

A polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs

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Abstract

We describe a polynomial algorithm for the Hamiltonian cycle problem for semicomplete multipartite digraphs. The existence of such an algorithm was conjectured in [16] (see also [15]).

1 Introduction

A *semicomplete multipartite digraph* is a digraph $D = (V(D), A(D))$ for which the vertices of $V(D)$ can be partitioned into a number $k \geq 2$ of subsets (called *colour classes*) such that every pair of vertices from the same colour class are non-adjacent and every pair of vertices from different colour classes are adjacent (i.e. there is at least one arc between them). In Figure 1, a semicomplete 3-partite digraph H is depicted. The colour classes are $\{x_1, x_2, x_3\}, \{y_1, y_2, y_3\}, \{z\}$. The following is a Hamiltonian cycle in H : $zx_3y_3x_1y_1x_2y_2z$.

Two well-known special cases are *semicomplete digraphs* (when $k = |V(D)|$) and *semicomplete bipartite digraphs* (when $k = 2$). It is well-known that every strongly connected semicomplete digraph is Hamiltonian [9] and that a Hamiltonian cycle in a strong semicomplete digraph on n vertices can be found in time $O(n^2)$ [18]. For semicomplete bipartite digraphs strong connectivity by itself is not enough to guarantee the existence of a Hamiltonian cycle. Here we need to require another, obviously necessary, condition that D has a *factor*, i.e., a spanning collection of vertex-disjoint cycles. It was shown in [13] (see also [17]) that a semicomplete bipartite digraph is Hamiltonian if and only if it is strong and has a factor. It follows from the proof in [13] (see also [19]) that the existence of a Hamiltonian cycle in a semicomplete bipartite digraph on n vertices can be checked in time $O(n^2 + M(n))$ where $M(n)$ is the time required to decide the existence of a perfect matching in a bipartite graph and find one if it exists.

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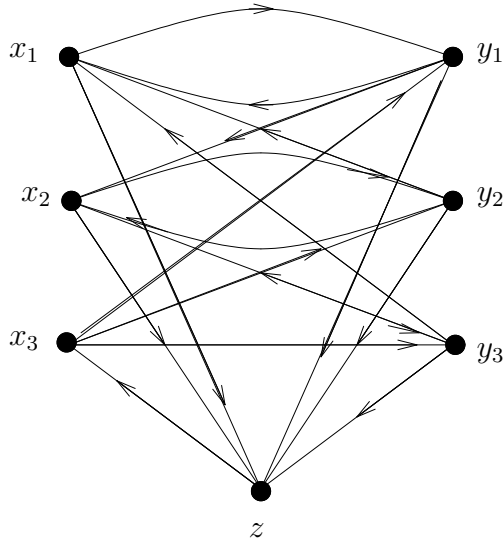


Figure 1: A semicomplete 3-partite digraph H .

Another special class of semicomplete multipartite digraphs for which a characterization of Hamiltonicity as well as a polynomial algorithm to find such a cycle (if it exists) was known, is the class of *extended semicomplete digraphs* (also called *ordinary semicomplete multipartite digraphs* in [14]). This is the class of digraphs one can obtain by starting from some semicomplete digraph by replacing each vertex x by an independent set of vertices I_x and adding all possible arcs from one such set I_x to another I_y if and only if the original digraph contained the arc $x \rightarrow y$. For extended semicomplete digraphs the same characterization as for semicomplete bipartite digraphs holds and a Hamiltonian cycle can be found within the same time bound as for semicomplete bipartite digraphs [14].

The complexity of the Hamiltonian cycle problem for general semicomplete multipartite digraphs has attracted the interest of several researchers and has been open for quite some time (it was first brought to the attention of the first author in a private communication with C. Thomassen in 1987). While this problem has resisted a number of attempts to solve it (see e.g. [3, 22]), the Hamiltonian path problem has a surprisingly elegant solution for semicomplete multipartite digraphs. It was shown in [12] that a semicomplete multipartite digraph D has a Hamiltonian path if and only if it has a path P such that $D - V(P)$ has a factor and a Hamiltonian path can be found if one exists in time $O(n^2 + M(n))$ where n is the number of vertices and $M(n)$ is as defined above (we remark that the proof of this result is non-trivial). Another characterization which is valid only for semicomplete multipartite digraphs with at most two vertices in each colour class can be found in [10]. In contrast to the nice characterization of Hamiltonian paths in semicomplete multipartite digraphs, it was shown in [3] that there is no k such that every k -strong semicomplete multipartite digraph with a factor is Hamiltonian (a digraph D is k -strong if D has at least $k + 1$ vertices and $D - X$ is strongly connected for every subset $X \subset V(D)$, where $|X| \leq k - 1$).

It turns out that the key to solving the Hamiltonian cycle problem for semicomplete

multipartite digraphs lies in understanding the structure of reducible factors. A factor $C_1 \cup C_2 \cup \dots \cup C_k$ of a digraph D is *reducible* if D has some other factor $C'_1 \cup C'_2 \cup \dots \cup C'_r$, $r < k$ such that for every C_i , there is some C'_j with $V(C_i) \subseteq V(C'_j)$. (In Figure 1 the digraph H contains a factor $\mathcal{F} = C_1 \cup C_2 \cup C_3$, where $C_i = x_i y_i x_i$, $i = 1, 2$, $C_3 = x_3 y_3 z x_3$ and the factor $\mathcal{F}' = C'_1 \cup C'_2$ where $C'_1 = x_1 y_1 x_2 y_2 x_1$ and $C'_2 = x_3 y_3 z x_3$. The factor \mathcal{F} can be reduced to the factor \mathcal{F}' which in turn can be reduced to a Hamiltonian cycle of H). In [3] the first such step was taken and a sufficient condition for Hamiltonicity of semicomplete multipartite digraphs, formulated in terms of reducible factors, was obtained (see Theorem 3.4). Recently the third author was able to refine the condition of Theorem 3.4 in such a way that we now have a very detailed understanding of reducible factors in semicomplete multipartite digraphs [21] (see Theorem 3.7).

