\gamma} z^\gamma \right\} , \end{aligned} \tag{5.22}$$

where

$$Q(z) := Q_{\alpha\beta} z^\alpha z^\beta , \tag{5.23}$$

and we have decomposed

$$P_{\alpha\beta} =: P \varepsilon_{\alpha\beta} + S_{\alpha\beta} , \quad S_{[\alpha\beta]} = 0 , \tag{5.24}$$

i.e.

$$P = \left[1 + \mathbf{B}^2 c^2(\tau) r^2 \right]^{-\frac{3}{2}} , \quad S_{\alpha\beta} = -\left[1 + \mathbf{B}^2 c^2(\tau) r^2 \right]^{-\frac{3}{2}} \mathbf{B}\varsigma c(\tau) r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} . \tag{5.25}$$

The integrand of (5.22) can be converted into a total derivative of τ:

$$\left.\left(\mathring{V}^{(G)(1)}_{n=0;\sigma,m}\right)_\alpha\right|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0}$$
$$= -\frac{\varsigma b_0\mathring{\nu}_\sigma}{2}\mathbf{A}z_\alpha\int_{-1}^{1}d\tau\frac{\partial}{\partial\tau}\left\{\frac{(e^{Q(z)}-1)(Q(z)P-z^\alpha S_\alpha{}^\beta Q_{\beta\gamma}z^\gamma)}{\frac{\partial Q^2(z)}{\partial c}}\right\}, \tag{5.26}$$

where, more explicitly,

$$\frac{(e^{Q(z)}-1)(Q(z)P-z^\alpha S_\alpha{}^\beta Q_{\beta\gamma}z^\gamma)}{\frac{\partial Q^2(z)}{\partial c}}$$
$$= -\frac{\sqrt{1+\mathbf{B}^2c^2(\tau)\,r^2}\left[1-\exp\left(\frac{\mathbf{B}c(\tau)^2r\left(z^1+z^2\right)\left(z^1-z^2\right)-2\varsigma c(\tau)z^1z^2}{2[1+\mathbf{B}^2c^2(\tau)r^2]}\right)\right]}{\mathbf{B}\varsigma c(\tau)\,r\left(z^1+z^2\right)\left(z^1-z^2\right)-2z^1z^2}. \tag{5.27}$$

Thus, assigning the singularity in the interior of the integration domain its principal value, and using separate analytical continuations above and below the singularity, one finds that it does not contribute, and hence

$$\left.\left(\mathring{V}^{(G)(1)}_{n=0;\sigma,m}\right)_\alpha\right|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0} = \frac{b_0\mathring{\nu}_\sigma}{2}z_\alpha\mathbf{A}\frac{1-\exp\left[\frac{1}{2\mathbf{B}r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\left(z^1+z^2\right)\left(z^1-z^2\right)}. \tag{5.28}$$

We note that the limit $z^\alpha \to 0$ must be taken after the integration over τ has been performed. This yields a well-defined limit, such that the twistor space connection is indeed real-analytic at $z^\alpha = 0$. If one instead takes the limit $z^\alpha \to 0$ under the integral, one ends up with a divergent integral; this divergence cannot, however, be interpreted as any pole or other singularity at $z^\alpha = 0$. Thus, the prescription that we use is the unique one leading to a sensible result.[29] We note, however, that in the holomorphic gauge the corresponding operations commute, and, correspondingly, the twistor space connection is non-real-analytic at $z^\alpha = 0$ in this gauge; see section 3.3.

Similarly, we can also calculate for $n = 1$:

$$\left.\left(\mathring{V}^{(G)(1)}_{n=1;\sigma,m}\right)_\alpha\right|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0} = \frac{b_0\mathring{\nu}_\sigma}{2}z_\alpha\mathbf{A}\frac{1-\exp\left[\frac{\mathbf{B}}{2r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\mathbf{B}^2\left(z^1+z^2\right)\left(z^1-z^2\right)}. \tag{5.29}$$

Finally, using the analog of eq. (5.1) in Vasiliev gauge, i.e. replacing the label (L) with $(G)(1)$ and substituting (5.28) and (5.29), we obtain

$$\left.\mathring{V}^{(G)(1)}_\alpha\right|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0}$$
$$= \frac{b_0}{4}z_\alpha\sum_{\sigma,m,\tilde{m}}\mathbf{A}\nu_{\sigma,m,\tilde{m}}\left[\frac{1-\exp\left[\frac{1}{2\mathbf{B}r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\left(z^1+z^2\right)\left(z^1-z^2\right)}\right.$$
$$\left.+\sigma\frac{1-\exp\left[\frac{\mathbf{B}}{2r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\mathbf{B}^2\left(z^1+z^2\right)\left(z^1-z^2\right)}\right]. \tag{5.30}$$

[29] This suggests that in more general perturbatively defined solutions to Vasiliev's equations obtained by repeated homotopy integration [1, 13] (see also [2]), the resulting auxiliary integrals should be performed prior to taking the limit $z^\alpha \to 0$; whether this prescription is actually unique and correct, remains to be investigated.

