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Supertwistor description of the AdS pure spinor string

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Abstract

We describe the pure spinor string in the $AdS_5 \times S^5$ using unconstrained matrices first used by Roiban and Siegel for the Green-Schwarz superstring.

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1 Introduction

The superstring sigma model on AdS spaces is usually described in terms of the supergroup coset $PSU(2, 2|4)/SO(1, 4) \times SO(5)$. The classical Green-Schwarz and pure spinor formulations are both well understood in terms of this coset. However for some applications, the usual exponential parametrization of the coset elements becomes cumbersome.

In [1] Roiban and Siegel introduced another parametrization for the $AdS_5 \times S^5$ coset in terms of the supergroup $GL(4|4)$. The usefulness of this new formulation is in the fact that the coordinates can be represented in terms of unconstrained matrices. Furthermore, the coordinates transform in the fundamental representation of the superconformal group, like supertwistors.

Depending on the application intended, different sets of coordinates are more useful than others. In the same way that global AdS coordinates and Poincaré patch are useful for different applications. This also extends to the full superspace, *e.g.* chiral vs. nonchiral. The construction of explicit vertex operators for string states depends heavily on these choices. Since the beginning of the formalism vertex operators for AdS have been discussed [2]. The first nontrivial example was introduced in [3, 4]. Further developments can be found in [5, 6]. The most complete description in the case of supergravity states was given in [7]. In this work the authors show that the ghost number two cohomology can be written in terms of harmonic superspace and a direct dictionary to the dual CFT single trace operators was obtained. The derivation is very lengthy due to the usual exponential parametrization of the coset elements. As advocated by Siegel [8], those results could be simplified using the $GL(4|4)$ description. This is one of the motivations to adapt the pure

spinor formalism for this new coset. In this paper we will describe in detail how to achieve this.

This paper is organized as follows. In Section 2 we describe the coset and its basic properties. In Section 3 the symmetries of AdS are discussed in terms of the new coset. The full pure spinor superstring action is constructed in Section 4. In Section 5 we make a few comments on the construction of the vertex operator related to the β -deformations. In Section 6 we conclude the paper and discuss future lines of investigation.

2 The $GL(4|4)/(GL(1) \times Sp(2))^2$ coset

Roiban and Siegel proposed a description of the $AdS_5 \times S^5$ sigma model in terms of a coset that can be described by standard matrices [1]. The observation is that the $PSU(2, 2|4)$ group is a coset by itself (not caring about reality conditions) $GL(4|4)/(GL(1) \times GL(1))$, where the two $GL(1)$ groups are defined by scalar multiplication in the upper and lower blocks. Note that the super-determinant is invariant under the action of both $GL(1)$'s combined. Up to reality conditions (*i.e.* signature) $AdS_5 \times S^5$ can be described by ¹

$$\frac{GL(4|4)}{(GL(1) \times Sp(2))^2}. \quad (2.1)$$

Note that $Sp(2) = Spin(5)$ (under Wick rotation, $Sp(1, 1) = Spin(1, 4)$.) Since we have a model with spinors, it is much more natural to work with groups where the spinors transform in the fundamental representation.

The coset elements are denoted by Z_M^A where the local $\Lambda_A^B (GL(1) \times Sp(2))^2$ transformations act on the right by simple matrix multiplication. The index M is a global $GL(4|4)$ index. We divide both indices under bosonic and fermionic elements $M = (m, \bar{m})$ and $A = (a, \bar{a})$. The $Sp(2)$ invariant matrices will be denoted by Ω_{ab} and $\bar{\Omega}_{\bar{a}\bar{b}}$. There are analog matrices with indices up, which will be denoted by the same symbol. They all satisfy $\Omega\Omega = -\mathbb{I}$ where \mathbb{I} is the identity matrix with appropriate indices. We will omit explicit indices most of the time, only making them explicit when necessary.

The left-invariant currents (invariant under global transformations) are defined by

$$J_A^B = Z_A^M dZ_M^B, \quad (2.2)$$

where $Z_A^M = (Z_M^A)^{-1}$.

A variation of the group element Z around a background Z_0 is given by

$$\delta Z_M^A = Z_M^B X_B^A, \quad (2.3)$$

where X_B^A is given by

$$X_B^A = \begin{pmatrix} X_b^a & \Theta_b^{\bar{a}} \\ \Theta_b^{\prime a} & Y_b^{\bar{a}} \end{pmatrix}. \quad (2.4)$$

¹In our notation $Sp(n)$ describes $2n \times 2n$ matrices, *e.g.* $Sp(1) \simeq SU(2)$.

For these variations to be in the coset, the matrices X and Y must satisfy

$$X^T = -\Omega X \Omega, \quad Y^T = -\bar{\Omega} Y \bar{\Omega}. \quad (2.5)$$

Since these conditions do not imply that X and Y are traceless, we further impose

$$\text{Tr } X = \text{Tr } Y = 0. \quad (2.6)$$

Doing this, we ensure that we work only with variations that are orthogonal to the gauge group.

We want to relate the elements described with the Roiban-Siegel formulation and the ones in the description using the $PSU(2, 2|4)/(SO(5) \times SO(1, 4))$ coset for the pure spinors. Our notation is closely related to the one adopted in [9]. By construction, it is not hard to see the equivalence between Z and the element $g \in PSU(2, 2|4)/(SO(5) \times SO(1, 4))$,

$$g \equiv Z. \quad (2.7)$$

In order to establish the equivalence between the content of the current in both formalism, we first identify the gauge content in our matrix formalism. Writing the block components of J as

$$J = \begin{pmatrix} J_X & K_1 \\ K_3 & J_Y \end{pmatrix}, \quad (2.8)$$

we split the diagonal elements into three irreducible components using the $Sp(2)$ metric Ω . Define for a matrix M_a^b its three irreducible components,

$$\langle M \rangle = \frac{1}{2} [M - \Omega M^T \Omega] - \frac{1}{4} \mathbb{I} \text{Tr } M, \quad (2.9)$$

$$(M) = \frac{1}{2} [M + \Omega M^T \Omega], \quad (2.10)$$

$$\text{Tr } M. \quad (2.11)$$

Usually, for any matrix, one can split it in its antisymmetric, its symmetric traceless and its trace part. Here $\langle M \rangle$ is the Ω -antisymmetric, Ω -traceless part of M , (M) is the Ω -symmetric part of M , and of course $\text{Tr } M$ is the Ω -trace of M . Using those independent structures, we can separate the element of J (2.2) that are pure gauge. we will define

$$K_X = \langle J_X \rangle, \quad A_X = (J_X), \quad a_X = \frac{1}{4} \text{Tr } J_X, \quad (2.12)$$

$$K_Y = \langle J_Y \rangle, \quad A_Y = (J_Y), \quad a_Y = \frac{1}{4} \text{Tr } J_Y. \quad (2.13)$$

A_X and a_X are $Sp(2)$ and $GL(1)$ connections respectively. By definition,

$$J_X = K_X + A_X + \mathbb{I} a_x. \quad (2.14)$$

By checking its transformation property, we can now relate the diagonal elements in (2.8) with the gauge part of current in the $\mathfrak{psu}(2, 2|4)$ algebra,

$$J_0^i \equiv \begin{pmatrix} A_X + \mathbb{I}a_X & 0 \\ 0 & A_Y + \mathbb{I}a_Y \end{pmatrix}. \quad (2.15)$$

