

# Asymmetric Exclusion Model and Weighted Lattice Paths.

R. Brak<sup>†</sup> and J. W. Essam<sup>‡\*</sup>

<sup>†</sup>Department of Mathematics and Statistics,  
The University of Melbourne  
Parkville, Victoria 3010,  
Australia

<sup>‡</sup>Department of Mathematics and Statistics,  
Royal Holloway College, University of London,  
Egham, Surrey TW20 0EX,  
England.

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## Abstract

We show that the known matrix representations of the stationary state algebra of the Asymmetric Simple Exclusion Process (ASEP) can be interpreted combinatorially as various weighted lattice paths. This interpretation enables us to use the constant term method (CTM) and bijective combinatorial methods to express many forms of the ASEP normalisation factor in terms of Ballot numbers. One particular lattice path representation shows that the coefficients in the recurrence relation for the ASEP correlation functions are also Ballot numbers. Additionally, the CTM has a strong combinatorial connection which leads to a new “canonical” lattice path representation and to the “ $\omega$ -expansion” which provides a uniform approach to computing the asymptotic behaviour in the various phases of the ASEP. The path representations enable the ASEP normalisation factor to be seen as the partition function of a more general polymer chain model having a two parameter interaction with a surface.

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\* email: [r.brak@ms.unimelb.edu.au](mailto:r.brak@ms.unimelb.edu.au), [j.essam@rhul.ac.uk](mailto:j.essam@rhul.ac.uk)

We show, in the case  $\alpha = \beta = 1$ , that the probability of finding a given number of particles in the stationary state can be expressed via non-intersecting lattice paths and hence as a simple determinant.

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# 1 Background and notation.

The Asymmetric Simple Exclusion Process (ASEP) is a simple hard core hopping particle model. It consists of a line segment with  $r$  sites. Particles are allowed to hop on to site 1 if it is empty, with rate  $\alpha$ . Any particles on sites 1 to  $r - 1$  hop on to a site to their right if it is empty, with rate 1. A particle on site  $r$  hops off with rate  $\beta$  – as illustrated in figure 1. The state of the system at any time is defined by the set of indicator variables  $\{\tau_1, \dots, \tau_r\}$ , where

$$\tau_i = \begin{cases} 1 & \text{if site } i \text{ is occupied} \\ 0 & \text{otherwise} \end{cases}$$

and the probability,  $P(\vec{\tau}; s)$  of the system being in state  $\vec{\tau} = (\tau_1, \dots, \tau_r)$  at time  $s$ , given some initial state at  $s = 0$ , satisfies a master equation (see [1] for details).

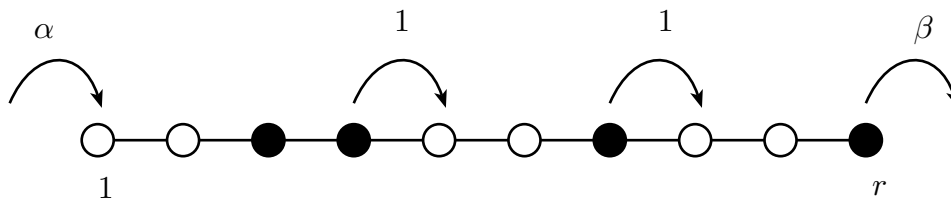


Figure 1: *The ASEP model.*

The model goes back at least as far as Macdonald, Gibbs and Pipkin [2] who used it in their study of kinetics of protein synthesis. For reviews of the extensive physics literature, including applications and analytical techniques, for this model and further developments of the model, see Schülz [3, 4], Derrida and Evans [5], Derrida [6], Stinchcombe [7]. The mathematical literature placing the model in the context of Markov chain theory may be found in the Liggett's book [8].

A significant step forward in the understanding of the mathematical aspects of the model was made with the realization, by Derrida et al [1], that a stationary state solution,  $P_S(\vec{\tau})$ , of the master equation could be determined by the matrix product Ansatz

$$P_S(\vec{\tau}) = \langle W | \prod_{i=1}^r (\tau_i D + (1 - \tau_i) E) | V \rangle / Z_{2r} \quad (1.1)$$

with normalisation factor  $Z_{2r}$  given by

$$Z_{2r} = \langle W | (DE)^r | V \rangle \quad (1.2)$$

provided that  $D$  and  $E$  satisfy the DEHP algebra

$$D + E = DE \quad (1.3a)$$

where  $\langle W|$  and  $|V\rangle$  are the eigenvectors

$$\langle W|E = \frac{1}{\alpha}\langle W|, \quad D|V\rangle = \frac{1}{\beta}|V\rangle. \quad (1.3b)$$

These equations are sufficient to determine  $P_S(\vec{\tau})$  but Derrida et al [1] also gave several interesting matrix representations of  $D$  and  $E$  and the vectors  $|V\rangle$  and  $\langle W|$ , any one of which may be used to determine  $P_S(\vec{\tau})$ .

The primary result of this paper is to show that each of the three matrix representations of the DEHP algebra can be interpreted as a transfer matrix for a different weighted lattice path problem. This then allows the normalisation, correlation function and other properties of the ASEP model to be interpreted combinatorially as certain weighted lattice path configuration sums – see section 2. One of the path connections is similar to that discussed in Derrida et. al. [9]

The lattice path interpretation has two primary consequences: The first is that it provides a starting point for a new method (the “constant term” or CT method) for calculating the normalisation and correlation functions – see section 3. This reproduces several existing results (but by a new method) and also provides several new results. One of note, the “ $\omega$ -expansion”, arises from a rearrangement of the constant term expression which leads to a form of the normalisation in terms of the variables  $\omega_c \equiv \alpha(1 - \alpha)$  and  $\omega_d \equiv \beta(1 - \beta)$ . The coefficients in this expansion are Catalan numbers, the asymptotic form of which enables a uniform approach to computing the asymptotic behaviour of  $Z_{2r}$  as  $r \rightarrow \infty$  in the various phases of the ASEP model – see section 5. The results agree with those found in [1], by steepest descent methods.

A bonus following from the lattice path interpretation of the algebra representations is that one of the lattice path interpretations (a slight variation of representation 3) has a natural interpretation as a polymer chain having a two parameter  $(\kappa_1, \kappa_2)$  interaction with a surface. In this context  $Z_{2r}$  is a partition function for the “two-contact” polymer model – see section 4. We also obtain recurrence relations (on the length variable) for the partition function of this polymer model and hence also for the ASEP normalisation. Our formulae for the single polymer chain may be used to construct the partition function for a polymer network interacting with a surface (see conclusion). In particular, in a subsequent paper we will discuss the application to vesicles and compact percolation clusters [10] near a damp wall.

The second primary consequence of the lattice path interpretation follows from the CT method itself, as the CT method has very natural combinatorial interpretations. For example,

the normalisation can be written in several different polynomial forms depending on which variables you use:

$$Z_{2r} = \sum_{n,m} p_{n,m}^{(1)} \bar{\alpha}^n \bar{\beta}^m \quad (\text{see - (4.24)}) \quad (1.4)$$

$$Z_{2r} = \sum_{n,m} p_{n,m}^{(2)} c^n d^m \quad (\text{see - (4.31)}) \quad (1.5)$$

$$Z_{2r} = \sum_{n,m} p_{n,m}^{(3)} \kappa_1^n \kappa_2^m \quad (\text{see - (4.20)}) \quad (1.6)$$

$$Z_{2r} = \sum_{n,m} p_{n,m}^{(4)} \bar{\kappa}_1^n \bar{\kappa}_2^m \quad (\text{see - (4.21)}) \quad (1.7)$$

where all the polynomial coefficients,  $p_{n,m}^{(i)}$ , are integers. (All the variables in these polynomials are related by simple equations eg.  $\bar{\alpha} = 1/\alpha$ ,  $c = \bar{\alpha} - 1$  etc. – see above and in later sections). Since, in each of these cases, the normalisation arises from a weighted lattice configuration sum, all the above coefficients have a direct combinatorial interpretation as enumerating a particular subset of the paths (eg. those with exactly  $m$  steps with the first weight and exactly  $n$  steps with the second weight).

However, we show that each of the above polynomial coefficients has an alternative combinatorial interpretation which corresponds to enumerating a different, *unweighted*, set of lattice paths, eg. in [1] the coefficient  $p_{m,n}^{(1)}$  was given as

$$p_{n,m-n}^{(1)} = \frac{m(2r - m - 1)!}{r!(r - m)!}$$

which, with a simple rearrangement, can be seen to be a particular “Ballot number”

$$p_{n,m-n}^{(1)} = B_{2r-m-1,m-1}$$

where  $B_{s,h}$  enumerates Ballot paths (see section 3.1 for details) of length  $s$  and height  $h$ . Thus  $p_{n,m-n}^{(1)}$  which enumerates a special set of paths with  $m$  weights of type  $\bar{\alpha}$  and  $n - m$  weights of type  $\bar{\beta} = 1/\beta$  is seen to be determined in terms of the much simpler combinatorial problem of enumerating *unweighted* Ballot paths of length  $2r - m - 1$  and height  $m - 1$ . This correspondence between the two combinatorial problems (one pair for each coefficient) arises as a bijection between the two path problems. This result may be turned around: If a bijection between a particular Ballot path problem and the weighted path problem can be proved then it provides an alternative derivation of the normalisation polynomial.

In section 6, the recurrence relations for the ASEP correlation functions derived in [1] are shown to follow from the lattice path interpretations. The coefficients of the terms in the recurrence relation are also seen to be various Ballot numbers.

Finally, in section 7, for the case  $\alpha = \beta = 1$ , we show that the probability of finding the system in some particular state  $\vec{r}$  is related to non-intersecting pairs of paths and the probability of finding the system in a state with exactly  $k$  particles is related to a simple determinant. This provides a connection with Derrida et. al. Brownian excursions [9].

## 2 Matrix Representations and Lattice Path Transfer Matrices.

Derrida et al [1], provided three different matrix representations for the ASEP algebra. Representation one,

$$D_1 = \begin{pmatrix} \bar{\beta} & \bar{\beta} & \bar{\beta} & \bar{\beta} & \bar{\beta} & \dots \\ 0 & 1 & 1 & 1 & 1 & \dots \\ 0 & 0 & 1 & 1 & 1 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \quad E_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \quad (2.1)$$

$$\langle W_1 | = (1, \bar{\alpha}, \bar{\alpha}^2, \bar{\alpha}^3, \dots) \quad |V_1\rangle = (1, 0, 0, 0, \dots)^T \quad (2.2)$$

where

$$\bar{\alpha} = 1/\alpha, \quad \bar{\beta} = 1/\beta,$$

representation two,

$$D_2 = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ 0 & 0 & 0 & 1 & 1 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \quad E_2 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & \dots \\ 1 & 1 & 0 & 0 & 0 & \dots \\ 0 & 1 & 1 & 0 & 0 & \dots \\ 0 & 0 & 1 & 1 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \end{pmatrix} \quad (2.3)$$

$$\langle W_2 | = \kappa(1, c, c^2, c^3, \dots) \quad |V_2\rangle = \kappa(1, d, d^2, d^3, \dots)^T \quad (2.4)$$

where

$$c = \bar{\alpha} - 1, \quad d = \bar{\beta} - 1,$$

and representation three

$$D_3 = \begin{pmatrix} \bar{\beta} & \kappa & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 1 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \quad E_3 = \begin{pmatrix} \bar{\alpha} & 0 & 0 & 0 & 0 & \cdots \\ \kappa & 1 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 1 & 0 & 0 & \cdots \\ 0 & 0 & 1 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & & \ddots \end{pmatrix} \quad (2.5)$$

$$\langle W_3 | = (1, 0, 0, 0, \dots) \quad |V_3\rangle = (1, 0, 0, 0, \dots)^T \quad (2.6)$$

where

$$\kappa^2 = \bar{\alpha} + \bar{\beta} - \bar{\alpha}\bar{\beta} = 1 - cd.$$

Since these matrices and vectors satisfy the algebraic relations (1.3), the normalisation factor (1.2) for the ASEP model, can be evaluated using any of the three formulae

$$Z_{2r}^{(j)} = \langle W_j | (D_j E_j)^r | V_j \rangle, \quad j = 1, 2, 3. \quad (2.7)$$

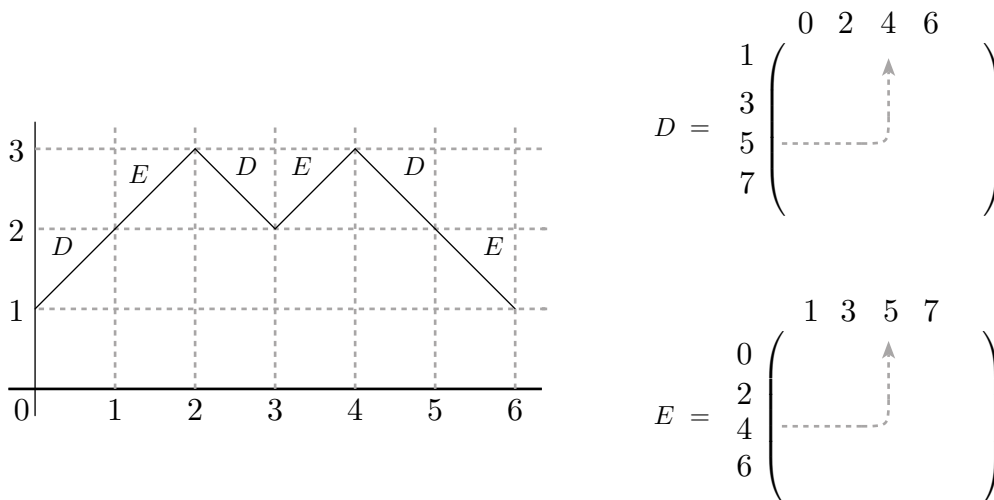


Figure 2: a) Action of  $D$  and  $E$  matrices on the square lattice, b) the labelling of the matrix elements.

Each of these three representations can be interpreted as the transfer matrix of a particular weighted lattice path problem. If the rows of the  $D_j$  matrix and the columns of the  $E_j$  matrix are labelled with odd integers  $\mathbb{Z}_{\text{odd}} \equiv \{1, 3, 5, \dots\}$  and the columns of  $D_j$  and rows of  $E_j$  are labelled with even integers  $\mathbb{Z}_{\text{even}} \equiv \{0, 2, 4, \dots\}$ , then  $(D_j)_{k,\ell}$  is the weight of a step from an odd height  $k$  to even height  $\ell$  and  $(E_j)_{k,\ell}$  is the weight of a step from an even height  $j$  to an odd

height  $\ell$  – see figure 2. Since the rows and columns are labelled with non-negative integers the steps are only in the upper half of  $\mathbb{Z}^2$ .

Similarly, the elements of  $\langle W_j |$  and  $|V_j\rangle$  are labelled by  $\mathbb{Z}_{\text{odd}}$  and are the weights attached to the initial and final vertices of the paths. The matrices  $D_j$  and  $E_j$  act successively to the left on the initial vector  $\langle W_j |$  and  $\langle W_j |(D_j E_j)^r |V_j\rangle$  is the weighted sum over all paths of length  $2r$  which begin and end at odd height above the  $x$ -axis.

An example path for each of the three representations is shown in figure 3. Notice that for the first representation the paths with non-zero weight begin at any odd height and end at unit height, for the second they both begin and end at any odd height and for the third they begin and end at unit height. All these paths are defined explicitly below.

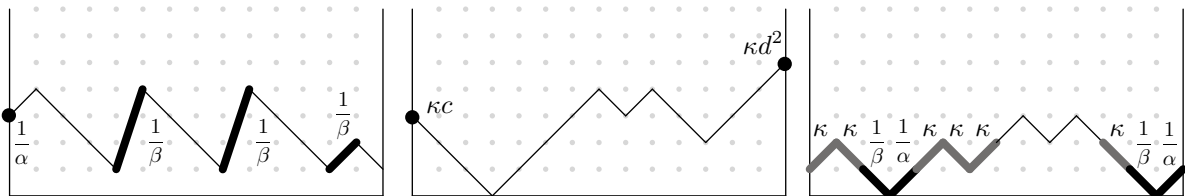


Figure 3: *Examples of the paths for each of the three representations a) representation one, “Jump step paths”, b) representation two, “Cross paths” and c) representation three, “One up paths”.*

Derrida et. al. gave another expression for the normalisation factor (see equation (39) of [1]). This form does not arise directly from any of the above three representations, however, we will show (corollary 5) that it is the partition function  $Z_{2r}^{(5)}$ , defined in (2.16), corresponding to a “canonical” path representation (see figure 6 for an example). In [11] we provide a combinatorial derivation of the equivalence of the above three and a number of other path representations of the normalisation factor to the “canonical” representation.

## 2.1 The Lattice Path Definitions.

We consider paths whose steps are between the vertices of the half plane square lattice  $\Xi = \{(x, y) | x \in \mathbb{Z}, y \in \mathbb{Z}^+\}$ , where  $\mathbb{Z}$  (resp.  $\mathbb{Z}^+$ ) is the set of integers (resp. non-negative integers).

**Definition 1 (Lattice paths).** *A **lattice path**,  $\omega$ , of length  $t \geq 0$  is a sequence of vertices  $(v_0, v_1, \dots, v_t)$  where  $v_i \equiv (x_i, y_i) \in \Xi$ , and for  $t > 0$ ,  $v_i - v_{i-1} \in \mathcal{S}_i^p$  where  $\{\mathcal{S}_i^p, i = 1, \dots, t\}$  is*



the set of allowed steps. For a particular path,  $\omega$ , denote the corresponding sequence of steps by  $\mathcal{E}(\omega) = e_1 e_2 \dots e_t$ ,  $e_i = (v_{i-1}, v_i)$ . A **subpath** of length  $k$  of a lattice path,  $\omega$ , is a path defined by a subsequence of adjacent vertices,  $(v_i, v_{i+1}, \dots, v_{i+k-1}, v_{i+k})$ , of  $\omega$ .

