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Pontryagin Term and Magnetic Mass in 4D AdS Gravity

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Abstract. In the context of the anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence, 4D AdS gravity is suitably renormalized by adding the Gauss Bonnet term to the Einstein-Hilbert action. The subsequent addition of the Pontryagin term, with a specific coupling, allows to write the on-shell action in terms of the Weyl tensor and its dual, such that the action becomes stationary for asymptotic (anti) self-dual solutions in the Weyl tensor. The addition to the action of both topological invariants mentioned above does not modify the bulk dynamics, but it does modify the expression of the Noether current and, therefore, the conserved quantities of the theory. Here, we show that the method of Iyer and Wald leads to a fully-covariant Noether charge, which contains both the electric and magnetic parts of the Weyl tensor. For configurations which are globally (anti) self-dual in the Weyl tensor, both the action and the Noether charge identically vanish. This means that, for such spacetimes, the *magnetic mass* is equal to the *electric mass*.

1. Introduction

It has been shown that in asymptotically anti-de Sitter (AAdS) gravity in four dimensions the infrared divergences are cancelled by the Gauss-Bonnet (GB) term with a given coupling [1]. The addition of this topological invariant cancels divergent terms in the asymptotic region, such as those present in the Euclidean gravity action, only for the specific value of coupling constant mentioned above [2],

$$I_4 = \frac{1}{16\pi G} \int_M d^4x \sqrt{g} \left[R + \frac{6}{\ell^2} + \frac{\ell^2}{4} \left(R_{\mu\nu\alpha\beta} R^{\mu\nu\alpha\beta} - 4R_{\mu\nu} R^{\mu\nu} + R^2 \right) \right]. \quad (1)$$

With the addition of this term, the action has an extremum for AAdS spacetimes. On the other hand, the renormalized action (1) can be rewritten in terms of the Weyl tensor [3],

$$I_4 = \frac{\ell^2}{64\pi G} \int_M d^4x \sqrt{g} W_{\alpha\beta}^{\mu\nu} W_{\mu\nu}^{\alpha\beta}, \quad (2)$$



for any Einstein space. The GB invariant is not the only topological invariant of the Lorentz group in four dimensions. There exists yet another topological term, of opposite parity, known as Pontryagin term,

$$\mathcal{P}_4 = -\frac{1}{4} \epsilon^{\mu\nu\alpha\beta} R_{\mu\nu}^{\sigma\lambda} R_{\sigma\lambda\alpha\beta}, \quad (3)$$

which is locally equivalent to the derivative of the gravitational Chern-Simons term in three dimensions. The addition of \mathcal{P}_4 to the gravity action does not induce additional divergences in the asymptotic region. To our best knowledge, the only way to fix the Pontryagin coupling is to consider (in the Euclidean sector) an asymptotic (anti) self-duality condition in the Weyl tensor,¹

$$W_{\alpha\beta\mu\nu} = \pm {}^*W_{\alpha\beta\mu\nu}, \quad (4)$$

such that the total action is stationary under arbitrary variations of the fields.

Then, the renormalized action (1) acquires an additional term which can be written in terms of the Weyl tensor and its dual,

$$I = I_4 \pm \frac{\ell^2}{32\pi G} \int_M d^4x \sqrt{g} W_{\alpha\beta}^{\mu\nu} {}^*W_{\mu\nu}^{\alpha\beta}. \quad (5)$$

This is analogous to the form in which the Pontryagin density for $U(1)$ group may be added to the Euclidean action in the Maxwell electromagnetism,

$$I^E = \frac{1}{4} \int_M (F^{\mu\nu} F_{\mu\nu} \mp {}^*F^{\mu\nu} F_{\mu\nu}) d^4x, \quad (6)$$

where the $F_{\mu\nu}$ is the Abelian field strength. Due to the Bianchi identity ($\partial_\mu {}^*F^{\mu\nu} = 0$), the second term in the Lagrangian does not modify the equations of motion. The Pontryagin coupling in eq.(6) is a consequence of an asymptotic (anti) self-dual condition for F ,

$$F^{\mu\nu} = \pm {}^*F^{\mu\nu} \quad \text{at } \partial M. \quad (7)$$

The analogy between AdS gravity and self-dual electromagnetism can be pushed forward, as both theories portray striking similarities at the level of the conserved charges.