The rest of the bosonic components are related as,

$$J_2^m \equiv \begin{pmatrix} K_X & 0 \\ 0 & K_Y \end{pmatrix}. \quad (2.16)$$

Before we continue, we have to make clear that the $\langle \cdot \rangle$ and (\cdot) operations need to be treated with care when there is a product of fermionic matrices. Take two fermionic matrices A and B , is easy to see that

$$\text{Tr} \left(\frac{1}{2} [AB - \Omega B^T A^T \Omega] - \frac{1}{4} \mathbb{I} \text{Tr} AB \right) = -\text{Tr} AB \neq 0, \quad (2.17)$$

$$\text{Tr} \left(\frac{1}{2} [AB + \Omega B^T A^T \Omega] \right) = \text{Tr} AB \neq 0. \quad (2.18)$$

The solution to this problem is to add a $(-)$ sign when transposing fermionic matrices. Thus, for a product of two fermionic matrices A and B , our three irreducible components read

$$\langle AB \rangle = \frac{1}{2} [AB + \Omega B^T A^T \Omega] - \frac{1}{4} \mathbb{I} \text{Tr} AB, \quad (2.19)$$

$$(AB) = \frac{1}{2} [AB - \Omega B^T A^T \Omega] \quad \text{and} \quad \text{Tr} AB. \quad (2.20)$$

It is not so obvious how to relate the fermionic part of $PSU(2, 2|4)$, J_1^α and $J_3^{\hat{\alpha}}$, with the nondiagonal terms in (2.8), K_1 and K_3 , because they do not have the right \mathbb{Z}_4 charge. The matrices which do have the right \mathbb{Z}_4 charge are F_1 and F_3 which define K_1 and K_3 as

$$K_1 = \frac{1}{\sqrt{2}} (F_1 - F_3^*) E^{-1/4} \quad \text{and} \quad K_3 = \frac{1}{\sqrt{2}} (F_1^* + F_3) E^{1/4}, \quad (2.21)$$

where

$$F_1^* = \bar{\Omega} F_1^T \Omega, \quad F_3^* = \Omega F_3^T \bar{\Omega}, \quad (2.22)$$

and $E = \text{Sdet} Z$. Now the identification is

$$J_1^\alpha \equiv F_1 \quad J_3^{\hat{\alpha}} \equiv F_3. \quad (2.23)$$

Following the same idea, we define the Θ s as functions of elements with the right \mathbb{Z}_4 charge,

$$\Theta = \frac{1}{\sqrt{2}} (\theta_1 - \theta_3^*) E^{-1/4} \quad \text{and} \quad \Theta' = \frac{1}{\sqrt{2}} (\theta_1^* + \theta_3) E^{1/4}. \quad (2.24)$$

In the same way as we related the components of the currents generated by g and Z , we can relate variations of g and Z by

$$x_2^m \equiv \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad (2.25)$$

$$x_1^\alpha \equiv \theta_1, \quad (2.26)$$

$$x_3^{\hat{\alpha}} \equiv \theta_3. \quad (2.27)$$

It is easy to check that there are the correct number of bosonic and fermionic variations.

Finally, we need to define the ghosts fields that are essential for the construction of the BRST operator. We define the right and left ghost, along with their conjugated momenta, as $\lambda_a^{\bar{a}}, \omega_a^{\bar{a}}, \bar{\lambda}_a^{\bar{a}}, \bar{\omega}_a^{\bar{a}}$. The indices in the ghost terms are such that λ has the same indices as F_1 and $\bar{\lambda}$ the same as F_3 . The crucial point to construct the right BRST operator is the pure spinor condition for λ and $\bar{\lambda}$. Originally, the pure spinor condition was written in term of gamma matrices [10],

$$(\lambda\gamma\lambda)^m = (\bar{\lambda}\gamma\bar{\lambda})^m = 0, \quad (2.28)$$

which in turns implies

$$\lambda^\alpha\lambda^\beta = \frac{1}{16 \cdot 5!} \gamma_{mnpq}^{\alpha\beta} (\lambda\gamma^{mnpq}\lambda), \quad \bar{\lambda}^{\hat{\alpha}}\bar{\lambda}^{\hat{\beta}} = \frac{1}{16 \cdot 5!} \gamma_{mnpq}^{\hat{\alpha}\hat{\beta}} (\bar{\lambda}\gamma^{mnpq}\bar{\lambda}). \quad (2.29)$$

These constraints reduce the elements of λ ($\bar{\lambda}$) from 16 to 11.

The pure spinor constraints in this matrix formulation read

$$\langle \lambda\lambda^* \rangle = 0, \quad \langle \lambda^*\lambda \rangle = 0, \quad (2.30)$$

$$\langle \bar{\lambda}\bar{\lambda}^* \rangle = 0, \quad \langle \bar{\lambda}^*\bar{\lambda} \rangle = 0. \quad (2.31)$$

One can check that there are actually 5 constraints for λ ($\bar{\lambda}$). Therefore,, our ghosts have 11 independent components, as expected. In a similar way to (2.29), (2.30) implies

$$\lambda_a^{\bar{a}}\lambda_b^{\bar{b}} = -\frac{1}{16}\Omega_{ab}\bar{\Omega}^{\bar{a}\bar{b}}\text{Tr}[\lambda\lambda^*] + \lambda_{(a}^{\bar{a}}\lambda_b)^{\bar{b}} + \lambda_{(a}^{\bar{a}}\lambda_b)^{\bar{b}}, \quad (2.32)$$

and a similar condition for the $\bar{\lambda}$ s. Note that

$$\lambda_{(a}^{\bar{a}}\lambda_b)^{\bar{b}} = \lambda_a^{(\bar{a}}\lambda_b^{\bar{b})} = \lambda_{(a}^{\bar{a}}\lambda_b)^{\bar{b}}, \quad (2.33)$$

and the same is true for $\langle \rangle$.

The ghost Lorentz currents are defined as

$$N_X = \frac{1}{2}(\lambda\omega - \omega^*\lambda^*), \quad \bar{N}_X = \frac{1}{2}(\bar{\omega}\bar{\lambda} - \bar{\lambda}^*\bar{\omega}^*), \quad (2.34)$$

$$N_Y = \frac{1}{2}(\omega\lambda - \lambda^*\omega^*), \quad \bar{N}_Y = \frac{1}{2}(\bar{\lambda}\bar{\omega} - \bar{\omega}^*\bar{\lambda}^*). \quad (2.35)$$

These definitions ensure that the N and \bar{N} terms transform as a gauge term.

Now we can make the identification between the ghost fields in the two descriptions:

$$\omega_\alpha \equiv \omega, \quad \hat{\omega}_{\hat{\alpha}} \equiv \bar{\omega}, \quad \lambda^\alpha \equiv \lambda, \quad \hat{\lambda} \equiv \bar{\lambda}, \quad (2.36)$$

$$N^i \equiv \begin{pmatrix} N_X & 0 \\ 0 & N_Y \end{pmatrix} \quad \text{and} \quad \hat{N}^i \equiv \begin{pmatrix} \bar{N}_X & 0 \\ 0 & \bar{N}_Y \end{pmatrix}. \quad (2.37)$$

3 Symmetries of the AdS Space

The main aim of this article is to write a BRST-invariant superstring action embedded on a $AdS_5 \times S^5$ target space in this formalism of unconstrained matrices. Since such action has to be invariant under the symmetries of a $AdS_5 \times S^5$ space, we first proceed to understand how those symmetries act in this formalism and then we find the structures that are invariant under such symmetries.

