Directed Compact Percolation near a wall: III. Exact results for the mean length and number of contacts.

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Abstract

Existing exact results for the percolation probability and mean cluster size for compact percolation near a dry wall are extended to the mean cluster length and the mean number of contacts with the wall. The results are derived from our previous work on vesicles near an attractive wall and involve elliptic integrals as opposed to the simple rational forms found for the percolation probability and cluster size below p_c. The results for the cluster length satisfy previously conjectured differential equations. A closed expression is conjectured for the mean size above p_c in terms of a hypergeometric function.

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1 Background and notation.

Directed compact percolation near a wall which restricts the lateral growth of the cluster has been described in I of this series [6] where earlier work is referred to and a picture of a typical cluster may be found. The atoms of a cluster occupy the sites of a directed square lattice which are the points of the t, x plane such that $t \geq 0$, $x \geq 0$ and $t + x$ even. The two bonds which are directed away from (t, x) connect to the sites $(t + 1, x \pm 1)$. The wall is represented by the sites $x = -1$ and t odd and here we consider the dry wall problem in which these sites are unoccupied. The special case of a random cluster which grows from a seed consisting of a single atom at $(0, 0)$ will be considered. The growth rule is that site (t, x) is occupied (or wet) with certainty if both its predecessors $(t-1, x \pm 1)$ are occupied, with probability p if just one of these sites is occupied and is otherwise unoccupied. Thus the clusters remain compact or free from the holes which would occur in ordinary directed percolation clusters.

At a given growth stage, in the absence of a wall, the width of the cluster increases by one with probability p^2, decreases with probability $(1 - p)^2$ or stays the same in two ways each with probability $p(1 - p)$. Since the cluster terminates with a single atom the number of stages in which the width increases must be the same as the number of stages in which it decreases so that the probability of occurrence of a given cluster having t growth stages before termination is $q^2(pq)^t$, where $q = 1-p$. The factor q^2 is the probability that the cluster having reached unit width finally terminates. The probability that t growth stages will take place before termination is therefore simply obtained by multiplying the above probability by the number of different clusters for which only a single atom is added at the t^{th} stage. This is how Domany and Kinzel [4] solved the bulk problem and obtained the scaling form of the cluster length distribution.

In the presence of a wall the probability associated with a given cluster must be modified depending on how many atoms are in contact with the wall. If the site $(t, 0)$ is occupied then with no wall the site $(t + 1, -1)$ would be unoccupied with probability q but with a dry wall this happens with probability one. Thus the probability of occurrence of a given cluster with t growth stages and c wall contacts, not counting the seed, is $q(pq)^t q^{-c}$. To determine the cluster length distribution we must therefore count the number of such clusters. Until recently [2] this had not been done but some progress had been made using difference equations. Lin [7] determined the percolation probability and found that the critical exponent changed to $\beta = 2$ instead of the bulk value $\beta = 1$. In I it was shown that this was only the case for unbiased growth and that a bias towards or away from the wall resulted in the bulk exponent. In II [5] the mean cluster size below the critical probability $p_c = \frac{1}{2}$ was shown to be a simple rational function with exponent $\gamma = 1$ instead of the bulk value $\gamma = 2$ thereby showing that the scaling size exponent $\Delta = 3$ is unchanged by the presence of a wall. It was also conjectured on the basis of differential approximants that above p_c the mean size of finite clusters satisfies a linear second order inhomogeneous differential equation with polynomial coefficients of degree four. No closed form solution was obtained but the critical behaviour was determined from the singular points of the differential equation. In the case of the mean cluster length no closed form solution was found for any value of p but it was conjectured to satisfy a differential equation similar in nature to the mean size with an inhomogeneous term depending on the position of p in relation to p_c. From the singular points the critical behaviour of the mean length was determined and found to have a logarithmic divergence at p_c. The exponent $\tau = 0$, compared with $\tau = 1$ in the bulk, therefore changes in such a way as to preserve the value of the length scale exponent $\nu_{\|} = 2$.

In this paper we obtain an exact closed form expression for the mean cluster length and also for the expected number of wall contacts (for finite clusters in the case $p > p_c$). These expressions involve complete elliptic integrals as in the case of the two-dimensional Ising model specific heat. The logarithmic divergence of the mean length is confirmed and a conjecture [II] concerning the constants involved in the asymptotic form is proven. The mean number of contacts is found to be non-divergent but has an infinite derivative at p_c just before its maximum value occurs. This behaviour is reminiscent of the susceptibility of the antiferromagnetic Ising model.

