Making a tournament k-arc-strong by reversing or deorienting arcs

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Abstract

We prove that every tournament $T = (V, A)$ on $n \geq 2k + 1$ vertices can be made k-arc-strong by reversing no more than $k(k + 1)/2$ arcs. This is best possible as the transitive tournament needs this many arcs to be reversed. We show that the number of arcs we need to reverse in order to make a tournament k-arc-strong is closely related to the number of arcs we need to reverse just to achieve in- and out-degree at least k. We also consider, for general digraphs, the operation of deorienting an arc which is not part of a 2-cycle. That is we replace an arc xy such that yx is not an arc by the 2-cycle xyx. We prove that for every tournament T on at least $2k+1$ vertices, the number of arcs we need to reverse in order to obtain a k-arc-strong tournament from T is equal to the number of arcs one needs to deorient in order to obtain a k-arc-strong digraph from T. Finally, we discuss the relations of our results to related problems and conjectures.

Key words: Digraphs, tournament, semicomplete digraph, k-arc-strong, k-strong, connectivity, arc reversal, flows, submodular flows, deorienting arcs.

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1 Introduction

The digraphs in this paper may have multiple arcs but no loops. In general the notation follows [1].

We denote an arc from x to y by xy and also sometimes by $x \rightarrow y$. We call x the **tail** and y the **head** of the arc xy. We denote by $N^+(x)$ ($N^-(x)$) the set of those vertices y (z) such that $x \rightarrow y$ ($z \rightarrow x$). For every pair U, W of not necessarily disjoint subsets of $V(D)$ we denote by $\#(U, W)$ the number of arcs with tail in U and head in W (such arcs are called (U, W)-arcs). The **out-degree** (**in-degree**) of a set X of vertices in D is the number $d^+(X) = \#(X, V - X)$ ($d^-(X) = \#(V - X, X)$). If there are several digraphs in play at the same time, we will use $d_H^+(x)$ to denote the out-degree of the vertex x in the subdigraph H. A digraph $D = (V, A)$ is k**-arc-strong** if $d^+(X) \geq k$ for every proper non empty subset X of V. The **arc-strong connectivity**, $\lambda(D)$, of D is the maximum integer k for which D is k-arc-strong.

For a given digraph D we denote by $UG(D)$ the undirected (multi)graph that we obtain by suppressing the orientations of the arcs. An **oriented graph** is a digraph with no parallel arcs and no 2-cycles. A **tournament** is an oriented graph D for which $UG(D)$ is a complete graph.

Clearly a digraph $D = (V, A)$ can be made k-arc-strong by reversing some arcs if and only if the edges of $UG(D)$ can be oriented such that the resulting digraph D' is k-arc-strong. By Nash-Williams' orientation theorem, such an orientation of $UG(D)$ exists if and only if $UG(D)$ is $2k$ edge-connected (see e.g. [1, Section 8.6]).

Denote by $\delta^0(D)$ the minimum over all in- and out-degrees of vertices in D. Let $r_k^{deg}(D)$ be the minimum number of arcs one needs to reverse in a digraph D in order to obtain a digraph D' with $\delta^0(D') \geq k$. Analogously define $r_k^{arc-strong}(D)$ to be minimum number of arcs one needs to reverse in D in order to obtain a k-arc-strong digraph. By the remark above, $r_k^{arc-strong}(D) < \infty$ if and only if $UG(D)$ is $2k$-edge-connected. It is well known that, by reducing the above reversal problem to a minimum cost submodular flow problem, one can determine, in polynomial time, a minimum cardinality set of arcs in D whose reversal gives a k-arc-strong digraph or detect that $r_k^{arc-strong}(D) = \infty$. We refer the reader to [1, Section 8.8.4] for a detailed account on how to do this. For an arbitrary digraph D the size of $r_k^{arc-strong}(D)$ may depend on n, the number of vertices of D. We prove in this paper that for tournaments $r_k^{arc-strong}(T)$ is always bounded by a quadratic function of k. It follows from our proofs that one can determine $r_k^{arc-strong}(T)$ for an arbitrary tournament T using standard minimum cost flows rather than submodular flows.