Indeed, starting in Vasiliev gauge, one can integrate the equations of motion for the linearized twistor space connection directly without factorizing the inner Klein operator κ, with the result [1]

$$V_\alpha^{(G)(1)} = -\frac{ib_0}{2} z_\alpha \int_0^1 d\tau\ \tau e^{iy^\alpha z_\alpha \tau} \left(\Phi^{(G)(1)} \Big|_{y \to -z\tau} \right) . \tag{5.31}$$

We note that, unlike the solution for the twistor space connection obtained starting in the holomorphic gauge, which refers to a splitting of z^α into $z^\pm$ as in section 3.2, the above expression does not refer to any auxiliary spinor frame in Z space. From $\Phi^{(G)(1)} = \Phi^{(L)(1)} = \Phi^{(L)}$ it follows that (5.31) implies that

$$\begin{aligned} &\mathring{V}_\alpha^{(G)(1)} \Big|_{\lambda=1;\ \vartheta=\frac{\pi}{2},\ t=\phi=0;\ y=\bar{y}=0} \\ &= \frac{b_0}{2} z_\alpha \sum_{m,\tilde{m}} \mathbf{A} \left[\nu_{1,m,\tilde{m}} \frac{1-\exp\left[\frac{1}{2\mathbf{B}r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\left(z^1+z^2\right)\left(z^1-z^2\right)} \right. \\ &\qquad \left. + \nu_{2,m,\tilde{m}} \frac{1-\exp\left[\frac{\mathbf{B}}{2r}\left(z^1+z^2\right)\left(z^1-z^2\right)\right]}{\mathbf{B}^2\left(z^1+z^2\right)\left(z^1-z^2\right)} \right] , \end{aligned} \tag{5.32}$$

which one can readily identify with (5.30) upon using (4.3).

6 Conclusion

In this paper, we have given a new class of bi-axially symmetric solutions to Vasiliev's bosonic higher spin gravity model using an Ansatz based on gauge functions and separation of the dependence on the coordinates in twistor space.

This facilitates the construction of perturbatively exact solutions in a holomorphic gauge. In this gauge, the spacetime connection vanishes, the Weyl zero-form is constant, i.e. it depends only on the fiber coordinates, while the twistor space connection depends on the twistor space via a universal holomorphic function on Z-space with singularties at $z^\alpha = 0$ that we have exhibited in the Weyl order in section 3.3, and on the fiber coordinates via the zero-form integration constants. We have then expanded the dependence on the fiber coordinates in terms of the basis of a group algebra generated by the exponents of θE and $\tilde{\theta} J$, where E and J are the generators the time-translational and rotational symmetries of the solutions.

We have then switched on the spacetime dependence using a vacuum gauge function L. In the resulting gauge, which we refer to as L-gauge, the spacetime connetion describes an anti-de Sitter spacetime. The terms containing the unity of the internal algebra need to be removed, in order for the Weyl zero-form in L-gauge to be real-analytic on twistor space. The resulting generalized spin-s Weyl tensor, which thus obeys the Bargmann-Wigner equation, is given by a sum of generalized Petrov type-D tensors that are asymptotically Kerr-like or 2-brane-like. For special values of the parameters, including the symmetry enhanced cases, the spin-s Weyl tensor is of generalized Petrov type D.

We have also shown that the twistor space connection in L-gauge, provided that the group algebra is truncated to a non-unital semigroup algebra as summarized in the table (5.5), is real-analytic in finite spacetime regions for a number of choices of parameters. In particular, in the spherically symmetric case, it is real-analytic everywhere away from the equatorial plane. The Ansatz introduces a fixed frame in Z-space that breaks the manifest spherical symmetry upon going to the normal order in master fields with a Z-dependence. At this plane, singularities may appear in auxiliary integrals, whose treatment requires analytical continuations in twistor space. We have not spelled out the nature of the resulting contributions to the twistor space connection in L-gauge in this work.