Domany and Kinzel [4] enumerated the number of clusters with t growth stages in the bulk by noting a correspondence with random walks on the dual lattice. For a cluster which terminates at (t, x), one walk starts at $(0, 1)$ and follows the motion of the upper edge of the cluster and terminates at $(t, x + 1)$. The other starts at $(0, -1)$ and follows the motion of the lower edge arriving at $(t, x - 1)$. The number of such pairs of directed parallel non-intersecting walks was known [3]. By extending the walks to start at $(-1, 0)$ and terminate at $(t + 1, x)$ a closed "staircase polygon" is formed which can be used as a simple "vesicle model". The problem is therefore one of enumerating vesicles on a directed square lattice. In the presence of a wall the number of wall contacts made by the cluster is the number of times the lower walk of the surrounding vesicle revisits $x = -1$. In [2] the problem of a vesicle in contact with an attractive wall was solved. In this problem a partition function $V_t(\kappa)$ is calculated as the weighted sum over all vesicle configurations $\mathcal{V}(t)$ formed from t-step walks, giving weight κ^c to a configuration the lower walk of which revisits the wall c times.

$$V_t(\kappa) := \sum_{v \in \mathcal{V}(t)} \kappa^{c(v)} \tag{1}$$

where $c(v)$ is the number of revisits made by the vesicle v which is also the number wall contacts made by the cluster which it bounds excluding the source. The variable κ is known as the wall fugacity and is found to have a critical value $\kappa_c = 2$ above which the vesicle sticks to the wall.

The required properties of compact clusters may be obtained from the vesicle grand partition function

$$Z(u,\kappa) := \sum_{t=0}^{\infty} V_t(\kappa) u^t \tag{2}$$

where u is known as the length fugacity. The probability $Q(p)$ that a compact cluster will be of finite length is given by $qZ(pq, 1/q)$ from which the percolation probability $P(p) = 1 - Q(p)$ may be derived. Thus compact percolation is a special case of the vesicle problem such that as p varies we move along the "percolation line"

$$u = (\kappa - 1)/\kappa^2. \tag{3}$$

The value of u corresponding to the critical wall fugacity is $u_c = \frac{1}{4}$ and since $u = p(1-p)$ this corresponds to $p_c = \frac{1}{2}$. Notice that u as a function of p achieves its maximum value at p_c so that for compact percolation $0 \leq u \leq \frac{1}{4}$. The expected length and number of wall contacts for compact clusters are determined by the derivatives of Z with respect to u and κ respectively and hence to determine these functions it is necessary to know the grand partition function off the percolation line.

We begin by recalling the results of [2] for vesicles near an attractive wall. In the next section the wall will be moved to $x = 0$ so that the walks start at $(0,0)$ and $(0,2)$ and weight κ is given to each return of the lower walk to $x = 0$.

2 The Vesicle Partition Function.

In this section we summarise the derivation of the vesicle partition function $V_t(\kappa)$ given in [2]. Since the result depends on the single chain partition function we begin with its derivation.

2.1 The single chain partition function.

Let $U_t(x,\kappa)$ be the partition for a single chain of length t with one end attached to the surface and the other fixed at distance x from the surface. In the partition sum a weight κ^c is given to a configuration in which the chain makes c contacts with the surface other than the point of attachment. $U_t(x,\kappa)$ is the solution of the following equations

$$\begin{aligned} U_t(x) &= U_{t-1}(x-1)+U_{t-1}(x+1) \quad \text{for} \quad x,t>0 \\ U_t(0) &= \kappa U_{t-1}(1) \quad \text{for} \quad t>0 \\ U_0(x) &= \delta_{x,0} \quad \text{for} \quad x\geq 0 \end{aligned} \tag{4}$$

and may be written in the form

$$U_t(x,\kappa) = CT[\Lambda^t z^x G(z)] \tag{5}$$

where $\Lambda = z + z^{-1}$ and

$$G(z) = \frac{1-z^2}{1-(\kappa-1)z^2}. \tag{6}$$

The *constant term* notation $CT[\cdot]$ means that the contents of the bracket is to be expanded about $z=0$ in powers of z (including negative powers) and the coefficient of z^0 selected. Equation (5) was derived in [2] but may be verified by substitution in the defining equations above.

2.2 The partition function for two chains.

The vesicle partition function is a special case of the two-chain partition function $U_t(x_1,x_2,\kappa)$ where the chains start at $x=0$ and $x=2$ and end at x_1 and x_2 after t steps. The chains are constrained to avoid one another so that only the first chain can contact the surface and a configuration with c contacts, excluding the first, is given weight κ^c as above. The equations to be solved are

$$\begin{aligned} U_t(x_1,x_2) &= U_{t-1}(x_1-1,x_2-1)+U_{t-1}(x_1+1,x_2-1)+ \\ &\quad U_{t-1}(x_1-1,x_2+1)+U_{t-1}(x_1+1,x_2+1) \quad \text{for} \quad t,x_1,x_2>0 \\ U_t(0,x_2) &= \kappa(U_{t-1}(1,x_2-1)+U_{t-1}(1,x_2+1)) \quad \text{for} \quad t,x_2>0 \\ U_t(x_1,x_1) &= 0 \quad \text{for} \quad t>0 \\ U_0(x_1,x_2) &= \delta_{x_1,0}\delta_{x_2,2} \quad \text{for} \quad x_1\geq 0, x_2\geq 2. \end{aligned} \tag{7}$$

It may be verified by substitution, using equations (4), that the solution is expressible in terms of the one-chain function as follows.

$$U_t(x_1,x_2,\kappa) = U_t(x_1,\kappa)U_{t+2}(x_2,\kappa) - U_t(x_2,\kappa)U_{t+2}(x_1,\kappa) \tag{8}$$

and using (5) gives the constant term formula

$$U_t(x_1,x_2,\kappa) = CT[(\Lambda_1\Lambda_2)^t z_1^{x_1} z_2^{x_2}(\Lambda_2^2-\Lambda_1^2)G(z_1)G(z_2)] \tag{9}$$

where $\Lambda_i = z_i + z_i^{-1}$.

