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Observations on interfacing loop quantum gravity with cosmology

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A simple idea of relating the loop quantum gravity (LQG) and loop quantum cosmology (LQC) degrees of freedom is introduced and used to define a relatively robust interface between these theories in context of toroidal Bianchi I model. The idea is an expansion of the construction originally introduced by Ashtekar and Wilson-Ewing and relies on explicit averaging of a certain subclass of spin networks over the subgroup of the diffeomorphisms remaining after the gauge fixing used in homogeneous LQC. It is based on the set of clearly defined principles and thus is a convenient tool to control the emergence and behavior of the cosmological degrees of freedom in studies of dynamics in canonical LQG. The constructed interface is further adapted to isotropic spacetimes. Relating the proposed LQG-LQC interface with some results on black hole entropy suggests a modification to the *area gap* value currently used in LQC.

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I. INTRODUCTION

Loop quantum gravity [1–3] (LQG)—one of the leading attempts to provide a solid framework unifying general relativity (GR) with quantum aspects of reality—has matured over the past years to the level at which extracting the concrete dynamical predictions out of it has become a technically feasible task [4,5]. Despite this success the concrete results regarding the evolution of quantum spacetime in LQG are yet to appear. On the other hand, in the past ten years a series of dynamical predictions have been made (with various levels of rigor) within the symmetry reduced framework originating from LQG known as loop quantum cosmology [6] (LQC). There, the set of already known results ranges from establishing a singularity resolution [7] through qualitative changes of the standard early Universe dynamics picture (found on the genuine quantum level) [8–10] to predictions of the behavior of cosmological perturbations [11] and (in some cases) nonperturbative inhomogeneities [12].

The results obtained within the LQC framework cannot be treated as final, as LQC was never derived from LQG in any systematic way. Instead, it is a stand-alone theory constructed by applying the methods of LQG to cosmological models [13] further enhanced via parachuting some results and properties of LQG on the phenomenological level [14]. Therefore, it is not *a priori* clear to what extent (if any) the predictions mentioned above reflect the true features of full LQG. Addressing these issues has brought considerable interest within the loop community. Its pioneering studies [15] were aimed towards controlling the so-called *inverse volume corrections* in LQC and as tools to control the heuristic effective descriptions of inhomogeneous extensions of LQC [16]. Presently, the attempts to provide a precise connection between LQG and LQC are directed in three main areas: (i) direct embedding of the LQC framework within the LQG one, (ii) approximation of cosmological (symmetric) solutions in LQG, and (iii) emergence of cosmological (LQC) degrees of freedom in appropriate scenarios within LQG.

The approach (i), realized on the level of mathematical formalism, focuses on embedding the elements of the LQC formalism (for example straight holonomies) as a proper subclass of their analogs in LQG. So far, however, the most natural ways of constructing such embeddings have proved to lead to inconsistencies and resulted in several *no-go* statements [17]. The main problem encountered in the attempts is the inherent diffeomorphism invariance of LQG [18] and the fact that the symmetries characterizing the cosmological solutions are a subgroup of the diffeomorphism group. In consequence the formalism of the theory is (by construction) insensitive to the very components distinguishing the cosmological spacetimes. On the other hand, recently, extending the standard LQC holonomy-flux algebra by all holonomies along piecewise analytic curves and imposing the symmetries on the classical level led to a viable embedding of LQC in LQG [19].

To overcome the problems of (i) another route [listed as point (ii)] was explored. There, instead of encoding the symmetries in the formalism one considered "the operational approach"—defining the symmetries as the relations between (the expectation values of) the LQG observables corresponding to appropriately selected spacetime geometry quantities. This idea has been implemented through the construction of the coherent states peaked about symmetric spacetimes (see for example [20]). Assuming that the

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evolution of such coherent states can be sufficiently well controlled on the level of full LQG, such description can provide a definition of a (n effective) reduced framework on its own—the true cosmological limit of LQG. That formalism would then have a much stronger foundation in its LQG origin than LQC. So far, however, in the genuine LQG no such framework has been constructed in a complete and technically manageable form. On the other hand, such a cosmological limit framework would lose its anchor in LQC (as it is derived in an entirely independent way); thus, it would not be able to provide a solid connection with the existing formulation of LQC or justification for the particular constructions implemented in it. As a consequence the utility of this approach as a test for LQC or its results would be limited at best.

To mend this gap one needs to build an interface between LQG and LQC, constructing in particular a precise dictionary between objects native to these two formalisms, while keeping the formalisms themselves autonomous [the approach (iii)]. Any such dictionary should associate selected (possibly emergent) degrees of freedom of LQG with cosmological ones—working as fundamental for LQC. In that aspect it is not necessary to restrict to symmetric or even near symmetric states in LQG. Building this dictionary would be just a process of extracting some (global) degrees of freedom and as such could, in principle, be well defined for *any* quantum spacetime (or at least for a family of such that is sufficiently large to cover physically interesting scenarios).

A good example of such a procedure is a simple toy model used in [21] to fix certain ambiguities in quantizing the Bianchi I spacetimes in LQC. There, as an example of the spin network one considered a regular (cubic) lattice of which edges have been excited to the first state above the ground one. Much simpler models attempting to mimic cosmology, playing the role analogous to the one above, are also being constructed in spin-foam formalism (often considered as the covariant formulation of LQG) [22,23].

It is also worth recalling that apart from the *top-down* approaches presented above there exists considerable literature on *bottom-up* approaches, where the components of LQC are cast onto LQG structures. An example of such is the so-called *lattice LQC* [24], where the structure of degrees of freedom and elementary operators inherent to perturbative cosmology are defined on the regular lattice, again playing the role of an example of an LQG spin network. Another, slightly more distant example is testing the Belinsky-Khalatnikov-Lifshitz conjecture in the context of LQC [25].

In this work we explore further the direction (iii), in particular proposing an LQG-LQC interface which is

(1) sufficiently general to be applicable to the largest possible class of specific quantization prescriptions in LQG, while
(2) being sufficiently robust and precise to serve as
    (a) the control tool for the heuristic components of LQC, and
    (b) the consistency filter for particular LQG quantization prescriptions and/or Hamiltonian constraint constructions.

The proposed interface will be defined on the kinematical level—without controlling the dynamical evolution on either the LQG or LQC side. This choice is necessary if we want to keep the studies largely independent of the particular prescription choices. While being relatively limited in comparison with any derived (dynamical) LQC limit of LQG, it will be able to serve as the control tool for any such limits possibly derived in the future. On the other hand, one can expect that the requirement of the consistency of the specified interface can be sufficient to provide useful insights into the properties of either LQC (in particular corrections to/invalidations if its heuristic input) or LQG (for example restrictions on possible statistics of spin networks). As we will see further in the article, this expectation is indeed met and the consistency of the interface itself is sufficient to obtain interesting results.

In the process of building the interface we restrict the space of LQG physical states to a certain (particular) subspace—a step performed to satisfy two requirements: on one hand we need in our description a sufficient number of degrees of freedom to be able to incorporate a class of inhomogeneities sufficient to be able to represent physically interesting spacetimes. On the other hand the selected states need to permit defining the auxiliary structures of LQC in a natural way.

The particular construction proposed here is strongly motivated by the (already mentioned) Bianchi I toy model of [21]. It shares with this model the choice of spin-network topology. Outside of this initial choice, however, we keep the construction as general as possible (avoiding the dependence on particular prescription in either LQG or LQC), at the same time keeping full control over the assumptions entering the construction. In particular, as the unreduced side of the interface we use *genuine* LQG without any simplifications. The dictionary will be provided by objects having a precise physical interpretation that are well defined in both theories.

Having such a simple tool available has recently become more than a matter of convenience, as there is an increasing amount of effort towards making the preliminary dynamical predictions of conservative LQG [26,27], its modifications [28,29], or its simplifications [in particular the simplifications of $SU(2)$ gauge to $U(1)^3$ [30–32]] via semiclassical approximations. Interesting developments in that direction also appear within the covariant approaches to (loop) quantum gravity, in particular the group field theory framework [33] and the spin foams [22,23]. One can thus confront the results of these projects with the predictions of LQC to confirm/falsify the latter.

Before proceeding with the construction of the specific interface let us briefly recall those element of both theories that will be relevant for this interface.

II. ELEMENTS OF LQC AND LQC

Since for all practical purposes both LQG and LQC are independent theories, just sharing common quantization methodology [13,18] we proceed with presenting them separately. Let us start with LQG.

A. Loop quantum gravity

LQG is a quantization of canonical general relativity, owing a lot of its mathematical components to Yang-Mills theories on the lattice. Its starting point is a $3+1$ canonical splitting, with the phase space coordinated by the Ashtekar-Barbero variables: $su(2)$ valued connection A^i_a and the desensitized triad E^a_i, where A^i_a is a combination of the Levi-Civita connection and exterior curvature $A^i_a = \Gamma^i_a + \gamma K^i_a$ (with γ being the Barbero-Immirzi parameter) [34]. As any representation of GR it is a constrained theory with the algebra of constraints generated by the Gauss constraint, the spatial diffeomorphisms, and the Hamiltonian constraint. To deal with them the Dirac program is implemented: theory is quantized without constraints (the so-called kinematical level), which are next solved on the quantum level (with solutions forming the *physical* sector of the theory).

The basic objects of the theory are the holonomies of A along the piecewise analytic curves $U_\gamma(A) = \mathcal{P}\exp(\int_\gamma A^i_a \tau_i \mathrm{d}x^a)$ and the fluxes of E^a_i across surfaces $K^i = \int_S E^{ai}\mathrm{d}\sigma_a$. Together they form the holonomy-flux algebra, which is the fundamental object in constructing quantum theory. An application of the Gelfand-Naimark-Segal (GNS) construction to this algebra leads to the kinematical Hilbert space $\mathcal{H}^{\mathrm{LQG}}_{\mathrm{kin}}$ spanned by the cylindrical functions supported on the graphs embedded in three-dimensional differential manifold. These functions are conveniently labeled by the $su(2)$ representations (on each edge of the graph plus the internal edges within the graph vertices)—enumerated by half integers. The quantum representation of holonomy-flux algebra provided by this construction is unique [35]. This space is nonseparable, and defining a separable physical Hilbert space structure in further steps of the Dirac program requires a nontrivial effort (see for example the discussion in [36] and references therein).