3.1 Local

A local (gauge) transformation is given by

$$\delta_L Z = ZL + Z \begin{pmatrix} \mathbb{I} \frac{l_X}{4} & 0 \\ 0 & \mathbb{I} \frac{l_Y}{4} \end{pmatrix}, \quad (3.1)$$

where

$$L = \begin{pmatrix} L_X & 0 \\ 0 & L_Y \end{pmatrix}. \quad (3.2)$$

and $(L_{X/Y}) = L_{X/Y}$. The constraints for the L matrices restrict them to be in $Sp(2) \times Sp(2)$, and the l_X and l_Y are the remaining terms of the stability group.

Thus, a local transformation on the current reads,

$$\delta_L \begin{pmatrix} J_X & K_1 \\ K_3 & J_Y \end{pmatrix} = \begin{pmatrix} [J_X, L_X] + \partial L_X + \mathbb{I} \frac{\partial l_X}{4} & K_1 L_Y - L_X K_1 - K_1 \frac{l_X - l_Y}{4} \\ K_3 L_X - L_Y K_3 + K_3 \frac{l_X - l_Y}{4} & [J_Y, L_Y] + \partial L_Y + \mathbb{I} \frac{\partial l_Y}{4} \end{pmatrix}. \quad (3.3)$$

Using that $L_X K_X = \Omega L_X^T K_X^T \Omega$ we find

$$\delta_L K_X = [K_X, L_X], \quad \delta_L K_Y = [K_Y, L_Y], \quad (3.4)$$

$$\delta_L A_X = [A_X, L_X] + \partial L_X, \quad \delta_L A_Y = [A_Y, L_Y] + \partial L_Y, \quad (3.5)$$

$$\delta_L a_X = \partial l_X, \quad \delta_L a_Y = \partial l_Y, \quad (3.6)$$

$$\delta_L F_1 = -L_X F_1 + L_Y F_1, \quad \delta_L F_3 = -L_Y F_3 - F_3 L_X, \quad (3.7)$$

which is expected due to the coset properties.

The first invariant structures that we find are

$$\delta_L \text{Tr} [K_X \bar{K}_X] = \delta_L \text{Tr} [K_Y \bar{K}_Y] = \delta_L \text{Tr} [K_1 \bar{K}_3] = 0. \quad (3.8)$$

The first attempt to construct a Wess-Zumino term will be to use $[K_1 \bar{K}_1^*]$ and $[K_3 \bar{K}_3^*]$. Note that the trace acts on two different spaces. It turns out that those structures are not invariants:

$$\delta_L \ln \text{Tr} [K_1 \bar{K}_1^*] = -\delta_L \ln \text{Tr} [K_3 \bar{K}_3^*] = -2(l_X - l_Y). \quad (3.9)$$

To solve this issue we note that $\delta_L E = (l_X - l_Y) E$. Therefore the right local invariant structures are

$$\delta_L \text{Tr} [K_1 \bar{K}_1^* E^{1/2}] = \delta_L \text{Tr} [K_3 \bar{K}_3^* E^{-1/2}] = 0. \quad (3.10)$$

Since we are equipped with a gauge transformations we can define a covariant derivative,

$$\nabla Z = \partial Z - ZA - Za/4, \quad (3.11)$$

where, as expected,

$$A = \begin{pmatrix} A_X & 0 \\ 0 & A_Y \end{pmatrix}, \quad a = \begin{pmatrix} \mathbb{I}a_X & 0 \\ 0 & \mathbb{I}a_Y \end{pmatrix}, \quad (3.12)$$

and $(A_{X/Y}) = A_{X/Y}$.

Since $[l, A] = [l, a] = 0$, is straightforward to show

$$\delta_L \nabla Z = \nabla Z (L + l/4). \quad (3.13)$$

This is the expected property for the covariant derivative. Finally, just to make everything explicit

$$\nabla Z^{-1} = \partial Z^{-1} + AZ + aZ/4, \quad (3.14)$$

$$\nabla E = 0. \quad (3.15)$$

The covariant derivative of the global invariant current is

$$\begin{aligned} \nabla J &= \partial J - \left[J, A + \frac{\mathbb{I}a}{4} \right] \\ &= \begin{pmatrix} \partial J_X & \partial K_1 \\ \partial K_3 & \partial K_Y \end{pmatrix} \\ &\quad + \begin{pmatrix} [A_X, J_X] & A_X K_1 - K_1 A_Y + \frac{1}{4}(a_X - a_Y) K_1 \\ A_X K_3 - K_3 A_X - \frac{1}{4}(a_X - a_Y) K_3 & [A_Y, J_Y] \end{pmatrix}. \end{aligned} \quad (3.16)$$

Thus, for the F s matrices we obtain

$$\nabla F_1 = \partial F_1 + A_X F_1 - F_1 A_Y, \quad (3.17)$$

$$\nabla F_3 = \partial F_3 + A_Y F_3 - F_3 A_X. \quad (3.18)$$

For the ghosts we require that $\lambda, \bar{\omega}$ behave as F_1 , and $\bar{\lambda}, \omega$ as F_3 . The local invariance of $\text{Tr} [\omega \bar{\nabla} \lambda]$ and $\text{Tr} [\bar{\omega} \nabla \bar{\lambda}]$ requires that

$$\delta_L \lambda = -L_X \lambda + \lambda L_Y, \quad \delta_L \omega = -L_Y \omega + \omega L_X, \quad (3.19)$$

$$\delta_L \bar{\lambda} = -L_Y \bar{\lambda} + \bar{\lambda} L_X, \quad \delta_L \bar{\omega} = -L_X \bar{\omega} + \bar{\omega} L_Y. \quad (3.20)$$

3.2 Global

As stated above, the currents J are invariant under global transformations

$$\delta_G Z = MZ, \quad (3.21)$$

where M is any global matrix. The ghosts fields are, by construction, invariant under global transformation, *i.e.*

$$\delta_G(\text{Ghosts}) = 0. \quad (3.22)$$

If we compute the global transformation under all the terms constructed above, we find that neither $[K_1 \bar{K}_1^* E^{1/2}]$ nor $[K_3 \bar{K}_3^* E^{-1/2}]$ are invariants for a general M :

$$\delta_G \ln \text{Tr} [K_1 \bar{K}_1^* E^{1/2}] = -\delta_G \ln \text{Tr} [K_3 \bar{K}_3^* E^{-1/2}] = \frac{1}{2} \text{STr} M. \quad (3.23)$$

Therefore, we require

$$\text{STr} M = 0. \quad (3.24)$$

4 BRST transformation and BRST invariant action

In [1] the relation between the GL -formalism with the PSU -formalism constructed in [11] of the Green-Schwarz superstring was established. So far we have established a relation between the elements of the pure spinor string [10] in both the GL -formalism and the PSU -formalism. We have also found all structures invariant under the global and local symmetries of the $AdS_5 \times S^5$ space. In order to construct an action for the pure spinor superstring, we are missing one important ingredient the BRST operator. Below, we will establish the BRST symmetry and then find a BRST invariant action. Before doing so, we will review the BRST symmetry in the PSU -formalism. Then we will construct the BRST symmetry for the GL -formalism and construct the BRST invariant action, using the previous construction as a guide.