Finally, we have examined the problem of transforming the master fields from the L-gauge to Vasiliev gauge at the linearized level. It is trivial in the case of the Weyl zero-form. As for the linearized twistor space connection, we have argued that the transformation exists at spacetime points where the connection is real-analytic in twistor space in L-gauge. Among the remaining cases, we have focused on the potential divergence at the equatorial plane in the spherically symmetric case, which should be removed by the transformation, as the twistor space connection in Vasiliev's gauge does not refer to any fixed frame in Z-space. Indeed we have verified that this is the case at the origin of the fiber space (i.e. at $Y^{\underline{\alpha}} = 0$), for general z^{α}, and consequently we have found agreement with the expression for the twistor space connection in Vasiliev gauge obtained by direct integration.

Thus, more briefly, we have found families of exact bi-axially symmetric solutions in the holomorphic and L-gauges, and we have verified that they can be brought to Vasiliev gauge at the linearized level in a special case, leaving the more general case as well as higher order perturbation for future study.

We end our conclusions by commenting on future directions. We have left a number of technical details unattended, that we would like to examine more carefully. Besides the issues related to real-analyticity of the linearized master fields in Vasiliev gauge, there is the intriguing degenerate case $\mathbf{B} = \mathbf{C}$. Moreover, by taking limits for θ and ν-parameters it is possible to make contact with the solutions found in [26], and more general Kerr-like extensions thereof by expanding the fiber subalgebra using a combination of group algebra elements and endomorphisms in Fock spaces.

More generally, we recall that the importance of Vasiliev's gauge at linearized level is that, when combined with normal order, the linearized spacetine connection $W_{\mu}^{(G)(1)}$ has a Y-expansion at $Z = 0$ in terms of unfolded Fronsdal tensors and the initial data $H^{(1)}|_{Z=0}$ modulo gauge transformations.[30] Exact solutions, however, are easier to find in Weyl order using the gauge function method. As far as we can see from the results here and elsewhere, we expect there to be an agreement at the linearized level between the holomorphic and Vasiliev gauges for a fairly large class of linearized zero-form initial data $\Phi'^{(1)}(Y)$, and it would be desirable to establish this correspondence more precisely, e.g. by expanding $\Phi'^{(1)}(Y)$ in terms of twistor space plane waves.

[30] If $H^{(1)}|_{Z=0}$ and the gauge parameters belong to the same class of functions then $H^{(1)}|_{Z=0}$ describes pure gauge degrees of freedom.

Turing to higher order perturbations, the next step is to compute the first subleading corrections to all master fields in Vasiliev gauge, and examine whether real-analyticity in twistor space for generic spacetime points constrains the initial data $\Phi'^{(n)}(Y)$ for the zero-form and $H^{(n)}|_{Z=0}$ for the gauge function, for $n = 1, 2$. This may lead to modified asymptotic boundary conditions in AdS_4 and corresponding corrections to the zero-form charges. In particular, as proposed in [24], the zero-form charge $\mathcal{I}_2$ is a contribution to the free energy functional. The corresponding sesqui-linear form is not definite on the representation space of the underlying higher spin symmetry algebra containing the initial data of our solutions. There are additional contributions to the free energy, however, that may lead to an interesting phase diagram.

The above analysis can also be performed for the closely related Kerr-like solutions outlined above. More generally, one may consider relaxing the Vasiliev gauge as well as the smoothness conditions in twistor space, which may lead to more general noncommutive geometries with interesting properties.

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A The $\star$-exponent

Let Y^α, $\alpha = 1, \dots, N$, be oscillator variables obeying

$$[Y^\alpha, Y^\beta]_\star = 2iC^{\alpha\beta}\,, \tag{A.1}$$

where N is even and $C^{\alpha\beta}$ is invertible. Denote

$$w = \frac{1}{4} K_{\alpha\beta} Y^\alpha \star Y^\beta\,, \tag{A.2}$$

where $K_{\alpha\beta}$ is a constant matrix obeying

$$K_{\alpha\beta} = K_{\beta\alpha}\,, \qquad K_{\alpha\beta} K^{\beta\gamma} = \delta_\alpha{}^\gamma\,, \tag{A.3}$$

where indices are raised and lowered using the conventions $Y^\alpha = C^{\alpha\beta} Y_\beta$, $Y_\beta = Y^\beta C_{\beta\alpha}$, and $C^{\alpha\beta} C_{\alpha\gamma} = \delta_\gamma{}^\beta$. The $\star$-exponent is defined by the Taylor series of exponential function with $\star$-products replacing ordinary products. In what follows we will compute the symbol in Weyl order of the $\star$-exponent

$$g \equiv e_\star^{-2tw}\,. \tag{A.4}$$

From (A.4) we can derive

$$w \star g = -\frac{1}{2}\frac{\partial g}{\partial t}\,, \tag{A.5}$$

and to proceed we will compute the symbol of $w \star g$. To do so we use the identity

$$Y_\alpha \star f(Y) = Y_\alpha f(Y) + i\frac{\partial}{\partial Y^\alpha} f(Y)\,. \tag{A.6}$$