It is not difficult to find examples of tournaments T for which the number of arcs we need to reverse in order to obtain a k-arc-strong tournament from T is strictly larger than the number of new arcs we need to add to T in order to obtain a k-arc-strong directed multigraph. Note that here we allow the creation of parallel arcs. See Section 6 for such an example. Instead we consider the operation of deorienting an arc. Let xy be an arc of a digraph D which is not in a 2-cycle (that is D does not contain the arc yx). By **deorienting** xy we mean the operation which replaces xy by the 2-cycle xyx (or equivalently, adds the arc yx to D). Let $DEO_k^{deg}(D)$ denote the minimum number of arcs we need to deorient in D in order to obtain a digraph D' with $\delta^0(D') \geq k$. Clearly $DEO_k^{deg}(D) < \infty$ if and only if each vertex of D has degree at least k in $UG(D)$ and $DEO_k^{deg}(D) \leq r_k^{deg}(D)$ for every oriented graph. Analogously define $DEO_k^{arc-strong}(D)$ to be the minimum number of arcs one needs to deorient in D in order to obtain a k-arc-strong digraph. It is easy to see that $DEO_k^{arc-strong}(D) < \infty$ if and only if $UG(D)$ is k-edge-connected. Furthermore if D is an oriented graph (in particular if D is a tournament) then we have $DEO_k^{arc-strong}(D) \leq r_k^{arc-strong}(D)$ since instead of reversing an optimal set A' of arcs we may deorient these arcs and obtain a digraph with in- and out-degree at least k.

For arbitrary digraphs we don't know how to determine $DEO_k^{arc-strong}(D)$ efficiently, but as we show in this paper, when T is a tournament, we can determine $DEO_k^{arc-strong}(T)$ efficiently and furthermore we have $r_k^{arc-strong}(T) = DEO_k^{arc-strong}(T)$.

2 Determining $r_k^{deg}(D)$ efficiently

We start by observing that the problem of determining $r_k^{deg}(D)$ and finding an optimal reversing set can be solved using flows in networks for any given digraph D.

Let $D = (V, A)$ be an arbitrary digraph and let $N = (V, A, l \equiv 0, u \equiv 1)$ be the corresponding flow network in which every arc has capacity (i.e. upper bound) one and lower bound zero. By a **flow** in N we mean simply a function $x : A \rightarrow R$ such that $0 \leq x_{ij} \leq 1$ for every arc $ij \in A$. We call x an integer flow in N if $x_{ij} \in \{0, 1\}$ for every arc $ij \in A$. Starting from any digraph D' which was obtained from D by reversing some arcs we can define an integer flow x in N by taking $x_{ij} = 1$ precisely if the arc ij was reversed when going from D to D'. It is easy to see that we may also go the other way. Hence we may study reversals of arcs in D through flows in N. Given an **integer** flow x in N let D' be obtained from D by reversing those arcs ij that have $x_{ij} = 1$. The in-degree of a vertex i in D' is given by $d^-_{D'}(i) = d^-_D(i) + \sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji}$.

Hence, in order for D' to have $\delta^0(D') \geq k$ we must have

$$k \leq d^-_D(i) + \sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji} \leq d^+_D(i) + d^-_D(i) - k, \qquad (1)$$

where the last inequality ensures that $d^+_{D'}(i) \geq k$. The condition above is equivalent to requiring that the flow x satisfies $0 \leq x_{ij} \leq 1$ and

$$d^+_D(i) - k \geq \sum_{ij \in A} x_{ij} - \sum_{ji \in A} x_{ji} \geq k - d^-_D(i). \qquad (2)$$

This is just a feasibility problem for flows and hence can be solved in polynomial time using any algorithm for finding a maximum flow in a network (See e.g. [1, Exercise 3.32]). By introducing the cost 1 on every arc and solving a minimum cost flow problem, in polynomial time, we can determine $r^{deg}_k(D)$ and find an optimal reversing set, or determine that $r^{deg}_k(D) = \infty$ which corresponds to the case when there is no feasible flow in N.

Note that if D has 2-cycles then the optimal reversal may involve the creation of parallel arcs. We can exclude the reversal of arcs in 2-cycles by letting $l_{ij} = u_{ij} = 0$ for every arc which is part of a 2-cycle.