2.3 The vesicle partition function.

The vesicle partition function $V_t(x, \kappa)$ restricted to configurations ending at $x_1 = x, x_2 = x + 2$ is given by $U_t(x, x + 2, \kappa)$. Expanding $G(z_1)$ and $G(z_2)$ about $\kappa = 1$ it was shown in [2] (equation 4.16) that

$$V_t(x, \kappa) = \sum_{n=0}^{\infty} (\kappa - 1)^n U_t(x, x + 2n + 2, 1). \tag{10}$$

For given t this is a polynomial in κ since $U_t(x, x + 2n + 2, 1)$ is zero for $n > \frac{1}{2}(t - x)$. Substituting from (9), noting that $G(z) = 1 - z^2$ when $\kappa = 1$, and carrying out the sum over n gives

$$V_t(x, \kappa) = CT[(\Lambda_1\Lambda_2)^t (z_1 z_2)^x z_2^2 (\Lambda_2^2 - \Lambda_1^2)(1 - z_1^2) G(z_2)]. \tag{11}$$

Going from (9) to (11) is a major step forward since there is now only one denominator and κ appears in just one factor instead of two. For even $t = 2r$, further summation over even $x \geq 0$ gives the unrestricted vesicle partition function $v_r^{even}(\kappa) := V_{2r}(\kappa)$

$$\begin{aligned} v_r^{even}(\kappa) &= CT[(\Lambda_1\Lambda_2)^{2r} \frac{z_2^2(\Lambda_2^2 - \Lambda_1^2)}{1 - z_1^2 z_2^2}(1 - z_1^2)G(z_2)] \\ &= CT[(\Lambda_1\Lambda_2)^{2r} z_1^{-2}(z_1^2 - z_2^2)(1 - z_1^2)G(z_2)] \end{aligned} \tag{12}$$

Separating the terms involving z_1 and z_2 gives

$$v_r^{even}(\kappa) = U_{2r}(0, \kappa) CT[\Lambda_1^{2r}(1 - z_1^2)] + U_{2r}(2, \kappa) CT[\Lambda_1^{2r}(1 - z_1^{-2})] \tag{13}$$

$$= C_r(U_{2r}(0, \kappa) + U_{2r}(2, \kappa)) \tag{14}$$

$$= C_r U_{2r+1}(1, \kappa) = C_r U_{2r+2}(0, \kappa)/\kappa \tag{15}$$

where C_r is the Catalan number

$$C_r = \frac{1}{r+1}\binom{2r}{r} \tag{16}$$

and in the last two steps we have used (4). For vesicles of odd length let $v_r^{odd}(\kappa) := V_{2r+1}(\kappa)$ and summing over odd values of x instead of even gives an additional factor of $z_1 z_2$ with the result

$$v_r^{odd}(\kappa) = C_{r+1} U_{2r+2}(0, \kappa)/\kappa. \tag{17}$$

2.4 The ω expansion

In our analysis of the compact percolation problem it will be useful to have $V_t(\kappa)$ expressed in powers of $\omega = (\kappa - 1)/\kappa^2$. This may be achieved by first noting the identity

$$(\kappa - 1)\Lambda^2 - \kappa^2 = (1 - (\kappa - 1)z^2)((\kappa - 1)z^{-2} - 1) \tag{18}$$

which on using (5) leads to the difference equation

$$\begin{aligned}(\kappa-1)U_{2r+2}(0,\kappa)-\kappa^2U_{2r}(0,\kappa) &= -\kappa CT[\Lambda^t(1-z^{-2})]+CT[\Lambda^t(z^2-z^{-2})] \\ &= -\kappa C_r. \end{aligned} \tag{19}$$

Defining $S_r(\kappa) := U_{2r+2}(0,\kappa)/\kappa$ the vesicle partition is determined by

$$v_r^{even}(\kappa)/C_r = v_r^{odd}(\kappa)/C_{r+1} = S_r(\kappa) \tag{20}$$

where $S_r(\kappa)$ satisfies the difference equation

$$(\kappa-1)S_r - \kappa^2 S_{r-1} = -C_r \tag{21}$$

which, for $\kappa \neq 1$, has the solution

$$S_r(\kappa) = \omega^{-r}\left(1-\frac{1}{\kappa-1}\sum_{s=1}^{r} C_s\omega^s\right). \tag{22}$$

The following alternative form of $S_r(\kappa)$ will be used in the application to compact percolation. For $\omega \leq \frac{1}{4}$

$$\begin{aligned}\sum_{s=1}^{\infty} C_s\omega^s &= \frac{1}{2\omega}-1-\frac{\sqrt{1-4\omega}}{2\omega} \\ &= \frac{1}{2\omega}-1\pm\left(\kappa-\frac{1}{2\omega}\right) \qquad (23) \\ &= \begin{cases} \kappa-1 & \kappa\leq 2 \\ \frac{1}{\kappa-1} & \kappa>2 \end{cases} \qquad (24)\end{aligned}$$

and hence

$$S_r(\kappa) = \omega^{-r}\frac{\kappa(\kappa-2)}{(\kappa-1)^2}\theta(\kappa-2)+\frac{1}{\kappa-1}\sum_{s=r+1}^{\infty} C_s\omega^{s-r} \tag{25}$$

where $\theta(\cdot)$ is the unit step function.