Thus

$$Y^\alpha \star Y^\beta = Y^\alpha Y^\beta + iC^{\alpha\beta}\,. \tag{A.7}$$

Hence (A.2) can also be written as

$$w = \frac{1}{4} K_{\alpha\beta} Y^\alpha Y^\beta\,. \tag{A.8}$$

Using this, we can show that

$$w \star g = wg - \frac{N}{8}\frac{\partial g}{\partial w} - \frac{1}{4} w \frac{\partial^2 g}{\partial w^2}\,. \tag{A.9}$$

Proof.

$$\begin{aligned}
w \star g &= \frac{1}{4} K^{\alpha\beta} Y_\alpha \star Y_\beta \star g \\
&= \frac{1}{4} K^{\alpha\beta} Y_\alpha \star \left(Y_\beta g + i \frac{\partial g}{\partial Y^\beta} \right) \\
&= \frac{1}{4} K^{\alpha\beta} \left(Y_\alpha Y_\beta g + i Y_\alpha \frac{\partial g}{\partial Y^\beta} + i \frac{\partial (Y_\beta g)}{\partial Y^\alpha} - \frac{\partial^2 g}{\partial Y^\alpha \partial Y^\beta} \right) \\
&= wg + \frac{i}{2} K^{\alpha\beta} Y_\alpha \frac{\partial g}{\partial Y^\beta} - \frac{1}{4} K^{\alpha\beta} \frac{\partial^2 g}{\partial Y^\alpha \partial Y^\beta}\,.
\end{aligned}$$

The last two terms can be further converted:

$$\begin{aligned}
\frac{i}{2} K^{\alpha\beta} Y_\alpha \frac{\partial g}{\partial Y^\beta} &= \frac{i}{2} K^{\alpha\beta} Y_\alpha \frac{\partial g}{\partial w} \frac{\partial w}{\partial Y^\beta} \\
&= \frac{i}{2} K^{\alpha\beta} Y_\alpha \left(\frac{1}{2} K_{\beta\gamma} Y^\gamma \right) \frac{\partial g}{\partial w} \\
&= \frac{i}{4} Y_\alpha Y^\alpha \frac{\partial g}{\partial w} \\
&= 0\,, \\
-\frac{1}{4} K^{\alpha\beta} \frac{\partial^2 g}{\partial Y^\alpha \partial Y^\beta} &= -\frac{1}{4} K^{\alpha\beta} \frac{\partial}{\partial Y^\alpha} \left(\frac{\partial g}{\partial w} \frac{\partial w}{\partial Y^\beta} \right) \\
&= -\frac{1}{4} K^{\alpha\beta} \frac{\partial}{\partial Y^\alpha} \left(\frac{1}{2} K_{\beta\gamma} Y^\gamma \frac{\partial g}{\partial w} \right) \\
&= -\frac{1}{8} \frac{\partial}{\partial Y^\alpha} \left(Y^\alpha \frac{\partial g}{\partial w} \right) \\
&= -\frac{1}{8} \delta_\alpha{}^\alpha \frac{\partial g}{\partial w} - \frac{1}{8} Y^\alpha \frac{\partial^2 g}{\partial w^2} \frac{\partial w}{\partial Y^\alpha} \\
&= -\frac{N}{8} \frac{\partial g}{\partial w} - \frac{1}{8} Y^\alpha \left(\frac{1}{2} K_{\alpha\beta} Y^\beta \right) \frac{\partial^2 g}{\partial w^2} \\
&= -\frac{N}{8} \frac{\partial g}{\partial w} - \frac{1}{4} w \frac{\partial^2 g}{\partial w^2}\,.
\end{aligned}$$

Thus (A.9) is proven. □

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By substituting (A.9), (A.5) can be converted to

$$wg - \frac{N}{8}\frac{\partial g}{\partial w} - \frac{1}{4}w\frac{\partial^2 g}{\partial w^2} = -\frac{1}{2}\frac{\partial g}{\partial t}\,. \tag{A.10}$$