4.1 PSU -formalism

The BRST transformation for the group element is given by

$$\epsilon \delta_B g = g \epsilon \left(\lambda + \hat{\lambda} \right). \quad (4.1)$$

When acting on the global invariant current we obtain,

$$\epsilon \delta_B J = \partial \epsilon \left(\lambda + \hat{\lambda} \right) + \left[J, \epsilon \left(\lambda + \hat{\lambda} \right) \right]. \quad (4.2)$$

It is useful to write the transformation for the different \mathbb{Z}_4 -elements of the current,

$$\epsilon\delta_B J_0 = [J_1, \epsilon\hat{\lambda}] + [J_3, \epsilon\lambda] , \quad (4.3)$$

$$\epsilon\delta_B J_1 = \nabla\epsilon\lambda + [J_2, \epsilon\hat{\lambda}] , \quad (4.4)$$

$$\epsilon\delta_B J_2 = [J_1, \epsilon\lambda] + [J_3, \epsilon\hat{\lambda}] , \quad (4.5)$$

$$\epsilon\delta_B J_3 = \nabla\epsilon\hat{\lambda} + [J_2, \epsilon\lambda] , \quad (4.6)$$

where, as usual, the covariant derivative is defined as $\nabla = \partial + [J_0, \cdot]$. The λ and $\hat{\lambda}$ ghosts are invariants under the BRST transformation, but not the ω and $\hat{\omega}$. Thus the BRST transformation for the ghosts is given by,

$$\epsilon\delta_B \omega = -J_3\epsilon, \quad \epsilon\delta_B \lambda = 0 , \quad (4.7)$$

$$\epsilon\delta_B \hat{\omega} = -\bar{J}_1\epsilon, \quad \epsilon\delta_B \hat{\lambda} = 0 . \quad (4.8)$$

The ghosts currents were already defined as²

$$N = \{\omega, \lambda\} \quad \text{and} \quad \hat{N} = \{\hat{\omega}, \hat{\lambda}\} . \quad (4.9)$$

Their BRST transformation are

$$\epsilon\delta_B N = -[J_3, \epsilon\lambda] \quad \text{and} \quad \epsilon\delta_B \hat{N} = -[\bar{J}_1, \epsilon\hat{\lambda}] . \quad (4.10)$$

In order to prove the BRST invariance of the action we will use the Maurer-Cartan equations. They read

$$\partial\bar{J}_0 - \bar{\partial}J_0 + [J_0, \bar{J}_0] + [J_1, \bar{J}_3] + [J_2, \bar{J}_2] + [J_3, \bar{J}_1] = 0 , \quad (4.11a)$$

$$\nabla\bar{J}_1 - \bar{\nabla}J_1 + [J_2, \bar{J}_3] + [J_3, \bar{J}_2] = 0 , \quad (4.11b)$$

$$\nabla\bar{J}_2 - \bar{\nabla}J_2 + [J_1, \bar{J}_1] + [J_3, \bar{J}_3] = 0 , \quad (4.11c)$$

$$\nabla\bar{J}_3 - \bar{\nabla}J_3 + [J_1, \bar{J}_2] + [J_2, \bar{J}_1] = 0 . \quad (4.11d)$$

Now we can show that the action

$$S_{\text{PSU}} = \int d^2z \text{Tr} \left[\frac{1}{2} J_2 \bar{J}_2 + \frac{1}{4} J_1 \bar{J}_3 + \frac{3}{4} \bar{J}_1 J_3 + \omega \bar{\nabla}\lambda + \hat{\omega} \nabla\hat{\lambda} - N\hat{N} \right] , \quad (4.12)$$

is BRST invariant.

²There is a minus sign of difference between our definition and the definition in [9].

Applying the BRST transformation given by (4.3)-(4.8) to (4.12) we obtain,

$$\begin{aligned}
\epsilon\delta_B S_{\text{PSU}} &= \int d^2z \text{Tr} \left\{ \frac{1}{2} \left([J_1, \epsilon\lambda] + [J_3, \epsilon\hat{\lambda}] \right) \bar{J}_2 + \frac{1}{2} \left([\bar{J}_1, \epsilon\lambda] + [\bar{J}_3, \epsilon\hat{\lambda}] \right) J_2 \right. \\
&\quad + \frac{1}{4} \left(\nabla\epsilon\lambda + [J_2, \epsilon\hat{\lambda}] \right) \bar{J}_3 + \frac{1}{4} J_1 \left(\bar{\nabla}\epsilon\hat{\lambda} + [\bar{J}_2, \epsilon\lambda] \right) + \frac{3}{4} \left(\bar{\nabla}\epsilon\lambda + [\bar{J}_2, \epsilon\hat{\lambda}] \right) J_3 \\
&\quad + \frac{3}{4} \bar{J}_1 \left(\nabla\epsilon\hat{\lambda} + [J_2, \epsilon\lambda] \right) - J_3\epsilon\bar{\nabla}\lambda - \bar{J}_1\epsilon\nabla\hat{\lambda} + \omega \left[[\bar{J}_1, \epsilon\hat{\lambda}] \right] \\
&\quad + \left[[\bar{J}_3, \epsilon\lambda], \lambda \right] + \hat{\omega} \left[[J_1, \epsilon\hat{\lambda}] + [J_3, \epsilon\lambda], \hat{\lambda} \right] + [J_3, \epsilon\lambda] \hat{N} + N [\bar{J}_1, \epsilon\hat{\lambda}] \left. \right\} \\
&= \int d^2z \text{Tr} \left\{ \frac{\epsilon\lambda}{4} \left(\bar{\nabla}J_3 - \nabla\bar{J}_3 + [\bar{J}_1, J_2] - [J_1, \bar{J}_2] \right) \right. \\
&\quad \left. + \frac{\epsilon\hat{\lambda}}{4} \left(\nabla\bar{J}_1 - \bar{\nabla}J_1 + [J_2, \bar{J}_3] + [J_3, \bar{J}_2] \right) - \epsilon\lambda [N, \bar{J}_3] - \epsilon\hat{\lambda} [\hat{N}, J_1] \right\}.
\end{aligned} \tag{4.13}$$

Using the pure spinor condition (2.28) and the Maurer-Cartan equations (4.11) in the second equality, we can easily show,

$$\epsilon\delta_B S_{\text{usual}} = 0. \tag{4.14}$$

Before ending this section, we note that (4.1) is not actually nilpotent,

$$\epsilon\delta_B \epsilon' \delta_B g = g\epsilon\epsilon' (\lambda\lambda + \bar{\lambda}\bar{\lambda} + \{\lambda, \bar{\lambda}\}). \tag{4.15}$$

Using the pure spinor condition (2.28) we can see that $d_B^2 \sim \{\lambda, \bar{\lambda}\}$. Therefore the BRST transformation is nilpotent up to a gauge transformation. The reason for this is that we are ignoring the BRST transformation for the ghosts. It was shown by Chandía in [12] that in a general curved space the pure spinor ghosts acquire a nonvanishing BRST transformation. The case of *AdS* background was discussed in more detail in [13]. It is straightforward to adapt these results to the present case.