3 The percolation probability for compact percolation.

The percolation probability is given by $P(p) = 1 - Q(p)$ where $Q(p)$ is the probability that the compact cluster seeded with a single atom on the surface is finite. Using the duality relation between clusters and vesicles, the probability that the cluster grows for t stages and then terminates is $qV_t(1/q)(pq)^t$ which using (2) gives

$$Q(p) := q\sum_{t=0}^{\infty} V_t(1/q)(pq)^t = qZ(pq,1/q) \tag{26}$$

where the vesicle grand partition function $Z(u,\kappa)$ is given by (2). Using (20)

$$\begin{aligned} Z(u,\kappa) &= \sum_{r=0}^{\infty} u^{2r}\left(v_r^{even}(\kappa)+uv_r^{odd}(\kappa)\right) \\ &= \sum_{r=0}^{\infty} u^{2r}(C_r+uC_{r+1})S_r(\kappa) \\ &= Z^+(u,\kappa)\theta(\kappa-2)+\frac{1}{\kappa-1}\sum_{r=0}^{\infty}u^{2r}(C_r+uC_{r+1})\sum_{s=r+1}^{\infty}C_s\omega^{s-r} \end{aligned} \tag{27}$$

where

$$Z^+(u,\kappa) = \frac{\kappa(\kappa-2)}{(\kappa-1)^2}\left[1+\left(1+\frac{\omega}{u}\right)\left(\frac{\omega}{2u^2}-1-\frac{\sqrt{\omega(\omega-4u^2)}}{2u^2}\right)\right]. \tag{28}$$

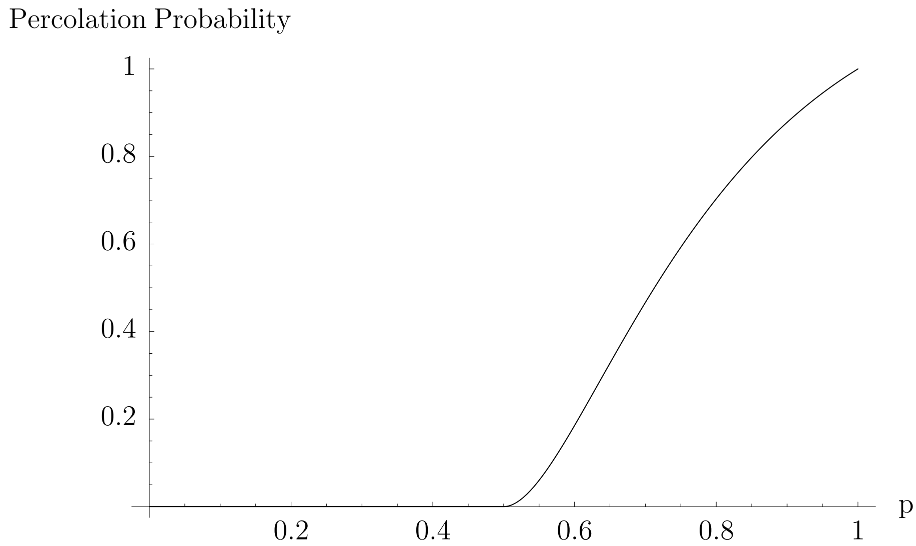

Figure 1: *A plot of the percolation probability*

Using (20) and (22) and noting that when $\kappa = 1/q$, $\omega = pq = u$,gives

$$Z(pq,1/q) = \frac{(2-p)(2p-1)}{p^3}\theta(p-\frac{1}{2})+\frac{q}{p}\left[\left(\sum_{r=0}^{\infty}C_ru^r\right)^2-\sum_{r=0}^{\infty}C_ru^r\right] \tag{29}$$

and using (23)

$$\sum_{r=0}^{\infty} C_r (pq)^r = \begin{cases} 1/q & p \le p_c \\ \\ 1/p & p > p_c \end{cases} \tag{30}$$

where the critical probability $p_c = \frac{1}{2}$. Substituting in (26) rederives the result of [7]

$$P(p) = \begin{cases} 0 & p \le p_c \\ \\ \frac{(2p-1)^2}{p^3} & p > p_c. \end{cases} \tag{31}$$

A plot of the percolation probability is shown in figure 1.