3 Reversals of arcs to achieve high in- and out-degree in tournaments

In this section we consider $r^{deg}_k(T)$ when T is a tournament and prove that this number is always bounded by a (quadratic) function in k. Define the deficiency $DEF_k(T)$ of a tournament T with respect to the degree requirement k as follows:

$$DEF_k(T) = \Sigma_{x \in V(T)} max\{0, k - d^+(x)\} + \Sigma_{x \in V(T)} max\{0, k - d^-(x)\}.$$

Thus $\delta^0(T) \geq k$ if and only if $DEF_k(T) = 0$.

Lemma 3.1 *If T is a tournament, with $|V(T)| \geq 2k+1$, then $r^{deg}_k(T) \leq DEF_k(T)$.*

Proof: We will show that if $DEF_k(T) > 0$ and we cannot reverse one arc, such that the deficiency drops by at least one, then we can reverse two arcs, such that the deficiency drops by two after these two reversals. This will imply the claim by induction on $DEF_k(T)$. So assume that $DEF_k(T) > 0$ and we cannot reverse any arc, such that the deficiency drops by at least one.

Without loss of generality let $w \in V(T)$ have $d_T^+(w) \leq k-1$. Let $X = N_T^-(w)$, $Y = N_T^+(w)$. Since $d_T^+(w) \leq k-1$ we get that $|X| = |V(T)| - |Y| - 1 \geq 2k+1-(k-1)-1 = k+1$.

For all $x \in X$ we must have $d_T^+(x) \leq k$, since if there was some $x' \in X$ with $d_T^+(x') > k$, then we could reverse $x'w$, obtaining a contradiction. Therefore we have

$$|X|k \geq \sum_{x \in X} d_T^+(x) = |X|(|X|-1)/2 + \#(X,Y) + |X|. \tag{3}$$

If some $y \in Y$ has $d_T^-(y) < k$ then there exists some vertex $z \in X$, such that $y \to z \to w$. Now reversing the arcs yz and zw, we note that the deficiency drops by two. Therefore we may assume that $d_T^-(y) \geq k$, for all $y \in Y$, which implies that

$$|Y|k \leq \sum_{y \in Y} d_T^-(y) = |Y|(|Y|-1)/2 + \#(X,Y) + |Y|. \tag{4}$$

Isolating $\#(X,Y)$ in (3) and (4) we obtain the following.

$$k|X| - |X| - |X|(|X|-1)/2 \geq \#(X,Y) \geq k|Y| - |Y| - |Y|(|Y|-1)/2$$

$$\Downarrow$$

$$k|X| - k|Y| - \tfrac{|X|^2-|Y|^2}{2} - \tfrac{|X|-|Y|}{2} \geq 0$$

$$\Downarrow$$

$$(2k-1)(|X|-|Y|) - (|X|^2 - |Y|^2) \geq 0$$

As $|X| > k > |Y|$ we may divide by $|X|-|Y|$ in the last line, whereby we obtain that $2k-1-(|X|+|Y|) \geq 0$, which is a contradiction, as $|X|+|Y|+1 = |V(T)| \geq 2k+1$. $\diamond$

Lemma 3.2 *For every tournament T on at least $2k+1$ vertices*

$$\max\{\sum_{x \in V(T)} \max\{0, k-d^+(x)\}, \sum_{x \in V(T)} \max\{0, k-d^-(x)\}\} \leq k(k+1)/2.$$

Proof: Let $X = \{x | d_T^-(x) < k\}$ and $Y = \{y | d_T^+(y) < k\}$. Note that $\sum_{x \in X}[k - d_T^-(x)] \leq k|X| - |X|(|X|-1)/2 = |X|(2k+1-|X|)/2$. It is easy to show that the later is never larger than $k(k+1)/2$ since $|X|$ is an integer (differentiate it, and note that the maximum is found in $|X| = k$ or $|X| = k+1$, when $|X|$ is an integer). Analogously we see that $k(k+1)/2 - \sum_{y \in Y}[k - d_T^+(y)] \geq 0$. $\diamond$

It follows from Lemmas 3.1 and 3.2 that $r_k^{deg}(T) \leq DEF_k(T) \leq k(k+1)$, but we need to count more detailed in order to prove Corollary 3.4. In order to do so, we consider a set of arcs, A', in T, such that $DEF_k(T') = DEF_k(T) - 2|A'|$, where T' is the tournament obtained from T, by reversing all the arcs in A'. Let $dd_k(T)$ denote the maximum number of arcs possible in such a set of arcs A'. Hence $dd_k(T)$ is the maximum size of a set of arcs such that reversing these arcs in any order will decrease the deficiency by 2 per arc.