4.2 *GL*-formalism

Now that we are familiar with the original BRST procedure, we can construct the right BRST transformation and the BRST invariant action using a $\frac{GL(4|4)}{(GL(1) \times Sp(2))^2}$ coset. Our ansatz for the BRST transformation of Z is

$$\epsilon\delta_B Z_M^A = Z_M^B \epsilon\Lambda_B^A \quad \epsilon\delta_B Z_A^M = -\epsilon\Lambda_A^B Z_B^M. \tag{4.16}$$

At first one would expect a Λ of the form

$$\Lambda = \begin{pmatrix} 0 & \lambda \\ \bar{\lambda} & 0 \end{pmatrix}. \tag{4.17}$$

But a quick computation shows that δ_B^2 is not 0, nor even proportional to a gauge term. A correct form for Λ is

$$\Lambda = \begin{pmatrix} 0 & \lambda_1 \\ \lambda_3 & 0 \end{pmatrix}, \quad (4.18)$$

$$\lambda_1 = \frac{1}{\sqrt{2}} (\lambda - \bar{\lambda}^*) E^{-1/4}, \quad (4.19)$$

$$\lambda_3 = \frac{1}{\sqrt{2}} (\lambda^* + \bar{\lambda}) E^{1/4}. \quad (4.20)$$

This is of the right form since we want that $\epsilon \delta_B \ln E = \text{STr} \epsilon \Lambda = 0$, and also that $\delta_B^2 \sim \text{gauge}$, as discussed at the end of the previous subsection. Also, the form of λ_1 and λ_3 are such that λ and $\bar{\lambda}$ transform as F_1 and F_3 , respectively.

The transformation for the global invariant currents are

$$\epsilon \delta_B K_X = \langle F_1 \epsilon \lambda^* + \epsilon \bar{\lambda}^* F_3 \rangle, \quad (4.21)$$

$$\epsilon \delta_B A_X = (F_1 \epsilon \bar{\lambda} - \epsilon \lambda F_3), \quad (4.22)$$

$$\epsilon \delta_B K_Y = \langle F_1^* \epsilon \lambda + \epsilon \bar{\lambda} F_3^* \rangle, \quad (4.23)$$

$$\epsilon \delta_B A_Y = (F_3 \epsilon \lambda - \epsilon \bar{\lambda} F_1), \quad (4.24)$$

$$\epsilon \delta_B F_1 = \nabla \epsilon \lambda + \epsilon \bar{\lambda}^* K_Y - K_X \epsilon \bar{\lambda}^*, \quad (4.25)$$

$$\epsilon \delta_B F_3 = \nabla \epsilon \bar{\lambda} - \epsilon \lambda^* K_X + K_Y \epsilon \lambda^*, \quad (4.26)$$

and for the ghosts

$$\epsilon \delta_B \omega = -\epsilon F_3, \quad \epsilon \delta_B \lambda = 0, \quad (4.27)$$

$$\epsilon \delta_B \bar{\omega} = \epsilon, \quad \epsilon \delta_B \bar{\lambda} = 0. \quad (4.28)$$

Finally, we are able to show that

$$S_{\text{GL}} = \int d^2 \tilde{\text{Tr}} \left[\frac{1}{2} K_X \bar{K}_X - \frac{1}{2} K_Y \bar{K}_Y + \frac{1}{4} F_1 \bar{F}_3 + \frac{3}{4} \bar{F}_1 F_3 + \omega \bar{\nabla} \lambda + \bar{\omega} \nabla \bar{\lambda} + N_X \bar{N}_X - N_Y \bar{N}_Y \right] \quad (4.29)$$

is BRST invariant. Before we do that, a few comments are in order. $\tilde{\text{Tr}}$ is defined in such a way to avoid confusion on which space the trace acts on. Since Tr acts in either a or \bar{a} indices, we cannot write a term like $\text{Tr} (\lambda \omega + \bar{\lambda} \bar{\omega})$. To avoid further confusion, we define an operation $\tilde{\text{Tr}}$ such that $\tilde{\text{Tr}} (\lambda \omega + \bar{\lambda} \bar{\omega})$ means $\lambda_a^{\bar{a}} \omega_{\bar{a}}^a + \bar{\lambda}_{\bar{a}}^a \bar{\omega}_a^{\bar{a}}$. Note that the trace of K_Y has a minus sign, that is because $\text{STr} M = M_X - M_Y$. Also, while $\epsilon \delta_B \omega$ has a minus sign, $\epsilon \delta_B \bar{\omega}$ does not. That is because F_3 is related to $-J_3^{\hat{\alpha}}$, and we did that only for aesthetic reasons. Finally, in S_{usual} the ghost current term is $\text{Tr} -N \hat{N}$, and here is $\tilde{\text{Tr}} N \bar{N}$. The difference in sign is because $\bar{\omega}$ is equivalent to $-\hat{\omega}_{\hat{\alpha}}$. In both actions we want that the kinetic term of the ghost to be positive defined. To obtain that, we need to define $\hat{\omega} = -\hat{\omega}_{\hat{\alpha}} \eta^{\hat{\alpha} \hat{\alpha}} T_{\hat{\alpha}}$, and this in turn implies that $\text{Tr} -N \hat{N} = N^i \hat{N}^j g_{ij}$ which is equivalent to $\text{STr} N \bar{N}$.

We are going to need the following Maurer-Cartan equation:

$$\bar{\nabla}F_1 - \nabla\bar{F}_1 - \bar{K}_X F_3^* + K_X \bar{F}_3^* - \bar{F}_3^* K_Y + F_3^* \bar{K}_Y = 0, \quad (4.30)$$

$$\bar{\nabla}F_3 - \nabla\bar{F}_3 + \bar{K}_Y F_1^* - K_Y \bar{F}_1^* + \bar{F}_1^* K_X - F_1^* \bar{K}_X = 0. \quad (4.31)$$