4 The Mean Length of Finite Compact Clusters

We define the cluster length to be the number of particles in the shortest path from the seed to the terminal point, including the seed (i.e. t+1). By definition, for $p < p_c$, the mean cluster length is given in terms of the vesicle partition $V_t(\kappa)$, defined in (1), by

$$\bar{L}(p) := q \sum_{t=0}^{\infty} (t+1) V_t(1/q)(pq)^t = q \frac{\partial}{\partial u} (uZ(u,\kappa)) \Big|_{\kappa=1/q, u=pq} \tag{32}$$

The unweighted sum over t gives $Q(p)$ (see eq. 26), the probability that the cluster is finite, and for $p > p_c$, $Q(p) < 1$. Thus above p_c we call $\bar{L}(p)$ the "unnormalised" mean length and define the normalized mean length by $L(p) := \bar{L}(p)/Q(p)$. This is the mean cluster length given that the cluster is finite. Using (27)

$$\bar{L}(p) = \theta(p - p_c) \frac{q(3-2p)}{p^3} + L^*(p) \tag{33}$$

where

$$\begin{aligned} L^*(p) &= \frac{q^2}{p} \sum_{r=0}^{\infty} \Big((2r+1) C_r u^r + (2r+2) C_{r+1} u^{r+1} \Big) \sum_{s=r+1}^{\infty} C_s u^s & (34) \\ &= \frac{q^2}{p} \sum_{k=1}^{\infty} (a_k u^k + b_k u^{k+1}) & (35) \end{aligned}$$

with

$$a_k = \sum_{r=0}^{\lfloor \frac{1}{2}(k-1) \rfloor} (2r+1) C_r C_{k-r} \quad \text{and} \quad b_k = \sum_{r=0}^{\lfloor \frac{1}{2}(k-1) \rfloor} (2r+2) C_{r+1} C_{k-r}. \tag{36}$$

The sums depend on the parity of k and we find, using Zeilberger's algorithm [10] as implemented by Paule and Schorn [8],

$$a_{2s+1} = \binom{2s+1}{s}\binom{2s+2}{s+1} - \frac{1}{2s+3}\binom{4s+3}{2s+1} \quad (37)$$

$$a_{2s+2} = \frac{1}{2}\binom{2s+3}{s+1}^2 - \frac{1}{2s+4}\binom{4s+5}{2s+2} \quad (38)$$

and

$$b_{2s+1} = \frac{2s^2+8s+7}{(2s+3)(s+2)}\binom{2s+1}{s}\binom{2s+4}{s+2} - \frac{2(4s+5)}{(2s+3)(s+2)}\binom{4s+3}{2s+1} \quad (39)$$

$$b_{2s+2} = \binom{2s+3}{s+1}\binom{2s+4}{s+2} - \frac{2(4s+7)}{(2s+5)(s+2)}\binom{4s+5}{2s+2}. \quad (40)$$

Using Mathematica to perform the summations yields, after further manipulations,

$$L^*(p) = \frac{1}{8p^3}\Big\{-5+4\,u+6\sqrt{1-4u} - \frac{8\,E(16\,u^2)}{\pi} + \frac{2\,(3-4\,u)\,(1+4\,u)\,K(16\,u^2)}{\pi}\Big\}. \quad (41)$$

The asymptotic form of this function near p_c is dominated by the logarithmic singularity of the Elliptic integral $K(m)$ near $m=1$ and using $\lim_{m_1\to 0}[K(1-m_1) - \frac{1}{2}\log(16/m_1)] = 0$ and $E(1) = 1$ [1] gives

$$\bar{L}(p) \cong B\log|1-2p| + C^{\pm} \quad (42)$$

where the superscripts refer to the approach from below and above p_c. Here

$$B = -\frac{8}{\pi} \qquad \text{and} \qquad C^{\pm} = \frac{4(3\log 2 - 2)}{\pi} \mp 4 = 0.101148\ldots \mp 4 \quad (43)$$

in agreement with the results conjectured in [5] based on numerical work.

By substitution, $\bar{L}(p)$, as given by (33), may be shown to satisfy both the low and high density inhomogeneous second order differential equations (16) and (18) of [5] which were found empirically by fitting power series expansion coefficients. The part of the solution involving elliptic integrals satisfies the homogeneous part of the differential equations and the algebraic parts are particular solutions of the inhomogeneous equations.

The normalized mean length is shown in figure 2.