This differential equation can be solved by substituting the Ansatz

$$g = a\,(t)\,e^{b(t)w}\,, \tag{A.11}$$

which gives

$$a\,(t)\,we^{b(t)w} - \frac{N}{8}a\,(t)\,b\,(t)\,e^{b(t)w} - \frac{1}{4}a\,(t)\,b^2\,(t)\,we^{b(t)w} = -\frac{1}{2}a'\,(t)\,e^{b(t)w} - \frac{1}{2}a\,(t)\,b'\,(t)\,we^{b(t)w}\,, \tag{A.12}$$

and this equation requires the following set of ordinary differential equations for $a\,(t)$ and $b\,(t)$ to be satisfied:

$$-\frac{1}{2}a'\,(t) = -\frac{N}{8}a\,(t)\,b\,(t)\;, \tag{A.13}$$

$$-\frac{1}{2}a\,(t)\,b'\,(t) = a\,(t)\left(1 - \frac{1}{4}b^2\,(t)\right)\,. \tag{A.14}$$

The general solution is given by

$$a\,(t) = C_2\,[\mathrm{sech}\,(t + C_1)]^{\frac{N}{2}}\;, \tag{A.15}$$

$$b\,(t) = -2\mathrm{tanh}\,(t + C_1)\;, \tag{A.16}$$

where C_1 and C_2 are constants. These are determined by requiring that (A.11) and (A.4) stand for the same solution of (A.10). It is obvious that

$$g|_{t=0} = e_\star^{0w} = 1\,, \tag{A.17}$$

and hence

$$\left(wg - \frac{N}{8}\frac{\partial g}{\partial w} - \frac{1}{4}w\frac{\partial^2 g}{\partial w^2}\right)\Bigg|_{t=0} = w \star g|_{t=0} = w\,. \tag{A.18}$$

Consequently we have

$$a\,(0)\,e^{b(0)w} = 1\,, \tag{A.19}$$

$$a\,(0)\,we^{b(0)w} - \frac{N}{8}a\,(0)\,b\,(0)\,e^{b(0)w} - \frac{1}{4}a\,(0)\,b^2\,(0)\,we^{b(0)w} = w\,. \tag{A.20}$$

Therefore,

$$a\,(0) = 1 \text{ and } b\,(0) = 0\;\;. \tag{A.21}$$

By substituting them into (A.15) and (A.16) we can determine that

$$C_1 = 0 \text{ and } C_2 = 1\,. \tag{A.22}$$

Then we derive that

$$a\,(t) = [\mathrm{sech}\,(t)]^{\frac{N}{2}}\;, \tag{A.23}$$

$$b\,(t) = -2\mathrm{tanh}\,(t)\;. \tag{A.24}$$

In this way we conclude

$$e_\star^{-2tw} = g = [\mathrm{sech}\,(t)]^{\frac{N}{2}}\,e^{-2\mathrm{tanh}(t)w}\,. \tag{A.25}$$

B Van der Waerden symbols and gamma matrices

To simplify some of the calculations in this paper, one can use a set of explicit matrix expressions of Pauli matrices and gamma matrices, which for example is given in this appendix.

B.1 Pauli matrices

We define the σ-matrices with two lower spinor indices

$$(\sigma_0)_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\sigma_1)_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\sigma_2)_{\alpha\dot{\alpha}} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad (\sigma_3)_{\alpha\dot{\alpha}} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{B.1}$$

where σ_0 is the identity matrix and $\sigma_{1,2,3}$ are the usual Pauli matrices. We also define their complex conjugate:

$$(\bar{\sigma}_0)_{\dot{\alpha}\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (\bar{\sigma}_1)_{\dot{\alpha}\alpha} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad (\bar{\sigma}_2)_{\dot{\alpha}\alpha} = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad (\bar{\sigma}_3)_{\dot{\alpha}\alpha} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{B.2}$$

Then obviously we have

$$(\sigma_a)_{\alpha\dot{\alpha}} = (\bar{\sigma}_a)_{\dot{\alpha}\alpha} . \tag{B.3}$$

Furthermore, we use

$$\varepsilon^{\alpha\beta} = \varepsilon_{\alpha\beta} = \varepsilon^{\dot{\alpha}\dot{\beta}} = \varepsilon_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{B.4}$$

to raise or lower indices (by NW-SE rules).