Lemma 3.3 *If T is a tournament, with $|V(T)| \geq 2k+1$, then $DEF_k(T) - dd_k(T) \leq k(k+1)/2$.*

Proof: Let T be a tournament, with $|V(T)| \geq 2k+1$ and let $X = \{x | d_T^-(x) < k\}$ and let $Y = \{y | d_T^+(y) < k\}$. We now build a flow network, N, with $V(N) = X \cup Y \cup \{s, s', t', t\}$ and $A(N) = \{xy | x \in X, y \in Y, xy \in A(T)\} \cup \{sx, xt' | x \in X\} \cup \{s'y, yt | y \in Y\} \cup \{ss', s't', t't\}$. Let the capacities in N be defined as follows.

(i) $u(sx) = k - d_T^-(x)$	for all $x \in X$
(ii) $u(xy) = 1$	for all $x \in X$, $y \in Y$ and $xy \in A(T)$
(iii) $u(yt) = k - d_T^+(y)$	for all $y \in Y$
(iv) $u(ss') = \frac{k(k+1)}{2} - \sum_{x \in X}[k - d_T^-(x)]$	
(v) $u(t't) = \frac{k(k+1)}{2} - \sum_{y \in Y}[k - d_T^+(y)]$	
(vi) $u(pq) = \infty$	all other arcs in N.

Note that we do not include in N those arcs in T that go from Y to X or are inside X or Y, nor do we include any of the vertices of $V - X - Y$ in N. However, the arcs that we have deleted in this way still contribute to the capacities in N.

It follows from Lemma 3.2 that all capacities are greater than or equal to zero. We will now show that there exists a feasible (s,t)-flow in N, of value $k(k+1)/2$. Let $u(U, W)$ denote the sum of the capacities of the arcs with tail in U and head in W. By the max-flow-min-cut theorem (See e.g. [1, Theorem 3.5.3]), we just have to show that $u(S, \bar{S}) \geq k(k+1)/2$ for every (s,t)-cut $(S, \bar{S})$ in N (An (s,t)-cut in N is a partition of $V(N)$ into two sets S, T such that $s \in S$ and $t \in T$). Now let $(S, \bar{S})$ be chosen such that $u(S, \bar{S})$ is minimum over all (s,t)-cuts. If $s' \in S$, then we may assume that $Y \subseteq S$ and $t' \in S$, by (vi) above. As $t \in \bar{S}$, (iii) and (v) imply that the capacity across $(S, \bar{S})$ is at least $k(k+1)/2$. So we may assume that $s' \in \bar{S}$. Analogously we may assume that $t' \in S$. Let $X_s = X \cap S$, $X_t = X \cap \bar{S}$, $Y_s = Y \cap S$ and $Y_t = Y \cap \bar{S}$ and observe that $u(Y_t, t) \leq k|Y_t| - |Y_t|(|Y_t|-1)/2 - \#(Y_t, X)$ and $u(s, X_s) \leq k|X_s| - |X_s|(|X_s|-1)/2 - \#(Y, X_s))$. Now we can estimate $u(S, \bar{S})$ as follows:

$$\begin{aligned}
u(S,\bar{S}) &= u(X_s, Y_t) + u(Y_s, t) + u(t', t) + u(s, X_t) + u(s, s') \\
&= (|X_s||Y_t| - \#(Y_t, X_s)) + u(V(N) - Y_t, t) + u(s, V(N) - X_s) \\
&= |X_s||Y_t| - \#(Y_t, X_s) + (\tfrac{k(k+1)}{2} - u(Y_t, t)) + (\tfrac{k(k+1)}{2} - u(s, X_s)) \\
&\geq |X_s||Y_t| - \#(Y_t, X_s) + \tfrac{k(k+1)}{2} - (k|Y_t| - |Y_t|(|Y_t|-1)/2 - \#(Y_t, X)) \\
&\quad + \tfrac{k(k+1)}{2} - (k|X_s| - |X_s|(|X_s|-1)/2 - \#(Y, X_s)) \\
&\geq \tfrac{k(k+1)}{2} + \tfrac{1}{2}(k(k+1) + (|X_s| + |Y_t|)^2 - (2k+1)(|X_s| + |Y_t|))
\end{aligned}$$

As $k(k+1) + (|X_s|+|Y_t|)^2 - (2k+1)(|X_s|+|Y_t|) \geq 0$, unless $k < |X_s|+|Y_t| < k+1$, we see that $u(S, \bar{S}) \geq k(k+1)/2$, as desired (as $|X_s|+|Y_t|$ is an integer). So there exists a feasible integer valued (s,t)-flow in N, of value $k(k+1)/2$. Note that such a flow is also a maximum (s,t)-flow in N since the (s,t)-cut $(s, V(N) - s)$ has capacity exactly $k(k+1)/2$.