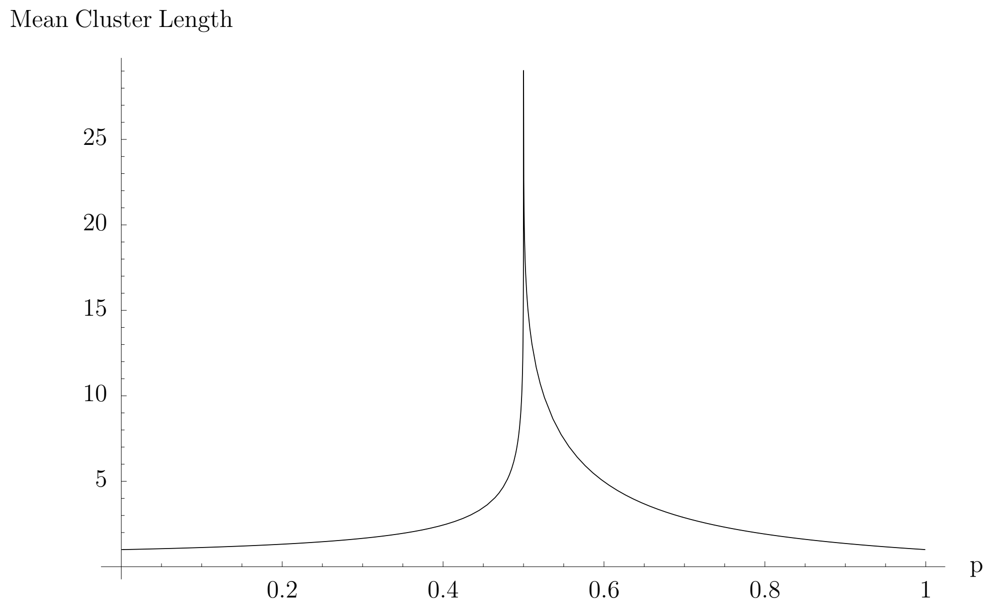

Figure 2: *A plot of the normalized mean cluster length.*

5 The Mean Number of Wall Contacts for Compact Clusters

In this section the expected number of contacts with the wall is obtained, including the initial contact. The stages of the calculation are similar to those for the cluster length. Above p_c the average is taken over only finite clusters and $\bar{N}(p)$ denotes the unnormalized average defined by

$$\bar{N}(p) := q \frac{\partial}{\partial \kappa} \left(\kappa Z(u, \kappa) \right) \Big|_{\kappa=1/q, u=pq} \tag{44}$$

where the κ-derivative gives weight $c+1$ to a cluster having c contacts with the wall other than the seed. Using (27) gives

$$\bar{N}(p) = \theta(p - p_c) \frac{q(1-2q^3)}{p^4} - \frac{q}{p} Q(p) + N^*(p). \tag{45}$$

Here

$$N^*(p) = \frac{q^3(1-2p)}{p} \sum_{r=0}^{\infty} (C_r u^r + C_{r+1} u^{r+1}) \sum_{s=r+1}^{\infty} (s-r) C_s u^{s-1} \tag{46}$$

$$= \frac{q^3(1-2p)}{p} \sum_{k=1}^{\infty} (c_k u^{k-1} + d_k u^k) \tag{47}$$

with

$$c_k = \sum_{r=0}^{\lfloor \frac{1}{2}(k-1)\rfloor} (k-2r) C_r C_{k-r} \qquad \text{and} \qquad d_k = \sum_{r=0}^{\lfloor \frac{1}{2}(k-1)\rfloor} (k-2r) C_{r+1} C_{k-r}. \tag{48}$$

Using Zeilberger's algorithm [10] we find

$$c_{2s+1} = \binom{4s+3}{2s+1} - \binom{2s+1}{s}\binom{2s+2}{s+1} \tag{49}$$

$$c_{2s+2} = \binom{4s+5}{2s+2} - \frac{s+2}{2s+3}\binom{2s+3}{s+1}^2 \tag{50}$$

and

$$d_{2s+1} = \frac{4s^2+16s+13}{(2s+3)(s+2)}\binom{4s+3}{2s+1} - \frac{4s^2+14s+11}{2(2s+3)(s+2)}\binom{2s+1}{s}\binom{2s+4}{s+2} \tag{51}$$

$$d_{2s+2} = \frac{2(2s^2+10s+11)}{(2s+5)(s+2)}\binom{4s+5}{2s+2} - \binom{2s+3}{s+1}\binom{2s+4}{s+2}. \tag{52}$$

Use of Mathematica to evaluate the sums in (47) gives

$$N^*(p) = \frac{(1-2p)}{8p^4}\Big\{1-4u-2(1-2u)\sqrt{1-4u}+ \frac{4u(1+2u)}{\sqrt{1-4u}} + \frac{8E(16u^2)}{\pi} - \frac{2\,(3-4u)\,(1+4u)\,K(16u^2)}{\pi}\Big\}. \tag{53}$$

Combining (44) and (53) we obtain the asymptotic form near p_c as

$$\bar{N}(p) \cong 2 + \frac{16}{\pi}(1-2p)\log|1-2p|. \tag{54}$$

We note that the discontinuity arising from the fourth term in the bracket is balanced by that from the first term of (44) but that the critical point is marked by an infinite derivative which occurs just before $\bar{N}(p)$ passes through its maximum value. A graph of $\bar{N}(p)/Q(p)$, the mean number of contacts given that the cluster is finite, is shown in figure 3. This surprisingly never exceeds three which implies that the "dry wall" condition acts as a strong repulsion away from the wall. We defer consideration of a "damp wall" condition which would increase the number of contacts to a later date.

6 Differential equations satisfied by the cluster properties.

In [5] it was conjectured on the basis of differential approximant techniques that the mean cluster length functions above and below p_c satisfy inhomogeneous second order differential equations with polynomial coefficients. The