We now check that

Applying the BRST transformation to (4.29),

$$\begin{aligned} \epsilon\delta_B S_{\text{RS}} = & \int d^2z \tilde{\text{Tr}} \left\{ \frac{1}{2} (F_1 \epsilon \lambda^* + \epsilon \bar{\lambda}^* F_3) \bar{K}_X + \frac{1}{2} K_X (\bar{F}_1 \epsilon \lambda^* + \epsilon \bar{\lambda}^* \bar{F}_3) \right. \\ & - \frac{1}{2} (F_1^* \epsilon \lambda + \epsilon \bar{\lambda} F_3^*) \bar{K}_Y - \frac{1}{2} K_Y (\bar{F}_1^* \epsilon \lambda + \epsilon \bar{\lambda} \bar{F}_3^*) + \frac{1}{4} (\nabla \epsilon \lambda + \epsilon \bar{\lambda}^* K_Y - K_X \epsilon \bar{\lambda}^*) \bar{F}_3 \\ & + \frac{1}{4} F_1 (\bar{\partial} \epsilon \bar{\lambda} - \epsilon \lambda^* \bar{K}_X + \bar{K}_Y \epsilon \lambda^*) + \frac{3}{4} (\bar{\nabla} \epsilon \lambda + \epsilon \bar{\lambda}^* \bar{K}_Y - \bar{K}_X \epsilon \bar{\lambda}^*) F_3 \\ & + \frac{1}{4} \bar{F}_1 (\partial \epsilon \bar{\lambda} - \epsilon \lambda^* K_X + K_Y \epsilon \lambda^*) - \epsilon F_3 \bar{\nabla} \lambda + \epsilon \bar{F}_1 \nabla \bar{\lambda} + (\bar{F}_1 \epsilon \bar{\lambda} - \epsilon \lambda \bar{F}_3) N_X \\ & - (F_1 \epsilon \bar{\lambda} - \epsilon \lambda F_3) \bar{N}_X - (\bar{F}_3 \epsilon \lambda - \epsilon \bar{\lambda} \bar{F}_1) N_Y + (F_3 \epsilon \lambda - \epsilon \bar{\lambda} F_1) \bar{N}_Y \\ & \left. - \epsilon \lambda F_3 \bar{N}_X + N_X \epsilon \bar{F}_1 \bar{\lambda} + \epsilon F_3 \lambda \bar{N}_Y - N_Y \bar{\lambda} \epsilon \bar{F}_1 \right\}. \end{aligned} \quad (4.32)$$

The pure spinor condition ensures that $N_X \lambda - \lambda N_Y = 0$ and $\bar{N}_Y \bar{\lambda} - \bar{\lambda} \bar{N}_X = 0$. The only terms that survive are

$$\epsilon\delta_B S_{\text{RS}} = \int d^2z \frac{1}{4} \tilde{\text{Tr}} \left[\epsilon \bar{\lambda} (\bar{\nabla} F_1 - \nabla \bar{F}_1 - \bar{K}_X F_3^* + K_X \bar{F}_3^* - \bar{F}_3^* K_Y + F_3^* \bar{K}_Y) \right] \quad (4.33)$$

$$+ \epsilon \lambda (\bar{\nabla} F_3 - \nabla \bar{F}_3 + \bar{K}_Y F_1^* - K_Y \bar{F}_1^* + \bar{F}_1^* K_X - F_1^* \bar{K}_X) \quad (4.34)$$

which are identically 0 because of the Maurer-Cartan equation,

$$S_{WZ} = -\frac{1}{4} \int d^2z \text{Tr} \left[K_3^* \bar{K}_3 E^{-1/2} - K_1 \bar{K}_1^* E^{1/2} \right] = -\frac{1}{4} \int d^2z \text{Tr} \left[F_1 \bar{F}_3 - \bar{F}_1 F_3 \right]. \quad (4.35)$$

As we saw in the previous section, (4.29) is both local and global invariant if and only if the global transformation is generated by a supertraceless matrix.

4.3 Vectors

In [14, 15] a systematic construction of vertex operators for a supersphere sigma model was developed. An important ingredient for such construction was vectors describing the target spaced. The existence of such vectors describing the bosonic coordinates of the $AdS_5 \times S^5$ superspace was discussed in [1]. It is an interesting question whether we can construct all the matter part of (4.29) with such vectors. We will now discuss how to obtain this. The first set of vectors we can construct are

$$W^{MN} = Z_a^M \Omega^{ab} Z_b^N E^{1/4}, \quad W_{MN} = Z_M^a \Omega_{ab} Z_N^b E^{-1/4}, \quad (4.36)$$

$$W'^{MN} = Z_{\bar{a}}^M \Omega^{\bar{a}\bar{b}} Z_{\bar{b}}^N E^{-1/4}, \quad W'_{MN} = Z_M^{\bar{a}} \Omega_{\bar{a}\bar{b}} Z_N^{\bar{b}} E^{1/4}. \quad (4.37)$$

Being careful with the indices and product of fermionic matrices, the only terms that we can construct are,

$$\nabla W^{MN} \bar{\nabla} W_{NM} = \text{Tr} [4K_X \bar{K}_X + 2K_1 \bar{K}_3] , \quad (4.38)$$

$$\nabla W'^{MN} \bar{\nabla} W'_{NM} = \text{Tr} [4K_Y \bar{K}_Y + 2K_3 \bar{K}_1] . \quad (4.39)$$

Now we are able to construct part of the matter part of (4.29),

$$\frac{1}{8} [\nabla W^{MN} \bar{\nabla} W_{NM} - \nabla W'^{MN} \bar{\nabla} W'_{NM}] = \tilde{\text{Tr}} \left[\frac{1}{2} K_X \bar{K}_X - \frac{1}{2} K_Y \bar{K}_Y + \frac{1}{4} K_1 \bar{K}_3 + \frac{1}{4} \bar{K}_1 K_3 \right] . \quad (4.40)$$

In order to obtain the right factor for the K -terms, we need to introduce another group of vectors,

$$U^{MN} = Z_a^M \Omega^{ab} \overleftrightarrow{\nabla} Z_b^N E^{1/4} , \quad U_{MN} = Z_M^a \Omega_{ab} \overleftrightarrow{\nabla} Z_N^b E^{-1/4} , \quad (4.41)$$

$$U'^{MN} = Z_{\bar{a}}^M \Omega^{\bar{a}\bar{b}} \overleftrightarrow{\nabla} Z_{\bar{b}}^N E^{-1/4} , \quad U'_{MN} = Z_M^{\bar{a}} \Omega_{\bar{a}\bar{b}} \overleftrightarrow{\nabla} Z_N^{\bar{b}} E^{1/4} . \quad (4.42)$$

We define $A \overleftrightarrow{\nabla} B = A \nabla B - \nabla A B$, and ∇ is the covariant derivative defined in (3.11). A direct computation shows that the product (in this case, the STr) of any two different vectors is always 0.

Using (4.38) and (4.39) we can construct,

$$U^{MN} \bar{U}_{NM} = \text{Tr} [-2K_1 \bar{K}_3] , \quad (4.43)$$

$$\bar{U}'^{MN} U'_{MN} = \text{Tr} [-2K_3 \bar{K}_1] . \quad (4.44)$$

With all those ingredients, we can construct the matter part of (4.29) without the Wess-Zumino term:

$$\frac{1}{8} [\nabla W^{MN} \bar{\nabla} W_{NM} - U^{MN} \bar{U}_{NM} - \nabla W'^{MN} \bar{\nabla} W'_{NM} + U'^{MN} \bar{U}'_{NM}] \quad (4.45)$$

$$= \frac{1}{2} \tilde{\text{Tr}} [K_X \bar{K}_X - K_Y \bar{K}_Y + K_1 \bar{K}_3 + \bar{K}_1 K_3] . \quad (4.46)$$

The question now is how can we write the Wess-Zumino term of the action. First we remember that the Wess-Zumino term is

$$\mathcal{L}_{\text{WZ}} = -\frac{\kappa}{2} \text{Tr} [F_1 \bar{F}_3 - \bar{F}_1 F_3] = \frac{\kappa}{2} \text{Tr} [K_1 \bar{K}_1^* E^{1/2} - K_3^* \bar{K}_3 E^{-1/2}] . \quad (4.47)$$