On $\mathcal{H}^{\mathrm{LQG}}_{\mathrm{kin}}$ the constraints are solved in hierarchy (in the following order: Gauss, diffeomorphism, and Hamiltonian). The Gauss constraints distinguish (through the kernel of an operator corresponding to it) a subspace by selecting out the *gauge-invariant* kinematical basis elements. These elements are characterized by the restrictions on the representation labels on the edges converging on a vertex (for each vertex of the graph), restrictions that can be thought of as analogs of the angular momentum addition rules in quantum mechanics.

Next, the diffeomorphism constraint is solved by the procedure of averaging [18] over a group of finite diffeomorphisms, which act on the basis elements by modifying the embedding of the graphs supporting them, but without modifying the topology of the graphs or their quantum labels. In particular, in case the framework used involves a single particular graph the group averaging would simply lift the graph from the embedded one to the abstract one (see for example [28]).

The diffeomorphism-invariant Hilbert space $\mathcal{H}^{\mathrm{LQG}}_{\mathrm{diff}}$ provided by this procedure serves next as a main kinematical space, on which the (diffeomorphism-invariant) observables are defined and the Hamiltonian constraint is solved. The action of many geometry observables is explicitly known. An interesting property of many of them, among others the area, volume, angle or length operators is that their spectra are purely discrete, composed of (generically) isolated points. In this meaning it is often stated that in LQG the space(time) is discrete. These operators are, however, not physical observables since one more constraint remains—the Hamiltonian one.

In the last step of the Dirac program one identifies the physical Hilbert space as a kernel of the Hamiltonian constraint operator and builds the physical observables out of the diffeomorphism-invariant kinematical ones. The latter is achieved through the so-called *partial observable* framework [37]. In solving the Hamiltonian constraint several approaches are explored. Among them two approaches are considered as the most promising: *the master program* [38] and the matter deparametrization. In the first approach, to avoid mathematical complications related to the structure of the constraint algebra one constructs a single non-negative definite operator out of all the constraints—the so-called master constraint. Then the physical Hilbert space is again given as a kernel of this operator. The last step was however not completed due to complicated mathematical structure of the constraint.

In the second approach one uses the matter reference frames to provide the time variable (missing in the formalism) [4,5]. This allows one to reformulate the (originally constrained) theory as the free theory with a true Hamiltonian and where the original diffeomorphism-invariant Hilbert space itself or its proper subspace (depending on the matter frame used) becomes the physical Hilbert space. The Hamiltonian is either a former Hamiltonian constraint operator [5] or its square root [4]. In the former case its action on $\mathcal{H}^{\mathrm{LQG}}_{\mathrm{phy}} \equiv \mathcal{H}^{\mathrm{LQG}}_{\mathrm{diff}}$ is explicitly known and feasible to compute [39]. In this formulation, the diffeomorphism-invariant kinematical observables become the physical ones.

Among the components of the theory two particular objects play the crucial role in constructing our dictionary. These are the diffeomorphism-invariant area operator and the Euclidean part of the Hamiltonian constraint. Let us focus our attention on the area first.

1. The area operator

The properties of this operator are known in detail (see for example [1,3]). Its action on $\mathcal{H}^{\rm LQG}_{\rm diff}$ (i.e. on its spin network basis elements) is relatively simple [40]: the area of chosen (arbitrary) 2-surface S depends only on the j labels of the edges of a spin network reaching (or intersecting) this surface,

$$\mathrm{Ar(S)}\Psi[\mathrm{A}] = 4\pi\gamma\ell_{\rm Pl}^2 \times \Big[\sum_{e^+}\sqrt{j_{e^+}(j_{e^+}+1)} + \sum_{e^-}\sqrt{j_{e^-}(j_{e^-}+1)}\Big], \tag{2.1}$$

where e^+ are the incoming edges of the graph supporting $\Psi[A]$ that terminate on the surface and e^- are the ones starting at the surface. $j_e^{\pm}$ are their respective $su(2)$ representation labels. The edges intersecting (piercing) the surface are counted as both incoming and outgoing (i.e. in that case a trivial 2-valent node is temporarily introduced on the surface); thus, their contribution is twice the terminating ones. The form of (2.1) immediately implies that the spectrum of Ar(S) is discrete. In particular the first nonzero value of the area is isolated from zero and equals

$$A_1 = 2\sqrt{3}\pi\gamma\ell_{\rm Pl}^2. \tag{2.2}$$

If the surface is distinguished in the diffeomorphism-invariant way, for example, the surface edges correspond to the edges of a graph supporting the spin network (which will be the case in the applications to the construction of the dictionary; see Sec. III B), the area operator is automatically diffeomorphism invariant.

2. The Euclidean part of the Hamiltonian

Classically the Hamiltonian constraint (or, more precisely, its part corresponding to gravity) is of the form

$$C = \int \mathrm{d}^3x\sqrt{-q}\,\mathcal{C}, \tag{2.3a}$$

$$\mathcal{C} = \frac{\gamma^2}{2\sqrt{\det E}}E^a_iE^b_j[\epsilon^{ij}{}_kF^k_{ab} + 2(1+\gamma^2)K^i_{[a}K^j_{b]}] \tag{2.3b}$$

where the field strength F_{ab} is the curvature of the connection A. The first term in $\mathcal{C}$ is the so-called *Euclidean term* of the Hamiltonian density. The loop quantization procedure requires one to express $\mathcal{C}$ in terms of the holonomies and fluxes—the process known as Thiemann regularization [26]. In particular to express the curvature term F one implements the known classical identity

$$F^i_{ab}X^aY^b(x) = \lim_{\mathrm{Ar}(\Delta(x))\to 0}\frac{U_{\Delta(x)}-1}{\mathrm{Ar}(\Delta(x))}, \tag{2.4}$$

where $\Delta(x)$ is the closed, piecewise analytic loop such that the vectors X, Y are tangent to it at the point x, $U_{\Delta(x)}$ is the holonomy along this loop, and $\mathrm{Ar}(\Delta(x))$ is the physical area of the loop. The right-hand side of this identity can be quantized directly in LQG, as the operators corresponding to the holonomy and the area are well defined.

The particular implementation of this identity (and the action of the resulting regularized $\hat{F}$ operator) depends on the specific construction (or prescription) of the Hamiltonian constraint. In the original construction by Thiemann [26] the (Euclidean part of the) Hamiltonian constraint added at the vertices of the spin network a small (planar) triangular loops adjacent to a given vertex, with the new edge (closing the loop) carrying the label of the fundamental $su(2)$ representation ($j = 1/2$). Then the limit of shrinking this triangular loop to a point (at the node) was taken in the sense of the embedding, which nonetheless led to a diffeomorphism-invariant result due to the nature of operator components in the approximation of F (for details, see [27]).

In the alternative construction (see for example [28]), where the Hamiltonian constraint does not generate new edges, the loop has to be formed by existing edges of the spin network. It is defined by a requirement to form a plaquet—the minimal closed surface of the interior not intersected by any edges. In particular, when the spin networks are supported on the regular lattice, these loops are the minimal squares.

The particular construction of the Hamiltonian constraint also affects critically the structure of the physical Hilbert space. Below we briefly discuss the issues related to it.

3. Physical Hilbert space structure

Since in the master program $\mathcal{H}^{\rm LQG}_{\rm phy}$ is defined only abstractly (formally) to probe its properties we focus on the deparametrization picture. To start with, we note that the kinematical Hilbert space that arises from the GNS construction is nonseparable. The reason behind it is that the induced inner product on $\mathcal{H}^{\rm LQG}_{\rm kin}$ makes the states supported on disjoint graphs orthogonal. Each subspace of states supported on the chosen graph is separable, with a discrete inner product; however, the complete $\mathcal{H}^{\rm LQG}_{\rm kin}$ contains a continuum of (disjoint) graphs. Unfortunately, the gauge-invariant and diffeomorphism-invariant Hilbert spaces retain this property: The former puts only the restrictions on the labels of the graph without significantly decreasing the possible graph structures. The latter still allows for the continuum of distinct (orthogonal) graphs with a discrete inner product between them. Since in the deparametrization picture the space $\mathcal{H}^{\rm LQG}_{\rm diff}$ becomes the physical one, this deficiency is transmitted directly to the physical sector. As the nonseparability can affect significantly the construction and properties of the coherent states and the statistical ensembles (see [41] for discussion of these issues on a simple quantum-mechanical example), this problem requires a certain amount of care.

The particular treatment depends on whether the action of the (true in the deparametrization picture) Hamiltonian changes the graph topology. If the graph is fixed (see for example [28]) the Hamiltonian distinguishes subspaces invariant with respect to its action (supported on an unchanging graph). If the relevant observable operators are defined carefully and also preserve these subspaces, they become the superselection sectors, each of them being separable (as supported on one specific graph). The standard treatment calls then for a restriction of the studies to just one such sector.

If the Hamiltonian is graph changing, this procedure becomes less straightforward, although often superselection sectors can be distinguished due to the fact that (in specific prescriptions) the Hamiltonian changes the graph in a specific controlled way. This happens for example in the case of the original construction of [26]. However, those superselection sectors can become already nonseparable.

On the other hand, our experience from LQC shows (see the discussion in [36]) that for certain models such restriction might be insufficient to provide a sufficiently large semiclassical sector reproducing general relativity dynamics in small gravitational field regimes. In that case an alternative construction may be needed. Such an alternative is provided for example in [36].[1] There one makes use of the available Lebesgue[2] measure on the space of superselection sectors. Then the inner product is defined as the integral with respect to that measure of inner products $\langle\cdot|\cdot\rangle_\epsilon$ on the single superselection spaces $\mathcal{H}_\epsilon$ (with ϵ being an abstract superselection sector label),

$$\forall\ \psi,\phi\in\mathcal{H}\colon\ \langle\psi|\phi\rangle = \int \mathrm{d}\mu(\epsilon)\langle\psi_\epsilon|\phi_\epsilon\rangle_\epsilon, \tag{2.5}$$

where $\psi_\epsilon,\ \phi_\epsilon\in\mathcal{H}_\epsilon$ are the restrictions (projections) of the states to the single sector.[3] Action of the operators preserving the sectors extends in a straightforward way. The integral Hilbert space $\mathcal{H}$ is again separable.