Theorem 3.7 has many important consequences [21]:

- Every regular semicomplete multipartite digraph is Hamiltonian.
- If the size of the largest independent set in a k -strong semicomplete multipartite digraph D is at most $2k$ and D has a factor, then D is Hamiltonian.
- Any set of k vertices in a k -strong semicomplete multipartite digraph lie on a common cycle.

For yet another application of Theorem 3.7 see [4].

The purpose of this paper is to give a constructive proof for the existence of a polynomial algorithm for the Hamiltonian cycle problem for semicomplete multipartite digraphs (Theorem 5.3). The algorithm is highly non-trivial and relies heavily on Theorem 3.7, the main result of [21]. We have not tried to optimize the complexity of the algorithm since we found it was quite complicated to describe and prove the correctness of the algorithm, even without this goal in mind. For technical reasons we prove the existence of a polynomial algorithm for a slightly more general problem (see note 4.3).

2 Terminology

We shall assume that the reader is familiar with the standard terminology on digraphs and refer to [8] for terminology not discussed here. Every *cycle* and *path* is assumed simple and directed.

In this section, D denotes a digraph (in the whole paper we use n to denote the number of vertices in the digraph currently studied), x and y are distinct vertices of D , X and Y are disjoint sets of vertices in D . D is *Hamiltonian* if it contains a *Hamiltonian cycle*, namely a cycle with n vertices. A *k -path-cycle subgraph* \mathcal{F} of D is a collection of k paths P_1, P_2, \dots, P_k and a number, $s \geq 0$, of cycles Z_1, Z_2, \dots, Z_s in D such that no two of $P_1, P_2, \dots, P_k, Z_1, Z_2, \dots, Z_s$ have a vertex in common. We shall write $\mathcal{F} = P_1 \cup P_2 \cup \dots \cup P_k \cup Z_1 \cup Z_2 \cup \dots \cup Z_s$ (we always list the paths before the cycles.) If $k = 0$ then \mathcal{F} is a *cycle subgraph*. A *factor* is a spanning cycle subgraph, i.e one that covers all vertices of D .

If there is an arc from x to y then we say that x *dominates* y and write $x \rightarrow y$. We write $X \rightarrow Y$ if $x \rightarrow y$ for every pair $x \in X$, $y \in Y$ of adjacent vertices. If $X \rightarrow Y$, then

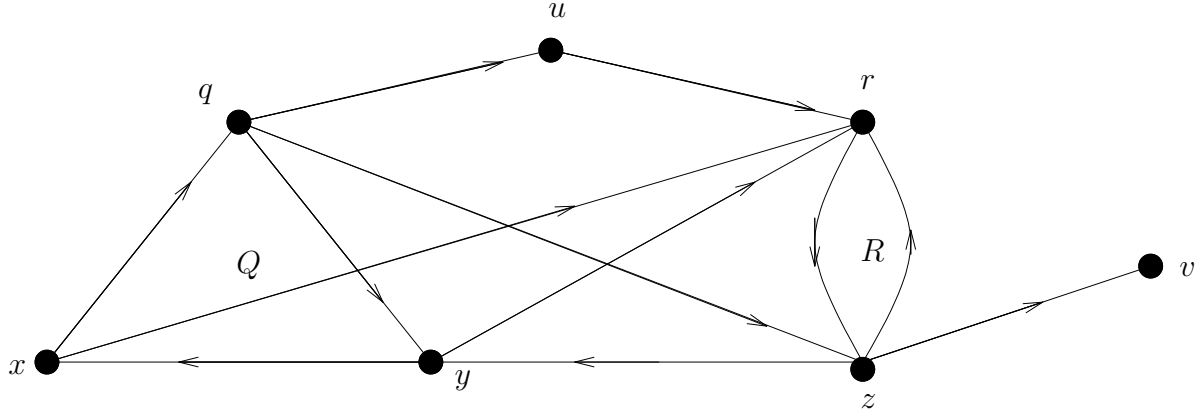


Figure 2: A digraph D with two subgraphs Q, R such that $Q \rightsquigarrow R$. Here $Q = \{q, x, y\}$ and $R = \{r, z\}$.

there may be a pair $x \in X, y \in Y$ such that both $x \rightarrow y$ and $y \rightarrow x$. Sometimes, we wish to exclude that possibility: We write $X \Rightarrow Y$ (X *strongly dominates* Y) if $X \rightarrow Y$ and there is no arc from Y to X .

Let R and Q be disjoint subdigraphs of D . We write $Q \rightsquigarrow R$ if

- 1) Q and R contain vertices q and r , respectively, such that $q \Rightarrow V(R)$ and $V(Q) \Rightarrow r$;
- 2) every vertex in Q dominates a vertex in R and every vertex of R is dominated by a vertex of Q (see Figure 2 for an example).

A path starting at x and terminating at y is an (x, y) -path. An (x, y) -path P is a (Q, R) -path if $x \in V(Q), y \in V(R)$ and $|V(P) \cap (V(Q) \cup V(R))| = 2$. Let \mathcal{F} be a subgraph of D containing both Q and R . A (Q, R) -path P is a $(Q, R)_{\mathcal{F}}$ -path if $|V(P) \cap V(\mathcal{F})| = 2$, i.e. only the first and last vertex of P is in $V(\mathcal{F})$. For example, in Figure 2 the path qur is a $(Q, R)_{\mathcal{F}}$ -path, where \mathcal{F} is the subdigraph induced by $V(Q) \cup V(R)$.

D is *strongly connected* (or just *strong*) if there exists an (x, y) -path in D for every ordered pair of distinct vertices $\{x, y\}$ of D . We say that D is k -strong if $|V(D)| \geq k + 1$ and $D - X$ is strongly connected for all $X \subset V(D)$ with $|X| \leq k - 1$.

