

SWITCHING NETWORKS FOR BIDIRECTIONAL TELEPHONE SYSTEMS

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Abstract

We consider two-stage bidirectional switching networks which have a minimum number of switches. Results on the structure of such networks in terms of the number of switches per crosswire are established.

1. Introduction

Newbury and Raby [2] have considered two-stage switching arrangements in bidirectional telephone systems. In this kind of network each subscriber line is connected via switches to some (or all) of a set of crosswires. Thus a connection between two subscribers is made along one crosswire via two switches. We call such an arrangement of subscriber lines, crosswires and switches connecting them, a two-stage bidirectional switching network.

Such a network may be represented as an incidence structure (P, B, S) where $P = \{ P_1, \dots, P_m \}$ is a set labelling the subscriber lines, $B = \{ x_1, \dots, x_b \}$ is a set labelling the crosswires and $S \subseteq P \times B$ with $(P_i, x_j) \in S$ if and only if subscriber line P_i is connected to crosswire x_j by a switch. We only consider networks in which each crosswire has at least two switches on it.

Since pairs of subscriber lines must be connected via distinct crosswires and we would like to be able to connect as many subscribers in pairs as possible, we also require that $b \geq \lceil m/2 \rceil$. If $m = 2n + \delta$, where δ is 0 or 1, and $b = n + t$, where $t \geq 0$, we use $2BSN(n, t, \delta)$ to denote such a two-stage bidirectional switching network. The two-stage bidirectional switching networks with $2n$ subscriber lines

and n crosswires denoted by $2BSN(n)$ in [1] are here denoted by $2BSN(n,0,0)$.

Of interest is the minimum number of switches required to ensure that for any n disjoint pairs of subscriber lines there is an assignment of distinct crosswires to the pairs such that each pair may be connected via the crosswire assigned to it. A $2BSN(n,t,\delta)$ with this property is called rearrangeably non-blocking (or rearrangeable) and is denoted $R2BSN(n,t,\delta)$. In this paper we consider $R2BSN(n,t,\delta)$ with $t = 0$ and $t = 1$, and show that the minimum number of switches requirement puts strong conditions on the number of switches that the individual crosswires may contain.

2. $R2BSN(n,0,\delta)$

We extend the results of [1] to cover the case δ

LEMMA 2.1. *Let $D = (P,B,S)$ be an $R2BSN(n,0,\delta)$. Then for each crosswire in B there are at most $n-1$ subscriber lines not connected to it by a switch.*

Proof. Suppose $x_j \in B$ is not connected to n subscriber lines. Without loss of generality we may assume these are P_1, \dots, P_n . Then $(P_1, P_{n+1}), \dots, (P_n, P_{2n})$ is a collection of n disjoint pairs of subscriber lines such that crosswire x_j does not connect any of them. Since B contains only n crosswires these n disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, so no such x_j exists. \square

LEMMA 2.2. *Let $D = (P,B,S)$ be an $R2BSN(n,0,\delta)$. Then there is at most one crosswire in B which is not connected to $n-1$ subscriber lines by a switch.*

Proof. Suppose there are two such crosswires, x_1, x_2 say. Without loss of generality we may assume that x_1 is not connected to P_1, \dots, P_{n-1} and x_2 is not connected to $P_1, \dots, P_s, P_n, \dots, P_{2n-2-s}$, where $0 < s < n-1$. Then

$(P_1, P_{2n-1-s}), \dots, (P_s, P_{2n-2})$ and $(P_{s+1}, P_n), \dots, (P_{n-1}, P_{2n-2-s})$ form a collection of $n-1$ disjoint pairs of subscriber lines such that neither x_1 nor x_2 connects any of them. Since B contains only n crosswires these disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, and so there can exist at most one such crosswire. \square

PROPOSITION 2.3. Let $D = (P, B, S)$ be an $R2BSN(n, 0, \delta)$. Then $|S| \geq n^2 + (2+\delta)n - 1$. If equality holds then exactly one crosswire contains $n + 1 + \delta$ switches, and the other $n-1$ crosswires each contain $n + 2 + \delta$ switches.