B. Loop quantum cosmology

LQC, even when applied to the description of the inhomogeneous spacetimes, always relies on the reorganization of the geometry and matter degrees of freedom onto the quasiglobal ones, for example, the Fourier or spherical harmonic modes of the inhomogeneities/gravitational waves/matter (see [11,42]). As a consequence in this description there are always distinguished degrees of freedom corresponding to the “background” homogeneous spacetime. This distinction is achieved by partial gauge fixing, which is naturally distinguished in the cases of homogeneous spacetimes and in perturbative approaches. The remaining (inhomogeneous) degrees of freedom are then treated as the objects “living” on that homogeneous background on the equal footing with the matter fields. Thus, in all of the models a proper handling of the homogeneous spacetimes is an essential first step.

Here, we focus on the simplest model representing such a spacetime—the model of Bianchi I universe. For most of the paper we further fix the topology of its spatial slices to 3-torus. The precise mathematical formulation of the LQC quantization of this model has been presented in [13] (isotropic spacetimes) and [21] (actual quantization of the model following the procedures of [13]). It is performed via direct repetition of the procedure developed for LQG, although here the symmetries distinguish additional structure, which plays an essential role in the process.

First, the homogeneity distinguishes the natural partial gauge, in which the spacetime metric takes the form

$$g = -N^2(t)\mathrm{d}t^2 + a_1^2(t)\mathrm{d}x + a_2^2(t)\mathrm{d}y + a_3^2(t)\mathrm{d}z \tag{2.6}$$

where $N(t)$ is the lapse function, (a_1,a_2,a_3) are the scale factors in three orthogonal directions (in which the metric is diagonal), and ${}^oq = \mathrm{d}x^2+\mathrm{d}y^2+\mathrm{d}z^2$ is the isotropic fiducial metric constant in comoving coordinates (x,y,z). This choice fixes all the spacetime diffeomorphisms up to a global time reparametrization, and the (rigid in ${}^oq$ metric) global spatial translations.

Similarly to the general GR case, we select the Ashtekar-Barbero variables, although here the fiducial metric distinguishes the orthonormal triad ${}^oe^a_i$ of vectors pointing in eigendirections of the physical metric and preserving the spatial symmetries of the system. That structure again allows one to partially gauge fix the variables by selecting

$$A^i_a = c^i(L_i)^{-1}\,{}^o\omega^i_a, \qquad E^a_i = p_iL_iV_o^{-1}\sqrt{{}^oq}\,{}^oe^a_i, \tag{2.7}$$

where ${}^o\omega^i_a$ is a cotriad dual to ${}^oe^a_i$, V_o is the fiducial (with respect to ${}^oq$) volume of the homogeneous spatial slices, and L_i are their (also fiducial) linear dimensions. The global coefficients c^i and p_i are the so-called connection and triad coefficients. They form the canonical set with Poisson bracket $\{c^i,p_j\} = 8\pi G\gamma\delta^i_j$.

The free length parameters L_i reflect the symmetry of the model corresponding to the freedom of rescaling the coordinates of ${}^oq$. This fact becomes relevant if one considers the noncompact Bianchi I model (see [21,43] for the details). However, due to chosen scaling of variables (c^i,p^i) with respect to L_i presented in (2.7) the model is invariant with respect to this symmetry. Here, we set the

[1]The construction there is presented on the example of the simple quantum-mechanical system—a harmonic oscillator; however, the applications to LQG are also discussed there.

[2]The construction can also be extended to many cases with a singular measure.

[3]This projections are known in the literature as the so-called *shadow states*.

length parameters to $L_i = 1$, which corresponds to the choice of the coordinates $(x, y, z) \in [0, 1)^3$.

In the next step one constructs the holonomy-flux algebra. Here, however, one notices that upon the choice (2.7) one can just select the subalgebra of holonomies $U_i^{(\lambda)}$ along the straight edges in direction ${}^o e_i$ and the fluxes S_i along the 2-tori[4] orthogonal to ${}^o e_i$ as they suffice to separate the phase space points. Further, the fluxes S_i can be associated with the triad coefficients themselves as

$$S_i = p_i. \tag{2.8}$$

On such restricted (subalgebra of the) holonomy-flux algebra one implements the GNS construction, arriving at the unique quantum representation [44]. The kinematical Hilbert space resulting from this construction is a product of the square summable functions on the Bohr compactification of the real line

$$\mathcal{H}_{\rm kin}^{\rm LQC} = [\Sigma^2(\bar{\mathbb{R}}_{\rm Bohr}, d\mu_{\rm Bohr})]^3, \tag{2.9}$$

each copy of the space corresponding to one direction of ${}^o e_i^a$.

The basic operators—quantum counterparts of the holonomy-flux algebra elements—are the holonomy operators $\hat{U}_i^{(\lambda)}$ and the unit flux operators $\hat{p}^i = \hat{S}^i$. the latter are known in the literature as the *LQC triad operators* due to the simple classical relation (2.8). We implement the same naming policy here. One has to remember, however, that these operators represent *the fluxes*, not the triads. As in full LQG, in LQC the operators corresponding to the holonomies and triads themselves *do not exist*. In fact, the relation between p^i and the scale factors a_i in (2.6) (see [21]) shows that the operators $\hat{p}^i$ measure *the area of the maximal surface orthogonal to* ${}^o e_i^a$.

The kinematical states $|\psi\rangle \in \mathcal{H}_{\rm kin}^{\rm LQC}$ automatically satisfy the Gauss and diffeomorphism constraints. The only gauge transformations left after the partial gauge fixing—the global spatial translations—act on the elements of $\mathcal{H}_{\rm kin}^{\rm LQG}$ as an identity. The only nontrivial constraint remaining is the Hamiltonian one.

To construct the operator representing the (gravitational term of the) Hamiltonian constraint one repeats the Thiemann construction of LQG, partially discussed in Sec. II A 2. For the Bianchi I model, the Lorentzian (exterior curvature) term is proportional to the Euclidean one; thus, only the quantization of the latter is needed. For that one again has to deal with the field strength term, which is approximated by holonomies along a closed loop via (2.4). Since only the holonomies along the diagonal directions of q are available the loop is a rectangle oriented in directions of ${}^o e_i^a$.

[4]In case of noncompact spatial slice topology one takes unit squares.

Here however we see an important difference with respect to LQG. In the full theory either (depending on the formulation) one could move (shrink) the loop in the embedding manifold and the transformation has not modified the physical area of the loop. In LQC, the presence of the *background fiducial metric* ${}^o q$, an object responsible for the rigid relation between a_i and p_i, fixes a unique relation between the embedding (fiducial) area of the loop and its *physical area*. For a loop spanned by (holonomies along) vectors ${}^o e_j$, ${}^o e_k$ we have

$$\mathrm{Ar}(\Box_{\rm jk}) = \epsilon^{\rm i}{}_{\rm jk} \mathrm{p_i} \lambda_{\rm j} \lambda_{\rm k}, \tag{2.10}$$

where λ_i are the fiducial lengths of the straight edges in direction ${}^o e_i^a$ forming the loop.

This relation plays a crucial role in fixing the action of the Hamiltonian constraint in LQC [14]. In principle, in LQC one can set the fiducial edge lengths λ_i freely, which would allow one to construct the loop or arbitrarily small areas. On the other hand in LQG the spectrum of the area operator is purely discrete and the first nonzero value of the area is determined by the theory. Since the goal of formulating LQC was the construction of the simplified settings approximating or mimicking the full theory as close as possible, this particular property (discreteness of the area) has been *parachuted from LQG*. The fiducial lengths λ_i are fixed by the requirement that the physical area of the loop equals

$$\mathrm{Ar}(\Box) =: \Delta = 2\mathrm{A}_1, \tag{2.11}$$

where A_1 is provided by (2.2). The loop of these dimensions is then considered as the minimal loop realized in LQC. The reason why one takes as the minimal area $2A_1$ instead of A_1 [45] is that it should correspond to the area of the surface *pierced* by the edge, not just with the edge terminating on it. This requirement follows from the semiheuristic lattice construction in [21] and will became apparent in the process of constructing the dictionary in further sections.

The particular way in which the requirement (2.11) fixes the lengths λ_i is construction dependent and in the past led to several distinct prescriptions in quantizing the Bianchi I model in LQC (see for example [46]). Subsequently one choice has been distinguished by the construction of the lattice LQG states in [21] and by the requirement that in the noncompact case the theory should admit well-defined and nontrivial infrared regulator (namely the fiducial cell) removal limit.[5] Since it is hoped that the LQC dynamics

[5]Historically, this prescription has been fixed by the requirement that the so-called effective dynamics [47] approximating the genuine quantum dynamic is invariant with respect to the fiducial cell choice [48]. The requirement of the explicit invariance is, however, too strong to be implementable on the genuine quantum level [49] and thus had to be relaxed to the requirement of admitting the well-defined limit as the fiducial cell becomes infinite.

should mimic the LQG one and since the LQG Hamiltonian (constraint) is a quasilocal operator (local up to Planck size loops) it is natural to expect that the dynamics of a universe should not be strongly affected by the global topology of its spatial slices.[6] As a consequence the prescription distinguished in the noncompact case should also be the used in the compact one. Thus, at present the construction introduced in [21] is considered to be the unique consistent prescription.