If P is a path containing a subpath from x to y we let $P[x, y]$ denote that subpath. Similarly, if C is a cycle containing vertices x and y , $C[x, y]$ denotes the subpath of C from x to y . $D\langle Q \rangle$ is the subgraph induced by the vertices in Q . Let C be a cycle containing z and a subset $X \subseteq V(D)$. Then, z_C^+ (z_C^-) denotes the *successor* (*predecessor*, resp.) of z on C . Moreover, $X_C^+ = \{x_C^+ : x \in X\}$, $X_C^- = \{x_C^- : x \in X\}$. When the cycle C is determined from the context, we shall usually omit the subscript C .

If Y is a subset of the vertices of a digraph D and H is a subdigraph of D such that $Y \subset V(H)$, then H *covers* Y .

3 Preliminary results and outline of the main ideas

We start with a few results on efficient construction of cycle subgraphs covering specified vertices.

Lemma 3.1 *Given any digraph $D = (V(D), A(D))$ on n vertices and m arcs and a subset $W \subseteq V(D)$, in time $O(nm)$, we can either find a cycle subgraph \mathcal{F} which covers W or show that such a subgraph does not exist.*

Proof: Let $D = (V(D), A(D))$ and W be given. Form a network N with lower bounds and capacities on the arcs in the following way: $V(N) = \{x', x'' | x \in V(D)\}$. $A(N) = \{x'' \rightarrow y' | x \rightarrow y \in A(D)\} \cup \{x' \rightarrow x'' | x \in V(D)\}$. All arcs have capacity one. All arcs of the kind $w' \rightarrow w''$, where $w \in W$, have lower bound one. All other arcs have lower bound zero.

It is easy to see that D has the desired cycle subgraph if and only if N has an integer valued feasible circulation. Furthermore, in the case when all capacities are one and only $|W| \leq n$ arcs have lower bounds different from zero, we can find such a circulation in time $O(nm)$ (if one exists). For an explicit algorithm achieving this complexity see e.g. [1, page 195]. \diamond .

An easy consequence of Lemma 3.1 is the following:

Lemma 3.2 *Let $D = (V(D), A(D))$ be a digraph on n vertices and m arcs, and subsets $R, S, T \subseteq V(D)$ (not necessarily disjoint). In time $O(nm)$ we can either find a 1-path-cycle subgraph $P \cup C_1 \cup \dots \cup C_k$ which covers T such that P starts in R and terminates in S or show that such a subgraph does not exist.*

Proof: Let D' be obtained from D by adding a new vertex x and all possible arcs from x to vertices in R as well as all possible arcs from vertices in S to x . Now it is easy to see that D has a 1-path-cycle subgraph with the required properties if and only if D' has a cycle subgraph which covers T . Thus the claim follows from Lemma 3.1. \diamond .

Note 3.3 *In the rest of the paper the digraph D being considered is a semicomplete multipartite digraph of order n with colour classes V_1, V_2, \dots, V_c ; the colour class containing a vertex $x \in V(D)$ is denoted by $V_{col(x)}$, where $col(x)$ stands for the corresponding integer from $\{1, 2, \dots, c\}$.*

Note that if C, C' are disjoint cycles in a semicomplete multipartite digraph D , then (up to switching the role of the two cycles) at least one of the following four cases apply (see Figure 3):

- (A) Every vertex on C has an arc to and from C' .
- (B) There exist vertices $x \in V(C), y \in V(C')$ such that $x \Rightarrow V(C')$ and $y \Rightarrow V(C)$, or $V(C') \Rightarrow x$ and $V(C) \Rightarrow y$.
- (C) C contains distinct vertices x, y such that $x \Rightarrow V(C')$ and $V(C') \Rightarrow y$.

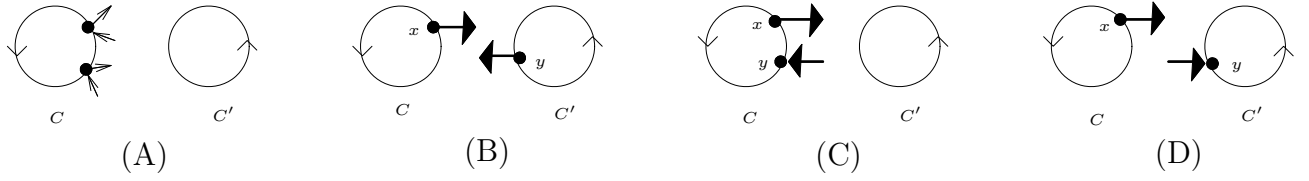


Figure 3: The four possible situations (up to switching the role of the two cycles or reversing all arcs) for arcs between two disjoint cycles in a semicomplete multipartite digraph. In (A) every vertex on C has arcs to and from C' . In (B)-(D) a fat arc indicates that all arcs go in the direction shown from or to the specified vertex (i.e. in (B) all arcs between x and C' leave x).

(D) C contains a vertex x such that $x \Rightarrow V(C')$ and C' contains a vertex y such that $V(C) \Rightarrow y$.

The following result was proved in [3]:

Theorem 3.4 *If D is a semicomplete multipartite digraph with disjoint cycles C_1, C_2 for which one of the alternatives (A)-(C) above holds, then in time $O(|V(C_1)| \times |V(C_2)|)$ one can find a cycle C in D with $V(C) = V(C_1) \cup V(C_2)$.*

Since we shall use the existence of this algorithm a number of times in the rest of the paper, we give it a name, **Merge**. The algorithm **Merge** takes as input the two cycles C_1, C_2 and if one of the alternatives (A)-(C) holds, it replaces these with one cycle C such that $V(C) = V(C_1) \cup V(C_2)$. If alternative (D) holds, then **Merge** leaves the cycles unchanged (note that this does not necessarily mean that there is no cycle C as above).

Let us return to the situation in Theorem 3.4. In the case when only alternative (D) holds the third author [21] proved (as part of a stronger result on minimal factors in semicomplete multipartite digraphs, see Theorem 3.7 below) that if there are arcs in both directions between C_1 and C_2 then one can still merge the cycles into a cycle C as above in the same time, unless the following holds for $i = 1$ or $i = 2$: there exists a colour class V_χ of D such that all arcs $x \rightarrow y$ from C_{3-i} to C_i satisfy that $x^+, y^- \in V_\chi$. Recall that x^+ is the successor of x on C_{3-i} and y^- is the predecessor of y on C_i .