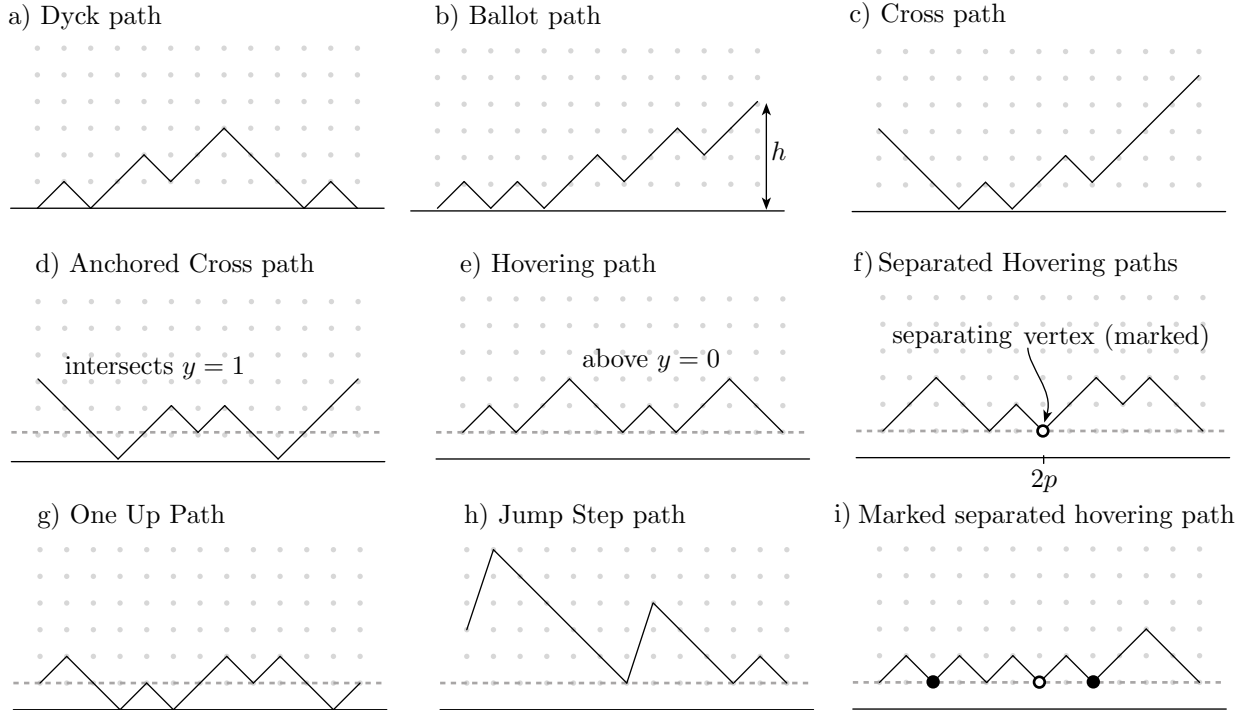


Figure 4: An example of a) a Dyck path, b) a Ballot path, c) a Cross path, d) an Anchored Cross path, e) a Hovering path, f) a Separated Hovering path, g) a One Up path, h) a Jump Step path and i) a Marked separated hovering path.

**Definition 2 (Dyck  $P_{2r}^{(D)}$ , Ballot  $P_{t,h}^{(B)}$ , & Cross  $P_{t,h_1,h_2}^{(C)}$  paths).** The set of length  $2r$  **Dyck** paths,  $P_{2r}^{(D)}$ ; length  $t$  and height  $h$  **Ballot** paths  $P_{t,h}^{(B)}$ , and length  $t$  and heights  $h_1$  and  $h_2$  **Cross** paths  $P_{t,h_1,h_2}^{(C)}$  all have step set  $\mathcal{S}_i^D = \{(1,1), (1,-1)\}$ . Dyck paths have  $v_0 = (0,0)$  and  $v_{2r} = (2r,0)$ , Ballot paths have  $v_0 = (0,0)$  and  $v_t = (t,h)$  and Cross paths have  $v_0 = (0,h_1)$  and  $v_t = (t,h_2)$ ,  $h_1, h_2 \geq 0$ . Denote,  $P_{2r}^{(C)} = \bigcup_{h_1, h_2 \in \mathbb{Z}_{\text{odd}}} P_{2r;h_1,h_2}^{(C)}$

**Definition 3 (Elevated Dyck path (Bubble)).** An **elevated Dyck path** or **Bubble**, is a subpath,  $(v_i, \dots, v_{i+k})$ ,  $k \geq 0$ , for which  $y_i = y_{i+k}$  and  $y_j \geq y_i$  for all  $i < j < i+k$ ,  $k > 0$ . If  $k = 0$  then the elevated Dyck path is a single vertex.

**Definition 4 (Anchored Cross  $P_{t,h_1,h_2}^{(aC)}$  paths).** The subset  $P_{t,h_1,h_2}^{(aC)} \subset P_{t,h_1,h_2}^{(C)}$  of **Anchored Cross Paths** is defined as all the paths in  $P_{t,h_1,h_2}^{(C)}$  with at least one vertex in common with the

line  $y = 1$ . Denote,  $P_{2r}^{(aC)} = \bigcup_{h_1, h_2 \in \mathbb{Z}_{\text{odd}}} P_{2r; h_1, h_2}^{(aC)}$

**Definition 5 (One Up  $P_{2r}^{(O)}$  paths).** The set of **One Up** paths,  $P_{2r}^{(O)}$  is the set of lattice paths of length  $2r$  which have step set  $\mathcal{S}_i^D = \{(1, 1), (1, -1)\}$ ,  $v_0 = (0, 1)$  and  $v_{2r} = (2r, 1)$ , i.e.  $P_{2r}^{(O)} = P_{2r; 1, 1}^{(C)}$ .

**Definition 6 (Jump Step  $P_{t; h}^{(J)}$  paths).** The set of **Jump Step** paths of length  $t$ ,  $P_{t; h}^{(J)}$ , is the set of lattice paths which have step set

$$\mathcal{S}_i^J = \begin{cases} \{(1, -1)\} & \text{for } i \text{ even} \\ \{(1, -1)\} \cup \{(1, 2\ell + 1) | \ell \in \mathbb{Z}^+\} & \text{for } i \text{ odd} \end{cases}$$

with  $v_0 = (0, h)$ ,  $h \in \mathbb{Z}_{\text{odd}}$ ,  $v_t = (t, 1)$ . The “height”,  $g_i$ , of a step  $e_i = (v_{i-1}, v_i)$  is defined as  $y_i - y_{i-1}$ . Odd steps with  $g_i \in \{3, 5, \dots\}$  will be called “jump” steps. If  $g_i = 1$  (resp.  $g_i = -1$ ) the step is called an “up” (resp. “down”) step. Denote,  $P_{2r}^{(J)} = \bigcup_{h \in \mathbb{Z}_{\text{odd}}} P_{2r; h}^{(J)}$

*Note:* Jump step paths never visit the  $x$ -axis since such a visit can only occur on an odd step and return to  $y = 1$  is impossible since all even steps are down.

**Definition 7 (Hovering  $P_{2r}^{(H)}$ , Separated Hovering  $P_{2r; 2p}^{(sH)}$  and Marked Separated Hovering  $P_{2r; 2p}^{(mH)}$  paths).** **Hovering** paths,  $P_{2r}^{(H)}$  are One Up paths with no vertex on the  $x$ -axis. **Separated Hovering** paths,  $P_{2r; 2p}^{(sH)}$ ,  $0 \leq p \leq r$ , are the subset of the Hovering paths which have  $v_{2p} = (2p, 1)$ . The vertex  $v_{2p}$  is marked (with an empty circle – see figure 4f) and known as the **separating vertex**. **Marked separated hovering** paths,  $P_{2r; 2p}^{(mH)}$ , are obtained from  $P_{2r; 2p}^{(sH)}$  by marking (with a solid circle – see figure 4i) subsets of the steps (or vertices) which return to  $y = 1$ .

*Note:* For all paths considered  $x_i + y_i$  is either odd for all  $i$  or even for all  $i$ , i.e. the paths are confined either to the odd sublattice or the even sublattice.

**Definition 8 (Contacts, Returns).** A vertex of a Ballot or Dyck path in common with the  $x$ -axis is called a **contact**. All contacts except the initial one are called **returns**. The polynomial  $R_t(h; \kappa) = \sum_{\omega \in P_{t; h}^{(B)}} \kappa^{\rho(\omega)}$ , where  $\rho(\omega)$  is the number of returns for the path  $\omega$ , is called the **return polynomial** for Ballot paths.

## 2.2 Weights and lattice path representations.

For each of the three different representations the following lemma converts the matrix formula (2.7) for the normalization factor into a sum over one of the path sets defined above, where the

summand is a product of the step weights  $w_j^{step}(e_i)$ , an initial vertex weight  $w_j^i(v_0)$ , and a final vertex weight  $w_j^f(v_t)$ .

**Lemma 1.** *The normalisation factor,  $Z_{2r}^{(j)}$  for each of the three matrix representations,  $j = 1, 2, 3$ , can be written as*

$$Z_{2r}^{(j)} = \sum_{\omega \in P_{2r}^{(j)}} W^{(j)}(\omega) \quad (2.8)$$

with

$$W^{(j)}(\omega) = w_j^i(v_0) \left[ \prod_{i=1}^{2r} w_j^{step}(e_i) \right] w_j^f(v_{2r}) \quad (2.9)$$

where  $P_{2r}^{(1)} = P_{2r}^{(J)}$ ,  $P_{2r}^{(2)} = P_{2r}^{(C)}$ , and  $P_{2r}^{(3)} = P_{2r}^{(O)}$ , and the weight  $W^{(j)}(\omega)$  of a particular path  $\omega$  with step sequence,  $\mathcal{E}(\omega) = e_1 \dots e_{2r}$  is defined, for each of the three cases, as follows.

For  $j = 1$

$$w_1^i((0, 2k + 1)) = \bar{\alpha}^k \quad k \in \mathbb{Z}^+ \quad (2.10a)$$

$$w_1^{step}(e_i) = \begin{cases} \bar{\beta} & \text{if } e_i = ((i - 1, 1), (i, 2k)), k \in \mathbb{Z}^+ \text{ and } i \text{ odd} \\ 1 & \text{otherwise} \end{cases} \quad (2.10b)$$

$$w_1^f((2r, 1)) = 1, \quad (2.10c)$$

for  $j = 2$

$$w_2^i((0, 2k + 1)) = \kappa c^k \quad k \in \mathbb{Z}^+ \quad (2.11a)$$

$$w_2^{step}(e_i) = 1 \quad (2.11b)$$

$$w_2^f((2r, 2k + 1)) = \kappa d^k \quad k \in \mathbb{Z}^+ \quad (2.11c)$$

and for  $j = 3$

$$w_3^i((0, 1)) = 1 \quad (2.12a)$$

$$w_3^{step}(e_i) = \begin{cases} \kappa & \text{if } e_i = ((i - 1, 1), (i, 2)) \text{ or } e_i = ((i - 1, 2), (i, 1)) \\ \bar{\beta} & \text{if } e_i = ((i - 1, 1), (i, 0)) \\ \bar{\alpha} & \text{if } e_i = ((i - 1, 0), (i, 1)) \\ 1 & \text{otherwise} \end{cases} \quad (2.12b)$$

$$w_3^f((2r, 1)) = 1 \quad (2.12c)$$

*Proof.* The above lemma is a direct consequence of the lattice path interpretation of the  $D_j$  and  $E_j$  matrices as transfer matrices.  $\square$

The equivalence of the different expressions is a consequence of the invariance of the normalisation factor under similarity transformations relating the different matrix representations of  $D$  and  $E$ .

In order to compute  $Z_{2r}^{(3)}$ , it turns out to be a little more convenient to rearrange the weights associated with representation three – see figure 5. If we do so we obtain the following corollary.

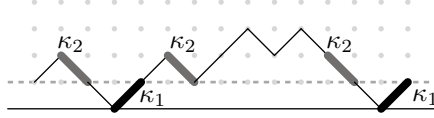


Figure 5: An example of a representation three, one up path, with the weights re-organised

**Corollary 1.** An equivalent set of weights for  $Z_{2r}^{(3)}$ , is

$$w_4^i((0,1)) = 1 \tag{2.13}$$

$$w_4^{step}(e_i) = \begin{cases} \kappa_1 = \bar{\alpha}\bar{\beta} & \text{if } e_i = ((i-1,0), (i,1)) \\ \kappa_2 = \kappa^2 & \text{if } e_i = ((i-1,2), (i,1)) \\ 1 & \text{otherwise} \end{cases} \tag{2.14}$$

$$w_4^f((0,1)) = 1 \tag{2.15}$$

*Note:* this rearrangement of the weights is only valid for computing  $Z_{2r}^{(3)}$ . For correlation functions one has less freedom in the weight rearrangement.

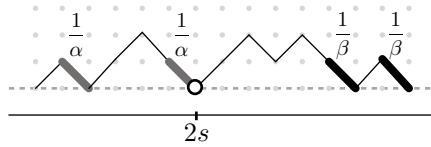


Figure 6: An example of a canonical path – a Separated Hovering path with a  $W^{(5)}$  weighting.

Later in the paper (corollary 5) we will show that the normalisation factor can also be expressed in terms of Separated Hovering paths.

$$Z_{2r}^{(5)} = \sum_{p=0}^r \sum_{\omega_p \in P_{2r;2p}^{(sH)}} W^{(5)}(\omega_p) \tag{2.16}$$

where  $W^{(5)}(\omega_p)$  is defined by (2.9) with

$$w_5^i((0, 1)) = 1 \tag{2.17a}$$

$$w_5^{step}(e_i) = \begin{cases} \bar{\alpha} & \text{if } e_i = ((i-1, 2), (i, 1)) \text{ and } i \leq 2p \\ \bar{\beta} & \text{if } e_i = ((i-1, 2), (i, 1)) \text{ and } i > 2p \\ 1 & \text{otherwise} \end{cases} \tag{2.17b}$$

$$w_5^f((0, 1)) = 1 \tag{2.17c}$$

Thus, for any particular path,  $\omega_p$ , all the  $\bar{\alpha}$  weighted steps (if any) occur to the left of vertex  $(2p, 1)$  and all the  $\bar{\beta}$  weighted steps (if any) occur to the right of  $(2p, 1)$ . We call this combination of paths and weights the ‘‘canonical’’ path representation of the normalisation factor. An example is shown in figure 6.

**Lemma 2.** *Let  $P_{2r;2p}^{(mH)}$  be the set of marked separated hovering paths obtained from  $P_{2r;2p}^{(sH)}$  by marking subsets of the steps which return to  $y = 1$  then*

$$Z_{2r}^{(2)} = \sum_{p=0}^r \sum_{\omega_p \in P_{2r;2p}^{(mH)}} W^{(2a)}(\omega_p) \tag{2.18}$$

where the weight  $W^{(2a)}(\omega_p)$  has a factor  $c$  for each marked return step which occurs to the left of  $v_{2p}$  and a factor  $d$  for the other marked return steps.

The lemma is proved in [11] in two stages. First an involution on  $P_{2r}^{(C)}$  for which  $P_{2r}^{(aC)}$  is the fixed set and then a bijection between  $P_{2r}^{(aC)}$  and  $\bigcup_{p=0}^r P_{2r;2p}^{(mH)}$ .

**Corollary 2.**

$$Z_{2r}^{(2)} = Z_{2r}^{(5)} \Big|_{\bar{\alpha}=1+c, \bar{\beta}=1+d}$$

*Proof.* Substituting  $\bar{\alpha} = 1 + c$ ,  $\bar{\beta} = 1 + d$  in the weight attached to a path  $\omega_p \in P_{2r;2p}^{(sH)}$  and expanding leads to a sum of terms obtained by weighting each return step to the right of  $v_{2p}$  (if any) with either 1 or  $d$  and the remaining returns (if any) by 1 or  $c$ . Marking the subsets of steps weighted  $c$  or  $d$  determines a set of paths belonging to  $P_{2r;2p}^{(mH)}$ .  $\square$

### 3 Methods.

In this section we briefly review methods to be used and results obtained previously [12] as they will be required in the next section. One additional new lemma is stated.

### 3.1 The constant term method.

**Definition 9.** *The constant term operation,  $CT[\cdot]$ , is defined by*

$$CT[f(z)] = \text{the constant term in the Laurent expansion of } f(z) \text{ about } z = 0.$$

The number of  $t$ -step lattice paths with step set  $S_i^D$  which begin at  $(0, 0)$  and end at  $(t, y)$  with no further constraint (i.e. replacing the constraint  $y \in \mathbb{Z}^+$  in the definition of  $\Xi$  by  $y \in \mathbb{Z}$ ) is the binomial coefficient  $\binom{t}{\frac{1}{2}(t-y)}$  for which the constant term formula is

$$\binom{t}{\frac{1}{2}(t-y)} = CT[(z + 1/z)^t z^y]. \quad (3.1)$$

By the reflection principle [13], the number of  $t$ -step Ballot paths of height  $h$  is obtained by subtracting the number of unrestricted paths which begin at  $(0, -2)$  from those beginning at  $(0, 0)$ , both ending at  $(t, h)$ .

$$B_{t,h} \equiv |P_{t,h}^{(B)}| = CT[(z + 1/z)^t z^h] - CT[(z + 1/z)^t z^{h+2}] = CT[\Lambda^t z^h (1 - z^2)] \quad (3.2)$$

where  $\Lambda = z + 1/z$ . Differencing the binomial coefficients expresses  $B_{t,h}$  in terms of factorials; for  $t + h$  even

$$B_{t,h} = \frac{(h+1)t!}{(\frac{1}{2}(t+h)+1)!(\frac{1}{2}(t-h))!}. \quad (3.3)$$

Dyck paths are Ballot paths ending at  $(t, 0)$  so the number Dyck paths with  $2r$  steps is

$$|P_{2r}^{(D)}| = CT[\Lambda^t (1 - z^2)] = \frac{1}{r+1} \binom{2r}{r} = C_r, \quad (3.4)$$

a Catalan number.