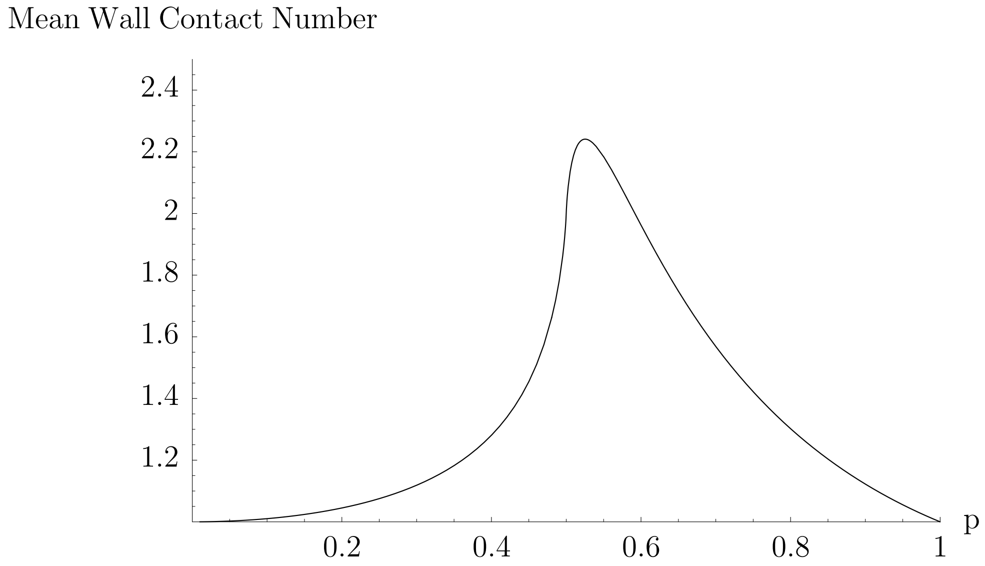

Figure 3: *A plot $\bar{N}(p)/Q(p)$, the mean number of wall contacts given that the cluster is finite.*

equations had the same homogeneous part in the two regions. The degree of the coefficients was such that it was thought to be impossible to express the mean length in terms of previously studied functions. For example the coefficient of the second derivative was of degree six. However the singular points and their exponents were obtained (Table 2 of [5]). The present work shows that we were unduly pessimistic and in fact it may be verified by direct substitution that the conjectured equations ((16) and (18) of [5]) are satisfied by (33).

The cluster size $S(p)$ in the region $p < p_c$ was found to be [5]

$$S(p) = \frac{1-p}{1-2p} \tag{55}$$

Above p_c no such simple solution was found but it was conjectured that the unnormalized mean size $\bar{S}(p)$ satisfies a differential equation similar to that for the mean length. Inspired by our exact solution for the mean length we show below the conjectured equation may also be solved in terms of hypergeometric functions.

We begin with the mean length and, guided by our exact solution, show that a sequence of substitutions reduces equation (16) of [5] to second order equation having only four regular singular points which is solved by a Heun function [9]. A further transformation then produces a hypergeometric equation.

Defining $\tilde{L} = p^3 \bar{L}$ we find that equation (16) of [5] takes the form

$$p(1-p)(1-2p)(1+4p-4p^2)\frac{d^2\tilde{L}}{dp^2} + (1-2p+2p^2)(1-12p+12p^2)\frac{d\tilde{L}}{dp} + \\ 8(1-2p)(1-2p+2p^2)\tilde{L} = p^2(9-12p+12p^2) \quad (56)$$

which has the polynomial solution $(1-8p-4p^2)/8$. Substituting

$$p = \frac{1-\sqrt{1-4u}}{2} \quad (57)$$

gives the algebraic part of (41). The remaining combination of elliptic integrals must therefore be a solution of the homogeneous part of (56). This is indeed the case as may be seen by changing the variable in (56) to $u = p(1-p)$ and cancelling a factor $\sqrt{1-4u}$, which gives

$$u(1-16u^2)\frac{d^2\tilde{L}^{(h)}}{du^2} + (1-16u+16u^2)\frac{d\tilde{L}^{(h)}}{du} + 8(1-2u)\tilde{L}^{(h)} = 0. \quad (58)$$

The solution of this equation which is regular at the origin is the Heun function [9] $F(-1,-2;-1,-1,1,-3;4u)$ and by matching at $u = 0$, equation (41) may be rewritten as

$$L^*(p) \;=\; \frac{1}{8p^3}\left\{-5 + 4\,p(1-p) + 6\sqrt{1-4p(1-p)} \right.\\ \left. - F(-1,-2;-1,-1,1,-3;4p(1-p))\right\}. \quad (59)$$

It may be verified by substitution that

$$F(-1,-2;-1,-1,1,-3;4u) = \frac{8\,E(16\,u^2)}{\pi} - \frac{2\;(3-4\,u)\;(1+4\,u)\;K(16\,u^2)}{\pi} \quad (60)$$

The parameters of the Heun function satisfy the conditions for the transformation [9]VII(9d) to a hypergeometric function thus

$$F(-1,-2;-1,-1,1,-3;4u) = (1+4u)\;{}_2F_1(-\frac{1}{2},\frac{3}{2};1;\frac{16u}{(1+4u)^2}) \quad (61)$$

A similar process applied to the $p > p_c$ differential equation (18) of [5] , which was conjectured from the high density series expansion, leads to (56) with p replaced by $1-q$ and a modified inhomogeneous part which takes account of the step function in (32).