We also define

$$(\sigma_{ab})_{\alpha\beta} = -(\sigma_{ba})_{\alpha\beta} = \left(\sigma_{[a}\right)_{\alpha}{}^{\dot{\gamma}} \left(\bar{\sigma}_{b]}\right)_{\dot{\gamma}\beta} , \tag{B.5}$$

$$(\bar{\sigma}_{ab})_{\dot{\alpha}\dot{\beta}} = -(\bar{\sigma}_{ba})_{\dot{\alpha}\dot{\beta}} = \left(\bar{\sigma}_{[a}\right)_{\dot{\alpha}}{}^{\gamma} \left(\sigma_{b]}\right)_{\gamma\dot{\beta}} . \tag{B.6}$$

To write them explicitly:

$$\begin{aligned}
&(\sigma_{01})_{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, && (\sigma_{02})_{\alpha\beta} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, && (\sigma_{03})_{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&(\sigma_{12})_{\alpha\beta} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, && (\sigma_{13})_{\alpha\beta} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, && (\sigma_{23})_{\alpha\beta} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}, \\
&(\bar{\sigma}_{01})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, && (\bar{\sigma}_{02})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} -i & 0 \\ 0 & -i \end{pmatrix}, && (\bar{\sigma}_{03})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\
&(\bar{\sigma}_{12})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, && (\bar{\sigma}_{13})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, && (\bar{\sigma}_{23})_{\dot{\alpha}\dot{\beta}} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.
\end{aligned} \tag{B.7}$$

As shown above, the pair of spinor indices are symmetric.

B.2 Gamma matrices

We construct the explicit expressions of gamma matrices in the following way:

$$(\Gamma_a)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & (\sigma_a)_\alpha{}^{\dot\beta} \\ (\bar\sigma_a)_{\dot\alpha}{}^\beta & 0 \end{pmatrix}, \tag{B.8}$$

whose explicit expressions are

$$(\Gamma_0)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad (\Gamma_1)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix},$$
$$(\Gamma_2)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}, \qquad (\Gamma_3)_{\underline{\alpha}}{}^{\underline{\beta}} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}, \tag{B.9}$$

One can check the property that

$$\left(\Gamma_{(a}\right)_{\underline{\alpha}}{}^{\underline{\gamma}} \left(\Gamma_{b)}\right)_{\underline{\gamma}}{}^{\underline{\beta}} = \eta_{ab}\delta_{\underline{\alpha}}{}^{\underline{\beta}}, \tag{B.10}$$

where $\eta_{ab} = \text{diag}\{-1, 1, 1, 1\}$.

We further use

$$C^{\underline{\alpha\beta}} = \begin{pmatrix} \varepsilon^{\alpha\beta} & 0 \\ 0 & \varepsilon^{\dot\alpha\dot\beta} \end{pmatrix} \text{ and } C_{\underline{\alpha\beta}} = \begin{pmatrix} \varepsilon_{\alpha\beta} & 0 \\ 0 & \varepsilon_{\dot\alpha\dot\beta} \end{pmatrix}, \tag{B.11}$$

to raise or lower the spinor indices of gamma matrices (by NW-SE rules). For example, by lowering the second spinor index, we get

$$(\Gamma_a)_{\underline{\alpha\beta}} = (\Gamma_a)_{\underline{\alpha}}{}^{\underline{\gamma}} C_{\underline{\gamma\beta}} = \begin{pmatrix} 0 & (\sigma_a)_{\alpha\dot\beta} \\ (\bar\sigma_a)_{\dot\alpha\beta} & 0 \end{pmatrix}. \tag{B.12}$$

One can check that in this way of construction, the pair spinor indices are symmetric, i.e. $(\Gamma_a)_{\underline{\alpha\beta}} = (\Gamma_a)_{\underline{\beta\alpha}}$.

We also define

$$(\Gamma_{ab})_{\underline{\alpha}}{}^{\underline{\beta}} = \left(\Gamma_{[a}\right)_{\underline{\alpha}}{}^{\underline{\gamma}} \left(\Gamma_{b]}\right)_{\underline{\gamma}}{}^{\underline{\beta}}. \tag{B.13}$$

One can easily check that

$$(\Gamma_{ab})_{\underline{\alpha\beta}} = \begin{pmatrix} (\sigma_{ab})_{\alpha\beta} & 0 \\ 0 & (\bar\sigma_{ab})_{\dot\alpha\dot\beta} \end{pmatrix}. \tag{B.14}$$

In this way of construction, $(\Gamma_{ab})_{\underline{\alpha\beta}} = (\Gamma_{ab})_{\underline{\beta\alpha}}$.