A quick glance to list of vectors shows that the only possible way is a product between W s and U s. Indeed

$$(-)^M \nabla W^{MN} \bar{U}'_{NM} = \text{Tr} [-2K_1 \bar{K}_1^* E^{1/2}] , \quad (4.48)$$

$$(-)^M \nabla W'^{MN} \bar{U}_{NM} = \text{Tr} [-2K_3 \bar{K}_3^* E^{-1/2}] . \quad (4.49)$$

Before we continue, a comment should be made: The product between vector is a STr between supermatrices,

$$W^{MN}W_{NM} = \text{STr}WW. \quad (4.50)$$

The product between W and U' should also be a STr. The product $\nabla W^{MN}\bar{U}'_{NM}$ is not, since

$$\nabla W^{MN}\bar{U}'_{NM} \neq \bar{U}'_{NM}\nabla W^{MN}. \quad (4.51)$$

The solution to this problem is the addition of the $(-)^M$ term. Now

$$(-)^M\nabla W^{MN}\bar{U}'_{NM} = \text{STr}\nabla W\bar{U}'. \quad (4.52)$$

We finally have all the ingredients to construct the matter part of S_{GL} and choosing $\kappa = \frac{1}{2}$, we get

$$\begin{aligned} \mathcal{L}_{\text{RS}} = & \frac{1}{8} \left[\nabla W^{MN}\bar{\nabla}W_{NM} - U^{MN}\bar{U}_{NM} - (-)^M\nabla W^{MN}\bar{U}'_{NM} - \nabla W'^{MN}\bar{\nabla}W'_{NM} \right. \\ & \left. + U'^{MN}\bar{U}'_{NM} + (-)^M\nabla W'^{MN}\bar{U}_{NM} \right] \end{aligned} \quad (4.53)$$

$$= \frac{1}{2} \tilde{\text{Tr}} \left[K_X \bar{K}_X - K_Y \bar{K}_Y + \frac{1}{2} K_1 \bar{K}_3 + \frac{3}{2} \bar{K}_1 K_3 \right]. \quad (4.54)$$

5 An application: vertex operator construction

Following [3, 4] we will construct an operator V such that $\epsilon\delta_B V = 0$. To achieve this, we will construct the conserved current j related to global symmetries of the action (4.29). Then we will construct V by applied the BRST transformation to j , $\delta_B j = \partial V$. This will be our first vertex operator in this formalism. In future works we will try to apply the procedure explained in sections 4.3 to the construction of vertex operators, as in [14].

5.1 Equation of Motions

As usual, in order to construct a conserved current, we need the equation of motions (EOM). To obtains such equations we will vary Z around a background field,

$$Z = Z_0 e^X, \quad (5.1)$$

where the components of X have been defined in (2.4). This leads to

$$\delta J = \partial X + [J, X]. \quad (5.2)$$

Writing this in components

$$\delta J_X = \partial X + [J_X, X] + K_1 \Theta_3 - \Theta_1 K_3, \quad (5.3a)$$

$$\delta J_Y = \partial Y + [J_Y, Y] + K_3 \Theta_1 - \Theta_3 K_1, \quad (5.3b)$$

$$\delta K_1 = \nabla \Theta_1 + K_X \Theta_1 - \Theta_1 K_Y + K_1 Y - X K_1, \quad (5.3c)$$

$$\delta K_3 = \nabla \Theta_3 + K_Y \Theta_3 - \Theta_3 K_X + K_3 X - Y K_3. \quad (5.3d)$$

Since we have written (4.29) in terms of F_1 and F_3 , we write the variation of those, using the above equations:

$$\delta F_1 = \nabla \theta_1 - K_X \theta_3^* + \theta_3^* K_Y - F_3^* Y + X F_3^*, \quad (5.4)$$

$$\delta F_3 = \nabla \theta_3 + K_Y \theta_1^* - \theta_1^* K_X + F_1^* X - Y F_1^*, \quad (5.5)$$

and the same for the K s and A s:

$$\delta K_X = \nabla X + \frac{1}{2} (F_1 \theta_1^* - \theta_1 F_1^* + \theta_3^* F_3 - F_3^* \theta_3) - \frac{\mathbb{I}}{4} \text{Tr} (F_1 \theta_1^* + \theta_3^* F_3), \quad (5.6)$$

$$\delta K_Y = \nabla Y + \frac{1}{2} (F_1^* \theta_1 - \theta_1^* F_1 + \theta_3 F_3^* - F_3 \theta_3^*) - \frac{\mathbb{I}}{4} \text{Tr} (F_1 \theta_1^* + \theta_3^* F_3), \quad (5.7)$$

$$\delta A_X = [K_X, X] + \frac{1}{2} (F_1 \theta_3 + \theta_3^* F_1^* - \theta_1 F_3 - F_3^* \theta_1^*), \quad (5.8)$$

$$\delta A_Y = [K_Y, Y] + \frac{1}{2} (F_3 \theta_1 + \theta_1^* F_3^* - \theta_3 F_1 - F_1^* \theta_3^*). \quad (5.9)$$

Using the variation of the action and the Maurer-Cartan equations we obtain,

$$\nabla \bar{K}_X + \frac{1}{2} (F_1 \bar{F}_1^* - \bar{F}_1 F_1^*) - \frac{\mathbb{I}}{4} \text{Tr} F_1 \bar{F}_1^* + [\bar{N}_X, K_X] - [N_X, \bar{K}_X] = 0, \quad (5.10a)$$

$$\bar{\nabla} K_X + \frac{1}{2} (F_3^* \bar{F}_3 - \bar{F}_3^* F_3) - \frac{\mathbb{I}}{4} \text{Tr} F_3 \bar{F}_3^* + [\bar{N}_X, K_X] - [N_X, \bar{K}_X] = 0, \quad (5.10b)$$

$$\nabla \bar{K}_Y + \frac{1}{2} (F_1^* \bar{F}_1 - \bar{F}_1^* F_1) - \frac{\mathbb{I}}{4} \text{Tr} F_1 \bar{F}_1^* + [\bar{N}_Y, K_Y] - [N_Y, \bar{K}_Y] = 0, \quad (5.10c)$$

$$\bar{\nabla} K_Y + \frac{1}{2} (F_3 \bar{F}_3^* - \bar{F}_3 F_3^*) - \frac{\mathbb{I}}{4} \text{Tr} F_3 \bar{F}_3^* + [\bar{N}_Y, K_Y] - [N_Y, \bar{K}_Y] = 0, \quad (5.10d)$$

$$\bar{\nabla} \bar{F}_1 - \bar{K}_X F_3^* + K_X \bar{F}_3^* + F_3^* \bar{K}_Y - \bar{F}_3^* K_Y - N_X \bar{F}_1 + \bar{N}_X F_1 + \bar{F}_1 N_Y - F_1 \bar{N}_Y = 0, \quad (5.10e)$$

$$\nabla F_1 - N_X \bar{F}_1 + \bar{N}_X F_1 + \bar{F}_1 N_Y - F_1 \bar{N}_Y = 0, \quad (5.10f)$$

$$\nabla \bar{F}_3 - \bar{K}_Y^* F_1 + K_Y^* \bar{F}_1 + F_1^* \bar{K}_X - \bar{F}_1^* K_X - N_Y \bar{F}_3 + \bar{N}_Y F_3 + \bar{N}_Y F_3 - F_3 \bar{N}_X = 0, \quad (5.10g)$$

$$\bar{\nabla} F_3 - N_Y \bar{F}_3 + \bar{N}_Y F_3 + \bar{F}_3 N_X - F_3 \bar{N}_X = 0, \quad (5.10h)$$

$$\bar{\nabla} \omega + \omega \bar{N}_X - \bar{N}_Y \omega = 0, \quad (5.10i)$$

$$\bar{\nabla} \lambda + \bar{N}_X \lambda - \lambda \bar{N}_Y = 0, \quad (5.10j)$$

$$\nabla \bar{\omega} - N_X \bar{\omega} + \bar{\omega} N_Y = 0, \quad (5.10k)$$

$$\nabla \bar{\lambda} + \bar{\lambda} N_X - N_Y \bar{\lambda} = 0. \quad (5.10l)$$

To obtain these equations we used the fact $\text{Tr} [XH] = \text{Tr} [\langle X \rangle H] = \text{Tr} [X \langle H \rangle]$. Thus, the right EOM for X is given by $\langle H \rangle = 0$.