The Noether theorem allows to obtain an expression of conserved quantities in a gauge theory and gravity, as well, though the symmetries present in each case are of different nature. As long as the Noether current is written (globally) as a total derivative, the conserved charges are given by surface integrals in the asymptotic region, without the need of introducing a source at the origin.

Gauge symmetry $U(1)$ in electromagnetism is realized by the transformation in the gauge field $\delta_\lambda A_\nu = \partial_\nu \lambda$. As this is an internal symmetry ($\delta x^\mu = 0$), the Lagrangian density (6) produces a Noether current

$$J^\mu = \partial_\nu [(F^{\mu\nu} + \gamma {}^*F^{\mu\nu}) \lambda]. \quad (8)$$

Due to the conservation law, $\partial_\mu J^\mu = 0$, and the fact that J^μ can be written as a total derivative, the conserved charge is an integral on a two-dimensional sphere

$$Q[\lambda] = - \int_{S^2} (F^{\mu\nu} \mp {}^*F^{\mu\nu}) \lambda d\Sigma_{\mu\nu}, \quad (9)$$

¹ The Hodge star dual is defined by ${}^*W_{\alpha\beta\mu\nu} = \frac{1}{2} \sqrt{g} \epsilon_{\alpha\beta\gamma\delta} W^{\gamma\delta}{}_{\mu\nu}$.

in terms of the electromagnetic tensor and its dual. The expression (9), when λ is covariantly constant, leads to total charge which is the sum of the electric and the magnetic charges. Therefore, the magnetic charge is understood as a topological charge which is the Noether charge associated to a topological invariant. The total charge, for a global (anti) self-dual condition in F , will be identically zero, and so the Euclidean action. This defines a subclass of Euclidean instantons which can be taken as ground states of the theory.

In what follows, we apply the method developed by Wald and Iyer [4] to the 4D AdS gravity action supplemented by the topological invariants discussed above. The resulting charge features striking similarities to the case of self-dual electromagnetism.

2. Noether-Wald charges and Topological Invariants

In Riemannian gravity, the action $\int d^4x L$ depends on the metric and its derivatives combined in the form of the Christoffel symbol. Invariance under coordinate transformations of the type ($\delta x^\mu = \xi^\mu(x)$) leads to a conserved current

$$\sqrt{g} J^\mu[\xi] = \Theta(\delta_\xi g)^\mu + \Theta(\delta_\xi \Gamma)^\mu + L\xi^\mu,$$

where we have assumed that the surface term Θ which appears in the arbitrary variation of the action is separable in contributions along the Lie derivative of both the metric and Christoffel connection.

When ξ^μ is a Killing vector, the diffeomorphic variation of the metric, $\delta_\xi g_{\mu\nu} = -(\nabla_\mu \xi_\nu + \nabla_\nu \xi_\mu)$, vanishes.

In an arbitrary gravity theory with a Lagrangian L given in terms of the metric, the Riemann tensor and its contractions,

$$L = L(g_{\mu\nu}, R_{\mu\nu\alpha\beta}, R_{\mu\nu}, R), \quad (10)$$

the conserved current turns out to be

$$\sqrt{g} J^\mu[\xi] = E^{\mu\nu\alpha\beta} g_{\alpha\lambda} \delta_\xi \Gamma_{\nu\beta}^\lambda + L\xi^\mu, \quad (11)$$

where the tensor $E^{\mu\nu\alpha\beta}$ corresponds to the variation of L with respect to the Riemann tensor $R_{\mu\nu\alpha\beta}$, and the Lie derivative of the Christoffel symbol takes the form

$$\delta_\xi \Gamma_{\nu\beta}^\lambda = -\frac{1}{2} \left(\nabla_\nu \nabla_\beta \xi^\lambda + \nabla_\beta \nabla_\nu \xi^\lambda \right) + \frac{1}{2} \left(R^\lambda_{\beta\nu\sigma} + R^\lambda_{\nu\beta\sigma} \right) \xi^\sigma.$$