Let x be a feasible integer valued (s,t)-flow in N, of value $k(k+1)/2$, and let $x(U, W)$ denote the sum of the flow values on the arcs from U to W. Now consider the (s,t)-cut, $(\{s,s'\} \cup X, \{t', t\} \cup Y)$, and note that since there are no arcs from $\{t', t\} \cup Y$ to $\{s, s'\} \cup X$ in N we have $k(k+1)/2 = x(\{s,s'\} \cup X, \{t', t\} \cup Y) = x(X, Y) + x(X, t') + x(s', Y) + x(s', t')$. This implies the following:

$$\begin{aligned}
\tfrac{k(k+1)}{2} &= x(X,Y) + (x(X,t') + x(s',t')) + (x(s',Y) + x(s',t')) - x(s',t') \\
&\leq x(X,Y) + (\tfrac{k(k+1)}{2} - \textstyle\sum_{x \in X}(k - d_T^-(x))) \\
&\quad + (\tfrac{k(k+1)}{2} - \textstyle\sum_{y \in Y}(k - d_T^-(y))) - x(s',t')
\end{aligned}$$

$$\Downarrow$$

$$\tfrac{k(k+1)}{2} \geq \textstyle\sum_{x \in X}(k - d_T^-(x)) + \sum_{y \in Y}(k - d_T^-(y)) - x(X,Y)$$

Let H be the arcs from X to Y on which x takes the value one. Let T' be the tournament obtained from T, by reversing the arcs in H and note that $DEF_k(T') = DEF_k(T) - 2|H|$. So by the definition of dd_k, and the fact that $k(k+1)/2 \geq DEF_k(T) - |H|$, we are done. $\diamond$

Corollary 3.4 *If T is a tournament, with $|V(T)| \geq 2k+1$, then $r_k^{deg}(T) \leq k(k+1)/2$.*

Proof: It is not difficult to see that $r_k^{deg}(T) \leq DEF_k(T) - dd_k(T)$, as we first choose a set A' as described in the definition of $dd_k(T)$, and then use Lemma 3.1 to see that after reversing all arcs in A' we need to reverse at most $DEF_k(T) - 2dd_k(T)$ further arcs. Now the claim follows from Lemma 3.3. $\diamond$

For general digraphs D there need not be any close relation between the numbers $r_k^{deg}(D)$ and $r_k^{arc-strong}(D)$. For example, let D be the digraph obtained by replacing each vertex of a directed path P_t on t vertices by a 3-cycle (That is $D = P_t[C_3, \ldots, C_3]$). Then $r_2^{deg}(D) = 6$ and it is easy to see that $r_2^{arc-strong}(D)$ is proportional to t and hence can be made much larger than $r_2^{deg}(D)$ by increasing t.

4 Reversals of arcs to achieve high arc-strong connectivity in tournaments

We now show that in the case of tournaments, the numbers r_k^{deg} and $r_k^{arc-strong}$ are closely related.

Theorem 4.1 *For every tournament T with $|V(T)| = n \geq 2k+1$ we have*

$$r_k^{arc-strong}(T) = max\{k - \lambda(T), r_k^{deg}(T)\}.$$

In particular, if $r_k^{deg}(T) \geq k - \lambda(T)$ then $r_k^{arc-strong}(T) = r_k^{deg}(T)$.

Proof: Let $q = max\{k - \lambda(T), r_k^{deg}(T)\}$ and let T' be a tournament obtained from T by reversing at most q arcs, such that the following holds:

(i). $\delta^+(T'), \delta^-(T') \geq k$.
(ii). $\sum_{x \in V(T')}(d^+(x))^2$ is minimum.