5.2 Construction of V

In order to compute the Noether current we first make a few observations. The first of them is noting that $\text{Tr} K_X \bar{K}_X = \text{Tr} K_X \bar{J}_X$, thus, instead of taking $\text{Tr} \bar{K}_X \langle Z_a^M \partial M_M^N Z_N^b \rangle$, we just take $\text{Tr} \bar{K}_X Z_a^M \partial M_M^N Z_N^b$. The same can be done for the ghost current, since

$N_X = (\lambda\omega)$. Using the EOM (5.10) and the global transformation studied in section 3.2, we find the left and right conserved currents,

$$j = \begin{pmatrix} K_X + 2N_X & \frac{1}{2\sqrt{2}} (F_1 - 3F_3^*) E^{-1/4} \\ \frac{1}{2\sqrt{2}} (F_1^* + 3F_3^*) E^{1/4} & K_Y + 2N_Y \end{pmatrix}, \quad (5.11)$$

$$\bar{j} = \begin{pmatrix} \bar{K}_X - 2\bar{N}_X & \frac{1}{2\sqrt{2}} (3\bar{F}_1 - \bar{F}_3^*) E^{-1/4} \\ \frac{1}{2\sqrt{2}} (3\bar{F}_1^* + \bar{F}_3^*) E^{1/4} & \bar{K}_Y - 2\bar{N}_Y \end{pmatrix}. \quad (5.12)$$

Since $\epsilon\delta_B\delta_G S_{RS} = 0$ one would expect $\epsilon\delta_B j = \partial V$ and $\epsilon\delta_B \bar{j} = -\bar{\partial}V$ as in the usual description. But here $\text{STr}M = 0$, thus, $\epsilon\delta_B j = \partial V + \mathbb{I}A$ and $\epsilon\delta_B \bar{j} = -\bar{\partial}V + \mathbb{I}B$ is the most general form, for any A and B . For the same reason, one would expect that $\epsilon\delta_B \epsilon' \delta_B \delta_G S = 0$ yields $\epsilon\delta_B V = 0$, but the most general possibility is $\epsilon\delta_B V = \mathbb{I}C$, for any C . Now, this $\mathbb{I}C$ term should be expected from the gauge group $(GL(1))^2$, since a the condition $A = \Omega A^T \Omega$, imposed to gauge terms, does not apply to the term proportional to the trace³ thus, it seems that we have eliminated those term. But this is not true, we did eliminated the a_X , a_Y gauge terms: we did it when writing the action proportional to the Tr. Therefore, the correct BRST invariant vector is $\text{STr}V$.

After a long calculation, for BRST transformation of the left current we find that

$$\epsilon\delta_B j = \frac{1}{2\sqrt{2}} \partial \epsilon V - \frac{\mathbb{I}}{4} \text{Tr} (F_1 \epsilon \lambda^* + \epsilon \bar{\lambda}^* F_3), \quad (5.13)$$

$$\epsilon V = Z \begin{pmatrix} 0 & \epsilon (\lambda + \bar{\lambda}^*) E^{-1/4} \\ \epsilon (\lambda^* - \bar{\lambda}) E^{1/4} & 0 \end{pmatrix} Z^{-1} = Z \epsilon \Lambda' Z^{-1}. \quad (5.14)$$

For the right current we find, as expected,

$$\epsilon\delta_B \bar{j} = -\frac{1}{2\sqrt{2}} \bar{\partial} \epsilon V - \frac{\mathbb{I}}{4} \text{Tr} (\bar{F}_1 \epsilon \lambda^* + \epsilon \bar{\lambda}^* \bar{F}_3). \quad (5.15)$$

Finally, we check that $\epsilon\delta_B \text{STr}V = 0$:

$$\epsilon' \delta_B \epsilon V = Z [\epsilon' \Lambda, \epsilon \Lambda'] Z^{-1} = Z \epsilon' \epsilon \{ \Lambda, \Lambda' \} Z^{-1} \quad (5.16)$$

$$= 2\epsilon' \epsilon Z \begin{pmatrix} \lambda \lambda^* + \bar{\lambda}^* \bar{\lambda} & 0 \\ 0 & \lambda^* \lambda + \bar{\lambda} \bar{\lambda}^* \end{pmatrix} Z^{-1} \quad (5.17)$$

$$= \frac{1}{2} \epsilon' \epsilon \mathbb{I} \text{Tr} (\lambda \lambda^* + \bar{\lambda}^* \bar{\lambda}), \quad (5.18)$$

therefore $\delta_B \text{STr}V = 0$. The vertex operator corresponding to the β -deformation discussed in [3, 4] can now be described as the tensor product of two V .

³Note that $\mathbb{I} = -\Omega \mathbb{I}^T \Omega$.

6 Conclusion and further directions

We have described the pure spinor superstring in $AdS_5 \times S^5$ using the $GL(4|4)/(Sp(2) \times GL(1))^2$ coset first used by Roiban and Siegel for the Green-Schwarz superstring in [1]. This formulation provides additional choices for the parametrization of the AdS coordinates. This additional choices have been shown to be useful in formulations different superspaces relevant to the AdS/CFT conjecture [8]. Recently, Schwarz described another parametrization for the GS string in $AdS_5 \times S^5$ [16]. As was shown by Siegel [17], this new formulation can also be used in the present case.

Furthermore, the complete superspace propagator for the entire tower of Kaluza-Klein modes was calculated in [18] using this new coset. This propagator was shown to be invariant under κ -symmetry. Since there is a close relation between κ -symmetry and the BRST transformations of the pure spinor formalism ⁴ it is likely that this propagator can be used to construct a BRST invariant ghost number two superspace function. Such function would be related to the unintegrated vertex operators of the supergravity modes in the pure spinor formulation. We are presently working in this direction. The ultimate goal is to have a systematic way to construct vertex operators at any mass level using the world sheet dilatation operator [20] to derive physical state conditions. Although BRST invariance should also be imposed, vanishing world sheet anomalous dimension may be enough to calculate the spacetime energy of the string states.

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⁴For example, demanding invariance under κ -symmetry of the GS action in a general curved supergravity background puts the background on-shell. The same is achieved in pure spinor formalism demanding BRST invariance [19].

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