Instead of using the reflection principle the Ballot numbers may be obtained as the solution of the equations,  $t, h \geq 1$ ,

$$B_{0,h} = \delta_{h,0} \quad (3.5a)$$

$$B_{t,0} = B_{t-1,1} \quad (3.5b)$$

$$B_{t,h} = B_{t-1,h-1} + B_{t-1,h+1}, \quad (3.5c)$$

all of which are solved by  $CT[\Lambda^t z^h (1 - z^2)]$ .

Historically, Ballot numbers arise in the combinatorial problem of a two candidate election. If you ask how many ways can  $t$  votes be cast such that the first candidate ends  $h$  votes ahead of the second candidate and at any stage of the voting never has fewer votes than the second candidate.

### 3.2 Partition function for the one contact model.

Previously [12] we proved the following proposition concerning the return polynomial (see definition 8).

**Proposition 1.** *The return polynomial for Ballot paths of length  $t$  and height  $h$  is given by*

$$R_t(h; \kappa) = CT\left[\frac{\Lambda^t z^h (1 - z^2)}{1 - (\kappa - 1)z^2}\right] \quad (3.6)$$

*Proof.* For  $t, h \geq 1$ ,  $R_t(h; \kappa)$  is determined by the recurrence relations

$$R_0(h; \kappa) = \delta_{h,0} \quad (3.7a)$$

$$R_t(0; \kappa) = \kappa R_{t-1}(1, \kappa) \quad (3.7b)$$

$$R_t(h; \kappa) = R_{t-1}(h - 1; \kappa) + R_{t-1}(h + 1; \kappa). \quad (3.7c)$$

The first and last equations are the same as for the unweighted Ballot paths and are satisfied by  $CT[\Lambda^t z^h (1 - z^2)g(z)]$  provided that on expansion  $g(z)$  has no negative powers and  $g(0) = 1$  (which will be the case). We have introduced the factor  $g(z)$  to allow the second equation to be satisfied. But  $z + 1/z = \kappa z + (1 - (\kappa - 1)z^2)/z$  and choosing  $g(z) = 1/(1 - (\kappa - 1)z^2)$

$$\begin{aligned} CT[\Lambda^t (1 - z^2)g(z)] &= CT[\Lambda^{t-1} (z + 1/z)(1 - z^2)g(z)] \\ &= \kappa CT[\Lambda^{t-1} z(1 - z^2)g(z)] + CT[\Lambda^{t-1} (1/z - z)]. \end{aligned}$$

The result follows since the last term may be evaluated to give zero. □

**Corollary 1 ([12]).**

$$R_t(h; \kappa) = \sum_{m=0}^{\frac{1}{2}(t-h)} B_{t, h+2m} (\kappa - 1)^m = \sum_{m=0}^{\frac{1}{2}(t-h)} B_{t-m-1, m+h-1} \kappa^m. \quad (3.8)$$

*Proof.* The first equality follows by expanding (3.6) in powers of  $(\kappa - 1)z^2$  and using (3.2). The second is obtained by rewriting the constant term formula as

$$R_t(h; \kappa) = CT\left[\frac{\Lambda^{t-1} z^{h-1} (1 - z^2)}{1 - \kappa z/\Lambda}\right] \quad (3.9)$$

and then expanding in powers of  $\kappa z/\Lambda$ . This result suggests a bijection between Ballot paths of length  $t$  with  $m$  returns and Ballot paths of length  $t - m - 1$  and height  $m + h - 1$  and hence suggests a combinatorial proof. Such a bijection was given in [14]. A Ballot path can be represented schematically as shown in figure 7. The Bubbles (see definition 3) represent a,

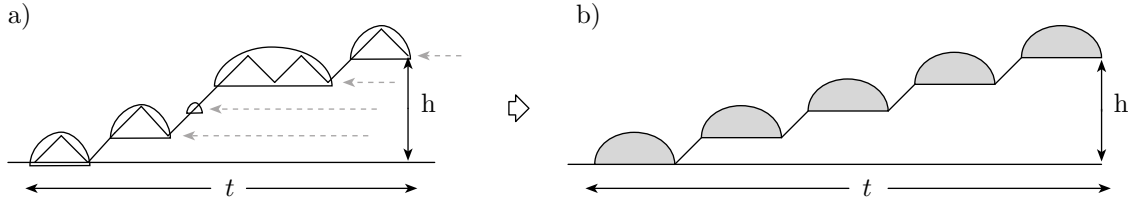


Figure 7: a) An example of a Ballot path showing the “terraces” which define the Bubbles shown schematically in b) which represents of a Ballot path of length  $t$  and height  $h = 4$ .

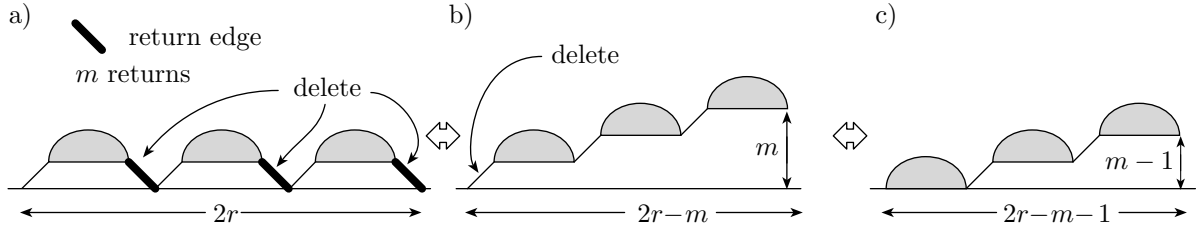


Figure 8: a) Schematic representation of a Dyck path with  $m = 3$  returns. b) All return steps deleted c) First step deleted produces a Ballot path of height  $m - 1$  and length  $2r - m - 1$ .

possibly empty, arbitrary elevated Dyck path. Using this schematic representation, proving (3.8) is then straightforward. For simplicity we show the bijection in the case  $h = 0$  in figure 8. The expansion in powers of  $\kappa - 1$  was also obtained by bijection in [14]. In this case the bijection is between Ballot paths with at least  $m$  returns,  $m$  of which are marked, and Ballot paths of the same length but of height  $h + 2m$ .  $\square$

The above methodology is typical of that used for the more complicated two parameter case in the next section. A constant term formula will be derived and then rewritten in four different ways each giving rise to an expansion in pairs of different variables the coefficients of which are Ballot numbers and are shown, by bijection, to enumerate various types of lattice path.

In the next section we will also need the following lemma.

**Lemma 3.** Let  $P_{2r;2s}^{(sH)}$  be the set of Separated Hovering paths (definition 7) then

$$\sum_{s=0}^r |P_{2r;2s}^{(sH)}| = |P_{2r+2}^{(D)}| \quad (3.10)$$

*Proof.* Since we can shift any Hovering path down on to the  $x$ -axis to give a Dyck path with vertex  $(2s, 0)$  marked, the lemma says the number of Dyck paths with one contact marked is equal to the number of (unmarked) Dyck paths two steps longer. A sketch of the bijective



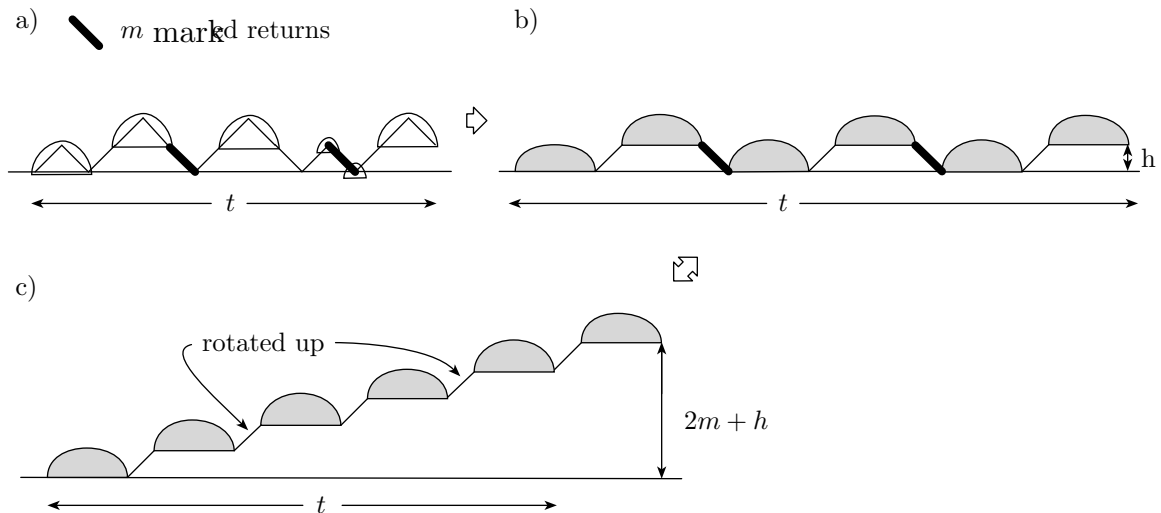


Figure 9: a) An example of a marked return Ballot path of height  $h = 1$  with  $m = 2$  marked returns and b) its schematic representation in terms of bubbles. c) Each marked return rotated through 90 degrees anti-clockwise (or equivalently replaced by an up step) producing a Ballot path of height  $2m + h$  and length  $t$ .

combinatorial proof is shown in figure 10. The construction is a bijection, since, given any length  $2r + 2$  Dyck path a unique length  $2r$  marked contact Dyck path is determined by deleting the rightmost step and the rightmost step from  $y = 0$  to  $y = 1$  (and marking the left vertex of the latter step).  $\square$

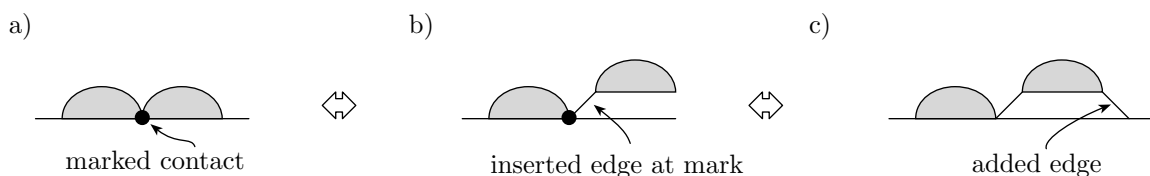


Figure 10: a) Dyck (or hovering) path with one marked contact, b) insert an up step at the mark produces a unique height one Ballot path (length  $2r + 1$ ) and c) adding a final down step produces a unique Dyck path of length  $2r + 2$

## 4 The two contact model.

We begin by computing  $Z_{2r}^{(3)}$  using the  $W^{(4)}$  weights – we will refer to this as the two contact model. We will generalise the model by allowing arbitrary starting and ending heights for the

paths. In particular, let the paths be of length  $t$ , start at  $(0, y^i)$  with  $y^i \in \mathbb{Z}_{\text{odd}}$  and terminate at  $(t, y^f)$ ,  $t + y^f \in \mathbb{Z}_{\text{odd}}$ , i.e. Cross paths in  $P_{t; y^i, y^f}^{(C)}$ .

This model can also be thought of as a polymer model with two different surface interactions, or “contacts”. The path interacts with the “thickened” surface  $y = 0, 1$  via two parameters,  $\kappa_1$  and  $\kappa_2$  and the partition function is defined by

$$Z_t(y^f | y^i; \kappa_1, \kappa_2) = \sum_{\omega \in P_{t; y^i, y^f}^{(C)}} W^{(4)}(\omega) \quad (4.1)$$

where the sum is over Cross paths of length  $t$  with given initial and final heights  $y^i$  and  $y^f$  and the weight  $W^{(4)}(\omega)$  is defined by (2.9) and (2.13). This is a generalisation of the ASEP partition function  $Z_{2r}^{(3)}$  thus

$$Z_{2r}^{(3)} = Z_{2r}(1|1; \bar{\alpha}\bar{\beta}, \kappa^2) \quad (4.2)$$

This generalisation allows this partition function to be determined by recurrence relations similar to those for the return polynomial of Ballot paths. By considering paths of length  $t - 1$  which can reach the point  $(t, y)$  by adding one more step the partition function  $Z_t(y^f | y^i; \kappa_1, \kappa_2)$  may be seen to satisfy the equations,

$$Z_1(1|y^i; \kappa_1, \kappa_2) = 0 \quad (4.3)$$

$$Z_0(y|y^i; \kappa_1, \kappa_2) = \delta_{y, y^i} \quad (4.4)$$

$$Z_t(0|y^i; \kappa_1, \kappa_2) = Z_{t-1}(1|y^i; \kappa_1, \kappa_2) \quad (4.5)$$

$$Z_t(1|y^i; \kappa_1, \kappa_2) = \kappa_1 Z_{t-1}(0|y^i; \kappa_1, \kappa_2) + \kappa_2 Z_{t-1}(2|y^i; \kappa_1, \kappa_2), \quad t \geq 2 \quad (4.6)$$

and for  $t = 1, 2, \dots$ ;  $y = 2, 3, \dots$  and  $t + y$  odd

$$Z_t(y|y^i; \kappa_1, \kappa_2) = Z_{t-1}(y - 1|y^i; \kappa_1, \kappa_2) + Z_{t-1}(y + 1|y^i; \kappa_1, \kappa_2). \quad (4.7)$$

These partial difference equations can be solved for  $Z_t(y^f | y^i; \kappa_1, \kappa_2)$  by using the constant term method.

**Proposition 2.** *With  $\bar{\kappa}_i = \kappa_i - 1$ ,  $\bar{z} = 1/z$  and  $\Lambda = z + \bar{z}$ , for  $y^i \in \mathbb{Z}_{\text{odd}}$  and  $y \geq 1$*

$$Z_t(y|y^i; \kappa_1, \kappa_2) = CT[\Lambda^t(z^{y-y^i} - z^{y+y^i-2})] + (\delta_{y^i, 1} + \kappa_2(1 - \delta_{y^i, 1}))CT[z^{y+y^i-2}\Lambda^t G(z)] \quad (4.8)$$

where

$$G(z) = \frac{(1 - z^2)z\Lambda}{1 - (\bar{\kappa}_1 + \bar{\kappa}_2)z^2 - \bar{\kappa}_2 z^4} \quad (4.9)$$

Note: For  $y^i = 0$  and  $y \geq 1$

$$Z_t(y|0; \kappa_1, \kappa_2) = \kappa_1 Z_{t-1}(y|1; \kappa_1, \kappa_2) \quad (4.10)$$

*Proof.* Substituting (4.8) into the partial difference equations and noting that  $G(z)$  may be expanded in even powers of  $z$  with no inverse powers shows that (4.3), (4.4) and (4.7) are satisfied. (4.5) may be taken as the definition of  $Z_t(0|y^i; \kappa_1, \kappa_2)$  and (4.6) may then be transformed into

$$Z_t(1|y^i; \kappa_1, \kappa_2) = \kappa_1 Z_{t-2}(1|y^i; \kappa_1, \kappa_2) + \kappa_2 Z_{t-1}(2|y^i; \kappa_1, \kappa_2), \quad t \geq 2 \quad (4.11)$$

To verify that this equation is also satisfied we note that

$$\Lambda^2 G(z) = \Lambda(1/z - z) + (\kappa_1 + \kappa_2 z \Lambda) G(z) \quad (4.12)$$

In the case  $y^i = 1$  this gives

$$Z_t(1|1; \kappa_1, \kappa_2) = CT[\Lambda^t G(z)] = CT[\Lambda^{t-1}(1/z - z)] + \kappa_1 Z_{t-2}(1|y^i; \kappa_1, \kappa_2) + \kappa_2 Z_{t-1}(2|y^i; \kappa_1, \kappa_2)$$

and (4.11) follows since the first term is zero.

Otherwise  $y^i = 3, 5, \dots$  in which case

$$Z_{t-1}(2|y^i; \kappa_1, \kappa_2) = CT[(z^{2-y^i} - z^{y^i})\Lambda^{t-1}] + \kappa_2 CT[z^{y^i} \Lambda^{t-1} G(z)]$$

and using this together with (4.12) and the fact that the first term in (4.8) vanishes when  $y = 1$

$$\begin{aligned} Z_t(1|y^i; \kappa_1, \kappa_2) &= \kappa_2 CT[z^{y^i-1} \Lambda^t G(z)] \\ &= \kappa_2 CT[z^{y^i-1}(1/z - z)\Lambda^{t-1}] + \kappa_1 Z_{t-2}(1|y^i; \kappa_1, \kappa_2) + \kappa_2^2 CT[z^{y^i} \Lambda^{t-1} G(z)] \\ &= \kappa_2 CT[(z^{y^i-2} - (1/z)^{y^i-2})\Lambda^{t-1}] + \kappa_1 Z_{t-2}(1|y^i; \kappa_1, \kappa_2) + \kappa_2 Z_{t-1}(2|y^i; \kappa_1, \kappa_2) \end{aligned}$$

and again (4.11) follows since the first term evaluates to zero.