Proof. By Lemma 2.1 each crosswire contains at least $2n + \delta - (n-1) = n + 1 + \delta$ switches, and by Lemma 2.2 at most one has exactly $n + 1 + \delta$ switches. Thus $|S| \geq (n+1+\delta) + (n-1)(n+2+\delta) = n^2 + (2+\delta)n - 1$. \square

In [1] an $R2BSN(n, 0, 0)$ with $n^2 + 2n - 1$ switches is constructed for every n . The following result shows that there also exist $R2BSN(n, 0, 1)$ with $n^2 + 3n - 1$ switches for every n .

PROPOSITION 2.4. Let $D = (P, B, S)$ be an $R2BSN(n, 0, 0)$. Let $P' = P \cup \{P_{2n+1}\}$, $B' = B$, and $S' = S \cup \{(P_{2n+1}, x_j) : x_j \in B\}$. Then $D' = (P', B', S')$ is an $R2BSN(n, 0, 1)$ with $|S'| = n^2 + 3n - 1$.

Proof. Let A_1, \dots, A_n be n disjoint pairs of subscriber lines in P' . First suppose $P_{2n+1} \notin A_1 \cup \dots \cup A_n$. Then A_1, \dots, A_n are n disjoint pairs of subscriber lines in P and so there are n distinct crosswires in $B = B'$ which connect them.

Now suppose $P_{2n+1} \in A_1 \cup \dots \cup A_n$. Without loss of generality we may assume that $A_1 = (P_1, P_{2n+1})$ and that $P' - (A_1 \cup \dots \cup A_n) = \{P_{2n}\}$. Put $A_1' = (P_1, P_{2n})$. Then A_1', A_2, \dots, A_n are n disjoint pairs of subscriber lines in P . Hence they are connected via distinct crosswires. Since the crosswire connecting P_1 and P_{2n} also connects P_1 and

P_{2n+1} , the n disjoint pairs A_1, \dots, A_n are also connected via distinct crosswires. It follows that D' is rearrangeable and so is an $R2BSN(n,0,1)$. \square

3. $R2BSN(n,1,\delta)$

We consider the structure of an $R2BSN(n,1,\delta)$ having at most $n^2 + (2+\delta)n - 1$ switches, i.e. the minimum number of switches for an $R2BSN(n,0,\delta)$.

LEMMA 3.1. Let $D = (P,B,S)$ be an $R2BSN(n,1,\delta)$. Then there is at most one crosswire in B which is not connected to n subscriber lines by a switch.

Proof. Suppose there are two such crosswires. Without loss of generality we may suppose that x_1 is not connected to P_1, \dots, P_n and x_2 is not connected to $P_1, \dots, P_s, P_{n+1}, \dots, P_{2n-s}$ where $0 \leq s \leq n$. Then $(P_1, P_{2n+1-s}), \dots, (P_s, P_{2n})$ and $(P_{s+1}, P_{n+1}), \dots, (P_n, P_{2n-s})$ are n disjoint pairs of subscriber lines none of which are connected via x_1 or x_2 . Since B contains only $n+1$ crosswires, these n disjoint pairs cannot be connected via distinct crosswires. But D is rearrangeable, so there is at most one such crosswire. \square

An $R2BSN(n,1,\delta)$ in which every crosswire contains at least $n+1+\delta$ switches has at least $(n+1)(n+1+\delta) = n^2 + (2+\delta)n + 1+\delta$ switches. Thus, in an $R2BSN(n,1,\delta)$ with at most $n^2 + (2+\delta)n - 1$ switches, some crosswire, x_1 say, is connected to $s \leq n-1$ subscriber lines. Moreover there must be at least $s+1$ other crosswires which contain exactly $n+1+\delta$ switches.

LEMMA 3.2. Let $D = (P,B,S)$ be an $R2BSN(n,1,\delta)$ with at most $n^2 + (2+\delta)n - 1$ switches. Suppose crosswire x_1 contains $s \leq n-1$ switches, and two crosswires, each containing $n+1+\delta$ switches are connected to r common subscriber lines. Then $3+\delta \leq r \leq n+\delta$ and $s \geq \min(r+1+\delta, n+2+\delta-r)$.