In addition, we can use the following identities for the Riemann tensor

$$[\nabla_\beta, \nabla_\nu] \xi_\alpha = R_{\beta\nu\alpha\sigma} \xi^\sigma \quad \text{and} \quad E^{\mu\nu\alpha\beta} (R_{\alpha\beta\nu\sigma} - 2R_{\alpha\nu\beta\sigma}) = 0, \quad (12)$$

where the last one is obtained by contracting the first type Bianchi identity with $E^{\mu\nu\alpha\beta}$ tensor. In doing so, one can isolate the contribution from the field equations, such that the Noether current can be finally rewritten as

$$\sqrt{g} J^\mu = 2E^{\mu\nu\alpha\beta} \nabla_\nu \nabla_\alpha \xi_\beta. \quad (13)$$

This is the key formula for the conserved current in any gravitational theory defined by the Lagrangian (10).

For the particular case of the AdS gravity action in eq.(5), the Noether current is

$$J^\mu[\xi] = \frac{\ell^2}{64\pi G} \nabla_\nu \left[\delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\sigma\lambda]} \nabla^\alpha \xi^\beta \left(W_{\sigma\lambda}^{\gamma\delta} \mp *W_{\lambda\sigma}^{\gamma\delta} \right) \right], \quad (14)$$

as it was shown in ref.[5]. It is an analog of the current (8) found in electromagnetism. As it is a total derivative, the corresponding Noether charge is a surface integral on a two-dimensional surface, given in terms of the Weyl tensor and its dual. The spacetime topology is $M \simeq \mathbb{R} \times \Sigma$, where Σ is the space section with unitary normal $u^\mu = (-\tilde{N}, 0, 0, 0)$, thus we can write the conserved charge as

$$Q[\xi] = \frac{\ell^2}{64\pi G} \int_{\partial\Sigma} d^2x \sqrt{\sigma} n_\mu u_\nu \delta_{[\alpha\beta\gamma\delta]}^{[\mu\nu\lambda\sigma]} \nabla^\alpha \xi^\beta \left(W_{\lambda\sigma}^{\gamma\delta} \mp *W_{\lambda\sigma}^{\gamma\delta} \right). \quad (15)$$

Here, $n^\mu = (0, N, 0, 0)$ is the unitary normal at the boundary $\partial M \simeq \mathbb{R} \times \partial\Sigma$ and σ is the determinant of the induced metric at the surface $\partial\Sigma$, which satisfies the relation $\sqrt{g} = N\sqrt{h} = N\tilde{N}\sqrt{\sigma}$.

As the first term in the Noether charge is a function of the Weyl tensor, it allows to connect the addition of the topological invariant of GB to the notion of conformal mass in AAdS gravity defined by Ashtekar, Magnon and Das [6], given in terms of the electric part of the Weyl tensor. In analogous way to the electromagnetic case, the form of eq. (15) makes manifest the fact that configurations globally (anti) self-dual in the Weyl tensor have a vanishing Noether charge. For these solutions, the magnetic mass is equal to the conformal mass, as it is the case for Taub-NUT-AdS solution [7, 8, 9].

3. Magnetic Mass

The discussion above is unnecessary for Schwarzschild-AdS or Kerr-AdS solutions as they have a vanishing Pontryagin density. Some of simplest examples to consider are Taub-NUT-AdS and Taub-Bolt-AdS solutions. Taub-NUT-AdS has an (anti) self-dual Weyl tensor and Taub-Bolt-AdS meets that condition in the asymptotic region. Indeed, both solutions present topological defects and have a non-vanishing magnetic mass [5].