An alternative constructive proof of this proposition is given in appendix B.  $\square$

**Corollary 2.** *The number of  $t$ -step Cross paths which begin at  $v^i = (0, h_1)$  and end at  $v^f = (t, h_2)$  is given by*

$$Z_t(h_2|h_1; 1, 1) = \binom{t}{\frac{1}{2}(t-h_2+h_1)} - \binom{t}{\frac{1}{2}(t-h_2-h_1)-1} \quad (4.13)$$

in terms of which we can write,  $Z_t(y|y^i; \kappa_1, \kappa_2)$  as

$$Z_t(y|y^i; \kappa_1, \kappa_2) = Z_t(y-2|y^i-2; 1, 1) + (\delta_{y^i, 1} + \kappa_2(1 - \delta_{y^i, 1})) Z_t^a(y|y^i; \kappa_1, \kappa_2) \quad (4.14)$$

where

$$Z_t^a(y|y^i; \kappa_1, \kappa_2) = CT \left[ \Lambda^t z^{y+y^i-2} G(z) \right] \quad (4.15)$$

is the partition function restricted to Anchored Cross paths except that for  $y^i > 1$  the step leading to the first visit to  $y = 1$  has weight 1.

In particular

$$Z_t(y|1; \kappa_1, \kappa_2) = Z_t^a(y|1; \kappa_1, \kappa_2) = CT \left[ \Lambda^t G(z) z^{y-1} \right]. \quad (4.16)$$

Notes:

- It follows from the constant term formula that

$$Z_t^a(y|y^i; \kappa_1, \kappa_2) = Z_t(y + y^i - 1|1; \kappa_1, \kappa_2) \quad (4.17)$$

- Setting  $\kappa_1 = \kappa_2 = 1$  gives the number of unweighted paths starting at height 1.

$$Z_t(y|1; 1, 1) = CT[\Lambda^t(1 - z^4)z^{y-1}] = CT[\Lambda^{t+1}(1 - z^2)z^y] = B_{t+1, y}$$

as expected since the paths biject to Ballot paths by adding an initial up step.

*Proof.* With  $\kappa_1 = \kappa_2 = 1$ ,  $G(z) = 1 - z^4$  and substituting in (4.8) gives

$$Z_t(y|y^i; 1, 1) = CT \left[ \Lambda^t (z^{y-y^i} - z^{y+y^i+2}) \right] \quad (4.18)$$

which yields (4.13).

$Z_t(y - 2|y^i - 2; 1, 1)$  is the number of Cross paths which avoid  $y = 1$ . This follows since such paths are in simple bijection with the paths starting at height  $y^i - 2$  and ending at height  $y - 2$ , eg. just push the whole path down (or up) two units. The second term in (4.14) is therefore the partition function for Anchored Cross paths. In the case  $y^i > 1$  Anchored Cross paths always have a first visit to  $y = 1$  and the step leading to this visit has a factor  $\kappa_2$ . Removing this factor leaves  $Z_t^a(y|y^i; \kappa_1, \kappa_2)$  which is therefore the partition function for Anchored Cross paths except that for  $y^i > 1$  the step leading to the first visit to  $y = 1$  has weight 1. Setting  $y^i = 1$  in (4.14) gives (4.16) since the first term vanishes.

□

**Corollary 3.**

$$Z_t^a(y|y^i; \kappa_1, \kappa_2) = CT \left[ \frac{(1-z^2)\Lambda^{t-1}z^{y+y^i-3}}{1-(\kappa_1+z\Lambda\kappa_2)(z/\Lambda)^2} \right] \quad (4.19)$$

$$= \sum_{j=0}^{(t-y-y^i)/2+1} \sum_{k=0}^{(t-y-y^i)/2-j+1} \kappa_1^j \kappa_2^k \binom{j+k}{k} B_{t-2j-k-1, y+y^i+k-3} \quad (4.20)$$

*Note:* When  $y = y^i = 1$  and  $j = t/2, k = 0$  the coefficient  $B_{-1,-1}$  is indeterminate but setting  $B_{-1,-1} = 1$  gives the correct answer. This corresponds to the path which alternates between  $y = 0$  and  $y = 1$ . Notice that replacing the factorials in the definition of  $B_{t,h}$  by Gamma functions gives  $B_{t,t} = 1$  for all  $t > -1$ .

*Proof.* Rewrite  $G(z)$  in the form

$$G(z) = \frac{1-z^2}{(z\Lambda)(1-(\kappa_1+z\Lambda\kappa_2)(z/\Lambda)^2)},$$

expand in powers of  $\kappa_1$  and  $\kappa_2$  and use the CT formula (3.2) for Ballot numbers.

Equation (4.20) may also be proved by a combinatorial argument. To simplify the proof we only consider the ASEP case  $y = y^i = 1$ . The extension to general  $y, y^i$  is straightforward. Recall that, from the weight definition, (2.13), the returns to  $y = 1$  are weighted with  $\kappa_1$  or  $\kappa_2$  depending on whether the return is from below or from above  $y = 1$ . The binomial coefficient corresponds to choosing a particular sequence of  $\kappa_1$  and  $\kappa_2$  weighted returns. For each particular sequence of returns we need to show there are  $B_{t-2j-k-1, k-1}$  possible path configurations.

We first represent a particular sequence schematically and then show any path corresponding to the schematic can be bijected to a Ballot path with the correct height and length. Schematically an example of a particular sequence of  $\kappa_1$  and  $\kappa_2$  returns is shown in figure 11.

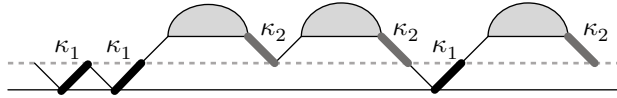


Figure 11: *Schematic representation of a one-up path corresponding to the return sequence  $\kappa_1^2\kappa_2^2\kappa_1\kappa_2$ .*

We now perform three operations to biject a given sequence into a Ballot path.

- First delete all  $\kappa_2$  return steps, see figure 12a). This produces a path of length  $t - k$  and height  $k + 1$ .

- Next, delete the first up step above  $y = 1$  (if any – which is the case if  $k > 0$ ), see figure 12b). This produces a path of length  $t - k - 1$  and height  $k$ .
- Finally, delete all  $2j$  steps originally below  $y = 1$ , see figure 12c). This produces a Ballot path of length  $t - k - 1 - 2j$  and height  $k - 1$  as required.

Given the sequence of  $\kappa_1$  and  $\kappa_2$  the reverse direction for the bijection is obtained by simply reversing the forward mapping – see figure 12.  $\square$

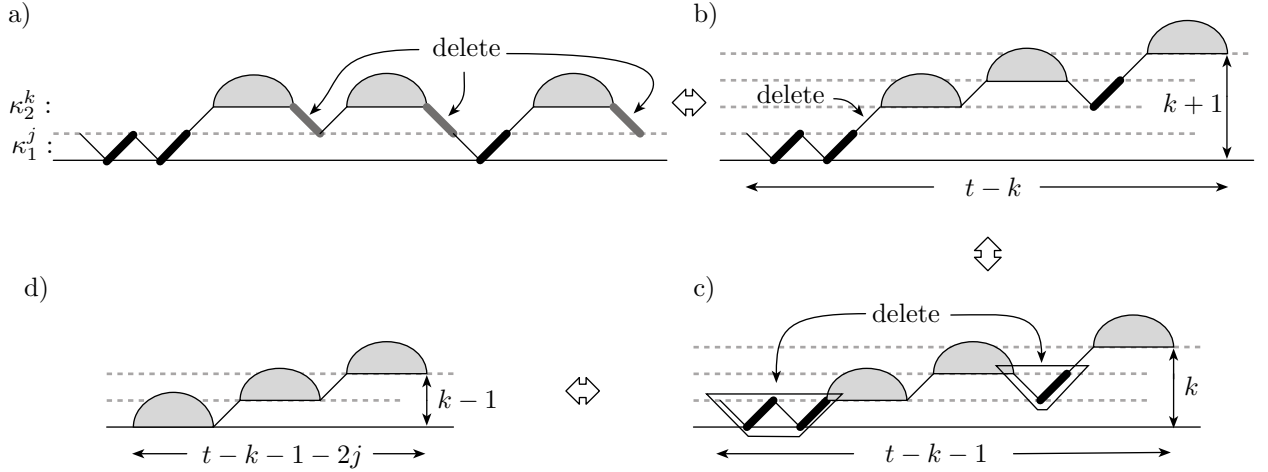


Figure 12: *Bijection of first example sequence in figure 11 to a Ballot path: a) Delete all  $\kappa_2$  return steps, then b) delete first up step above  $y = 1$ , then c) delete all steps originally below  $y = 1$ , to give d) the final Ballot path.*

**Corollary 4.**

$$Z_t^a(y|y^i; \kappa_1, \kappa_2) = \sum_{j=0}^{(t-y-y^i)/2+1} \sum_{k=0}^{(t-y-y^i)/2-j+1} \bar{\kappa}_1^j \bar{\kappa}_2^k \binom{j+k}{k} B_{t+k+1, y+y^i+2j+3k-1} \quad (4.21)$$

*Proof.* Rewrite  $G(z)$  in the form

$$G(z) = \frac{(1-z^2)z\Lambda}{1 - (\bar{\kappa}_1 + z\Lambda\bar{\kappa}_2)z^2},$$

expand in powers of  $\bar{\kappa}_1$  and  $\bar{\kappa}_2$  and use the CT formula (3.2) for Ballot numbers.

As with corollary 3, the result, (4.21), may also be proved combinatorially as follows. Again, in order to simplify the proof we consider only the ASEP case  $y = y^i = 1$ . The substitution

$\kappa_i = 1 + \bar{\kappa}_i$  means that a return which was weighted with  $\kappa_i$  is now weighted with either  $\bar{\kappa}_i$  or 1. The factor

$$\binom{j+k}{k} B_{t+k+1, 2j+3k+1} \quad (4.22)$$

is the number of paths with a subset of exactly  $j$  of the returns from below marked and exactly  $k$  of the returns from above marked corresponding to the choice of weight  $\bar{\kappa}_i$  or 1. The binomial coefficient in (4.22) is the number of ways of choosing a particular sequence of  $\bar{\kappa}_1$  and  $\bar{\kappa}_2$  weighted marks (reading the path from left to right), whilst, for a given sequence, the Ballot number represents the number of paths corresponding to the sequence. The most general path

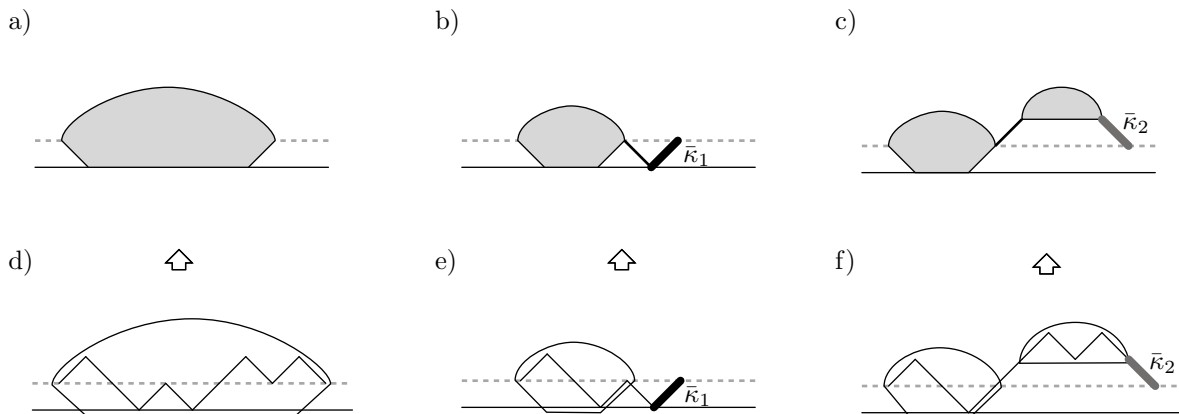


Figure 13: a) Schematic One Up path or “frying pan” with no steps marked. b) Schematic One Up path with only the last step  $\bar{\kappa}_1$  marked. c) Schematic One Up path with only the last step  $\bar{\kappa}_2$  marked. d) An example showing how the schematic frying pan represents a, possibly empty, One Up path. e) An example showing a frying pan followed by a down step and an up step (which forms a  $\bar{\kappa}_1$  marked return). f) An example showing a frying pan followed by an up step, then a Bubble then a final down step (which forms a  $\bar{\kappa}_2$  marked return).

corresponding to a given sequence can be represented schematically by concatenating the corresponding schematic sub-paths shown in figure 13b) and 13c) with a final “frying pan” shown in figure 13a). Examples of sub-paths corresponding to the three types of schematics are illustrated in figures 13d) – 13f). Note, the shaded regions of the schematics represent any number (possibly zero) of steps. Examples of two possible sequences are illustrated in figure 14.

Thus, for a given return sequence we need to show that there are  $B_{t+k+1, 2j+3k+1}$  return marked paths. Without loss of generality we choose a typical sequence and represent it schematically as shown in figure 14.

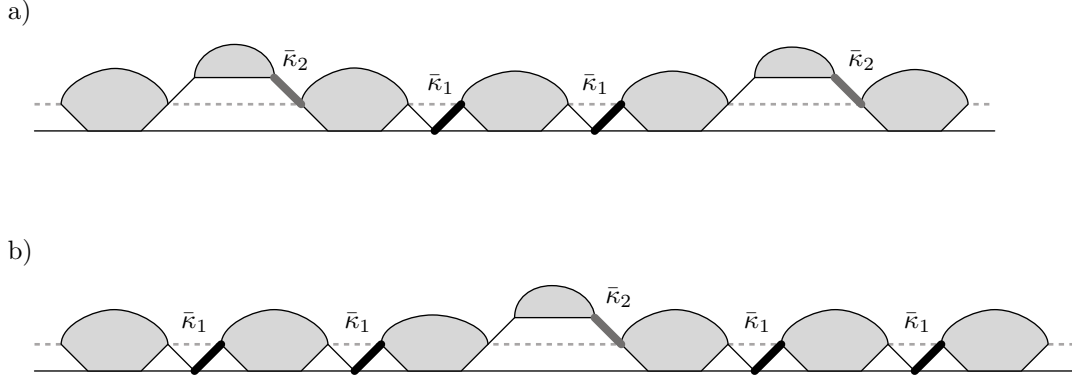


Figure 14: Schematic  $\bar{k}_1 - \bar{k}_2$  marked hovering path. a) For the sequence  $\bar{k}_2\bar{k}_1\bar{k}_1\bar{k}_2$  and b) for the sequence  $\bar{k}_1\bar{k}_1\bar{k}_2\bar{k}_1\bar{k}_1$ .

Thus to prove the Ballot number factor in (4.22) we need to biject any schematic marked return sequence to a Ballot path of length  $t + k + 1$  and height  $2j + 3k + 1$ . We do this by bijecting each schematic in the sequence to a sub-Ballot path (plus, possibly, an extra step) and then concatenate them all together.

- Thus, the last frying pan, of length say,  $2r'$ , bijects to a height 1, length  $2r' + 1$  Ballot path – see figure 15.

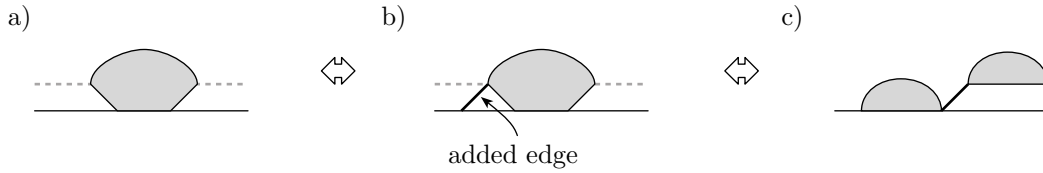


Figure 15: Bijection of a "frying pan" to a height one Ballot path. The final path is one step longer than the original.

- A marked  $\bar{k}_2$  schematic of length  $2r_1$  bijects to a height 2, Ballot path with an additional final up step (see figure 16). The final length is  $2r_1 + 1$  as an extra step has to be added.
- Finally, a marked  $\bar{k}_1$  schematic of length  $2r_2$  bijects to a height 1, Ballot path with an additional final up step (see figure 17). The final length is unchanged.

Putting these moves all together is illustrated in figure 18, which shows clearly that a Ballot path of length  $t + k + 1$  and height  $2j + 2k + k + 1$  is obtained.