In order to formulate Theorem 3.7 we need the following very important definition [21].

Definition 3.5 *Let D be a semicomplete multipartite digraph, let $X \subset V(D)$ and let $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ be a cycle subgraph of D . \mathcal{F} is called **X -singular** if every cycle of \mathcal{F} contains a vertex from X , and either $t = 1$, or \mathcal{F} satisfies the following properties:*

- (α) *The cycles in \mathcal{F} are labeled such that $C_i \rightleftharpoons C_j$ for every pair i, j ($1 \leq i < j \leq t$) (i.e. only alternative (D) from Figure 3 holds for C_i and C_j).*
- (β) *There are colours $\chi_R(C_i) \in \{1, 2, \dots, c\}$ for $i = 1, \dots, t-1$ and $\chi_L(C_j) \in \{1, 2, \dots, c\}$ for $j = 2, \dots, t$ such that every $(C_j, C_i)_{\mathcal{F}}$ -path with $j > i$ starting*

at some vertex $z \in V(C_j)$ and terminating at some vertex $y \in V(C_i)$ satisfies that $y^- \in V_{\chi_R(C_i)} \cap X$, $z^+ \in V_{\chi_L(C_j)} \cap X$.

(γ) For every pair of cycles C_i, C_j with $j > i$, if there exists a $(C_j, C_i)_{\mathcal{F}}$ -path, then $\chi_R(C_i) = \chi_L(C_j)$.

See Figure 4 for an illustration of the definition. In the rest of this paper, the labeling of the cycles of a singular cycle subgraph (with at least two cycles) will be always made according to (α).

For future reference, we state a useful consequence of Definition 3.5.

Proposition 3.6 *If $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ is X -singular, then*

1. For all $i = 1, \dots, t-1$: $V(C_i) \cap V_{\chi_R(C_i)} \Rightarrow V(C_{i+1}) \cup \dots \cup V(C_t)$
2. For all $j = 2, \dots, t$: $V(C_1) \cup \dots \cup V(C_{j-1}) \Rightarrow V(C_j) \cap V_{\chi_L(C_j)}$.
3. For all $j = 2, \dots, t$: $(\bigcup_{i=1}^{j-1} V(C_i)) \cap V_{\chi_L(C_j)} \Rightarrow V(C_j)$.

Proof: We only prove 1. as the proofs of 2. and 3. are analogous. Suppose there is an arc $x \rightarrow y$ such that $x \in V(C_j)$, $j \geq i+1$ and $y \in V(C_i) \cap V_{\chi_R(C_i)}$. Then it follows from Definition 3.5 (β) that $y^- \in V_{\chi_R(C_i)}$, hence, y^- and y belong to the same colour class; a contradiction. \diamond .

The importance of singular cycle subgraphs is evident from the following theorem.

Theorem 3.7 [20, 21] *Let D be a semicomplete multipartite digraph and X a specified subset of $V(D)$.*

- (a) *If D has a cycle subgraph covering X , then every cycle subgraph of D covering X with the minimum number of cycles is X -singular.*
- (b) *There is an $O(n^3)$ -time algorithm which given a cycle subgraph, $\mathcal{F}' = C'_1 \cup C'_2 \cup \dots \cup C'_l$, of D covering X , constructs an X -singular cycle subgraph, $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ covering X with the property that for every $i \in \{1, 2, \dots, l\}$ there is a $j \in \{1, 2, \dots, t\}$ such that $X \cap V(C'_i) \subseteq V(C_j)$ (i.e. vertices from X which are on one cycle in \mathcal{F}' remain on one cycle in \mathcal{F}).*

Note 3.8 *We shall use the name **Singular** for the algorithm of Theorem 3.7. **Singular** is an iterative algorithm. In each iteration, **Singular** starts with a cycle subgraph \mathcal{H} covering X and tries to merge two or more cycles of \mathcal{H} into one cycle, using among other subroutines the algorithm **Merge**. When we merge some cycles, we may lose some vertices of the cycles but only some which are not in X . If **Singular** succeeds in merging some cycles, then we obtain a new cycle subgraph covering X but with fewer cycles. The new cycle subgraph is the input for the next iteration. If however **Singular** has not succeeded to merge any set of two or more cycles in \mathcal{H} , then \mathcal{H} is X -singular and **Singular** returns \mathcal{H} .*

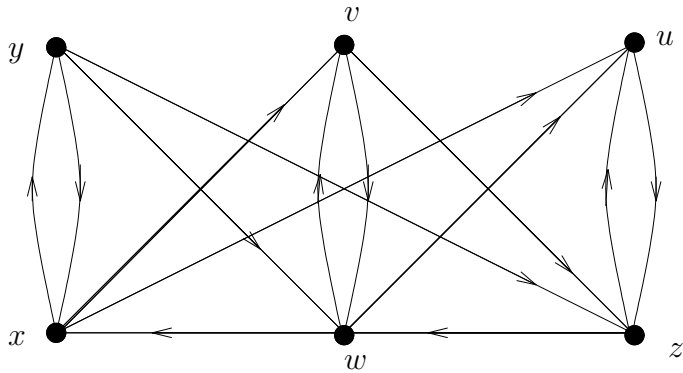


Figure 4: A semicomplete 3-partite digraph H with a X -singular cycle subgraph $\mathcal{F} = C_1 \cup C_2 \cup C_3$, where $C_1 = xyx$, $C_2 = vwv$, $C_3 = uzu$. Here $X = V(H)$

If **Singular** is given, as input, the digraph H in Figure 4 with $X = V(H) - \{v\}$ (instead of $X = V(H)$) and $\mathcal{F} = C_1 \cup C_2 \cup C_3$, as indicated in the figure, then it will return the cycle subgraph $\mathcal{F}' = C'_1 \cup C'_2$ with $C'_1 = xywx$, $C'_2 = uzu$. Note that this cycle subgraph is X -singular and contains the minimum number of cycles among all cycle subgraphs covering X .