Note that there exists such a T', by the definition of q. If T' is k-arc strong then we are done, so assume that $\lambda(T') < k$. Let S be chosen such that $\#(S, V-S) = \lambda(T')$ in T', and such that $|S|$ is minimum among all subsets S' with $\#(S', V-S') = \lambda(T')$. As $\delta^+(T'), \delta^-(T') \geq k$ and $\lambda(T') < k$, we note that $2 \leq |S| \leq |V(T')| - 2$.

If there exists a vertex $x \in S$, with $\#(S, x) \leq \#(x, V-S)$, then $S' = S - x$ is a contradiction against the choice of S. Therefore $\#(S, x) > \#(x, V-S)$, which implies that $d^+_{T'}(x) \leq |S| - 2$, for all $x \in S$. The minimality of $\#(S, V-S)$ implies that $\#(S, y) \leq \#(y, V-S)$ and hence we get $d^+_{T'}(y) \geq |S|$, for all $y \in V - S$.

If $|S| \leq 2k$, then

$$\sum_{x \in S} d^+_{T'}(x) = \#(S,S) + \#(S, V-S) < |S|(|S|-1)/2 + k \leq k(|S|-1) + k = k|S|,$$

which is a contradiction as $\delta^+(T') \geq k$. Therefore $|S| \geq 2k+1$. Analogously we can prove that $|V-S| \geq 2k+1$.

This implies that for all $x \in S$ we have $d^-_{T'}(x) \geq n-1-(|S|-2) \geq |V-S|+1 \geq 2k+2 > k+1$ and for all $y \in V-S$ we have $d^+_{T'}(y) \geq |S| \geq 2k+1 > k+1$

Note that reversing any arc yx which goes from $V-S$ to S in T' will maintain (i) and decrease (ii) as $d^+_{T'}(y) \geq d^+_{T'}(x)+2$. Hence it follows from the fact that $|S|, |V-S| \geq 2k+1$ (implying that T' contains arcs from $V-S$ to S) that we have reversed exactly q arcs in order to obtain T' and furthermore every arc from $V-S$ to S also goes from $V-S$ to S in T (otherwise we could improve (ii) by not reversing such an arc originally). Let R denote the arcs in T' which have an opposite direction to what they did in T (i.e. R are the arcs that have been reversed, and $|R| = q$). We will now show that all arcs in R go from S to $V-S$ in T'. It follows from the remark above that there is no $(V-S,S)$-arc in R.

If there exist an (S,S)-arc in R, then let vu be such an arc (i.e. $uv \in A(T)$). As $|V-S| \geq 2k+1$ and $\#_{T'}(S, V-S) < k$ (the number of arcs from S to $V-S$ in T'), there exists a vertex w in $V-S$, with $wv \in A(T')$. Now consider the tournament T'', obtained from T' by reversing vu and wv. Note that T'' also has q arcs reversed compared to T (uv is reversed back again). Compared to T' we see that all degrees stay the same except that $d^+(w)$ decreases by one and $d^-(u)$ decreases by one. Therefore we still have $\delta^+(T''), \delta^-(T'') \geq k$, and we obtain a contradiction against (ii).

If there exists a $(V-S, V-S)$-arc in R, we analogously obtain a contradiction. Therefore all arcs in R are $(S, V-S)$-arcs.

Since we have reversed q arcs, there are at least $\lambda(T) + q \geq k$ arcs in T' from S to $V-S$ contradicting the assumption that T' has fewer than k such arcs. $\diamond$

Note that the proof above can be turned into a polynomial algorithm for finding a set of q arcs whose reversal makes T k-arc-strong using just flows instead the more complicated of submodular flows (as we mentioned in the introduction, one can determine $r_k^{arc-strong}(D)$ for an arbitrary digraph D using minimum cost submodular flows). We leave the details to the interested reader.

Combining Corollary 3.4 with Theorem 4.1 we obtain the following upper bound on $r_k^{arc-strong}(T)$. Note that the transitive tournaments show that this is best possible.

Corollary 4.2 *For every tournament T with $|V(T)| = n \geq 2k+1$ we have $r_k^{arc-strong}(T) \leq k(k+1)/2$.*

5 Deorienting arcs of a tournament in order to achieve high in-and out-degree or high arc-strong connectivity

Lemma 5.1 *If T is a tournament, with $|V(T)| \geq 2k+1$, then $DEO_k^{deg}(T) \geq DEF_k(T) - dd_k(T)$.*

Proof: Let $X = \{x | d_T^-(x) < k\}$ and $Y = \{y | d_T^+(y) < k\}$ and let B be a set of $DEO_k^{deg}(T)$ arcs, such that the digraph $T \cup B$ has minimum out- and in-degree at least k.