□



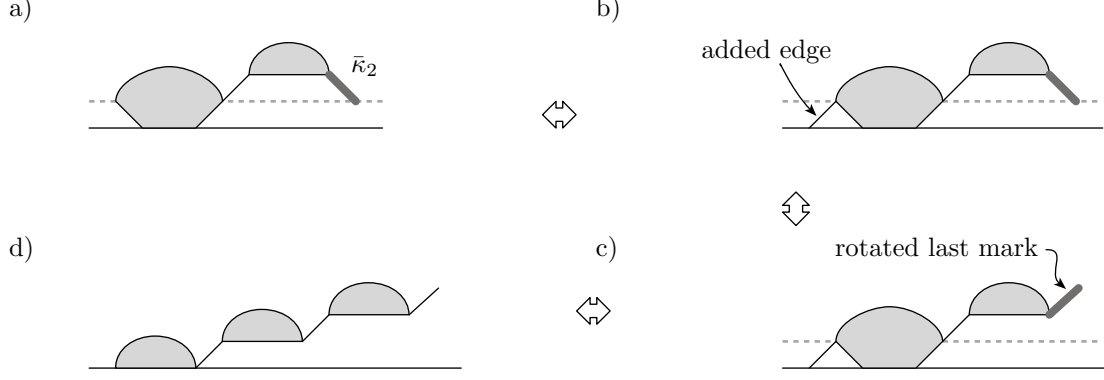


Figure 16: *Bijection of a  $\bar{\kappa}_2$  schematic to a height two Ballot path, with a final up step. The final path is one step longer than the original.*

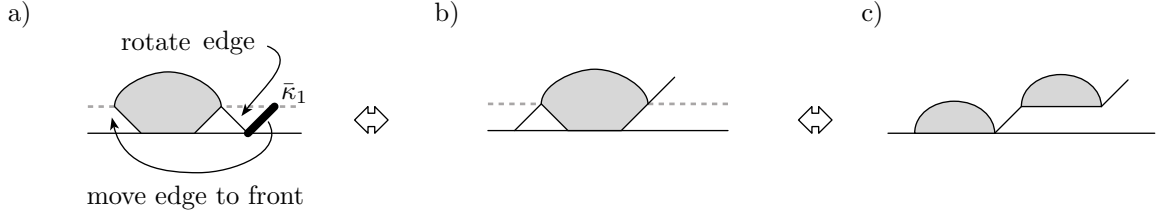


Figure 17: *Bijection of a  $\bar{\kappa}_1$  schematic to a height one Ballot path, with a final up step.*

An explicit evaluation of the partition function of the canonical path representation, defined by (2.16), is the case  $y^i = y^f = 1$  of the following result.

**Corollary 5.** *With  $\kappa_1 = \bar{\alpha}\bar{\beta}$  and  $\kappa_2 = \bar{\alpha} + \bar{\beta} - \bar{\alpha}\bar{\beta}$*

$$Z_t^a(y^i|y^f; \kappa_1, \kappa_2) = \sum_{m=0}^{\frac{1}{2}(t-y^i-y^f)+1} B_{t-m-1, m+y^i+y^f-3} \frac{\bar{\alpha}^{m+1} - \bar{\beta}^{m+1}}{\bar{\alpha} - \bar{\beta}} \quad (4.23)$$

$$= \sum_{m=0}^{\frac{1}{2}(t-y^i-y^f)+1} B_{t-m-1, m+y^i+y^f-3} \sum_{j=0}^m \bar{\alpha}^j \bar{\beta}^{m-j} \quad (4.24)$$

*Notes:*

- For the ASEP model ( $y^i = y^f = 1, t = 2r$ ) with  $r \geq 1$ , (4.24) reduces to the result of Derrida et al [1] equation (39)

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = \sum_{m=1}^r \frac{m(2r-m-1)!}{r!(r-m)!} \sum_{j=0}^m \bar{\alpha}^j \bar{\beta}^{m-j} = Z_{2r}^{(5)} \quad (4.25)$$

which they obtained directly from the algebra (1.3) ([1] appendix A1, equation (A12)) without the use of a matrix representation. An equivalent formula was also given by

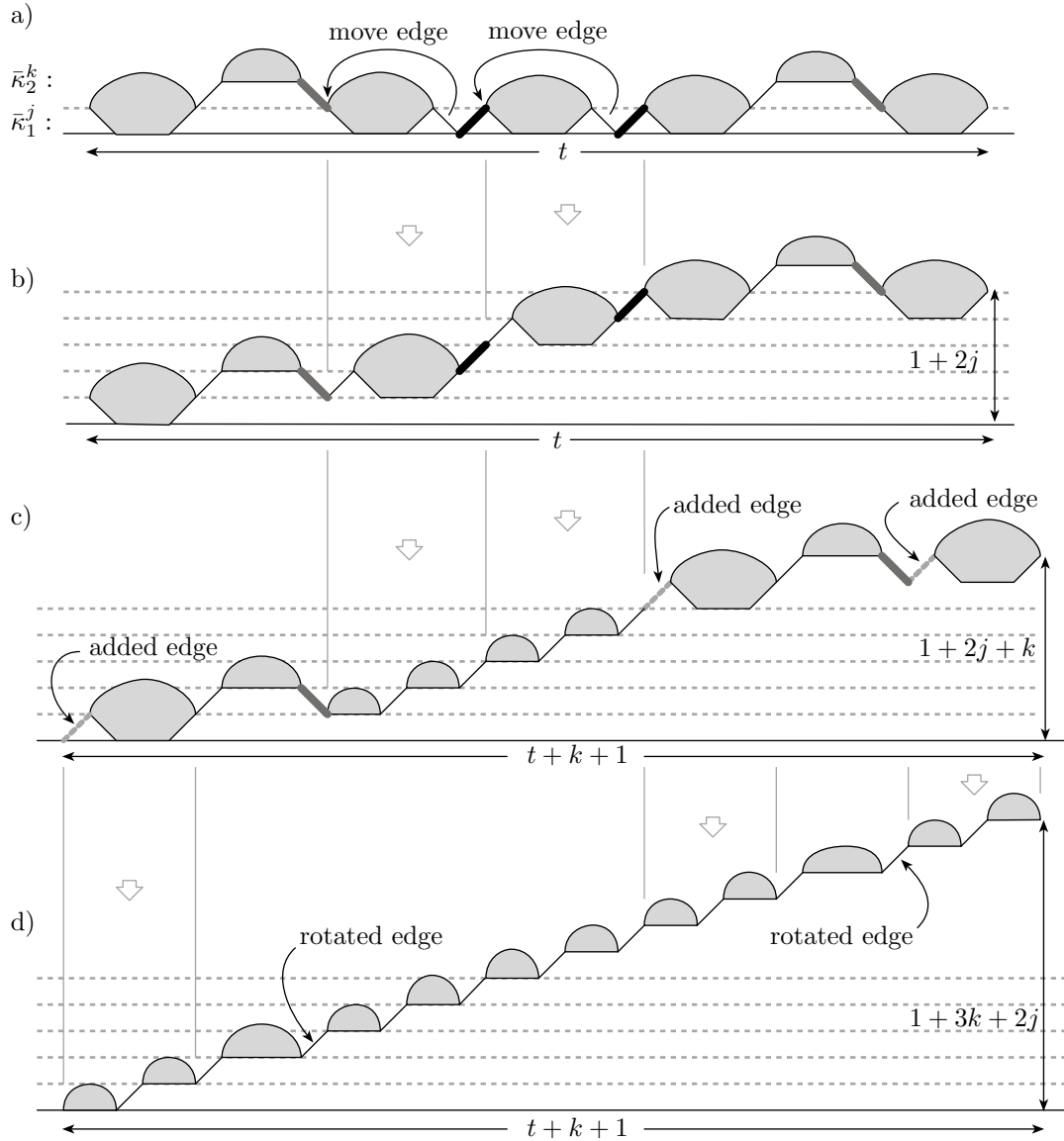


Figure 18: *Bijection between a One Up path with a given sequence of marked  $\bar{\kappa}_1$  and  $\bar{\kappa}_2$  steps and a Ballot path. a) The marked One Up path. b) After the application of move illustrated in figure 17. c) After the application of move illustrated in figure 16. e) Final Ballot path after rotating the remaining marks.*

Liggett [15], page 252, as part of his expression for the current [1]  $c_r = Z_{2r-2}/Z_{2r}$ . In his notation  $Z_{2r} = (\bar{\alpha}\bar{\beta})^n h_r(\alpha, \beta)$  and  $\alpha = \lambda, \beta = 1 - \rho$ .

- An equivalent formula was also given by Schutz and Domany [16] equation (2.17). They solved the recurrence relations of Derrida, Domany and Mukamel [17] who only obtained an exact solution in the special case  $\alpha = \beta = 1$ . A similar recurrence relation was given

by Liggett [15].

- The result for the ASEP model may be written in terms of the return polynomial for Ballot paths, thus

$$\begin{aligned} Z_{2r}(1|1; \kappa_1, \kappa_2) &= \sum_{j=0}^r \bar{\alpha}^j \sum_{\ell=0}^{r-j} B_{2r-j-\ell-1} j + \ell - 1 \bar{\beta}^\ell \\ &= \sum_{j=0}^r R_{2r-j}(j; \bar{\beta}) \bar{\alpha}^j. \end{aligned} \quad (4.26)$$

This formula may also be derived using a path representation based on the recurrence relations of Derrida, Domany and Mukamel [17].

*Proof.* By definition of  $\bar{\alpha}$  and  $\bar{\beta}$

$$1 - (\bar{\kappa}_1 + \bar{\kappa}_2)z^2 - \bar{\kappa}_2 z^4 = [1 - (\bar{\alpha} - 1)z^2][1 - (\bar{\beta} - 1)z^2] = z^2(\Lambda - \bar{\alpha}z)(\Lambda - \bar{\beta}z) \quad (4.27)$$

and from (4.9)

$$G(z) = \frac{(1 - z^2)\bar{z}\Lambda}{(\Lambda - \bar{\alpha}z)(\Lambda - \bar{\beta}z)} = \frac{(1 - z^2)\bar{z}^2}{\bar{\alpha} - \bar{\beta}} \left( \frac{1}{1 - \bar{\alpha}z/\Lambda} - \frac{1}{1 - \bar{\beta}z/\Lambda} \right) \quad (4.28)$$

The result follows by expanding in powers of  $\bar{\alpha}$  and  $\bar{\beta}$ , substituting in (4.15) and using the CT formula (3.2) for Ballot numbers.

In the ASEP case  $y^i = y^f = 1, t = 2r$  the coefficient  $B_{2r-m-1, m-1}$  in (4.24) is equal to the number of Dyck paths with  $m$  returns to  $y = 0$  (see (3.8)). The equality of  $Z_{2r}(1|1; \kappa_1, \kappa_2)$  with  $Z_{2r}^{(5)}$ , defined by (2.16), follows by raising the Dyck paths so that they become hovering paths  $\omega \in P_{2r}^{(H)}$  with returns to  $y = 1$ . The factor  $\bar{\alpha}^j \bar{\beta}^{m-j}$  in (4.24) corresponds to weighting the first  $j$  returns of  $\omega$  with  $\bar{\alpha}$  and the remainder with  $\bar{\beta}$ . The sum over  $s$  in (2.16) is obtained by partitioning the weighted paths according to the position  $(2s, 1)$  of the  $j^{\text{th}}$  return (ie. the separation vertex).

The equality of  $Z_{2r}^{(3)}$  and  $Z_{2r}^{(5)}$  is also shown directly in [11] by involution.  $\square$

**Corollary 6.** *Let  $\bar{\alpha} = c + 1$  and  $\bar{\beta} = d + 1$  then  $\kappa_1 = (c + 1)(d + 1)$ ,  $\kappa_2 = 1 - cd$  and*

$$Z_t^a(y|y^i; \kappa_1, \kappa_2) = \sum_{m=0}^{(t-y-y^i+2)/2} B_{t+1, y+y^i+2m-1} \sum_{j=0}^m c^j d^{m-j} \quad (4.29)$$

$$= \sum_{m=0}^{(t-y-y^i+2)/2} B_{t+1, y+y^i+2m-1} \frac{(c^{m+1} - d^{m+1})}{c - d} \quad (4.30)$$

in particular, in the ASEP case

$$Z_{2r}(1|1; (c+1)(d+1), 1-cd) = Z_{2r}^{(2)} = \sum_{m=0}^r B_{2r+1, 2m+1} \sum_{j=0}^m c^j d^{m-j} \quad (4.31)$$

*Note:* Equation (4.31) is not given in [1] but (34) and (35) of [1] together give the related formula

$$\langle W|C^r|V \rangle = (1-cd) \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \left( \binom{2r}{r+i-j} - \binom{2r}{r+i+j} \right) c^{i-1} d^{j-1} \quad (4.32)$$

This expression involves an infinite series whereas our expression is finite. By corollary 2 the coefficient in (4.32) is the number of Cross paths of length  $2r$  with  $h_1 = 2i - 1$  and  $h_2 = 2j - 1$  and the double sum extends over  $P_{2r}^{(C)}$ . The factor  $1 - cd$  restricts the sum to Anchored Cross paths. The equivalence of (4.32) and (4.31) is shown in [11] by constructing an involution on  $P_{2r}^{(C)}$  having  $P_{2r}^{(aC)}$  as its fixed point set.

*Proof.* From (4.27)

$$1 - (\bar{\kappa}_1 + \bar{\kappa}_2)z^2 - \bar{\kappa}_2 z^4 = (1 - cz^2)(1 - dz^2)$$

and substitution in (4.9) gives

$$G(z) = \frac{(1 - z^2)z\Lambda}{(1 - cz^2)(1 - dz^2)} \quad (4.33)$$

The result follows from (4.15) using the expansion

$$\frac{1}{(1 - cz^2)(1 - dz^2)} = \frac{1}{c-d} \left( \frac{c}{1 - cz^2} - \frac{d}{1 - dz^2} \right), \quad (4.34)$$

expanding in powers of  $z$  and using the CT formula (3.2) for Ballot numbers..

Again a combinatorial proof is possible which for simplicity we only give in the ASEP case. The equality with  $Z_{2r}^{(2)}$  follows from corollary 2. To obtain the Ballot number formula we obtain a bijection between  $\bigcup_{s=0}^r P_{2r; 2s}^{(mH)}$  and the set of Ballot paths of length  $2r+1$  and height  $h = 2m+1$ .

An example of the schematic representation of a path  $\omega_s \in P_{2r; 2s}^{(mH)}$  with a given number of  $c$  and  $d$  marked return steps and a separating vertex  $v_{2s}$  is shown in figure 19. As there are no steps below  $y = 1$  we have pushed the hovering path down by unit height and consider it as a Dyck path.

As in the proof of lemma 3, inserting an up step in each path at the position of  $v_{2s}$  and taking the union over all possible positions of  $v_{2s}$  starting at the last  $c$  return and up to but not including the first  $d$  return (or from the beginning if there are no  $c$  returns and to the end if there are no  $d$  returns) replaces the pair of Bubbles on either side of the separating vertex by

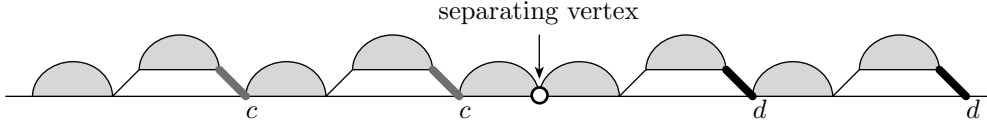


Figure 19: An example of a schematic representation of a Dyck path with a given number of  $c$  and  $d$  marked returns and a marked separating vertex  $v_{2s}$ .

the set Ballot paths of height one and one step longer, see figure 20a,b). Replacing each marked return step with an up step (and hence increasing the height of the path by 2 each time) then produces a Ballot path of length  $2r + 1$  and height  $2m + 1$  as required, see figure 20c).

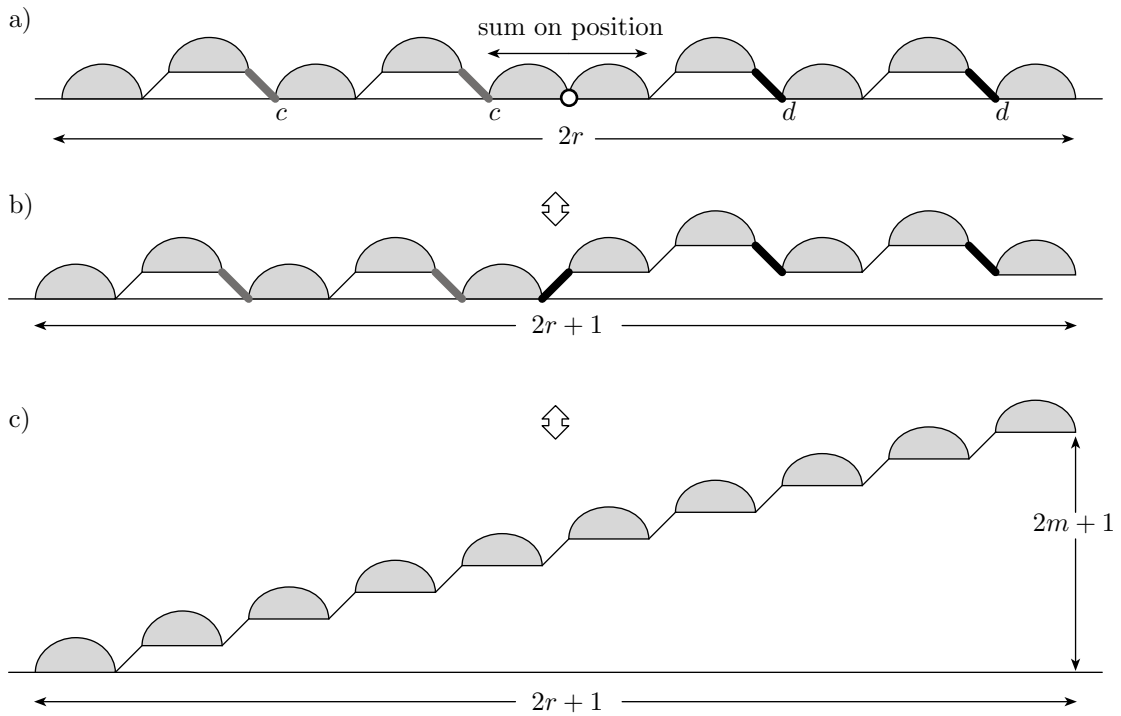


Figure 20: a) Taking the union over positions of the marked separating vertex produces a schematic height one Ballot path as in b), replacing each marked  $c$  and  $d$  return by an up step produces a Ballot path of length  $2r + 1$  and height  $2m + 1$  as in c).