With the lengths of holonomies fixed and the remaining components in the Hamiltonian constraint regularized via Thiemann construction, one arrives at the Hamiltonian constraint operator, which is a difference operator acting on the domain of elements of $\mathcal{H}^{\rm LQC}_{\rm kin}$ supported on the finite number of points $|p_1, p_2, p_3\rangle$ [see Eqs. (3.35)–(3.37) in [21]].[7] Given that, after coupling with appropriate matter fields the dynamical sector of the theory is determined by either group averaging [50] (see also [8,51] for applications in context of LQC) and partial observable formalism or via deparametrization.

Here, however, we encounter the same problem as in full LQG: the kinematical Hilbert space $\mathcal{H}^{\rm LQC}_{\rm kin}$ is nonseparable (due to the discrete inner product). Since in the deparametrization picture it becomes the physical space, the latter is also nonseparable. In isotropic LQC the procedure of dealing with this problem makes use of the fact that the Hamiltonian (or Hamiltonian constraint) distinguishes certain subsets (lattices) invariant under its action. The subspaces of states supported on those sets form then superselection sectors, each being separable. Subsequently one chooses just one sector to describe the dynamics.

An extension of this approach to the anisotropic LQC is nontrivial, since for example in the Bianchi I case the superselection sector lattices are formed out of families of sets dense on surfaces of codimension 1 in the configuration space [52]. Furthermore, it is not at all obvious that a single sector would admit a proper semiclassical regime, where low-energy dynamics conforms to GR.

In order to deal with such difficulties one can again implement the integral Hilbert space construction [36] following (2.5). In the case of known isotropic models [8,14] the results following from implementing this construction are (up to minor corrections) equivalent to the ones provided by treatment involving just one superselection sector.

III. THE DICTIONARY

The review in the previous section shows clearly that both LQG and LQC describe the geometry via very distinct sets of degrees of freedom. The principal difference between these frameworks is the presence of background (fiducial) geometry in LQC and absence of such in LQG. This LQC background structure is distinguished by the symmetries of the theory on the classical level. By its very construction, LQG does not "detect" these symmetries as there the symmetry transformations are just a specific class of finite diffeomorphisms, to which the framework is insensitive. As a consequence, building an LQG state representing the cosmological spacetime is a nontrivial task [20].

At this point it is worth recalling that the goal of this work is building the relation between the frameworks on the kinematical level, that is, without referring to the dynamics of the system in either theories. Answering the question of whether the LQC dynamics of a state representing a universe is a good approximation to the dynamics of the state representing the same universe in LQG is beyond the scope of this article. Instead, we explore the correspondence between the elements of both theories. In building that correspondence we focus on keeping the full control over the initial assumptions entering the construction and on determining the freedom left by these assumptions.

Our (principal) point of departure is the observation that the basic quantities characterizing the state in LQC—the areas p_i of maximal surfaces orthogonal to the basis triad vectors—correspond to observable quantities well defined for any physical LQG state. They are thus (especially being one of the most fundamental components in constructing the LQC) a natural candidate to use in defining the interface. We apply this observation to build the LQG ↔ LQC correspondence for flat Bianchi I universe. More precisely the type of spacetime and the construction of the physical sector of the theory is chosen as follows:

(i) Our objects of studies will be physical states corresponding to the (homogeneous but not isotropic) Bianchi I universe, of which the spatial slices have a 3-torus topology (although the results can be easily generalized to the noncompact flat case). Thus, the embedding manifold for the kinematical spin networks will be topologically T^3.

(ii) Since the dictionary we are constructing involves the correspondence between the geometry degrees of freedom only and does not employ the dynamics of the system, we can safely assume that it will not depend on the type of matter content coupled to gravity. Thus, we can work with a chosen particular type of matter and the results will automatically generalize to other types of matter. For that purpose we select the construction of the dynamical sector through the deparametrization with respect to the irrotational pressureless dust serving as the time frame in both LQG [5] and LQC [9]—a frame that on the classical level has been originally proposed in [53].

While there is no consensus in the area as to whether the particular matter content selected above is

[6] Here, we compare the topologies admitting the same curvature, like $\mathbb{R}^3$ vs T^3.

[7] For technical reasons (simplicity of the constraints) in [21] different labeling of the kinematical basis states is used.

well motivated physically, this choice provides a simple and precise framework, circumventing the difficulties present in other approaches. In particular the diffeomorphism-invariant Hilbert space in LQG and the kinematical Hilbert space of LQC automatically (and without any modifications or corrections) became the physical Hilbert spaces of their respective theories. The same applies to the area operators in LQG specified in Sec. II A 1 and the triad operators $\hat{p}^i$ in LQC—they become physical observables. As a consequence they can be used directly, when comparing *physical* areas.[8]

(iii) The selected physical states are chosen to be sharply peaked about the Bianchi I spacetime geometry, following the proposal of [20]. Since we allow for a nontrivial spread about the symmetric geometry our formalism *needs to allow* for the departures of the spin networks spanning the state from the homogeneity.

Given the above choices one can associate with each physical state $|\Psi\rangle \in \mathcal{H}^{\rm LQG}_{\rm phy}$ (where we do not require this state to be supported on just one spin network) a cosmological state $|\Phi\rangle \in \mathcal{H}^{\rm LQC}_{\rm phy}$ such that the expectation values of the area operators along the "flat and orthogonal" maximal surfaces agree with the expectation values $\langle\Phi|\hat{p}^i|\Phi\rangle$. Despite the lack of background metric, the notion of flatness and orthogonality[9] can be made precise in terms of the expectation values. Below we present one of possible constructions. It will not be directly used in construction of the LQG-LQC interface and it is presented solely as an example that making the above-mentioned association between LQG and LQC states is possible. One of many ways to define such a construction is the following:

(1) First one chooses on the embedding manifold a point p and a triad of vectors anchored on it.

(2) The angle operator in LQG is well defined [1]; thus, one can distinguish (also in the embedding manifold) a triple of 2-surfaces of T^2 topology—sections of the embedding manifold—such that the respective pairs of distinguished triad vectors are tangent to them (at their intersection). Their relative orientation is then fixed by the requirement that the expectation value of the angle operator corresponds to normal angles.

(3) Finally, the flatness of the surfaces is enforced by the requirement that the surfaces minimize their physical area (again in terms of the expectation values of the area operator).

Then the area of the distinguished surfaces is associated with respective areas $\langle\hat{p}_i\rangle$ in LQC.

This association between LQG and LQC states is far from unique: (i) it is fixed just by relation between expectation values of three observables, which is obviously insufficient to determine the state, (ii) the areas of the distinguished surfaces may depend on the point p and the chosen vector triad, and (iii) there may not be a global minimum of the areas in point 3 of the above construction. At this moment however we do not look for the uniqueness of the association. We just want to show that it is possible to define. It may not be particularly useful for constructing the LQC limit of LQG, especially because so far we have not introduced any notion of symmetry. Literally any (even very inhomogeneous) physical LQG state can be used in this construction.

On top of that deficiency, at present we are lacking any relation with the auxiliary structures in LQC, which is necessary to really understand the relation between the frameworks. Therefore, in further studies we are going to restrict the space of possible physical states by selecting a specific (yet sufficiently large to accommodate the physically interesting spacetimes) set of spin networks supporting the states. Once the appropriate subspace of physical states is selected, it will be easy to introduce the analog of the association presented above.

A. The lattice spin network

Our construction is heavily inspired by the semiheuristic construction introduced in [21]. There, one equipped the embedding $\mathbb{R}^3$ manifold with a fiducial metric ${}^oq$ and used it to define a regular lattice in it. This lattice has been next used to construct a specific spin network by associating with each edge (link) of the lattice a j label ($j = 1/2$) corresponding to the fundamental $su(2)$ representation—a minimal nonzero value allowed by the theory. Then the gauge-invariant state has been distinguished as supported on this spin network only.

Given that state, one could introduce a so-called fiducial cell—a compact region of space acting as the infrared regulator of the theory. It was chosen such that its edges were parallel (in the sense of ${}^oq$) to the edges of the lattice.

[8] Here, in principle, one may consider using also the connection coefficients c_i in the dictionary. However, one has to remember that the connection operator does not exist in either LQG or LQC. Even the triad coefficient operator is defined through the flux operator (see Sec. II B). One could, in principle, consider using the holonomies in its place; however, the action of a holonomy operator in LQG may be prescription dependent as (i) it differs between the graph-changing and graph-unchanging forms of the Hamiltonian, and (ii) it does not preserve gauge invariance and is not used in the Hamiltonian as a stand alone operator. Instead one forms the composite operators containing holonomies and next projects their image onto gauge-invariant subspace. Thus, any interface using it would need to be studied for particular prescriptions on a case by case basis. As a consequence its use in our studies is prevented by the requirement that the proposed interface (and the results following from its application) does not depend on the detailed prescription choices.

[9] We use these terms since for homogeneous spacetimes (of diagonal spatial metric) they coincide with the standard meaning of flatness and orthogonality. This agreement may however not extend beyond that class of spacetimes.

Because of fixing of the j labels the area of each face of the cell was then proportional to the number of the lattice edges piercing it. With these areas one subsequently associated the values of p_i in LQC.

Given that association, the regularity of the lattice allowed in turn to associate with each plaquet (minimal square loop of the lattice) a physical[10] area. Finally, the requirement that all these areas equal Δ (2.11) fixed the fiducial lengths of the edges of the plaquets, which in this construction are the curves along which the fundamental holonomies are taken.

In this article we expand on this idea, dropping however most of the assumptions made in [21]. To start with, we consider a single spin network, embedded in the T^3 manifold defined above. We further assume that our spin network is topologically equivalent to the regular lattice[11] or is a proper subgraph of such. This graph is next equipped with the $su(2)$ j labels on the graph edges (and internal edges at each node)*where in particular the value* $j = 0$ *is allowed.* If the original graph is the proper subgraph of the (topologically) regular lattice, it is completed to the lattice by adding appropriate edges with $j = 0$ and appropriate vertices. We further assume that the lattice is minimal: since two edges of $j = 0$ entering the 2-valent vertex can always be replaced with one single edge, given a lattice spin network we perform such a reduction, whenever it does not destroy the regular lattice topology. Such construction of a spin network, although abstract instead of embedded, is used for example to formulate the *algebraic quantum gravity* [28] framework.