Even though being able to convert a given cycle subgraph \mathcal{F}' covering the set X (which we wish to cover) into an X -singular cycle subgraph \mathcal{F} is quite useful, it does not seem to make finding a possible cycle covering X much easier. Consider for example the semicomplete multipartite digraph H' in Figure 5. Here $X = V(H')$ and the unique Hamiltonian cycle shows no nice pattern with respect to the last cycle in \mathcal{F} . Clearly more complicated examples with even worse structure of a Hamiltonian cycle can be constructed.

In order to simplify the search for a more suitable structure on our cycle subgraph covering X , we introduce the notion of a leftshifted cycle subgraph in the next section. As we shall see in Lemma 4.2, if we combine the two properties of being X -singular and X -leftshifted, then we obtain a very special structure on the last cycle C_t of our cycle subgraph. This property in turn allows us to look only for cycles covering X with a very special structure in terms of vertices from the cycle C_t , the so-called \mathcal{F} -canonical cycles (see Lemma 5.2 and Figure 6). Once we have all this structure we will show in Theorem 5.3 that the desired cycle can be found using a recursive approach (via redefining X). The main obstruction – how to create an X -singular and X -leftshifted cycle subgraph covering X – is handled in Section 6.

4 Leftshifted singular cycle subgraphs

As pointed out in Section 3, we need some tool to transform a given X -singular cycle subgraph into one which can be used efficiently when searching for a cycle covering X . As we shall see in Lemma 4.2 and Lemma 5.2 the notion of leftshifted defined below is the desired tool for this.

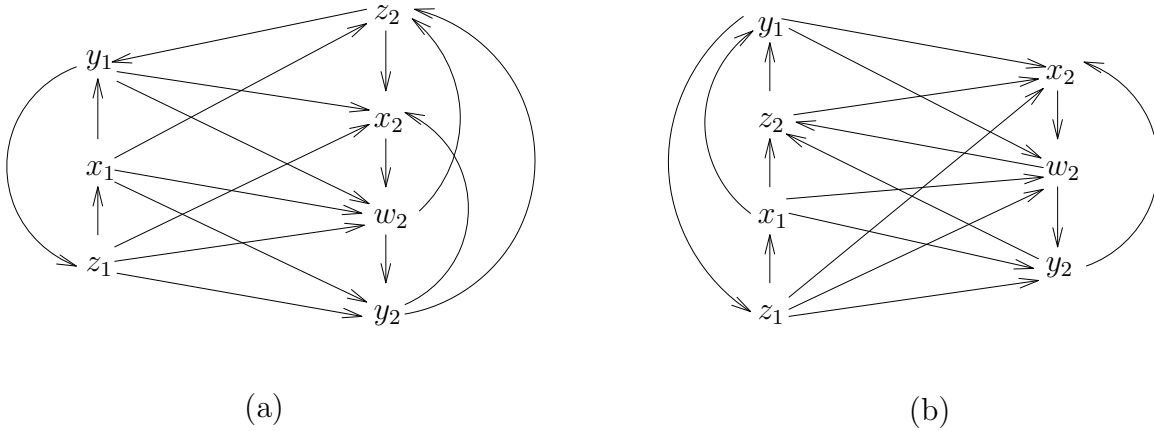


Figure 5: A semicomplete 4-partite digraph H' . Let $X = V(H')$. Part (a) shows an X -singular cycle subgraph $\mathcal{F} = C \cup C'$ with $C = x_1y_1z_1x_1$, $C' = z_2x_2w_2y_2z_2$. The unique Hamiltonian cycle in H is $y_1z_1x_1y_2x_2w_2z_2y_1$. Part (b) shows another cycle subgraph $\mathcal{F}' = C_1 \cup C_2$ with $C_1 = z_1x_1z_2y_1z_1$, $C_2 = x_2w_2y_2x_2$. Note that \mathcal{F}' is not X -singular, because $C_1 \rightsquigarrow C_2$, but $w_2 \rightarrow z_2$ and w_2^+ and z_2^- are adjacent.

Definition 4.1 A cycle subgraph $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ is X -leftshifted if either $t = 1$, or it has the following property for every choice of i, j with $1 \leq i < j \leq t$: There is no 1-path-cycle subgraph $\mathcal{F}' = P \cup Z_1 \cup \dots \cup Z_k$ such that $X \cap V(C_j) \subseteq V(\mathcal{F}') \cap V(\mathcal{F}) \subseteq V(C_j)$, P is a (u, v) -path and there exists $w \in V(C_i)$ such that $w \rightarrow u$ and $v \rightarrow w^+$.

Notice that if \mathcal{F} is not X -leftshifted then, for some choice of i, j with $1 \leq i < j \leq t$, there is a 1-path-cycle subgraph \mathcal{F}' with P as its path, such that the cycle C_i can be replaced by a cycle C'_i where $V(C_i) \cup V(P) = V(C'_i)$ and C_j can be replaced by the cycle subgraph $\mathcal{F}' - P$ such that $C'_i \cup (\mathcal{F} - C_i \cup C_j) \cup (\mathcal{F}' - P)$ is a new cycle subgraph which covers $V(\mathcal{F}) \cap X$. For example, if we let $X = V(H) - v$, then with respect to the ordering C_1, C_2, C_3 , the cycle subgraph \mathcal{F} in Figure 4 is not X -leftshifted since the vertex w can be inserted in the cycle xyx while leaving out the non-important vertex v . Similarly, with the ordering C, C' the cycle subgraph \mathcal{F} in Figure 5 is also not X -leftshifted, as the vertex z_2 can be inserted in C while leaving the cycle C_2 .

The following technical lemma plays an important role in our algorithm in Section 5 as it forms the basis for Lemma 5.2.