□

**Corollary 7.**

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = -\frac{1}{2}(1 - cd)CT \left[ \frac{\Lambda^{2r}(z^2 - \bar{z}^2)^2}{((1 + c)^2 - c\Lambda^2)((1 + d)^2 - d\Lambda^2)} \right]$$

Note: This converts to the integral formula of Derrida et al [1] (B10) with  $z^2 = e^{i\theta}$  and using a contour integral to pick out the constant term. It is related to the  $\omega$  expansion (see later). (B10) was obtained by finding the eigenvectors of  $C_2$  and is therefore an evaluation of  $Z^{(2)}$ .

*Proof.* With  $w \equiv z^2$  in (4.33) and using (4.16)

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = CT \left[ \frac{\Lambda^{2r}(1-w^2)}{(1-cw)(1-dw)} \right]$$

Now symmetrize the denominator by multiplying numerator and denominator by  $(1-c\bar{w})(1-d\bar{w})$ .

The result follows using

$$(1-w^2)(1-c\bar{w})(1-d\bar{w}) = 1 - cd - w^2 + cd\bar{w}^2 + (c+d)(w-\bar{w})$$

Because the rest of the expression is now symmetric the contribution from the last term vanishes by replacing  $\bar{w}$  by  $w$  and both  $w^2$  and  $\bar{w}^2$  can be replaced by  $(w^2 + \bar{w}^2)/2$ .  $\square$

## 5 The “ $\omega$ ” expansion and phase diagram of the ASEP model.

With  $\omega_c = c/(1+c)^2$  corollary 7 may be written in the form

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = \frac{\omega_c - \omega_d}{c - d} CT \left[ \frac{\Lambda^{2r+2}(1-z^2)}{(1-\omega_c\Lambda^2)(1-\omega_d\Lambda^2)} \right]. \quad (5.1)$$

which may be expanded to give

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = \frac{Z_{2r}(\omega_c) - Z_{2r}(\omega_d)}{c - d} \quad (5.2)$$

where

$$Z_{2r}(\omega) = CT \left[ \frac{\Lambda^{2r}(1-z^2)}{1-\omega\Lambda^2} \right]. \quad (5.3)$$

The asymptotic form of  $Z_{2r}(\omega)$  as  $r \rightarrow \infty$  was obtained in [18] and will now be used to study the phase diagram for the ASEP model. First we outline the method by which the asymptotic form was obtained.

Notice that expanding the factor  $(1-\omega\Lambda^2)^{-1}$  in (5.3) in powers of  $\omega$  gives an infinite series which is only valid for  $c \leq 1$  which is the point at which  $\omega$  as a function of  $c$  passes through its maximum value  $\frac{1}{4}$ . Instead we use (5.3) to obtain a recurrence relation. Thus noting that

$$\frac{\omega\Lambda^{2r}}{1-\omega\Lambda^2} = -\Lambda^{2r-2} + \frac{\Lambda^{2r-2}}{1-\omega\Lambda^2}$$

and substituting in (5.3) gives

$$\omega Z_{2r}(\omega) = -C_{r-1} + Z_{2r-2}(\omega). \quad (5.4)$$

where  $C_r$  is the Catalan number, (3.4). Solving (5.4) with  $Z_2 = (1+c)^2$  gives

$$Z_{2r}(\omega_c) = \omega_c^{-r} \left( 1 + c - \sum_{j=0}^{r-1} C_j \omega_c^j \right) \quad (5.5)$$

which on substituting for  $\omega_c$  in terms of  $c$  must give a polynomial in  $c$ . Comparing this with [18] equation (3.59) it may be seen that  $Z_{2r}(\omega_c)$  is the contact polynomial  $\hat{Z}_{2r}^S(\bar{\alpha})$  for Dyck paths of length  $2r$ .

For the ASEP model  $\omega_c = \alpha(1-\alpha)$  and

$$\begin{aligned} \sum_{j=0}^{\infty} C_j \omega_c^j &= \frac{1 - \sqrt{1 - 4\omega_c}}{2\omega_c} = \frac{1 - |1 - 2\alpha|}{2\alpha(1-\alpha)} \\ &= \begin{cases} \frac{1}{\alpha} & \alpha > \frac{1}{2} \\ \frac{1}{1-\alpha} & \alpha \leq \frac{1}{2} \end{cases} \end{aligned}$$

so

$$Z_{2r}(\omega_c) = \omega_c^{-r} \left[ \frac{1}{\alpha} - \left( \sum_{j=0}^{\infty} C_j \omega_c^j - \sum_{j=r}^{\infty} C_j \omega_c^j \right) \right] \quad (5.6)$$

$$= \omega_c^{-(r+1)} (1 - 2\alpha) \theta(1 - 2\alpha) + \sum_{j=r}^{\infty} C_j \omega_c^{j-r} \quad (5.7)$$

where  $\theta(\cdot)$  is the unit step function. This is [18] equation (3.61).

The asymptotic form for  $r \rightarrow \infty$  was obtained using

$$C_r \sim \frac{4^r}{\sqrt{\pi} r^{3/2}}$$

and may be written

$$Z_{2r}(\omega_c) \sim \begin{cases} f_{<}(\alpha) = \frac{1-2\alpha}{\omega_c^{r+1}} & \alpha < \frac{1}{2} \\ f_{=} = \frac{2}{\sqrt{\pi}} \frac{4^r}{r^{3/2}} & \alpha = \frac{1}{2} \\ f_{>}(\alpha) = \frac{4^r}{\sqrt{\pi} r^{3/2}} (1 - 4\omega_c)^{-1} & \alpha > \frac{1}{2} \end{cases} \quad (5.8)$$

These results agree with [1] equations (48)-(50). A similar analysis has been given by Liggett [8] using a recurrence relation equivalent to (5.4).

*Note:* In [18] the contact polynomial for Dyck paths was the partition function for a polymer chain attracted to a surface. There the factor  $1 - 4\omega_c$  in (5.8) was replaced by  $\log(1/4\omega_c)$  which arose from approximating the sum in (5.7) by an integral. Although this gives the correct scaling behaviour near the binding transition of the polymer chain which occurs at  $\bar{\alpha} = \frac{1}{2}$ , it breaks down near  $\bar{\alpha} = 1$ .

In the phase diagram there are therefore

- three special regions  $R_1 = \{\alpha > \frac{1}{2}, \beta > \frac{1}{2}\}$ ,  $R_2 = \{\alpha > \beta, \beta < \frac{1}{2}\}$ ,  $R_3 = \{\alpha < \beta, \alpha < \frac{1}{2}\}$

The partition function in  $R_3$  is obtained from that in  $R_2$  by interchanging  $\alpha$  and  $\beta$ .

- three special lines  $L_1 = \{\alpha = \beta < \frac{1}{2}\}$ ,  $L_2 = \{\beta = \frac{1}{2}, \alpha > \frac{1}{2}\}$ ,  $L_3 = \{\alpha = \frac{1}{2}, \beta > \frac{1}{2}\}$  The partition function on  $L_3$  is obtained from that on  $L_2$  by interchanging  $\alpha$  and  $\beta$ .
- a special point where the lines meet  $P = \{\alpha = \frac{1}{2}, \beta = \frac{1}{2}\}$

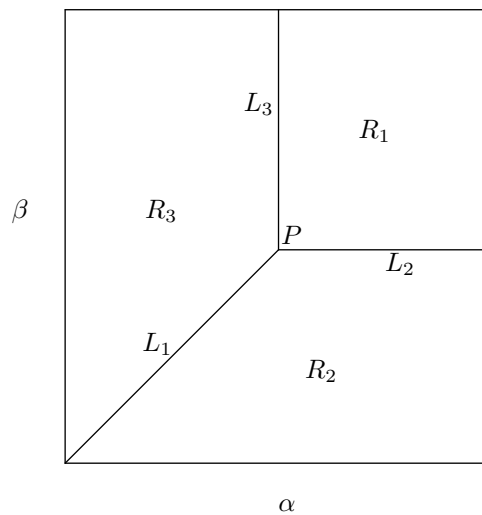


Figure 21: *The various regions of the ASEP phase diagram.*

Table 1 shows the asymptotic form of  $Z_{2r}(1|1; \kappa_1, \kappa_2)$  for the above cases



$R_3 : \frac{f_{<}(\alpha)}{\bar{\alpha}-\beta}$	$L_3 : \frac{f_{=}}{2-\beta}$	$R_1 : \frac{f_{>}(\alpha)-f_{>}(\beta)}{\bar{\alpha}-\beta}$
	$P : 4^r$	$L_2 : \frac{f_{=}}{2-\bar{\alpha}}$
$L_1 : -\alpha^2 f'_{<}(\alpha) \sim \frac{r(1-2\alpha)^2}{\omega_c^{r+2}}$		$R_2 : \frac{f_{<}(\beta)}{\beta-\bar{\alpha}}$

Table 1: Asymptotic forms of  $Z_{2r}(1|1; \kappa_1, \kappa_2)$  in the regions defined in the phase diagram in figure 21.

## 6 Recurrence relations for the partition function and correlation functions.

### 6.1 Recurrence relations for the partition function.

The various formulae for  $G(z)$  when substituted in (4.16) yield recurrence relations for  $Z_t(y|1; \kappa_1, \kappa_2)$ .

For example, using the identity

$$\frac{1}{(1-\omega_c\Lambda^2)(1-\omega_d\Lambda^2)} = 1 + \frac{(\omega_c + \omega_d)\Lambda^2 - \omega_c\omega_d\Lambda^4}{(1-\omega_c\Lambda^2)(1-\omega_d\Lambda^2)},$$

substitution in (5.1) and using the CT formula (3.4) for Catalan numbers leads to, for  $r = 0, 1, \dots$

$$Z_{2r}(1|1; \kappa_1, \kappa_2) = \frac{\omega_c - \omega_d}{c - d} C_{r+1} + (\omega_c + \omega_d) Z_{2r+2}(1|1; \kappa_1, \kappa_2) - \omega_c\omega_d Z_{2r+4}(1|1; \kappa_1, \kappa_2). \quad (6.1)$$

Substitution in terms of  $\kappa_1$  and  $\kappa_2$  yields, for  $r = 2, 3, \dots$

$$(\kappa_2 - 1)Z_{2r}(1|1; \kappa_1, \kappa_2) = (\kappa_2(\kappa_1 + \kappa_2) - 2\kappa_1)Z_{2r-2}(1|1; \kappa_1, \kappa_2) + \kappa_1^2 Z_{2r-4}(1|1; \kappa_1, \kappa_2) - \kappa_2 C_{r-1} \quad (6.2)$$

which may be initialised by  $Z_0(1|1; \kappa_1, \kappa_2) = 1$  and  $Z_2(1|1; \kappa_1, \kappa_2) = \kappa_1 + \kappa_2$ .

The following identity

$$\frac{1}{(1-\bar{\alpha}z/\Lambda)(1-\bar{\beta}z/\Lambda)} = 1 + \frac{(\bar{\alpha} + \bar{\beta})z/\Lambda - \bar{\alpha}\bar{\beta}z^2/\Lambda^2}{(1-\bar{\alpha}z/\Lambda)(1-\bar{\beta}z/\Lambda)}$$

when substituted in (4.28) gives, using the *CT* formula (3.2) for Ballot numbers, for  $y = 1, 2, \dots$  and odd  $t + y \geq 3$

$$Z_t(y|1; \kappa_1, \kappa_2) = B_{t-1, y-2} + (\bar{\alpha} + \bar{\beta})Z_{t-1}(y+1|1; \kappa_1, \kappa_2) - \bar{\alpha}\bar{\beta}Z_{t-2}(y+2|1; \kappa_1, \kappa_2) \quad (6.3)$$

which relates partition functions on lines of constant  $t + y$ . The partition function for  $y = 1$  is determined by (6.1) and the following proposition then determines  $Z_t(2|1; \kappa_1, \kappa_2)$  which provides the initial condition for (6.3).

**Proposition 3.** For  $r = 1, 2, \dots$

$$\bar{\alpha}Z_{2r-1}(2|1; \kappa_1, \kappa_2) = Z_{2r}(1|1; \kappa_1, \kappa_2) - Z_{2r}(1|1; 0, \bar{\beta})$$

*Proof.* From corollary 5, for  $r = 1, 2, \dots$

$$\begin{aligned} Z_{2r}^a(1|1; \kappa_1, \kappa_2) &= \sum_{m=1}^r B_{2r-m-1, m-1} \sum_{i=1}^m \bar{\alpha}^i \bar{\beta}^{m-i} + \sum_{m=1}^r B_{2r-m-1, m-1} \bar{\beta}^m \\ &= \bar{\alpha} \sum_{m=0}^{r-1} B_{2r-m-2, m} \sum_{i=0}^m \bar{\alpha}^i \bar{\beta}^{m-i} + Z_{2r}(1|1; 0, \bar{\beta}) \end{aligned}$$

and the result follows from corollary 5 with  $t = 2r - 1, y = 2$ .  $\square$

Finally, substituting the identity

$$\frac{1}{(1-cz^2)(1-dz^2)} = 1 + \frac{(c+d)z^2 - cdz^4}{(1-cz^2)(1-dz^2)} \quad (6.4)$$

in (4.33) and using (4.16) leads to the recurrence, for  $t = 0, 1, 2, \dots$ , and odd  $t + y \geq 1$

$$Z_t(y|1; \kappa_1, \kappa_2) = B_{t+1, y} + (c+d)Z_t(y+2|1; \kappa_1, \kappa_2) - cdZ_t(y+4|1; \kappa_1, \kappa_2). \quad (6.5)$$

which relates partition functions along lines of constant  $t$  and may be initialised using the above relations to find the partition functions for  $y = 1, 2, 3$  and 4.

## 6.2 Recurrence relations for the correlation functions of the ASEP model.

The probability of finding particles at positions  $i_1, i_2, \dots, i_n$  is, using (1.1),

$$Pr(\tau_{i_1} = 1, \tau_{i_2} = 1, \dots, \tau_{i_n} = 1) = \frac{G_n(i_1, i_2, \dots, i_n; r)}{Z_{2r}(1|1; \bar{\alpha}\bar{\beta}, \kappa^2)} \quad (6.6)$$

where the un-normalised  $n$ -point correlation function  $G_n(i_1, i_2, \dots, i_n; r)$  is given by [1]

$$G_n(i_1, i_2, \dots, i_n; r) = \langle W | C^{i_1-1} D C^{i_2-i_1-1} D \dots C^{i_n-i_{n-1}-1} D C^{r-i_n} | V \rangle. \quad (6.7)$$

where  $C = DE$ . This expression may be thought of as replacing  $C$  by  $D$  in  $\langle W|C^r|V \rangle$  at each of the positions  $i_k, k = 1, \dots, n$  which is equivalent to replacing the  $E_j$  matrix in a  $C_j = D_j E_j$  product by a unit matrix. Thus in the path representations  $G_n(i_1, i_2, \dots, i_n; r)$  is obtained by modifying the allowed step definition such that for  $k = 1, \dots, n$ , the step  $s_k \equiv e_{2i_k}$  (beginning at  $x = 2i_k - 1$  and ending at  $x = 2i_k$ ) is always an up step and has weight 1. We will say that  $s_k$  is a *forced up step*. This is illustrated in figure 22.

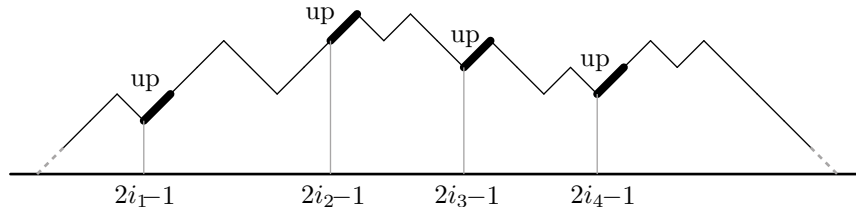


Figure 22: *The  $n$ -point correlation path interpretation ( $n = 4$  is shown above): each step starting at  $x = 2i_k - 1, k = 1, \dots, n$  must be an up step with weight one.*

### 6.2.1 The case $\alpha = \beta = 1$

In the case  $\alpha = \beta = 1$ , or  $\kappa_1 = \kappa_2 = 1$ , it is shown in [1] that

$$Z_{2r}(1|1; 1, 1) \equiv \langle W|C^r|V \rangle |_{\alpha=\beta=1} = C_{r+1}, \quad (6.8)$$

a Catalan number. This may also be seen in terms of the third path representation (corresponding to the third matrix representation of section 2).  $\langle W_3|(D_3 E_3)^r|V_3 \rangle |_{\alpha=\beta=1}$  is just the total number of One Up paths of length  $2r$  and these biject to Dyck paths of length  $2r + 2$  by adding an up step at the beginning and a down step at the end of each path (the dotted steps in figure 22). The result follows since number of Dyck paths of length  $2p$  is well known to be the Catalan number  $C_p$ . The result also follows from (4.29) by setting  $c = d = 0$  and noting that  $B_{2r+1,1} = C_{r+1}$  which is the  $m = 0$  term.

It is also shown in [1], equation (88) that

$$G_n(i_1, i_2, \dots, i_n; r) = \sum_{p_n=0}^{r-i_n} C_{p_n} G_{n-1}(i_1, i_2, \dots, i_{n-1}; r - p_n - 1) \quad (6.9)$$

This may be derived combinatorially as follows. Again we use the third path representation and to avoid special cases we imagine that the paths are extended to  $y = 0$  by a further down step.