Let $B' \subseteq B$ be defined, such that $|B'|$ is maximum and $DEF_k(T) = DEF_k(T \cup B') + 2|B'|$. Thus B' is a maximum cardinality subset among the arcs in B such that adding these arcs to T in any order will decrease the deficiency by 2 per arc.

Let $T' = T \cup B'$, and let $X' = \{x | d_{T'}^-(x) < k\}$ and $Y' = \{y | d_{T'}^+(y) < k\}$. Note that, by the maximality of $|B'|$, there is no arc in $B - B'$ which goes from Y' to X'. So there has to be at least $\sum_{x \in V(T)} \max\{0, k - d_{T'}^-(x)\}$ arcs in $B - B'$ going into a vertex in X', and there has to be at least $\sum_{y \in V(T)} \max\{0, k - d_{T'}^+(y)\}$ arcs in $B - B'$ going out of a vertex in Y'. As there was no $(Y', X') - arc$ in $B - B'$, this implies that $|B - B'| \geq DEF_k(T')$.

Now we see that $|B| \geq DEF_k(T') + |B'| = DEF_k(T) - |B'|$ (by the definition of B'). By the definition of $dd_k(T)$, we get that $DEO_k^{deg}(T) = |B| \geq DEF_k(T) - dd_k(T)$. ◇

Theorem 5.2 *Let T be a tournament on at least $2k+1$ vertices. Then we have $DEO_k^{deg}(T) = r_k^{deg}(T)$. In particular $DEO_k^{deg}(T) \leq k(k+1)/2$.*

Proof: We saw in the proof of Corollary 3.4 that $r_k^{deg}(T) \leq DEF_k(T) - dd_k(T)$. Thus we have $DEO_k^{deg}(T) \leq r_k^{deg}(T) \leq DEF_k(T) - dd_k(T)$. Lemma 5.1 now implies that equality must hold everywhere. Now it follows from Corollary 3.4 that $DEO_k^{deg}(T) \leq k(k+1)/2$. ◇

Since deorienting an arc xy in an oriented graph corresponds to adding the opposite arc yx and keeping xy, one might expect that $DEO_k^{arc-strong}(D) < r_k^{arc-strong}(D)$ for most digraphs. The next result shows that for tournaments the two numbers are equal and hence, with respect to increasing the arc-strong connectivity, there is no gain from deorienting arcs rather than reversing arcs.

Theorem 5.3 *For every tournament T on at least $2k+1$ vertices we have $DEO_k^{arc-strong}(T) = r_k^{arc-strong}(T)$.*

Proof: We saw in Theorem 4.1 that $r_k^{arc-strong}(T) = \max\{k - \lambda(T), r_k^{deg}(T)\}$. If $r_k^{arc-strong}(T) = r_k^{deg}(T)$ then we have by Theorem 5.2

$$\begin{aligned} DEO_k^{arc-strong}(T) &\leq r_k^{arc-strong}(T) \\ &= r_k^{deg}(T) \\ &= DEO_k^{deg}(T) \\ &\leq DEO_k^{arc-strong}(T), \end{aligned}$$

implying that $DEO_k^{arc-strong}(T) = r_k^{arc-strong}(T)$. So we may assume that $r_k^{arc-strong}(T) = k - \lambda(T)$. Now the claim follows from the easy fact that $DEO_k^{arc-strong}(T) \geq k - \lambda(T)$. $\diamond$

We argued in Section 4 that we can find, in polynomial time, a set of arcs $A' \subset A(T)$ of size $r_k^{arc-strong}(T)$ in a tournament T such that reversing the arcs of A' results in a k-arc-strong tournament. Thus it follows from Theorem 5.3 that, in polynomial time, we can determine $DEO_k^{arc-strong}(T)$ and find a set of $DEO_k^{arc-strong}(T)$ arcs to deorient such that the resulting semicomplete digraph is k-arc-strong (a digraph is semicomplete if it has no non adjacent vertices). One optimal set of arcs to deorient is simply a set that would form an optimal reversal.