Both Taub-NUT-AdS and Taub-Bolt-AdS solutions can be written, in the coordinates (τ, r, θ, ϕ) , as

$$ds^2 = f(r)(d\tau + 2n \cos\theta d\phi)^2 + \frac{dr^2}{f(r)} + (r^2 - n^2)(d\theta^2 + \sin^2\theta d\phi^2),$$

where

$$f(r) = \frac{r^2 - 2Mr + n^2 - \frac{3}{\ell^2} \left(n^4 + 2n^2 r^2 - \frac{r^4}{3} \right)}{r^2 - n^2}. \quad (16)$$

Here, n is a parameter related to a topological defect present in the geometry [11], and M is a mass parameter [12].

For either NUT or Bolt solution the largest root of the equation of $f(r) = 0$, namely $r = r_*$, defines a horizon. The difference between these two geometries is that, for NUT solution, the relation $r_* = |n|$ holds exactly while, for Bolt solution, it is the inequality $r_* = r_b > |n|$. Moreover, it must be satisfied $f'(r_*) = \frac{1}{2n}$ to avoid a singularity at r_* . The mass for the NUT solution, computed in ref.[1], is

$$M_{\text{NUT}} = \pm n (1 - 4\ell^{-2}n^2), \quad (17)$$

while for the Bolt solution

$$M_{\text{Bolt}} = \frac{r_b^2 + n^2}{2r_b} - \frac{3}{2\ell^2} \left(\frac{n^4}{r_b} + 2n^2 r_b - \frac{r_b^3}{3} \right).$$

As mentioned above, we are interested in analyzing the conserved charges as defined by eq.(15). For NUT solution, since its Weyl tensor is self-dual, we have

$$Q_{\text{NUT}}[\xi] = 0, \quad (18)$$

for any Killing vector $\xi = \xi^\mu \partial_\mu$. For Bolt solution, the only non-vanishing conserved charge in eq.(15) is the one associated to a timelike Killing vector

$$Q_{\text{Bolt}}[\partial_\tau] = M_{\text{Bolt}} \pm M_{\text{NUT}}. \quad (19)$$

4. Conclusions

In this article, we have shown that the addition of topological invariants (GB and Pontryagin terms), with specific couplings, modifies the value of the Noether charges of the theory. While the electric part of the Weyl tensor gives rise to the energy of AAdS black holes, the magnetic part produces a non-vanishing magnetic mass for gravitational instantons in AdS. Furthermore, we have proved that solutions with (anti) self-dual Weyl tensors are suitable candidates for ground states. Since the mass $Q[\partial_\tau]$ obtained from eq.(15) is a thermodynamic variable, we may expect a change in the thermodynamics of the system when solutions with a non vanish Pontryagin term, as Taub-NUT-AdS and Taub-Bolt-AdS, are considered.

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References

- [1] R. Aros, M. Contreras, R. Olea, R. Troncoso, and J. Zanelli 2000, Phys. Rev. Lett. **84**, 1647
- [2] R. Olea 2005, J. High Energy Phys. **06**, 023
- [3] O. Miskovic and R. Olea 2009, Phys. Rev. **D79**, 124020
- [4] V. Iyer and R. Wald 1994, Phys. Rev. D **50**, 846
- [5] R. Araneda, R. Aros, O. Miskovic and R. Olea 2016, Phys. Rev. D **93**, 084022
- [6] A. Ashtekar and A. Magnon 1984, Classical Quantum Gravity **1**, L39; A. Ashtekar and S. Das 2000, Classical Quantum Gravity **17**, L17
- [7] E. Newman, L. Tamburino and T. Unti 1963, J. Math. Phys. **4**, 915
- [8] D. N. Page and C. N. Pope 1987, Classical Quantum Gravity **4**, 213
- [9] D. Kramer, H. Stephani, M. MacCallum and E. Herlt, *Exact solutions of Einstein's field equations*, Cambridge University Press, 1980
- [10] A. Chamblin, R. Emparan, C.V. Johnson and R.C. Myers 1999, Phys. Rev. D **59**, 064010
- [11] T. Eguchi, P. B. Gilkey and A. J. Hanson 1980, Phys. Rept. **66**, 213
- [12] S. W. Hawking, C. J. Hunter and D. N. Page 1999, Phys. Rev. D **59**, 044033