Lemma 4.2 Let X be a subset of $V(D)$, and let $\mathcal{F} = C_1 \cup \dots \cup C_t$, $t \geq 2$ be a X -leftshifted X -singular cycle subgraph which covers X . Let also $R = C_1 \cup \dots \cup C_{t-1}$ and $W = \{w \in V(C_t) : \exists (C_t, R)\text{-path starting at } w \text{ in } D\}$. Suppose $W \neq \emptyset$. There exists a non-empty set $S \subset V(C_t)$ with the properties below:

$$S \subseteq V_{\chi_L(C_t)} \cap X \cap V(C_t), \quad (1)$$

$$S \Rightarrow V(C_t) - S^-, \quad (2)$$

$$W \subseteq S^-, \quad (3)$$

for every $u \in V(D) - (V(\mathcal{F}) \cup V_{\chi_L(C_t)})$, either $S \Rightarrow u$ or $D - V(R)$ has no (C_t, u) -path. (4)

Proof: For every $w \in W$ we define the following: $S_0^w = \emptyset$, $S_1^w = \{w^+\}$ and $S_i^w = \{r \in V(C_t) : \exists q \in S_{i-1}^w \text{ such that } r^- \rightarrow q\}$ for $i = 2, 3, \dots, |V(C_t)|$. Observe that there exists n_w ($w \in W$), such that $S_0^w \subset S_1^w \subset S_2^w \subset \dots \subset S_{n_w}^w = S_{n_w+1}^w = \dots = S_{|V(C_t)|}^w$. Let $S^w = S_{n_w}^w$ ($= S_{|V(C_t)|}^w$) and $S = \bigcup_{w \in W} S^w$. Let

$$T' = \{v \in V(C_t) : \exists q \in V(D) - V(\mathcal{F}) - V_{\chi_L(C_t)} \text{ such that } (V_{\chi_L(C_t)} \cap V(R)) \Rightarrow q \rightarrow v\}$$

and

$$T = \{v \in V(C_t) : v \notin V_{\chi_L(C_t)} \cap X\} \cup T'.$$

Claim: $T \cap S = \emptyset$

Proof of claim: Suppose that this is false, and let $w \in W$ be chosen such that $T \cap S^w \neq \emptyset$. We can now find a sequence of distinct vertices x_1, x_2, \dots, x_a ($x_i \in S_i^w - S_{i-1}^w$, $i = 1, \dots, a$, $a \geq 1$), such that $x_{i+1}^- \rightarrow x_i$ and $x_i \notin T$ for $i = 1, 2, \dots, a-1$ and $x_a \in T$.

A spanning 1-path-cycle subgraph $P \cup Z_1 \cup \dots \cup Z_c$ of $D\langle V(C_t) \rangle$ such that the path P starts in T and terminates in W can be constructed as follows: start with the cycle C_t , delete the arcs $x_i^- \rightarrow x_i$, $i = 1, 2, \dots, a$ and add the arcs $x_i^- \rightarrow x_{i-1}$, $i = 2, 3, \dots, a$. In the digraph obtained in this way all vertices except x_1^- and x_a have in- and out-degree one, x_1^- has out-degree zero and in-degree one and x_a has in-degree zero and out-degree one. Thus the digraph described is a spanning 1-path-cycle subgraph of $D\langle V(C_t) \rangle$ with the path P starting in $x_a \in T$ and ending in $x_1^- = w \in W$.

Let

$$U = \{u \in V(D) - V(R) - V_{\chi_L(C_t)} : V_{\chi_L(C_t)} \cap V(R) \Rightarrow u\},$$

$$Z = \{z \in V(D) - V(R) : (V_{\chi_L(C_t)} \cap V(R))^+ \not\Rightarrow z\},$$

Let $P = p_1 p_2 \dots p_b$. Since $p_b \in W$ there is a (C_t, R) -path, $U' = u'_1 u'_2 \dots u'_d$, with $p_b = u'_1$. By the definition of a (C_t, R) -path, $u'_{d-1} \in V(D) - V(R)$ and $(V_{\chi_L(C_t)} \cap R)^+ \not\Rightarrow u'_{d-1}$ (since $u'_d \in (V_{\chi_L(C_t)} \cap R)^+$ by Definition 3.5 and the fact that \mathcal{F} is X -singular), which implies that $u'_{d-1} \in Z$. Now consider the following possibilities:

If $p_1 \notin V_{\chi_L(C_t)}$, then $p_1 \in U$ by Proposition 3.6 (3) and $P' = PU'[u'_1, u'_{d-1}]$ is a path from U to Z .

If $p_1 \in V_{\chi_L(C_t)}$ **and** $p_1 \notin X$, then $p_2 \in U$, by Proposition 3.6 (3) and $P' = P[p_2, p_b]U'[u'_1, u'_{d-1}]$ is a path from U to Z (observe that $b \geq 2$, since $p_b \notin V_{\chi_L(C_t)}$ because $W \cap V_{\chi_L(C_t)} = \emptyset$ by Proposition 3.6 (2)).

If $p_1 \in V_{\chi_L(C_t)} \cap X$, then, by the definition of T , $p_1 \in T'$ and there is a vertex $q \in V(D) - V(\mathcal{F}) - V_{\chi_L(C_t)}$ such that $(V_{\chi_L(C_t)} \cap R) \Rightarrow q \rightarrow p_1$, thus $q \in U$ and $P' = qPU'[u'_1, u'_{d-1}]$ is a path from U to Z .

Observe that $P' \cup Z_1 \cup \dots \cup Z_c$ is a 1-path-cycle subgraph in $D - V(R)$ which covers $V(C_t) \cap X$ and whose path P' starts in U and terminates in Z . By the definitions of U and Z , there exists $y \in V(R)$ such that y dominates the first vertex of P' and y^+ is dominated by the last vertex of P' . Thus, the existence of $P' \cup Z_1 \cup \dots \cup Z_c$ is a contradiction against the assumption that \mathcal{F} is leftshifted. This concludes the proof of the claim.