To introduce the cosmological background structure we note that a large class of spatial diffeomorphic gauge fixings can be implemented via equipping the embedding manifold with a metric tensor. For every spin network one can define a fiducial isotropic metric ${}^oq = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ such that $\mathrm{x}_i := (x, y, z)$ are the functions defined along the edges of the graph, preserved by the discrete symmetries of the graph: for a cyclic permutation of the nodes along one lattice direction i[12] the function $\mathrm{x}_i(\vec{x})$ changes as follows,

$$\mathrm{x}_i \mapsto \mathrm{x}_i + \lambda_i, \qquad \lambda_i := 1/n_i, \tag{3.1}$$

where n_i is the number of graph edges forming a closed loop in direction i. The coordinates on the graph are next extended smoothly (nonuniquely) to the whole embedding manifold.

It is worth reiterating that neither the partial gauge fixing introduced here nor the auxiliary structure plays any role in describing the physics. All the physical geometry observables are insensitive to this choice, as their action depends only on the topology of the graph and its quantum labels.

Given the regular lattice defined above, one can precisely implement the construction of the LQG $\leftrightarrow$ LQC dictionary specified at the beginning of Sec. III. For that we choose the constancy surfaces S_i of the coordinates x_i (thus orthogonal to ${}^oe^a_i$). The areas of these surfaces [expectation value of the operator (2.1)] are then associated with LQC triad or flux coefficients

$$\langle \mathrm{Ar}(\mathrm{S_i}) \rangle = \mathrm{p_i} := \langle \hat{\mathrm{p}}_\mathrm{i} \rangle. \tag{3.2}$$

The form of (2.1) implies immediately that these areas do not depend on the way the coordinates x_i have been completed between the elements of the graph. The values of p_i depend however on the j labels of the edges intersecting, terminating, and contained within S_i, which may differ depending on the particular choice of the surface.[13] Thus, this association is not unique. To emphasize this fact, we further denote these values as p_{i,x_i} and the surfaces themselves as S_{i,x_i}. At this point we have to remember, however, that there is some residual diffeomorphism freedom left in the system: the rigid (with respect to ${}^oq$) translations in x_i. We exploit this freedom in the next subsection to complete the dictionary construction.

Before doing so, however, we have to address one issue: since the dictionary will rely on the auxiliary structure, it is critical to check how it will be affected by the dynamics.

In the case where the LQG Hamiltonian is graph preserving (like for example in [28]) there is no problem: the only elements characterizing the state affected by the Hamiltonian are the spin labels. The graph itself does not change; thus, its embedding in the manifold can be assumed to be constant in time. This in turn allows one to keep the auxiliary structure constant.

The situation complicates a bit when the Hamiltonian is *graph changing*. There, the preservation of the structure of the graph depends on the particular form of that Hamiltonian. For example, in original canonical LQG construction of [26] the Hamiltonian adds edges with $j = 1/2$ labels forming triangular loops with existing ones. This can be easily implemented in the construction considered here, if instead of triangle we add a square loop with two new $j = 1/2$ edges. The new spin network can be then easily be completed to a (topologically) regular lattice by adding $j = 0$ edges. Subsequently, the auxiliary structure can be easily rebuilt in one of two ways:

[10]On the formal level this value cannot be associated with the expectation value of the LQG area operator since no edge intersects such a plaquet. It can however be made precise with use of the so-called *dual graph*—a technique often applied in the spin-foam approaches.

[11]To be mathematically precise, we define the graph that admits a set of discrete symmetries of the regular (closed) lattice on T^3.

[12]The *direction* is defined here by topology of the graph: each direction is the set of the classes of equivalence of minimal closed loops not shrinkable to a point on the embedding manifold.

[13]We remind the reader that no restriction is made on the distribution of the values of j labels on the graph.

(1) The coordinates x_i and fiducial metric can be redefined so the new lattice becomes regular in them. This corresponds to shifting the vertices of the graph to new positions on the embedding manifold (passive diffeomorphism). The consecutive shifts will however be discontinuous in time. This discontinuity is however not a problem, as the Hamiltonian flow is not used in the construction of the dictionary and the auxiliary structure can be defined at each time slice independently.

(2) The new nodes can be placed in the center (with respect to ${}^oq$) of existing plaquets. Then the lattice would be a particular realization of the *dense spin network* [54]. It would however lose the regularity. The latter could, in principle, pose a problem as the fiducial length of the edges will be an essential component of the dictionary. In that case however one should average any quantity evaluated on the graph over the diffeomorphisms changing the fiducial lengths of the edges but preserving the directions (with respect to metric ${}^oq$) of the edges of the graph. For quantities that are averages weighted linearly by the fiducial lengths this averaging procedure yields exactly the same results as the "uniformization" defined in the point above (see Appendix A).

It is important to note that the regularity assumption can be replaced with the average over (passive) diffeomorphisms preserving the directions of the edges (with respect to ${}^oq$). Indeed, the group of such diffeomorphisms can be split onto three subgroups, each corresponding to (nonrigid) translations along one triad vector ${}^oe_i$. Then the physical area of the plaquet (or the entire maximal surface) is invariant with respect to the diffeomorphisms shifting the graph components in directions tangent to the plaquet (maximal surface). On the other hand, the (only nontrivial) averaging over diffeomorphisms shifting in the remaining orthogonal direction is mathematically equivalent (see Appendix A) to selecting the lattice regular in that direction.

B. The averaging procedure

At present a relevant difference remains between the LQG state constructed previously and its LQC analog. The LQC state is constructed with the implicit assumption of representing the homogeneous spacetime, whereas the LQG one can *a priori* be highly inhomogeneous. This implies that the association of the values of p_i has to involve some kind of averaging (over the inhomogeneities) procedure. In LQG the lack of background structure makes the definition of such averaging difficult. Here however the choice of the spin-network graph and partial gauge fixing is allowed to construct the necessary background.

To define the averaging, we now consider the remaining rigid translations as the *active transformations* shifting the surfaces S_{i,x_i} along the graph and define the values p_i as the averages with respect to this translation group. In the mathematically precise sense the variables p_i are chosen to equal the expectation values of the area operator (of each surface S_i) *averaged over the rigid spatial translation group*. A simple calculation using (2.1) shows then that

$$p_i = \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_i}\sum_{e\in\{e\}_i}\sqrt{j_e(j_e+1)} =: \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_i}\Sigma_i, \qquad (3.3)$$

where $\{e\}_i$ is the set of all the edges of the graph that point in the direction of ${}^oe_i$ and $n_i := 1/\lambda_i$ is the number of graph edges pointing in the direction of ${}^oe_i$ and forming a closed loop. Here, the edges terminating on the surface or contained within it do not contribute to the average, as having such a situation (edges terminating/embedded) would occur only for a discrete group of translations, forming the zero measure set within the distinguished translation group.

The next necessary component of our dictionary is the (physical) areas of the minimal loops (plaquets) needed to approximate the field strength operator. To evaluate these areas we proceed exactly as in the case of the surfaces S_i: we average the relevant area operators over the rigid (active) translation group. Again, a simple calculation yields (here we denote these areas by σ^i)

$$\sigma_i = \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1n_2n_3}\sum_{e\in\{e\}_i}\sqrt{j_e(j_e+1)} = p_i\frac{\lambda_i}{\lambda_1\lambda_2\lambda_3}. \qquad (3.4)$$

As a consequence, the ratio σ_i/p_i (no summation over i) does not depend on the j labels of the spin network. It depends *only on the number of edges of the graph*, which is an expected result since (once the averaging is implemented) each surface S_i can be simply composed out of $n_1n_2n_3/n_i$ plaquets. It is important to note that even though we are permitting the edges with $j=0$, the numbers n_j are invariant due to gauge invariance and the requirement that the lattice is minimal.

The standard procedure implemented in LQC would call in the next step for an association $\sigma_i = \Delta$, where Δ is defined via (2.11). This would fix λ_i as the functions of the phase space, leading exactly to the dependence found in [21]. Here however we do not implement this step, expecting in turn that the value of σ_i should follow from the properties of the spin network. Thus, the only property at our disposal is the relation of σ_i with the average (over the graph) of j labels associated with edges in direction ${}^oe_i$,

$$\sigma_i = 8\pi\gamma\ell_{\rm Pl}^2\overline{[\sqrt{j(j+1)}]_{e_i}} =: \Delta_i, \qquad (3.5)$$

where the symbol $\overline{[\cdot]}$ denotes the average over the graph. For the models aimed to reproduce the cosmological spacetime via specific semiclassical states the value of such an average depends on the details of the model and may, in principle, differ significantly from the value $\Delta_i = \sqrt{3/2}$ consistent with $\sigma_i = \Delta$ (see for example [31]). In such models, the values σ_i do not need to be fixed by any fundamental constant and *a priori* may depend on the state.

1. Alternative averaging procedure

The results of the averaging procedure implemented above can be easily understood on the intuitive level if we introduce a convenient decomposition of the group of rigid translations. First, one can introduce a discrete group $\mathbb{Z}^3_{n_1,n_2,n_3} := \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \mathbb{Z}_{n_3}$ of cyclic permutations of the graph vertices. The quotient of the group of translations over $\mathbb{Z}^3_{n_1,n_2,n_3}$ is the group of translations over the distances $\delta_i \in [0, \lambda_i)$ (modulo λ_i). The averaging procedure can now be split onto two steps:

(i) Averaging over $\mathbb{Z}^3_{n_1,n_2,n_3}$, which simply replaces the j label of a single edge with the average $\overline{[j]}_{e_i}$ of all the edges parallel to it.

(ii) Averaging over the quotient group. The result of this step follows directly from the observation that upon the action of the quotient group (with the exception of the group neutral element that forms a zero measure set) each plaquet of the graph is intersected by exactly one edge (now carrying the averaged j) orthogonal to it.

The above procedure leads immediately to (3.5) and (3.4) and further, after reassembling the surfaces S_i out of the plaquets, to (3.3).