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Now we define

$$E_{\underline{\alpha\beta}} = -\left(\Gamma_0\right)_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} , \tag{B.15}$$

$$J_{\underline{\alpha\beta}} = -\left(\Gamma_{12}\right)_{\underline{\alpha\beta}} = \begin{pmatrix} 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix} . \tag{B.16}$$

One can for instance check the properties (3.41)–(3.43) using the above explicit matrix expressions.

C Spacetime gauge function

The four-dimensional anti-de Sitter spacetime, AdS_4, of inverse radius λ, is the hyperbola

$$X^A X^B \eta_{AB} = -\lambda^{-2} \ , \tag{C.1}$$

in the five-dimensional space with coordinates X^A, $A = \{0,1,2,3,0'\}$ and flat metric η_{AB} =diag$\{-1,1,1,1,-1\}$. A set of global coordinates

$$(t,r,\vartheta,\phi) \ , \qquad 0 \leqslant \lambda t < 2\pi \, , \quad r \geqslant 0 \, , \quad 0 \leqslant \vartheta \leqslant \pi \, , \quad 0 \leqslant \phi < 2\pi \, , \tag{C.2}$$

can be introduced by taking

$$\begin{aligned} &X^0 = -\sqrt{\lambda^{-2}+r^2}\ \sin\lambda t \, , \qquad X^{0'} = -\sqrt{\lambda^{-2}+r^2}\ \cos\lambda t \, , \\ &X^1 = r\ \sin\vartheta\ \cos\phi \, , \qquad X^2 = r\ \sin\vartheta\ \sin\phi \, , \qquad X^3 = r\ \cos\vartheta \, . \end{aligned} \tag{C.3}$$

The resulting induced metric is

$$ds^2 = -\left(1+\lambda^2 r^2\right) dt^2 + \left(1+\lambda^2 r^2\right)^{-1} dr^2 + r^2 \left(d\vartheta^2 + \sin^2\vartheta\ d\phi^2\right) \, . \tag{C.4}$$

The stereographic coordinates

$$x^\mu \equiv \delta^\mu_a x^a = \frac{X^a}{1+|X^{0'}|} \, , \tag{C.5}$$

where $a = \{0,1,2,3\}$, η_{ab} = diag$\{-,+,+,+\}$ and $x^2 := x^a x^b \eta_{ab}$, maps the two halves $X^{0'} > 0$ and $X^{0'} < 0$ of AdS_4 to the region $-1 < \lambda^2 x^2 < 1$ of $\mathbb{R}^{3,1}$. From the inverse relation given by

$$X^a = \frac{2x^a}{1-\lambda^2 x^2} \, , \quad X^{0'} = \pm\lambda^{-1}\frac{1+\lambda^2 x^2}{1-\lambda^2 x^2} \, , \tag{C.6}$$

it follows that $X^{0'} \to -X^{0'}$ corresponds to $x^a \to -(\lambda^2 x^2)^{-1} x^a$. Thus, the extension of the stereographic coordinates x^a to the entire $\mathbb{R}^{3,1}$ provides a global coordinate of AdS$_4$; the boundary of AdS$_4$ is mapped to the hyperbola $\lambda^2 x^2 = 1$ in $\mathbb{R}^{3,1}$.

The gauge function

$$L\,(x;y,\bar{y}) = \frac{2h}{1+h}\exp\left(\frac{i\lambda}{1+h}x^{\alpha\dot{\alpha}}y_\alpha \bar{y}_{\dot{\alpha}}\right)\,, \qquad x_{\alpha\dot{\alpha}} := x^a\,(\sigma_a)_{\alpha\dot{\alpha}}\,, \quad h := \sqrt{1-\lambda^2 x^2}\,, \tag{C.7}$$

which is defined in the region $\lambda^2 x^2 < 1$, leads to

$$U_\mu = L^{-1} \star \partial_\mu L = -\frac{i}{2} e_\mu^{\alpha\dot{\alpha}} y_\alpha \bar{y}_{\dot{\alpha}} - \frac{i}{4}\left(\omega_\mu^{\alpha\beta} y_\alpha y_\beta + \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}}\right)\,, \tag{C.8}$$

where

$$e_\mu^{\alpha\dot{\alpha}} = -\frac{\lambda \delta_\mu^a\,(\sigma_a)^{\alpha\dot{\alpha}}}{h^2}\,, \tag{C.9}$$