We will now prove properties (1), (2), (3) and (4). Since $S \cap T = \emptyset$ we get $S \subseteq V_{\chi L(C_t)} \cap X \cap V(C_t)$, which proves (1). From the definition of S we get that $S \Rightarrow V(C_t) - S^-$ and $W \subseteq S^-$, which proves (2) and (3). To prove (4) suppose $u \in V(D) - V(\mathcal{F}) - V_{\chi L(C_t)}$ is a vertex such that $S \not\Rightarrow u$. Let $s \in S$ be arbitrary with $u \rightarrow s$. We must have $(V_{\chi L(C_t)} \cap R) \not\Rightarrow u$, since otherwise $s \in T'$ which is a contradiction by the claim. Let $q \in (V_{\chi L(C_t)} \cap R)$ be arbitrary such that $u \rightarrow q$. If there is a (C_t, u) -path, P , in $D - V(R)$, then Pq is a (C_t, R) -path, $p_1 p_2 \dots p_l u q$, with $q^- \notin V_{\chi L(C_t)}$ which is a contradiction against the fact that \mathcal{F} is singular, by Definition 3.5 (the definition of $V_{\chi L(C_t)}$). Therefore, if $S \not\Rightarrow u$, then there is no (C_t, u) -path in $D - V(R)$. This proves (4) and completes the proof of the lemma. \diamond

Note 4.3 *In the rest of this paper we distinguish a subset X of $V(D)$; let the vertices in X be called X -vertices and make the following assumptions about X if $\bar{X} = V(D) - X \neq \emptyset$:*

$$|\bar{X}| \geq 2, \quad (5)$$

$$X \rightarrow \bar{X} \quad (6)$$

$$\bar{X} \subseteq V_\chi \text{ for some colour } \chi \in \{1, 2, \dots, c\} \quad (7)$$

$$\text{There is a pair of vertices } x_\alpha, x_\beta \text{ in } X \text{ such that } x_\alpha \in V_{\text{col}(\bar{X})}, |V_{\text{col}(x_\beta)}| = 1, \quad (8)$$

and

$$X - \{x_\beta\} \Rightarrow x_\beta \Rightarrow \bar{X}, X - \{x_\beta\} \Rightarrow x_\alpha. \quad (9)$$

Notice, in particular, that by (7), every cycle of D contains a vertex from X .

For future reference we state two easy consequences of these assumptions.

Lemma 4.4 *If $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ is an X -singular cycle subgraph of D , then $V(C_1 \cup C_2 \cup \dots \cup C_{t-1}) \subseteq X$.*

Proof: Suppose there exists a vertex $q \in V(C_1 \cup C_2 \cup \dots \cup C_{t-1}) \cap \bar{X}$. Let $x \in X \cap V(C_t)$ be chosen such that $\text{col}(x) \neq \text{col}(q)$ (this is possible since $\bar{X} \subseteq V_\chi$ for some colour χ). Then $x \rightarrow q$, since $X \rightarrow \bar{X}$ (see Note 4.3). By Definition 3.5, x^+ , $q^- \in X \cap V_{\chi L(C_t)}$. Since $\text{col}(x^+) = \text{col}(q^-)$ and $x^+ \in X$, $x^+ \rightarrow q$. By Definition 3.5, we get $x^{++} \in V_{\chi L(C_t)}$. Thus we conclude that both x^+ and x^{++} are in $V_{\chi L(C_t)}$; a contradiction. \diamond

Lemma 4.5 *If $\bar{X} \neq \emptyset$ and $\mathcal{F} = C_1 \cup \dots \cup C_t$ is an X -singular cycle subgraph which covers X , with $R = C_1 \cup \dots \cup C_{t-1}$, then the following holds:*

$$\text{The arc } x_\alpha \rightarrow x_\beta \text{ is an arc of } C_t \tag{10}$$

$$V(R) \Rightarrow V(D) - V(\mathcal{F}) \tag{11}$$

Proof: As x_β is the only out-neighbour of x_α , by (8) and (9), the arc $x_\alpha \rightarrow x_\beta$ belongs to a cycle in \mathcal{F} and since the only out-neighbours of x_β are in \bar{X} it follows from Lemma 4.4 that $x_\alpha \rightarrow x_\beta$ is an arc of C_t . Assume that (11) is false and, thus, there exist $y \in V(D) - V(\mathcal{F})$, $z \in R$ such that $y \rightarrow z$. As y is not in X , $x_\beta \rightarrow y$. Hence, $x_\beta y z$ is a (C_t, R) -path. Thus, by Lemma 4.2, $x_\beta^+ \in X$; a contradiction.

Our goal is to describe a polynomial algorithm for deciding whether D has a cycle covering X . For this we need to be able to make a given cycle subgraph covering X X -singular and X -leftshifted. The highly technical proof of the following crucial result is postponed to Section 6:

Theorem 4.6 *Given any cycle subgraph in D which covers X , in $O(n^6)$ time, we can find an X -singular X -leftshifted cycle subgraph, $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ which covers X .*

5 The main algorithm

Below, we shall use the notation adopted in the formulation of Lemma 4.2.

In this section $\mathcal{F} = C_1 \cup \dots \cup C_t$ ($t \geq 2$) denotes an X -singular X -leftshifted cycle subgraph of D which covers X , $R = C_1 \cup \dots \cup C_{t-1}$ and S is the set determined in Lemma 4.2.

Definition 5.1 *A cycle C in D is called \mathcal{F} -canonical if there exists a subset $L = \{v_1, \dots, v_l\}$ of S^- such that $C_t[v_i, v_{i+1}]$ contains no vertices from $L - \{v_i, v_{i+1}\}$ for all $i = 1, \dots, l$ and with the property that C has the form*

$$C = C_t[v_l^+, v_1]T_1C_t[v_1^+, v_2]T_2 \dots T_{l-1}C_t[v_{l-1}^+, v_l]T_lv_l^+, \tag{12}$$

where T_i is a path in $D \setminus R$ for all $i = 1, 2, \dots, l$ (see Figure 6).

The following lemma shows the usefulness of \mathcal{F} -canonical cycles and also justifies the definition of an X -leftshifted cycle subgraph.