For each path which contributes to  $G_n(i_1, i_2, \dots, i_n; r)$  we determine a subpath  $\omega_n$  which starts with the last forced up step,  $s_n$ , and ends when the path returns to the same height for the first time. This subpath contains a Bubble which we suppose has length  $2p_n$  so that the subpath has length  $2p_n + 2$ — see figure 23a. Immediate return corresponds to  $p_n = 0$  and the maximum value of  $p_n$  is determined by the condition,  $(2i_n - 1) + (2p_n + 2) = 2r + 1$ , that there are no further steps beyond  $\omega_n$ .

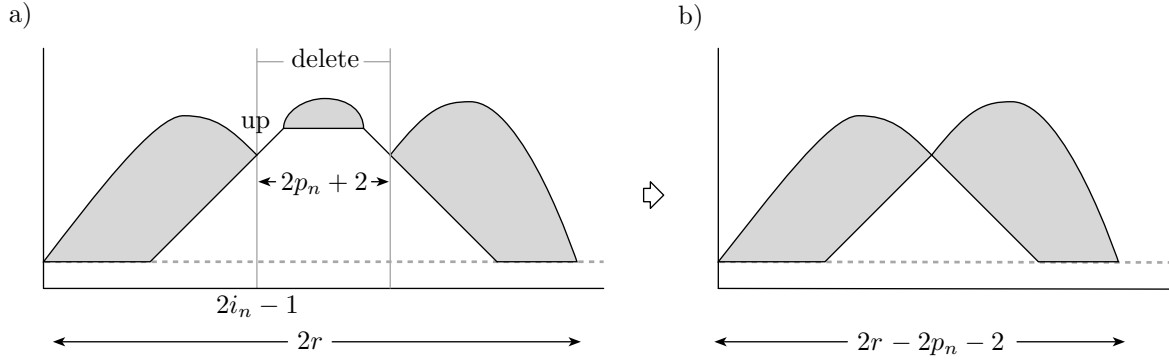


Figure 23: a) Schematic path picture for the correlation functions for  $\alpha = \beta = 1$ , showing the bubble following the last forced up step. b) Removing the bubble leaves a shorter Dyck path

We can now define a new path obtained by deleting  $\omega_n$  and joining the two (possibly empty) resulting sub-paths which remain. This path contributes to  $G_{n-1}(i_1, i_2, \dots, i_{n-1}; r - p_n - 1)$ . The result follows by partitioning the paths contributing to  $G_n$  according to the value of  $p_n$ . For a given value of  $p_n$  the number of configurations is therefore the product  $G_{n-1}$  and the number of configurations of  $\omega_n$  which is equal to the number,  $C_{p_n}$ , of Dyck paths of length  $2p_n$ .

For  $k = 1, 2, \dots, n$ , let

$$q_k = n - k + \sum_{j=k}^n p_j \quad (6.10)$$

then as pointed out in [1], equation (6.9) may be iterated or, combinatorially,  $n$  Bubbles may be removed, to give the explicit formula

$$G_n(i_1, i_2, \dots, i_n; r) = \sum_{p_1 \geq 0} \cdots \sum_{p_n \geq 0} C_{p_1} C_{p_2} \cdots C_{p_n} C_{r - q_1} \quad (6.11)$$

where the upper limits are  $p_k = r - i_k - q_{k+1}$  with  $q_{n+1} = 0$ .

This formula was previously conjectured by Derrida and Evans [19] on the basis of computer calculations up to  $r = 10$ .

### 6.2.2 General $\alpha$ and $\beta$ .

Equation (45) of [1] is the case  $n = 2$  of the following proposition which we now prove using a lattice path representation. An algebraic proof was given in [1].

**Proposition 4.** For  $1 \leq i_1 < i_2 < \dots < i_n \leq r - 1$

$$G_n(i_1, i_2, \dots, i_n; r) = \sum_{p=0}^{r-i_n-1} C_p G_{n-1}(i_1, i_2, \dots, i_{n-1}; r-p-1) + \bar{\beta} G_{n-1}(i_1, i_2, \dots, i_{n-1}; i_n - 1) \sum_{p=0}^{r-i_n} B_{2r-2i_n-p-1, p-1} \bar{\beta}^p \quad (6.12)$$

$$= \sum_{p=0}^{r-i_n-1} C_p G_{n-1}(i_1, i_2, \dots, i_{n-1}; r-p-1) + \bar{\beta} G_{n-1}(i_1, i_2, \dots, i_{n-1}; i_n - 1) R_{2(r-i_n)}(0; \bar{\beta}) \quad (6.13)$$

where  $R_t(h; \kappa)$  is the return polynomial for Ballot paths for which an explicit formula is given in (3.8).

$$(6.14)$$

*Notes:*

- When  $n = 1$ ,  $G_0(s)$  should be replaced by  $Z_{2s}(1|1; \bar{\alpha}\bar{\beta}, \kappa^2)$ . This yields [1] equation (43).
- The case  $i_n = r$  reduces to  $G_n(i_1, i_2, \dots, i_{n-1}, r; r) = \bar{\beta} G_{n-1}(i_1, i_2, \dots, i_{n-1}; r-1)$ , since the first sum is empty and  $R_0(0; \bar{\beta}) = 1$ . This reduces to [1] equation (44) when  $n = 1$ .
- When  $\alpha = \beta = 1$  this reduces to (6.9) since using

$$\sum_{p=0}^k B_{2k-p-1, p-1} = C_k$$

the second sum of (6.12) is just the missing term  $p = r - i_n$  of the first sum.

- The case  $n = 1$  was also considered by Schütz and Domany [16]. They solved the difference equations of [17] and used the solution to find a simple expression for the density gradient. Noting that their coefficient  $b_{NN}(r)$  is the Ballot number  $B_{N+r-1, N-r-1}$  it follows that equation (3.3) of [16] is equivalent to

$$G_1(i+1; r) - G_1(i; r) = (\bar{\alpha}\bar{\beta} - \bar{\alpha} - \bar{\beta}) R_{2i-1}(1; \bar{\alpha}) R_{2r-2i-1}(1; \bar{\beta}). \quad (6.15)$$

It may be seen that this expression follows immediately by noting a cancellation of the cross paths which occur in path representation two of  $G_1(i; r)$ .

- In constructing a proof it was found that the first representation in terms of Jump-Paths was simpler to use than the third which we used in the special case of the previous section. It was explained after definition 6 , that Jump-Step paths (representing  $Z_{2r}$ ) never intersect  $y = 0$ . However this is not the case in calculating  $G_n(i_1, i_2, \dots, i_n; r)$  since the paths are modified by the forced up steps. This allows  $y = 0$  to be visited and then a forced up step returns the path to  $y = 1$ . Notice that the down step leading to  $y = 0$  has a weight  $\bar{\beta}$ .

*Proof.* Partition the modified Jump-Step configurations according as the last forced up step,  $s_n$ , which starts at height  $y = 2k, k \geq 1$  (case A) or  $y = 0$  (case B) – see figure 24.

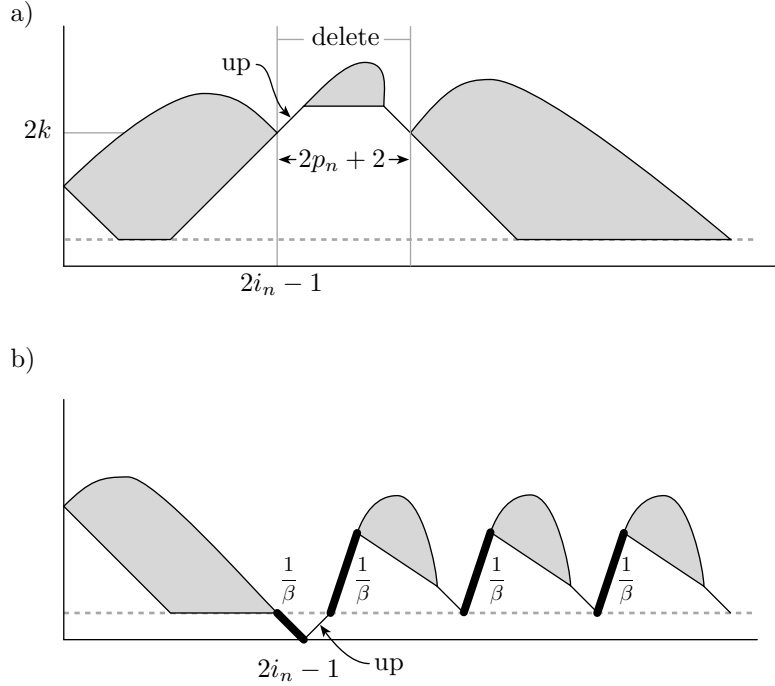


Figure 24: a) Case A: Last forced up step starts at  $y = 2k, k \geq 1$ . b) Case B: Last forced up step starts at  $y = 0$ . Note, the “bubbles” represent Jump Step subpaths.

**Case A:** A subpath  $\omega_n$  may be identified in exactly the same way as in the case  $\alpha = \beta = 1$  but in this case the Bubble of length  $2p$  obtained by deleting the first and last steps of  $\omega_n$  is not an elevated Dyck path since it may contain jump steps. However the number of configurations of  $\omega_n$  is still equal to  $C_p$  (see below) for any value of  $k$  and since  $\omega_n$  avoids  $y = 1$  its steps are unweighted. Thus on partitioning the paths according to the length  $(2p+2, p = 0, 1, \dots, r-i_n-1)$  of  $\omega_n$  a factor  $C_p$  may be removed from the sum over paths having the same value of  $p$ . When

$\omega_n$  is deleted the remaining steps form a weighted path of length  $2r - 2p - 2$  which has only  $n - 1$  forced up steps. Summing over configurations of this path for given  $k$  and then summing over  $k \geq 1$  gives  $G_{n-1}(i_1, i_2, \dots, i_{n-1}; r - p - 1)$ . The first term of the proposition formula is thus derived provided that we can show that the number of configurations of  $\omega_n$  is  $C_p$ .

Now the paths  $\omega_n$  of length  $2p + 2$  biject to  $P_{2p;1}^{(J)}$  by vertical translation through distance  $2k$  and from (2.10) and (2.13)

$$|P_{2p;1}^{(J)}| = Z_{2p}^{(1)}|_{\bar{\alpha}=0, \bar{\beta}=1} = Z_{2p}^{(3)}|_{\kappa_1=0, \kappa_2=1} = C_p. \quad (6.16)$$

The last equality follows since when  $\kappa_1 = 0$  the One Up paths which visit  $y = 0$  have zero weight and the remaining paths, which have weight 1 when  $\kappa_2 = 1$ , biject to Dyck paths by vertical translation through unit distance. This result is also proved combinatorially in [11].

**Case B:** The weighted sum over paths may be factorised into three parts.

- (i) A factor which arises from the subpath consisting of the first  $2i_n - 2$  steps. The subpath ends at  $y = 1$  and has only  $n - 1$  forced up steps, therefore the sum over these subpaths yields  $G_{n-1}(i_1, i_2, \dots, i_{n-1}; i_n - 1)$ .
- (ii) A factor  $\bar{\beta}$  which arises from the next two steps which visit  $y = 0$  and return to  $y = 1$ . The second of these is the last forced up step  $s_n$  having weight 1.
- (iii) A factor arising from the subpath consisting of the remaining  $2r - 2i_n$  steps which is a jump step path beginning and ending at  $y = 1$ , avoiding  $y = 0$ . The weighted sum over these paths is obtained by setting  $\bar{\alpha} = 0$  in the normalising factor for paths of length  $2r - 2i_n$ , thus using corollary 5

$$Z_{2r-2i_n}|_{\bar{\alpha}=0} = \sum_{m=1}^{r-i_n} B_{2r-2i_n-m-1, m-1} \bar{\beta}^m.$$

The product of these three factors yields the second term of the formula. □

## 7 Non-intersecting paths and the stationary state probability distribution: $\alpha = \beta = 1$

In this section we show, for the case  $\alpha = \beta = 1$ , that the probability of finding the system in a particular state  $\vec{r}$  is related to the combinatorial problem of enumerating the number of

configurations of a particular type of pairs of non-intersecting paths. If we do not specify the exact state, but only the number of particles, then the probability of finding the system in a stationary state with a fixed number of particles (in any position) is given by a simple determinant.

Most paths in this section will not be restricted to the upper half plane, such paths are called binomial paths. More precisely we have the following definition.

**Definition 1 (Binomial path).** *Let  $\omega = (u_0, u_1, \dots, u_t)$  where  $u_i \equiv (x_i, y_i) \in \mathbb{Z} \times \mathbb{Z}$  with  $u_i - u_{i-1} \in \{(1, -1), (1, 1)\}$ . The path  $\omega$  is called a binomial path.*

The number of binomial paths such that  $u_0 = (0, 0)$ ,  $u_t = (t, y)$  is given by

$$|\{\omega\}| = \begin{cases} \binom{t}{(t+y)/2} & \text{if } t + y \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \quad (7.1)$$

We will be interested in binomial paths defined by the state of the system. To this end we define a “state path” as follows.

**Definition 2 (State path).** *Let  $\vec{\tau}$  be the ASEP state on a line segment with  $r$  sites. A state path,  $\omega(\vec{\tau}) = (v_0, v_1, \dots, v_r)$ , is a particular binomial path where*

$$v_i - v_{i-1} = \begin{cases} (1, 1) & \text{if } \tau_i = 0, \\ (1, -1) & \text{if } \tau_i = 1. \end{cases} \quad (7.2)$$

Thus a particle contributes a down step and a vacancy contributes an up step. An example of a state path is shown in figure 25a). We will also need the following definition.

**Definition 3 (Non-intersecting path pair).** *Let  $\omega_1 = (u_0, u_1, \dots, u_t)$  and  $\omega_2 = (v_0, v_1, \dots, v_t)$  be two  $t$  length binomial paths with  $\omega_1$  starting at  $(0, 0)$  and ending at  $(t, y_{f_1})$  and  $\omega_2$  starting at  $(0, 2)$  and ending at  $(t, y_{f_2})$ . If there are no vertices in common between the two paths then they are said to be a non-intersecting path pair.*

An example of a pair of non-intersecting paths is shown in figure 25b). For convenience, if  $v_i - v_{i-1} = (1, 1)$  we will represent the up step by “U” and if  $v_i - v_{i-1} = (1, -1)$  we will represent the down step by “D”. An example of a state path and the corresponding “UD” word is shown in figure 25a).



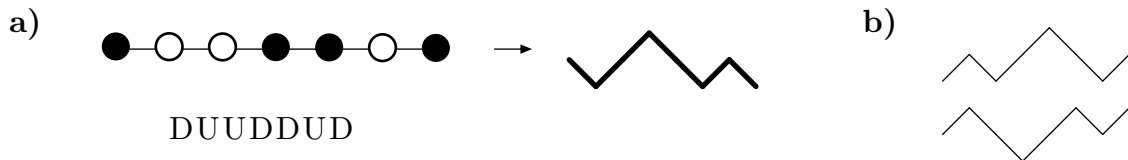


Figure 25: a) A state path and UD word,  $DUUDDUD$  corresponding to the state,  $\vec{\tau} = (1, 0, 0, 1, 1, 0, 1)$ . b) A pair of non-intersecting square lattice paths of length 7.

We now have the following proposition giving the probability of finding the system in some state.

**Proposition 5.** *Let  $\alpha = \beta = 1$  then, in the stationary state of the ASEP, the probability,  $P_r(\vec{\tau})$  of finding the system of  $r$  sites in configuration  $\vec{\tau}$  having  $k$  particles is given by*

$$P_r(\vec{\tau}) = |\{(\omega_1, \omega_2)\}| / C_{r+1} \quad (7.3)$$

where  $\{(\omega_1, \omega_2)\}$  is the set of all pairs of non-intersecting paths where  $\omega_2$  is any binomial path (with no vertices in common with  $\omega_1$ ) that ends at  $(r, 2r - k + 2)$ , and  $\omega_1$  is the state path corresponding to state  $\vec{\tau}$ .  $C_{r+1}$  is a Catalan number.

An example of the two non-intersecting path configurations giving rise to the numerator of  $P_3((0, 1, 0)) = 2/14$  is shown in in figure 26d).

*Proof.* As noted in section 6.2 the probability of finding particles at positions  $i_1, i_2, \dots, i_n$  is determined by enumerating weighted one-up paths where the steps  $s_k \equiv e_{2i_k}$  must be up steps with weight one. Similarly, if we require there to be *no* particle at position  $i$  (ie.  $\tau_i = 0$ ), then in the weighted one-up path the step ending at  $2i - 1$  must be a down step (ie.  $e_{2i-1}$  must be a down step) with weight one. Since, we only consider the case  $\alpha = \beta = 1$  all the weights are unity.