$$\omega_\mu^{\alpha\beta} = -\frac{\lambda^2 \delta_\mu^a x^b\,(\sigma_{ab})^{\alpha\beta}}{h^2}\,, \qquad \bar{\omega}_\mu^{\dot{\alpha}\dot{\beta}} = -\frac{\lambda^2 \delta_\mu^a x^b\,(\bar{\sigma}_{ab})^{\dot{\alpha}\dot{\beta}}}{h^2}\,, \tag{C.10}$$

are the vierbein and Lorentz connection of AdS$_4$ in stereographic coordinates, with flat indices converted to spinor ones using van der Waerden symbols. One also has

$$L^{-1} \star Y_{\underline{\alpha}} \star L = L_{\underline{\alpha}}{}^{\underline{\beta}} Y_{\underline{\beta}}\,, \tag{C.11}$$

with the matrix

$$L_{\underline{\alpha}}{}^{\underline{\beta}} = h^{-1}\begin{bmatrix} \delta_\alpha{}^\beta & \lambda x_\alpha{}^{\dot{\beta}} \\ \lambda x_{\dot{\alpha}}{}^\beta & \delta_{\dot{\alpha}}{}^{\dot{\beta}} \end{bmatrix}\,. \tag{C.12}$$

As an Sp(4;$\mathbb{R}$) group element, $L(x;y,\bar{y})$ corresponds to the transvection in AdS$_4$ that sends all the information of the classical solution encoded at the origin of the stereographic coordinate system to the point x^μ.

D Determination of Petrov type of spin-2 Weyl tensor

In this appendix, we briefly explain how to check (only for spin-2) the Petrov type of a Weyl tensor by using the eigenvalue method. For more details on this topic one can check [47].

The restricted Lorentz group SO$^+$(3,1,$\mathbb{R}$) is isomorphic to SO(3,$\mathbb{C}$), and a Weyl tensor can be converted into its equivalent form with SO(3,$\mathbb{C}$) indices. We can convert the Weyl tensor $C_{\alpha\beta\gamma\delta}$ with four symmetric SL(2;$\mathbb{C}$) indices into an equivalent tensor Q_{IJ} with two symmetric and traceless SO(3,$\mathbb{C}$) indices, simply by using the Pauli matrices:

$$Q_{IJ} = (\sigma_I)^{\alpha\beta}\,(\sigma_J)^{\gamma\delta}\,C_{\alpha\beta\gamma\delta}\,, \tag{D.1}$$

where $(\sigma_I)^{\alpha\beta} = \varepsilon^{\alpha\alpha'}\,(\sigma_I)_{\alpha'}{}^\beta$ and we can explicitly choose

$$(\sigma_1)_\alpha{}^\beta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \quad (\sigma_2)_\alpha{}^\beta = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\,, \quad (\sigma_3)_\alpha{}^\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\,. \tag{D.2}$$

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The indices $I, J = 1, 2, 3$ should be raised or lowered by the Kronecker delta, so whether they are upper or lower indices does not make a difference.

If we treat Q as a 3×3 matrix, then observing its eigenvalues and eigenvectors is sufficient for determining its Petrov type. Below we list all Petrov types and their corresponding Q-matrix criteria:

$$\begin{array}{cc} \text{Petrov types} & Q\text{-matrix criteria} \\ \text{I} & [Q-\lambda_1 I]\,[Q-\lambda_2 I]\,[Q-\lambda_3 I]=0 \\ \text{D} & \left[Q-\left(-\tfrac{1}{2}\lambda\right)I\right][Q-\lambda I]=0 \\ \text{II} & \left[Q-\left(-\tfrac{1}{2}\lambda\right)I\right]^2[Q-\lambda I]=0 \\ \text{N} & Q^2=0 \\ \text{III} & Q^3=0 \\ \text{O} & Q=0 \end{array} \tag{D.3}$$

In the list, $\lambda_{1,2,3}$, λ and $\left(-\frac{1}{2}\lambda\right)$ are eigenvalues of Q, $\lambda_1+\lambda_2+\lambda_3=0$ and I is the identity matrix. In particular, being Petrov type D means the matrix Q has three independent eigenvectors while two of them correspond to equal eigenvalues.

Using the explicit matrices and coordinates provided in appendices B and C, for spin $s=2$ we can evaluate the Weyl tensor (4.21) at a given spacetime point with a chosen set of parameters, and then we can evaluate the corresponding Q matrix to check its Petrov type. We have found that in general the Q matrix has three distinct eigenvalues (type I) and thus is not of type D, unless we choose some special parameters or consider only some special spacetime locations.

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