The potential application of the studies performed here to the models where the cosmological spacetime is defined by the semiclassical (often chosen to be coherent) state leads to another complication. So far we have considered the single graph. Although such a choice is perfectly fine to define a basis of a Hilbert (sub)space, it may be insufficient for such models. Therefore, one needs to extend the dictionary to incorporate a large number of such spin-network superselection sectors. We provide such an extension below, using the integral Hilbert space construction presented in Sec. II A 3.

C. The integral extension

As in the case of a single lattice space, here we define some subspace of $\mathcal{H}^{\rm LQG}_{\rm phy}$.

(1) We start with a single lattice spin network defined in Sec. III A (without introducing the background metric ${}^o q$).

(2) The plaquets of the spin network define three classes of surfaces of topology T^2 (maximal surfaces on the embedding manifold)[14] such that within one class the surfaces do not intersect each other and all the intersections of the surfaces of distinct classes are of S^1 topology.

(3) The discrete classes of surfaces are next completed to congruences of the embedding manifold (using the surfaces of the same topology) keeping the requirement that intersections between representants of different classes are S^1. This (nonunique) extension always exists.

(4) We extend the original lattice spin network to a class of disjoint spin networks of which edges are intervals of the intersections of surfaces defined above. This class is selected in such a way that
   (a) each point of the embedding manifold is a node of exactly one spin network in the class, and
   (b) given two graphs of the set, the maximal T^2 surfaces of point 2 are interlaced, that is, within each class of surfaces, between two surfaces of one graph there is exactly one surface of the other graph.

(5) Each spin network is completed to a (topologically) regular lattice by adding edges with $j = 0$.

In its essence this method produces a continuum of (topologically) regular lattices of which edges are parallel. They define a distinguished coordinate system x_i where the coordinates are functions constant on the T^2 surfaces from point 3. Given this coordinate system, one can equip the manifold with a background fiducial metric ${}^o q = \sum_{i=1}^3 (\mathrm{dx}_i)^2$. This construction is of course quite restrictive; however, it allows one to preserve the well-defined notion of principal (diagonal) directions of the single lattice.

One way to produce the specific example of such a set of spin networks is to start with one regular lattice, equip the embedding manifold with the fiducial metric ${}^o q$ (as in Sec. III A), and then act with the active rigid translations defined in Sec. III B. The family of possible continuous sets defined in points $1.-5.$ is however much bigger. In particular, the lattices do not need to be regular with respect to the metric ${}^o q$.

An important property of the selected set of spin networks is that it admits a well-defined Lebesgue measure $\mathrm{d}\sigma$ induced by the Lebesgue measure of a minimal cube of any (arbitrarily chosen) lattice within the set. This measure can now be used to construct the integral Hilbert space via (2.5) via setting $\mathrm{d}\mu(\epsilon) = \mathrm{d}\sigma$. Choosing different minimal cubes will lead to unitarily equivalent spaces.

Given the new (integral) Hilbert (sub)space we proceed with defining averaged quantities p^i, σ^i exactly as in Sec. III B. The only difference is an additional integration over the selected set of lattices. The calculations yield

$$p_i = \frac{8\pi\gamma\ell^2_{\rm Pl}}{n_i} \int \mathrm{d}\sigma(\vec{\epsilon}) \sum_{e\in\{e\}^i_{\Gamma(\vec{\epsilon})}} \sqrt{j_e(j_e+1)} =: \frac{8\pi\gamma\ell^2_{\rm Pl}}{n_i} \bar{\Sigma}_i, \tag{3.6a}$$

[14] One can introduce a notion of parallel edges terminating in a 6-valent node as the pair not being the edge of a single plaquet and next build the surface by selecting a plaquet and extending the surface by including plaquets whose least two edges are parallel to edges contained already by the surface.

$$\sigma_i = \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1 n_2 n_3}\int \mathrm{d}\sigma(\vec{\epsilon}) \sum_{e\in\{e\}^i_{\Gamma(\vec{\epsilon})}} \sqrt{j_e(j_e+1)}$$
$$= p_i \frac{\bar{\lambda}_i}{\bar{\lambda}_1\bar{\lambda}_2\bar{\lambda}_3} = 8\pi\gamma\ell_{\rm Pl}^2 \overline{[\sqrt{j(j+1)}]}_{e_i,\vec{\epsilon}} =: \bar{\Delta}_i, \quad (3.6b)$$

where $\vec{\epsilon}$ labels the superselection sectors and $\Gamma(\vec{\epsilon})$ is the graph corresponding to superselection sector $\vec{\epsilon}$. We see that the quantities λ_i, Δ_j are now simply replaced by their averages $\bar{\lambda}_i$, $\bar{\Delta}_j$ over the superselection sectors. The quantities $\bar{\lambda}_i$ are now *defined* by (3.6b) and do not necessary correspond to the average over fiducial lengths of the edges (which in turn may not be the constants of the graph). The average j label is however a proper average

$$\bar{\Delta}_j = \int \mathrm{d}\sigma(\vec{\epsilon})\Delta_j(\vec{\epsilon}). \quad (3.7)$$

The above result has been found under the assumption of a specific compact topology of the embedding manifold. However, in LQC consistency restrictions that actually do fix the theory originate in models of the noncompact universes, some of them already in the isotropic sector. Therefore, it is prudent to extend our dictionary to such a case and incorporate in it the notion of isotropy. For simplicity we consider single lattice states only, although the generalization to the integral states of Sec. III C is not difficult.

D. The noncompact extension and the isotropy

In order to keep the model as simple as possible we consider its extension to the flat space, assuming R^3 topology.

1. The extension to $\mathbb{R}^3$

In LQC the standard method of dealing with infinities due to Cauchy slice noncompactness is a selection of a compact region—the so-called fiducial cell—which then becomes the infrared regulator of the model. The physical predictions are then extracted within the regulator removal limit. Well definiteness of that limit is the first consistency condition imposed on LQC and is precisely the origin of the so-called *improved dynamics* prescription [14]. Here, we follow the same idea: We start with the construction of a single lattice spin network, as specified in Sec. III A, although now the lattice is open and infinite. We then introduce the background structure exactly as in the compact case and distinguish the regulator—a rectangular cube of edges pointing in directions of ${}^o e_i$. The expansion of the regulator is well defined in terms of the number of edges forming the interval of "straight lines" that is contained within the cube (or, equivalently, the number of elementary cells stacked along the edge of the cube).

The remaining spatial gauge freedom of the model is the same as in the compact case: the rigid translations. Now however we encounter a technical difficulty: the translation group is noncompact and does not preserve the regulator structure. We sidestep this problem, introducing the periodic boundary conditions on the faces of the regulator, thus restricting to a certain compact "cyclic translation" group. The new translations are obviously not elements of the original translation group; however, the proposed "trick" is well motivated on the heuristic level, as (i) their action will equal that of the original translation group elements on those spin networks that are composed of the set of identical "copies" of the portion contained within the regulator cell, and (ii) in the regulator removal limit we recover the original translation group.

On the technical level the above construction brings us exactly to a compact setting T^3 topology considered in previous subsections. Thus, we proceed with construction of the dictionary exactly as before. The results (3.3)–(3.5) remain true here. In the regulator removal limit the values p_i reach infinity; however, the plaquets areas are well defined,

$$\sigma_i = \lim_{n_1,n_2,n_3\to\infty} \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1 n_2 n_3} \sum_{e\in\{e\}^i} \sqrt{j_e(j_e+1)}$$
$$= 8\pi\gamma\ell_{\rm Pl}^2 \overline{[\sqrt{j(j+1)}]}_{e_i} =: \Delta_i, \quad (3.8)$$

provided that the limit on the right-hand side exists. It is however a reasonable expectation if we consider an asymptotically homogeneous state. As a consequence the relation (3.5) extends to the noncompact case.

2. The isotropic sector

In comparison to the homogeneous nonisotropic spacetimes considered so far, the isotropic ones admit an additional symmetry class (subgroup)—the rotations.

Implementing these symmetries in the compact T^3 case is not possible; thus, our starting point is the noncompact setting of the previous subsubsection. Here, we make one additional initial assumption: we restrict the auxiliary metric by requiring that the fiducial lengths of the edges in all three directions are the same. the flat metric ${}^o q$ now defines the group of rigid rotations. As in the case of the translations, we consider them as active diffeomorphisms and average over them the observables used to define the dictionary.

Because of noncompactness, the only meaningful element of the dictionary is the average area of plaquets σ_i. The rotational transformation of the spin network can be easily parametrized by the Euler angles. As in the case of translations, here we can distinguish a discrete group if the rotations by proper angles, which due to insensitivity of the area operator to the orientation of the edges can be replaced by a group Σ^3 of permutations of ${}^o e_i$. We can then distinguish a quotient group $SO(3)/\Sigma^3$.

Let us first consider the averaging over (the group of translations and) Σ^3. It follows immediately from (3.8) that

$$\sigma_\Sigma := \lim_{n_1,n_2,n_3\to\infty} \frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1n_2n_3}\frac{1}{3}\sum_{e\in E(\Gamma)}\sqrt{j_e(j_e+1)} = \frac{8}{3}\pi\gamma\ell_{\rm Pl}^2\overline{[\sqrt{j(j+1)}]}_{E(\Gamma)} = \frac{1}{3}[\Delta_1+\Delta_2+\Delta_3] =: \Delta_\star, \tag{3.9}$$

where this time we sum over *all* the edges of the spin network. This result is identical with the Friedmann-Robertson-Walker (FRW) limit of Bianchi I geometry (in LQC) studied in [21].