Proof. Put $u = n + 1 + \delta$. We note that at least $s+1 \geq 3$ crosswires contain u switches. Without loss of generality suppose that x_2 is connected to P_1, \dots, P_u and x_3 is connected to $P_1, \dots, P_r, P_{u+1}, \dots, P_{2u-r}$. We first show that $3 + \delta \leq r \leq n + \delta$.

As $2u-r = 2n + 2 + 2\delta - r \leq 2n + \delta$, we have $r \geq 2 + \delta$. If $r = 2 + \delta$ then for any permutation f of $\{u+1, \dots, 2n+\delta\}$ the n pairs $\{P_1, P_2\}, \{P_{3+\delta}, P_{f(u+1)}\}, \dots, \{P_u, P_{f(2n+\delta)}\}$ are disjoint, and only the pair $\{P_1, P_2\}$ is connected via x_2 or x_3 . Since D is rearrangeable the crosswire x_1 must connect one of the pairs $\{P_i, P_{f(n-1+i)}\}$. Since this is true for every f it follows that x_1 is connected to at least n of the subscriber lines $P_{3+\delta}, \dots, P_{2n+\delta}$. But x_1 contains only $s \leq n-1$ switches. Hence $r \geq 3 + \delta$.

Now $r \leq u = n + 1 + \delta$. If $r = n + 1 + \delta$ then for any $(2+\delta)$ -subset A of $\{1, \dots, u\}$ and any bijective mapping $f : \{1, \dots, u\} - A \rightarrow \{u+1, \dots, 2n+\delta\}$ the pairs $\{P_i, P_j\}$, $i, j \in A$, and $\{P_s, P_{f(s)}\}$, $s \in \{1, \dots, u\} - A$, are disjoint, and only the pair $\{P_i, P_j\}$ is connected via x_2 or x_3 . It follows that x_1 connects one of the pairs $\{P_s, P_{f(s)}\}$ for every A and f , and hence contains at least $n + 2 + \delta$ switches. But x_1 contains only $s \leq n-1$ switches and hence $r \leq n + \delta$.

Now suppose $3 + \delta \leq r \leq n + \delta$. Then for any $(2+\delta)$ -subset A of $\{1, \dots, r\}$ and bijective mappings $f : \{1, \dots, r\} - A \rightarrow \{2u-r+1, \dots, 2n+\delta\}$ and $g : \{r+1, \dots, u\} \rightarrow \{u+1, \dots, 2u-r\}$, the n pairs $\{P_i, P_j\}$, $i, j \in A$, $\{P_s, P_{f(s)}\}$, $s \in \{1, \dots, r\} - A$, and $\{P_s, P_{g(s)}\}$, $s \in \{r+1, \dots, u\}$, are disjoint, and only the pairs $\{P_i, P_j\}$ are connected via x_2 or x_3 . Hence for every A , f and g , the crosswire x_1 connects one of the pairs $\{P_s, P_{f(s)}\}$ or one of the pairs $\{P_s, P_{g(s)}\}$. It follows that x_1 is either connected to at least $r+1+\delta$ of the subscriber lines $P_1, \dots, P_r, P_{2u-r+1}, \dots, P_{2n+\delta}$, or to at least

$u-r+1 = n+2+\delta-r$ of P_{r+1}, \dots, P_{2u-r} . Thus
 $s > \min(r+1+\delta, n+2+\delta-r)$. \square

Lemma 3.2 says that in an $R2BSN(n,1,\delta)$ with at most $n^2 + (2+\delta)n - 1$ switches, in which x_1 contains $s \leq n-1$ switches, any two of the $s+1$ or more crosswires which contain $n+1+\delta$ switches are connected either to at most $s-1$ common subscriber lines or to at least $n+2+\delta-s$ common subscriber lines. An exhaustive search has shown that no such $R2BSN(n,1,\delta)$ exists for $n \leq 6$. The following questions remain to be answered:

1. Does there exist an $R2BSN(n,1,\delta)$ with fewer switches than the minimal $R2BSN(n,0,\delta)$ (which contains $n^2 + (2+\delta)n - 1$ switches)?
2. Does an $R2BSN(n,t,\delta)$ with the minimum number of switches have $t = 0$?

References

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