6 Related problems and conjectures

A digraph $D = (V, A)$ is k-**strong** if $D - X$ is strong for every $X \subset V$ with $|X| < k$. Denote by $r_k(D)$ the minimum number of arcs one needs to reverse in D in order to obtain a digraph which is k-strong. Contrary to $r_k^{arc-strong}(D)$ it is a very difficult problem to decide whether $r_k(D) < \infty$ for a given digraph D. This is equivalent to the problem of deciding whether a given undirected graph has a k-strong orientation, a problem which is again a special case of the problem of deciding whether a given digraph D has a k-strong orientation (an **orientation** of a digraph D without parallel arcs is any digraph that can be obtained from D by deleting one arc from each 2-cycle in D.)

Conjecture 6.1 *[3] Every $2k$-strong digraph contains a k-strong orientation.*

It is not even known whether there is any function $g = g(k)$ such that every $g(k)$-strong digraph has a k-strong orientation. Even the case when $k = 2$

and the digraph is symmetric (that is an undirected graph with every edge replaced by a 2-cycle) the problem is completely open.

Since there are k-strong tournaments on n vertices for every $n \geq 2k+1$, $r_k(T) < \infty$ for every tournament T on at least $2k+1$ vertices. It is not hard to prove that for tournaments r_k is in fact bounded by a function depending on k only. The key observations needed to show that every tournament can be made k-strong by reversing the orientation of at most $\frac{(4k-2)(4k-3)}{4}$ arcs are (see details in [1, page 379]):

(1) Every tournament on at least $4k-1$ vertices contains a vertex x with $\min\{d^+(x), d^-(x)\} \geq k$.
(2) If D is a k-strong digraph and D' is obtained from D by adding a new vertex z and new arcs from z to k distinct vertices $u_1, \dots, u_k \in V(D)$ and from k distinct vertices $v_1, \dots, v_k$ to z, then D' is also k-strong.

As every k-strong digraph is k-arc-strong Corollary 4.2 provides some support for the following Conjecture (again the transitive tournament shows that the bound would be best possible):

Conjecture 6.2 (Bang-Jensen, 1994) *If T is a tournament then $r_k(T) \leq \frac{k(k+1)}{2}$.*

Let $a_k(D)$ denote the minimum number of new arcs one must add to the digraph D in order to obtain a k-strong digraph. Since adding parallel arcs cannot increase the vertex-connectivity of a digraph it follows that an optimal augmenting set consisting of $a_k(D)$ new arcs will not contain any arc xy for which xy is already an arc of D. Hence in the case when D is a tournament all new arcs must form 2-cycles with existing arcs and hence adding these arcs corresponds to deorienting their opposites in D. If $|V(D)| \geq k+1$ then $a_k(D)$ is finite and it was shown in [2] that every semicomplete digraph on at least $3k-1$ vertices satisfies $r_k(D) = a_k(D)$.

Conjecture 6.3 *[2] For every tournament on at least $2k+1$ vertices $r_k(T) = a_k(T)$.*

Let $a_k^{arc-strong}$ be the analogous augmentation number for arc-strong connectivity. Here parallel arcs are allowed. It is not difficult to see that there exist tournaments T with arbitrarily many vertices and $r_k^{arc-strong}(T) > a_k^{arc-strong}(T)$. One such example is the tournament T_n which is obtained from a transitive tournament on the vertex set $\{1, 2, \dots, n\}$ with arcs $i \rightarrow j$ whenever $i < j$ by reversing the arcs $1 \rightarrow n$ and $2 \rightarrow n-1$. It is easy to check that $a_2^{arc-strong}(T_n) = 1 < r_2(T_n)$.

Let us finish with the following question for which we saw that the answer is yes when D is a tournament.

Problem 6.4 *Is there a polynomial algorithm which determines $DEO_k^{arc-strong}(D)$ of a given digraph D?*

References

[1] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Springer Verlag, London (2000) xxii+754 pp.

[2] J. Bang-Jensen and T. Jordán, Adding and reversing arcs in semicomplete digraphs, *Combin. Probab. Comput.* **7** (1998) 17-25.

[3] C. Thomassen, Configurations in graphs of large minimum degree, connectivity, or chromatic number, *Annals of the New York Academy of Sciences* **555** (1989) 402-412.