The averaging over (translations and) full $SO(3)$ is slightly more involved.

$$\sigma_R := \lim_{n_1,n_2,n_3\to\infty}\left[\frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1n_2n_3} \times \int_{SO(3)} \mathrm{d}\sigma \sum_{e\in E(\Gamma)} |{}^o q_{ab} n^a {}^o e(e)^b_i| \sqrt{j_e(j_e+1)}\right], \tag{3.10}$$

where $\mathrm{d}\sigma$ is the measure on $SO(3)$, n^a is a unit (in ${}^o q$) vector orthogonal to the plaquet, and ${}^o e^a_i$ is the fiducial triad element tangent to the graph edge e. The factor ${}^o q_{ab} n^a {}^o e(e)^b_i$ is a consequence of averaging over spatial translations as only the orthogonal (to ${}^o e_i$) "cross section" of the plaquet will contribute to the average over the translations along ${}^o e_i$. Using the known chart of $SO(3)$ defined by Euler angles [convention $Z(\alpha)X(\beta)Z(\gamma)$],[15] we get

$$\begin{aligned}\sigma_R &= \lim_{n_1,n_2,n_3\to\infty}\frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1n_2n_3}\frac{4}{\pi^2}\int_{\alpha\beta,\gamma\in[0,\ldots,\pi/2]} \\ &\quad\times \mathrm{d}\alpha\sin(\beta)\mathrm{d}\beta\mathrm{d}\gamma\frac{1}{3}[\sin(\beta)(\cos(\gamma) \\ &\quad + \sin(\gamma)) + \cos(\beta)]\sum_{e\in E(\Gamma)}\sqrt{j_e(j_e+1)} \\ &= 4\pi\gamma\ell_{\rm Pl}^2\overline{[\sqrt{j(j+1)}]}_{E(\Gamma)} = \frac{3}{2}\Delta_\star,\end{aligned} \tag{3.11}$$

where to write the first equality we split the rotation group onto Σ^3 and the quotient $SO(3)/\Sigma^3$.

As we can see, the contribution from the general angle rotations increases the physical plaquet area by a factor of $3/2$. This discrepancy is unexpected, since the critical energy density (upper bound of the matter energy density operator spectrum) is a bijective function of the area gap and both Bianchi I and FRW spacetime models in LQC provide the same value of that quantity (see [55] versus [8]). To address this discrepancy we investigate a bit closer the rotation group used in the averaging process: the rotations in fiducial metric ${}^o q$.

To do so, let us consider a large (classical size) cube of fiducial size L. Denote its surface area (averaged over the translation group and Σ^3 to mimic an isotropic spacetime as close as possible without rotating the graph) in the case when its edges are oriented along the triad vectors ${}^o e_i$, by $A_\Box$. By repeating the same calculation as in (3.11) one can show that upon rotating this cube by Euler angles (α,β,γ) the surface area changes as follows:

$$A_\Box(\alpha,\beta,\gamma) = A_\Box[\sin(\beta)(\cos(\gamma)+\sin(\gamma))+\cos(\beta)]. \tag{3.12}$$

This implies in particular that even if the j-label distributions in all the directions are the same this surface area *is not invariant under the rotations*. As a direct consequence of it we can formulate a no-go statement: one cannot build the isotropic spacetime using lattice spin network with the edges oriented in directions of one particular vector triad. The original construction proposed in Sec. III A has to be improved.

E. The improved lattice states

The lack of invariance of areas under the rotations noticed in the previous subsection can be cured in a straightforward way if instead of the spin network(s) oriented in particular directions one considers a large set of networks of edges oriented with respect to randomly chosen (randomly rotated) triads ${}^o e_i$. Implementing such a construction is quite easy, if for example one defines the integral Hilbert space structure (discussed in Sec. II A 3) using the integral measure of $SO(3)$ group. Unlike in the previous construction, this step is necessary, as the single superselection sector would correspond to the spin network (s) with edges pointing in directions of a fixed triad ${}^o e_i$, and thus would feature a deficient behavior under rotations (as established in previous subsection).

In this construction, however, even the anisotropic universe model will attain the modification due to integrating over the proposed structure. Indeed, in this case one can rotate the original triad ${}^o e_i$ and associate with each rotation a lattice state (orthogonal with respect to the original lattice state). The set of states supported on new (rotated) lattices forms now a set of superselection sectors [provided the inner product is introduced as in (2.5)]. Then

$$\sigma_i = \int_{SO(3)} \mathrm{d}\sigma_{SO(3)}\left[{}^o q_{ab} {}^o e^a_i M(g)^b{}_c {}^o e^c_j \times \lim_{n_1,n_2,n_3\to\infty}\frac{8\pi\gamma\ell_{\rm Pl}^2}{n_1n_2n_3}\int \mathrm{d}\sigma(\vec{\epsilon})\sum_{e\in\{e\}^j_{\Gamma(\vec{\epsilon},g)}}\sqrt{j_e(j_e+1)}\right], \tag{3.13}$$

[15] Instead of rotating the plaquet, we keep it fixed at $z=0$ and rotate the spin network.

where g is the finite rotation (parametrized by Euler angles) and $M(g)$ is the rotation matrix corresponding to it. The symbol $\Gamma(\vec{\epsilon}, g)$ denotes here graphs oriented along the fiducial triad ${}^o e_i$ rotated by g (that is, a support of a superselection sector of lattices oriented in the rotated triad) and belonging to the superselection sector labeled by $\vec{\epsilon}$. Using again the $SO(3)$ chart defined by Euler angles gives then

$$\sigma_i = \frac{\gamma \ell_{\mathrm{Pl}}^2}{\pi} \int_0^{2\pi} \mathrm{d}\alpha \int_0^{\pi} \sin(\beta)\mathrm{d}\beta \times \int_0^{2\pi} \mathrm{d}\gamma \left[\sum_{k=1}^{3} (e_i \cdot M(\alpha,\beta,\gamma) e_k) \times \overline{[\sqrt{j(j+1)}]}_{e_k,\vec{\epsilon}}(\alpha,\beta,\gamma) \right] =: \tilde{\Delta}_i, \qquad (3.14)$$

where the rotation matrix M are now expressed in terms of Euler angles and the averages over the translation superselection sectors of j labels (originally defined in Sec. III C) are now defined separately for each rotation superselection sector labeled by (α, β, γ).

In the case the averages of j labels of edges in the same direction over the graph (within a single superselection sector) are equal, a simple calculation shows that

$$\sigma_i = 4\pi\gamma \ell_{\mathrm{Pl}}^2 \sum_{i=1}^{3} \overline{[\sqrt{j(j+1)}]}_{e_i,\vec{\epsilon}} = \frac{1}{2} \sum_{i=1}^{3} \bar{\Delta}_i, \qquad (3.15)$$

where $\bar{\Delta}_i$ is defined in (3.6b). This leads exactly to the correction of the Bianchi I plaquet area by a multiplicative factor $3/2$ restoring the consistency with the isotropic limit.[16]

F. The physical consequences

For both constructions of separable physical Hilbert space (the single superselection sector and the integral one) we reached the same conclusion. Given a lattice LQG state specified in Sec. III A or III B (as well as its extension discussed in the previous subsection) representing the Bianchi I universe, if we want to mimic it in the Bianchi I LQC model, that model has to feature the area gap (single plaquet area) proportional to the average of j labels (in the appropriate direction) of the original LQG state. The time dependence of that average depends in turn on (i) the choice of the (initial) LQG state and (ii) the statistics of the particular Hamiltonian used to generate the time evolution. In particular, any model following from the strictly graph-preserving Hamiltonian[17] will have

$$\sigma_i \propto p_i, \qquad (3.16)$$

which will lead to original Bojowald's prescription in LQC [57]. On the other hand, the studies of the noncompact model show that in the low-energy limit σ_i should be constant. As a consequence, for the class of states considered in this article the average j label, more precisely Δ_j, should approach constant in the low curvature limit; thusm the expansion of the spacetime in the process of dynamical evolution should follow from increasing the number of spin-network nodes rather than the j labels.

This conclusion is supported by our intuitive understanding of the physical distance. Essentially, by examining the definitions of meter and second one sees that the definition of a distance unit can be recast as a certain number of the spatial oscillations of the electromagnetic field corresponding to the photon of certain energy (defined in turn by particular spontaneous emission process). On the other hand, the coupling of matter to gravity in LQG leads to theory where the matter degrees of freedom are represented by quantum labels living on the nodes (vertices) or the edges of the spin network (depending on the type of matter). This leads to an intuition that the physical distance should be proportional to the "number of inhomogeneities" a given interval is able to accommodate, and thus should be proportional to the number n_i of the spin-network edges.

The observations from the noncompact extensions and the presented intuitions imply that from physically viable models mimicking the cosmological spacetime by the LQG semiclassical state one should expect the average j label to remain constant at low curvatures at least in the leading order (see the discussion in Sec. II B). This consistency however does not fix the asymptotic value of Δ_i, which however is still subject to the constraints following from observations in high-energy particle physics, although the upper bounds following from it are too huge to be useful.[18] At present it seems that the more precise values can be provided only by statistical analysis of the dynamical evolution within specific frameworks of (or approximations to) LQG.

One of the ways of determining that value from genuine LQG is provided by the interface of Chern-Simons theory with LQG used to evaluate black hole entropy [59–66]. Indeed, the comparison of two statistical calculations in [59] and [60] shows that the edges with $j > 1/2$ provide a significant contribution to the surface areas; thus, the average "area gap" Δ_i should be detectably larger than Δ. On the other hand, more detailed combinatorial analysis of the relevant statistics [67] shows that for small areas the black hole entropy features a "stairlike" structure. It

[16] In this case the averaging over rotations can be performed as averaging of j labels between the rotation superselection sectors.

[17] We recall that we are considering the standard canonical LQG, where in particular the curvature is evaluated (approximated) by holonomies along closed loops formed of the edges of the spin-network graph. Outside of this context there exist different proposals of representing the curvature (see for example [23] expanding on the techniques of [56]) and the presented conclusion may not be applicable to prescriptions using them.

[18] The best to date estimate on the scale of quantum gravity effects follows from estimates on modifications to electromagnetic radiation dispersion relations. The results of statistical analysis of gamma ray bursts [58] provide (indirectly) the upper bound on j of the order 10^10.

indicates that the distribution of j labels is peaked about a certain value and increasing the number of edges intersecting the horizon by one statistically increases the horizon area by the area corresponding to that peak value of j label. The numerical simulations [61] determined it to approximately equal $\Delta' \approx 7.565\ell_{\rm Pl}^2$, which would give the average $\sqrt{j(j+1)} \approx 1.267$ (roughly corresponding to average $\bar{j} \approx 0.86$). The stairlike entropy structure dissipates for larger area due to dispersion in j and the complicated nature of the spectrum of area operator [68]. However, the statistical effects of adding one edge are expected to be the most prominent exactly for low areas (small number of graph edges intersecting the horizon).