These up and down step constraints are conveniently represented by a sequence of characters as illustrated in figure 26a). A particle contributes the pair “-U” and a vacancy the pair “D-”, where the dash represents an up or down step (all possibilities consistent with the one-up path constraint). An example of one choice of up and down steps for the dashes, and the corresponding one-up path, is shown in 26b). The constrained steps are shown in bold. From any sequence of U’s and D’s representing a path we can construct a pair of paths  $(\omega_1, \omega_2)$ . This process is shown in 26c).  $\omega_1$  is defined to start at  $(0, 0)$  and consist of the sub-sequence of constrained steps with up steps replaced by down steps and vice versa; this is just the state path  $\omega(\vec{\tau})$ .  $\omega_2$  is defined to

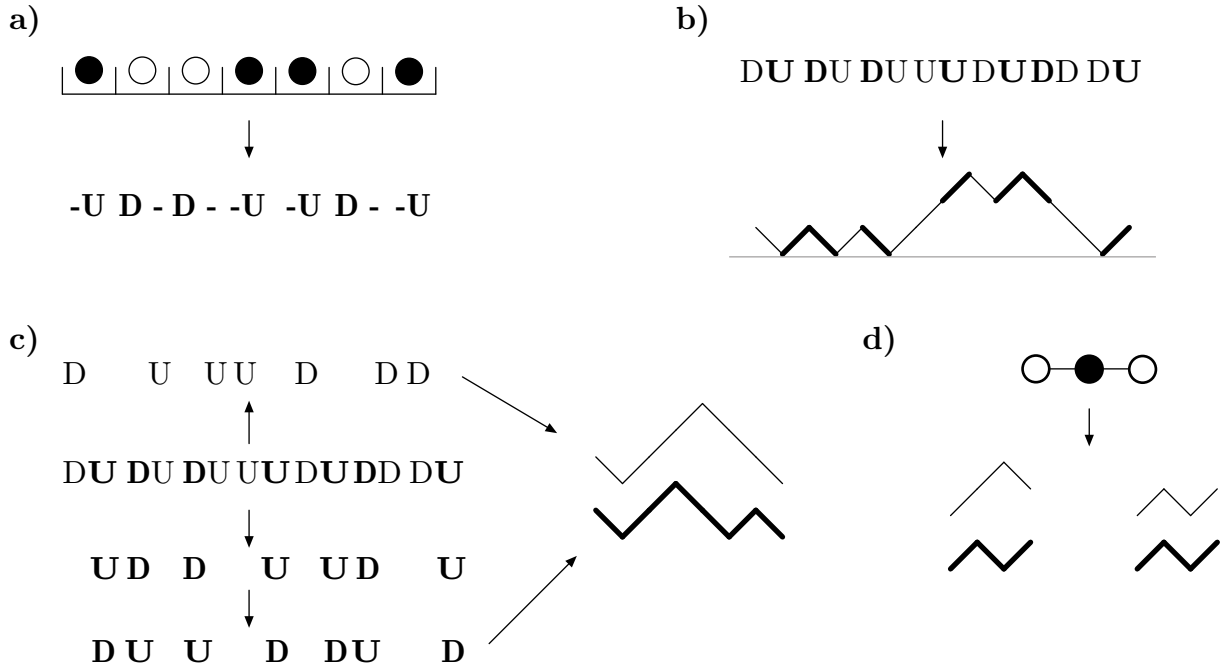


Figure 26: a) The correspondence with a particle configuration and a set of one-up lattice paths. The bold letter "U" and "D" are fixed steps, the dash represents an arbitrary step. b) An example of one possible choice of the "dashes" and the corresponding one-up path. c) The equivalent non-intersecting configuration of paths. The sub-sequence of bold (fixed) steps fixes the lower state path (note the "U" and "D"'s are interchanged first) and the subsequence of plain characters determine the upper path. d) The two non-intersecting path configurations giving rise to the numerator for  $P_3((0, 1, 0)) = 2/14$

start at  $(0, 2)$  and consist of the subsequence of unconstrained steps (arising from the "dashes"). We will refer to  $\omega_2$  as the "upper" path.

It now remains to prove that the path pair  $(\omega_1, \omega_2)$  is non-intersecting. Clearly the length of each path is  $r$ . The ending height of the state path is the difference between the number of vacancies (which contribute an up step) and the number of particles (which contribute a down step),  $y = (r - k) - k = r - 2k$ . Since the initial path is a one-up path the number of U's and D's must be equal and hence the difference between the number of unconstrained up and down steps is also  $r - 2k$ . Since we start the upper path at  $y = 2$ , it will end at  $y = r - 2k + 2$ . The non-intersection is proved as follows. Consider the case of a one-up Dyck path with a single constrained up step in some given column. Referring to figure 27 this up step is between A and B. Since it is a one-up path, it does not go below  $y = 0$ . Now, delete the up step, AB and

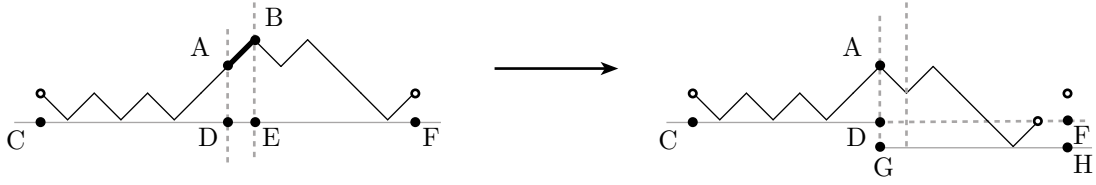


Figure 27: A one-up path with a single constrained up step, showing the process of deleting it and moving the “floor” down to  $y = -1$

shorten the path. The new path clearly does not go below CD and also does not go below GH (rather than DF), is one step shorter, and ends at  $(2r - 1, 0)$ .

Now, returning to the original one-up path (with half the steps constrained), we repeat, moving left to right, the process of deleting the constrained steps. If the deleted step is up, then the “floor” from that  $x$  coordinate on, moves *down* one unit (as in the GH of figure 27) and if the constrained step is down the floor, from that  $x$  coordinate on, moves *up* on unit. This deletion process is equivalent to replacing successive pairs of steps of the original one-up path by single steps (the “dashed” step) and simultaneously adjusting the floor as shown in figure 28. The new floor defines a second path, made of a sequence of,  $\lrcorner$  and  $\llcorner$  “steps”. The important point is that at the completion of this process the path (with half the steps removed) never steps below (ie. no edges below) the new floor. Thus if we push the upper path up one unit (so it starts at  $(0, 2)$ ) it will have no vertices in common with the new floor. Finally, replace a  $\llcorner$  pair with a single up step and  $\lrcorner$  with a single down step – see 27i). The resulting path is the precisely the state path and the upper path is a binomial path – see 27j) – with no vertices in common with the state path.  $\square$

**Proposition 6.** *Let  $\alpha = \beta = 1$  then, in the stationary state of the ASEP, the probability,  $P_{r,k}$ , of finding  $k$  particles is given by the determinant.*

$$P_{r,k} = \sum_{\vec{\tau}: k(\vec{\tau})=k} P_r(\vec{\tau}) = \frac{1}{C_{r+1}} \det \begin{vmatrix} \binom{r}{k} & \binom{r}{k-1} \\ \binom{r}{k+1} & \binom{r}{k} \end{vmatrix} = \frac{1}{(r+1)C_{r+1}} \binom{r+1}{k} \binom{r+1}{r-k}. \quad (7.4)$$

where the sum is over all states such that the number of particles in state  $\vec{\tau}$ ,  $k(\vec{\tau}) = k$ .

*Proof.* Summing over all possible positions of the  $k$  particles is equivalent to summing over all possible state paths that end at  $(r, r - 2k)$ . For each possible state path we are also summing over all possible positions of  $\omega_1$ , ending at  $(r, r - 2k + 2)$  which do not intersect the state path. This double sum is thus equivalent to summing over all non-intersecting path pairs where  $\omega_1$  and  $\omega_2$

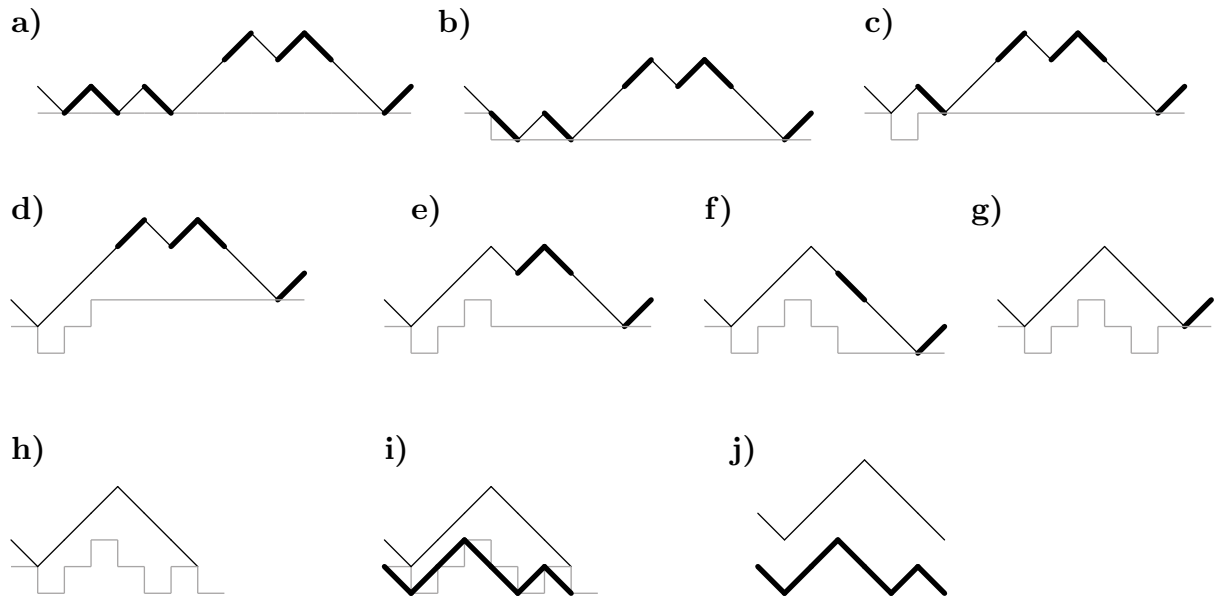


Figure 28: *a) – h) showing the action of repeatedly deleting the constrained up and down steps (shown bold) i) The horizontal and vertical steps of the new floor are replaced by up and down steps (shown bold). j) The floor steps have been removed and the upper path has been moved up one unit so it starts at  $(0, 2)$  giving a non-intersecting pair of binomial paths – the lower path is the state path*

are binomial paths. The number of such non-intersecting paths is given by the Gessel-Viennot determinant [21]. □

## 8 Conclusion

We have shown that the normalisation of the ASEP can be interpreted as various lattice path problems. The lattice path problems can then be solved using the constant term method (CTM). The combinatorial nature of the CTM enables us to interpret the coefficients of the normalisation polynomials as various un-weighted lattice path problems – usually as Ballot paths. One particular form has a natural interpretation as an equilibrium polymer chain adsorption model. The “ $\omega$ ” form of the normalisation is particularly suited to finding the asymptotic expansion of normalisation and hence the phase diagram. We also formulate a combinatorial interpretation of the correlation functions.

The lattice path interpretations enable us to make connections with many other models. In particular, because of the strong combinatorial nature of the CTM we are able to find a new

“canonical” lattice path representation. In a further paper [20] we show that this representation leads to an understanding of a non-equilibrium model (the ASEP model) in terms of a related equilibrium polymer model. Also, having extended the polymer chain model so that the end-points have arbitrary displacements from the surface the method of Gessel and Viennot [21], [22] (see also [23]) may be used to express the partition function of a network of non-intersecting paths as a determinant. In particular, the case of two paths gives the partition function for a vesicle model with a two parameter interaction with a surface. A bijection between these vesicles and compact percolation clusters [10] then enables an analysis of the properties of the clusters attached to damp wall to be made. These applications will also be the subject of a subsequent publication.

It is also of combinatorial interest to understand how the various path problems might be related. Clearly they are related algebraically as they are all just representations of the same algebra (and thus related by different similarity transformations). In a subsequent paper, [11], we show combinatorially (using bijections and involutions) how the various path representations are combinatorially equivalent.

## 9 Acknowledgements

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# APPENDIX

## A Specimen partition functions.

$$Z_0(1|1; \kappa_1, \kappa_2) = 1 \quad (\text{A.1})$$

$$Z_2(1|1; \kappa_1, \kappa_2) = \kappa_1 + \kappa_2 \quad (\text{A.2})$$

$$Z_4(1|1; \kappa_1, \kappa_2) = \kappa_1^2 + \kappa_2 + 2 \kappa_1 \kappa_2 + \kappa_2^2 \quad (\text{A.3})$$

$$Z_6(1|1; \kappa_1, \kappa_2) = \kappa_1^3 + 2 \kappa_2 + 2 \kappa_1 \kappa_2 + 3 \kappa_1^2 \kappa_2 + 2 \kappa_2^2 + 3 \kappa_1 \kappa_2^2 + \kappa_2^3 \quad (\text{A.4})$$

$$\begin{aligned} Z_8(1|1; \kappa_1, \kappa_2) = & \kappa_1^4 + 5 \kappa_2 + 4 \kappa_1 \kappa_2 + 3 \kappa_1^2 \kappa_2 \\ & + 4 \kappa_1^3 \kappa_2 + 5 \kappa_2^2 + 6 \kappa_1 \kappa_2^2 + 6 \kappa_1^2 \kappa_2^2 + 3 \kappa_2^3 + 4 \kappa_1 \kappa_2^3 + \kappa_2^4 \end{aligned} \quad (\text{A.5})$$

$$Z_0(3|1; \kappa_1, \kappa_2) = 0 \quad (\text{A.6})$$

$$Z_2(3|1; \kappa_1, \kappa_2) = 1 \quad (\text{A.7})$$

$$Z_4(3|1; \kappa_1, \kappa_2) = 2 + \kappa_1 + \kappa_2 \quad (\text{A.8})$$

$$Z_6(3|1; \kappa_1, \kappa_2) = 5 + 2 \kappa_1 + \kappa_1^2 + 3 \kappa_2 + 2 \kappa_1 \kappa_2 + \kappa_2^2 \quad (\text{A.9})$$

$$\begin{aligned} Z_8(3|1; \kappa_1, \kappa_2) = & 14 + 5 \kappa_1 + 2 \kappa_1^2 + \kappa_1^3 + 9 \kappa_2 + 6 \kappa_1 \kappa_2 + 3 \kappa_1^2 \kappa_2 + 4 \kappa_2^2 \\ & + 3 \kappa_1 \kappa_2^2 + \kappa_2^3 \end{aligned} \quad (\text{A.10})$$

$$Z_t(1|3; \kappa_1, \kappa_2) = \kappa_2 Z_t(3|1; \kappa_1, \kappa_2) \quad (\text{A.11})$$

$$Z_0(3|3; \kappa_1, \kappa_2) = 1 \quad (\text{A.12})$$

$$Z_2(3|3; \kappa_1, \kappa_2) = 2 \quad (\text{A.13})$$

$$Z_4(3|3; \kappa_1, \kappa_2) = 5 + \kappa_2 \quad (\text{A.14})$$

$$Z_6(3|3; \kappa_1, \kappa_2) = 14 + 4 \kappa_2 + \kappa_1 \kappa_2 + \kappa_2^2 \quad (\text{A.15})$$

$$Z_8(3|3; \kappa_1, \kappa_2) = 42 + 14 \kappa_2 + 4 \kappa_1 \kappa_2 + \kappa_1^2 \kappa_2 + 5 \kappa_2^2 + 2 \kappa_1 \kappa_2^2 + \kappa_2^3 \quad (\text{A.16})$$

## B Constructive proof of proposition 2.

The formula

$$Z_t(y|y^i; \kappa_1, \kappa_2) = CT[\Lambda^t z^y (\bar{z}^{y^i} + U(z) z^{y^i})]$$

clearly satisfies the general equation (4.7) and the initial condition (4.4) provided that

$$CT[z^{y+y^i}U(z)] = 0. \quad (\text{B.1})$$

In order to satisfy the boundary condition (4.11)

$$CT[\Lambda^t z(\bar{z}^{y^i} + U(z)z^{y^i})] = \kappa_1 CT[\Lambda^{t-2} z(\bar{z}^{y^i} + U(z)z^{y^i})] + \kappa_2 CT[\Lambda^{t-1} z^2(z^{y^i} + U(z)z^{y^i})]$$

or

$$CT[\Lambda^{t-2} \bar{z}^{y^i-1}(\Lambda^2 - \kappa_1 - \kappa_2 z \Lambda)] = -CT[\Lambda^{t-2} V(z)z^{y^i-1}]$$

where

$$V(z) = (1 - (\bar{\kappa}_1 + \bar{\kappa}_2)z^2 - \bar{\kappa}_2 z^4)U(z).$$

Replacing  $z$  by  $\bar{z}$  everywhere under the CT operation leaves the value unchanged so

$$V(z) = -(\Lambda^2 - \kappa_1 - \Lambda \bar{z} \kappa_2) = \bar{z}^2(\kappa_2 - 1 + (\bar{\kappa}_1 + \bar{\kappa}_2)z^2 - z^4)$$

and (B.1) is satisfied provided that  $y + y^i > 2$  or  $y^i > 1$  since  $y \geq 1$ . Hence in this case  $y^i > 1$

$$U(z) = \bar{z}^2(-1 + \kappa_2 G(z)).$$

When  $y^i = 1$  the boundary condition is satisfied if

$$CT[\Lambda^{t-2}(\Lambda^2 - \kappa_1 - \kappa_2 z \Lambda)] \equiv CT[\Lambda^{t-2}(\bar{z}^2 - 2 + z^2 - \kappa_1 - \kappa_2(1 + z^2))] = -CT[\Lambda^{t-2}V(z)]$$

and replacing  $\bar{z}^2$  by  $z^2$  in the second expression gives

$$V(z) = \bar{\kappa}_1 + \bar{\kappa}_2 + \bar{\kappa}_2 z^2 - z^2$$

which satisfies (B.1) and hence

$$U(z) = \bar{z}^2(-1 + G(z)).$$

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