Turning now to the unnormalized mean size of clusters $\bar{S}(p)$ we make a transformation similar to that for the mean length but allowing for a pole at $q = \frac{1}{2}$ as in the $p < p_c$ function. Thus substituting

$$\bar{S} = \frac{\tilde{S}}{(1-q)^3(1-2q)} \tag{62}$$

in equation (12) of [5] gives

$$\begin{aligned} q(1-q)(1-2q)(1+4q-4q^2)\frac{d^2\tilde{S}}{dq^2} + \\ 2(1-3q+23q^2-40q^3+20q^4)\frac{d\tilde{S}}{dq} + 4(1-2q)(1-8q+8q^2)\tilde{S} = \\ 2-2q-6q^2+4q^3+30q^4-48q^5+24q^6 \end{aligned} \tag{63}$$

which has the polynomial solution $\phi(q) = \frac{1}{2}(-1+4q-6q^2+2q^3+2q^4)$. Writing $\tilde{S} = \phi(q) + \tilde{S}^{(h)}$ and changing the variable to $u = q(1-q)$ in the homogeneous part of (63) gives

$$u(1-16u^2)\frac{d^2\tilde{S}^{(h)}}{du^2} + 2(1-4u+16u^2)\frac{d\tilde{S}^{(h)}}{du} + 4(1-8u)\tilde{S}^{(h)} = 0. \tag{64}$$

The solution which is regular at the origin is the Heun function

$$F(-1,-1;-2,-1,2,-3;4u) = (1+4u)\ {}_2F_1(-\frac{1}{2},\frac{3}{2};2;\frac{16u}{(1+4u)^2}). \tag{65}$$

Combining these results leads to the following conjecture for the cluster size above p_c

$$\begin{aligned} \bar{S}(q) &= \frac{1}{2(1-2q)(1-q)^3}\Big\{2\phi(q) \\ &\quad +(1+4q(1-q))\ {}_2F_1(-\frac{1}{2},\frac{3}{2};2;\frac{16q(1-q)}{(1+4q(1-q))^2})\Big\} \end{aligned} \tag{66}$$

which on expansion in powers of q is in agreement with the first fifty terms given in [5]. As in the case of the mean cluster length, the hypergeometric function in [66] can also be expressed [1] in terms of Elliptic Integrals via

$${}_2F_1(-\frac{1}{2},\frac{3}{2};2;x) = \frac{4}{3\pi x}\left\{(1-x)\mathrm{K}(x) - (1-2x)\mathrm{E}(x)\right\} \tag{67}$$

Using equation 15.3.11 of [1] the asymptotic form of (66) as the critical probability is approached from above is

$$\bar{S}(q) \cong \frac{1}{1-2q}\{A^- + B^-(1-2q)^4\log(1-2q)\} \tag{68}$$

where $A^- = \frac{32}{3\pi} - \frac{1}{2} = 2.895305453...$ and $B^- = \frac{8}{\pi}$. The estimate of A^- in [5] differs from the exact value by 1 in the 6th decimal place.

The normalized mean cluster size is shown in figure 4.

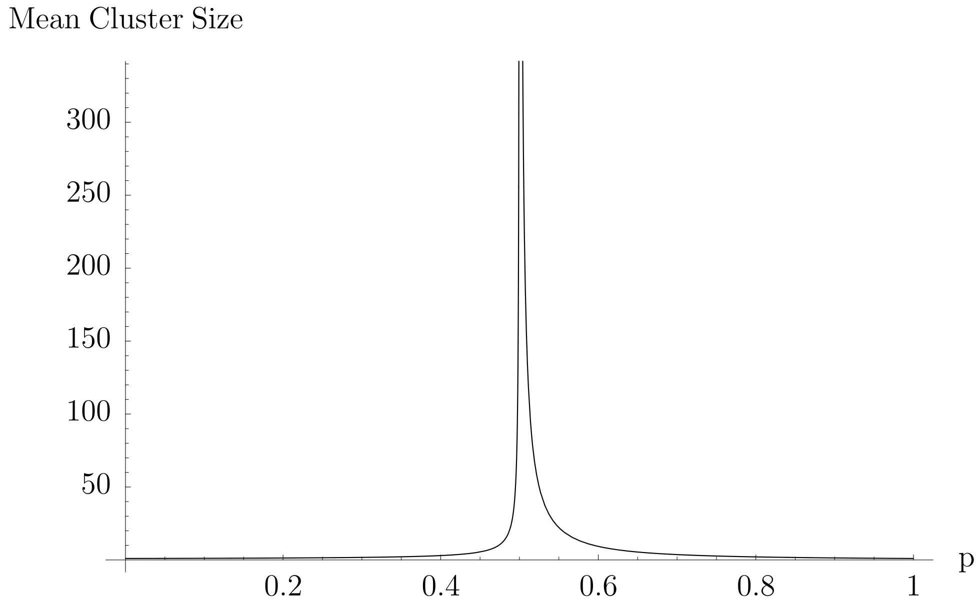

Figure 4: *A plot of the normalized mean cluster size*

7 Conclusions

We conclude by summarizing the new results obtained. The compact percolation probability previously solved is shown to be related to the partition function of a directed vesicle above a wall. By recasting the vesicle partition function in a new form and through a novel use of the Zeilberger algorithm we have been able to compute the first derivatives of the partition function and hence calculate the mean length of finite clusters and mean number of surface contacts .

These new results establish previously conjectured results for the differential equation satisfied by the mean cluster length. The form found for the mean length enabled the conjectured differential equation satisfied by the mean size function above p_c to be solved in terms of a hypergeometric function. The asymptotic behaviour of the mean size and mean length of clusters, in the neighbourhood of the critical point, deduced from our closed form solutions is in agreement with earlier numerical work.

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