While heuristic, the above argument provides a strong indication that, while the LQC area gap will not correspond exactly to Δ, it will remain of the same order. Given an interface constructed here it further suggests a specific correction to that area gap,[19] here proposed to be $\Delta \approx 11.3\ell_{\rm Pl}^2$.[20] Its value can be determined more precisely via use of the same statistical methods originally applied to find the distribution of the magnetic spins [59,60,70]. This is however beyond the scope of this article.

IV. CONCLUSIONS

In this work we explored the problem of relating the loop quantum gravity physical states possibly representing a homogeneous universe with their counterparts in loop quantum cosmology. Our goal was to provide a robust tool allowing one to verify/improve the heuristic components of LQC formalism as well as provide possible insights regarding the choice of physically viable quantization prescriptions in LQG, thus providing a useful control device for both LQG and LQC. For that reason, instead of attempting to construct the dynamical cosmological limit of LQG, we focused on building an interface between both formalisms on the kinematical level (that is, without controlling the dynamics on any side). In order to keep the interface applicable to as wide as possible family of existing/future constructions of cosmological limits of LQG, we based its construction solely on relations between LQG and LQC states that are natural to impose in the construction of any such limit. Furthermore, we avoided any possible simplifications to the formalism on the full theory side of the interface: LQG has been applied at its genuine level.

[19]The idea that the value Δ' should replace the LQC area gap was originally suggested in [69] by authors of [61]. It was however subsequently abandoned due to lack (at that time) of justification for such a choice.

[20]The value of Δ' is multiplied by the factor $3/2$ following from the construction in Sec. III E since originally Δ' corresponds to the quantum of the area of black hole horizon, which is a *fixed surface*. Active rotations are not symmetry transformations in this context.

In specification of the interface we focused our attention on Bianchi I spacetimes of toroidal spatial topology. To define the relations mentioned in the previous paragraph we used the observable quantities well defined both in loop quantum gravity and cosmology. Further, we used the specific subspace of genuine LQG states, distinguished by minimal selection criteria allowing one to define the necessary components of the LQC auxiliary structure for these states. The selection of this subspace was motivated by construction originally proposed in [21].

In order to allow for sufficiently large physical Hilbert subspace in LQG two constructions of states were considered: the states supported on a single spin network and the continuous linear combinations of states supported on distinct graphs. In the case the physical state is supported on one spin network we assumed that the spin networks supporting the state are topologically equivalent to subnetworks of the regular lattices. When the state is composed of a continuum of states on distinct spin networks we further provided a notion of congruence of the embedding manifold by parallel lattices. This criterion, being the sole restrictive condition on the LQG physical Hilbert space, allowed one to provide a precise notion of global orthogonal[21] directions on the spatial manifold.

This structure has proven to be sufficient to define all the remaining auxiliary LQC structure necessary to construct a dictionary. The embedding manifold has been equipped with LQC fiducial background metric via partial gauge fixing. Upon this fixing the spin network supporting the state became regular lattice. The choice of (metric) regularity of the lattice was however not relevant in further construction of the dictionary, being instead a matter of convenience. Indeed, it was shown that averaging over the diffeomorphisms preserving the global orthogonal directions leads to the same results.

Given the selected class of spin network and the auxiliary structure provided by the partial gauge fixing the precise dictionary was constructed, where to identify the states in two frameworks we used two classes of observables: areas of global T^2 slices of the spatial (embedding) manifold—the maximal surfaces and the areas of the square plaquets defined by the minimal loops of the LQG spin network. This was achieved by averaging the LQG area operators corresponding to these surfaces over the diffeomorphism transformations remaining after the gauge fixing—the rigid translations—which were considered as active transformations. In the process no restrictions regarding the distribution of the quantum numbers (j labels) on the spin-network graph have been made.

Despite the lack of control of the dynamics and very weak assumptions made when constructing the LQG states (for example no specific restriction to near-homogeneous

[21]The meaning of orthogonality has been defined subsequently by no longer restrictive partial gauge fixing.

spacetimes) the interface has been sufficient to provide a series of interesting results. In particular, the LQC area of each plaquet (small loop used to evaluate the curvature) has been associated (strictly and precisely) with the average of j labels (orthogonal to the plaquet) of the LQG spin network. No restrictions stronger that this relation (3.4)–(3.5) have been found. From there we concluded that the particular selections of the values of the plaquet areas as the LQC phase space functions (known as LQC prescription choices) depend solely on the statistical properties of the Hamiltonian (constraint) generating the evolution of the spin network and are not restricted by the principles chosen in the construction of the LQG-LQC interface. In particular, both the Bojowald's μ_o prescriptions and the so-called *improved dynamics* can be *a priori* realized.

The results were further extended to the case of flat Bianchi I universe of topologically $\mathbb{R}^3$ spatial slices, where the relation found in T^3 case persisted unmodified (3.8). This relation and the consistency requirements on the LQC framework (existence of well-defined infrared regulator removal limit) imply some restrictions on j-label statistics of LQG: for the class of states used to build the interface the averages of j labels have to remain constant on the low-energy (gravitational field) limit.

The extension to the noncompact universe was subsequently applied to study the flat (isotropic) FRW spacetime, where the plaquet area operators were further averaged over additional symmetries admitted by these classes of spacetimes—the rotations. This process related the (now unique) plaquet area with the average of j labels over *all* the edges of the spin network graph (3.9). Two levels of implementation of the symmetries were considered: averaging over a discrete group of rotations by proper angles and the full $SO(3)$. Studies of the former case lead to the FRW limit of Bianchi I cosmology consistent with the analogous limit found in [21]. In the latter it was found that due to contributions of all the graph j labels in the case the rotation angles differ from (multiples of) proper ones the area of the plaquet is larger by a factor $3/2$.

Because of the apparent discrepancy of the above result with the FRW limit of Bianchi I found in [21] the changes of areas in considered model have been investigated with more care. Consequently, it was found that the set of lattices oriented in directions of one distinguished triad is insufficient to support the states accurately reproducing an isotropic spacetime. Consequently, an extension of the Hilbert space using the integral structure defined by the group of rotations was proposed. It was further shown that on the extended space the discrepancy is cured and the single plaquet area in the models of Bianchi I universes is also increased by a factor $3/2$.

The found results have been finally confronted with the heuristic estimates of the j statistics following from studies of black hole entropy in LQG. The dictionary constructed in this article indicates that associating the plaquet areas with the minimal nonzero LQG area is accurate only in cases when the spin network statistics makes $j > 1/2$ nongeneric (zero measure contribution). On the other hand, the (known in literature) heuristic results following from numerical analysis of black hole entropy provide a natural (from the point of view of the constructed dictionary) estimate on the j-label averages. This estimate leads again to the *constant area gap* principle of improved dynamics. it indicates however a slightly different value of this area gap, corresponding to the LQC critical energy density $\rho_c \approx 0.19\rho_{\mathrm{Pl}}$. This value, while lower than the original LQC critical energy density ($\approx 0.41\rho_{\mathrm{Pl}}$), remains at the same level of magnitude.

The results listed above show that the interface is serving its intended purpose of providing a control over certain heuristic input or prescription choices in LQG and LQC formalisms. In consequence it provides a viable tool to analyze the cosmological limit of more advanced models aimed towards controlling or approximating the LQG dynamics (see for example [24,31]). Since (i) the (restrictive) selection criteria are precisely controlled here and the formalism remains relatively general, and (ii) the formalism is adaptable to the majority of prescriptions in defining the Hamiltonian (constraint) in LQG, it can be applied to a wide variety of models. It allows one to extract the cosmological degrees of freedom out of such models in a precise way, further providing a tool for validating the initial assumptions selected in their construction (like for example the statistical averages of j distributions). In particular, through the consistency conditions of LQC in the case of noncompact universes it provides a tool for consistency control of the LQG models: it implies that in the low-energy limit the average of the spin-network j labels needs to approach a constant.

At this point it is necessary to remember that the interface relies on quite strong restriction of the spin-network graph topology. In principle, no such restrictions should be made in order for the results to be completely robust. Any generalization, however, for example using the random graphs [71] is extremely difficult, as in such cases the LQC auxiliary structure (being a relevant part of the interface) has to emerge on the physical level (via observables) and may strongly depend on the j-label statistics of the physical states, which in turn is decided by the details in Hamiltonian (constraint) construction.

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APPENDIX: AVERAGING OVER DIFFEOMORPHISMS

Consider a one-dimensional lattice (n edges) on a circle (parametrized by a coordinate $x \in [0,1)$) with random distribution of the vertices of uniform probability measure. Suppose that each edge of this lattice is equipped with a label x_j taking the values in (the subset of) $\mathbb{R}$. Consider now the average of some function $f(x_i)$ weighted by the length of each edge

$$\bar{f} = \sum_{i=1}^{n} f(x_i) l_i. \tag{A1}$$

The probabilistic space of the edge length distribution is the n-dimensional romboid

$$\sum_{i=1}^{n} l_i = 1, \quad \forall\ i \in \{1, \ldots, n\}:\ l_i > 0, \tag{A2}$$

with the measure $\mathrm{d}\sigma = \mathrm{d}l_1 \ldots \mathrm{d}l_n$. The volume of this romboid is $V_n = 1/n!$. The average value $\langle l_i \rangle$ of l_i is the ratio of the volume of the romboid over the $n-1$-dimensional volume of its base, which is V_{n-1}. As a consequence we have

$$\langle \bar{f} \rangle = \sum_{i=1}^{n} f(x_i) \langle l_i \rangle = \frac{1}{n} \sum_{i=1}^{n} f(x_i), \tag{A3}$$

which corresponds precisely to the case where the vertices of the lattice are distributed uniformly along the interval.

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