Lemma 5.2 *Suppose that D contains a cycle covering X , and that $\mathcal{F} = C_1 \cup \dots \cup C_t$ ($t \geq 2$) is an X -leftshifted X -singular cycle subgraph of D covering X . Then D has an \mathcal{F} -canonical cycle covering X .*

Proof: Let $Q = V(D) - V(\mathcal{F})$ and $V^* = V_{XL(C_t)}$. Let Z be a cycle of D covering X and let $Z \setminus (D - V(R)) = P_1 \cup P_2 \cup \dots \cup P_l$, where P_1, \dots, P_l are pairwise vertex disjoint paths. Recall that, by Note 4.3,

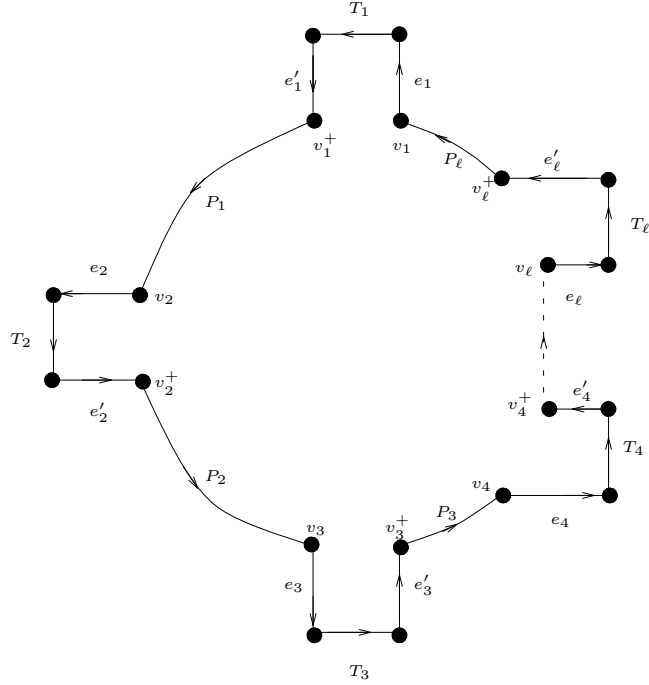


Figure 6: An example of an \mathcal{F} -canonical cycle for some X -singular X -leftshifted cycle subgraph $\mathcal{F} = C_1 \cup \dots \cup C_t$ which covers X . Each P_i is of the form $P_i = C_t[v_i^+, v_{i+1}]$, all e_i, e'_i denote arcs and each T_i is a path in $D\langle R \rangle$

all vertices of Q are from the same colour class $V_{col(\bar{X})}$. (13)

By (11), the last vertex v_i of P_i is from C_t and the successor w_i of v_i on Z must be from R . Therefore, by the definition of W in Lemma 4.2 and by (3), $v_i \in S^-$. Thus, since $|S| = |S^-|$ and $S \subseteq X$, there are l vertices from S whose predecessors on Z are not from S^- . Suppose $s \in S$ on P_i and is not the first vertex of P_i . We show that the predecessor u of s on P_i is in S^- . Assume that $u \notin S^-$. By (2), $u \in Q$. By (1) and (4), there is no (C_t, u) -path in $D - V(R)$. However, by (6), $s \rightarrow u$; a contradiction. This implies that the first vertex u_i of P_i is in S for each $i = 1, 2, \dots, l$. Observe that by the definition of the paths P_1, \dots, P_l , the predecessor r_i of u_i on Z is from R .

Without loss of generality, assume that v_1, v_2, \dots, v_l are labeled such that the subpath $C_t[v_i, v_{i+1}]$ contains no vertices from the set $\{v_1, \dots, v_l\} - \{v_i, v_{i+1}\}$ for all $i = 1, \dots, l$. Let v_i^+ be the successor of v_i on C_t . Since $r_i \rightarrow u_i$ and $u_i \in S \subseteq V^*$ (see (1)), r_i is not in V^* . As $v_i \in S^-$, $v_i^+ \in S \subseteq V^*$ and thus $r_i \rightarrow v_j^+$ for all i and j by Proposition 3.6 (2). Replace the subpaths $P_i[u_i, v_i]$ with the subpaths $C_t[v_{i-1}^+, v_i]$ in Z for all $i = 1, 2, \dots, l$. By the arguments above, the subgraph obtained after these substitutions is an \mathcal{F} -canonical cycle in D covering X . \diamond

Theorem 5.3 *We can verify whether D contains a cycle covering X (and construct one if it exists) in time $O(n^7)$.*

Proof: A short scheme of the algorithm is as follows: Using Lemma 3.1 try to construct a cycle subgraph \mathcal{F}' of D covering X . If none exists, then D has no cycle covering X . Otherwise, applying Theorem 4.6 we can transform \mathcal{F}' into an X -leftshifted X -singular cycle subgraph $\mathcal{F} = C_1 \cup \dots \cup C_t$ using the algorithms **Singular** and **Leftshift** (see the definition of **Leftshift** in Section 6). If $t = 1$, we are done. Otherwise, using Lemma 5.2, we reduce the problem of finding a cycle in D covering X into the problem of finding a cycle covering a new set X^* in a new semicomplete multipartite digraph D^* with fewer vertices. As the reduction takes $O(n^6)$ time (see Theorem 4.6) and we perform $O(n)$ reductions, the complexity of our algorithm is $O(n^7)$.

Note that by Definition 5.1 and Lemma 5.2 we may assume that \mathcal{F} is a spanning cycle subgraph, since no \mathcal{F} -canonical cycle contains vertices from $V(D) - V(\mathcal{F})$. Let $V^* = V_{\chi_L(C_t)}$ and let $Z = \{z \in S^- : R \not\rightarrow z\}$. If Z is empty, then by (3) there is no path from C_t to R , which means that D has no cycle covering X (recall that every cycle contains a vertex from X (see Note 4.3)). In the rest of the proof we assume that Z is not empty.

Case 1: $Z = \{z\}$. Set $V(D^*) = V(R) \cup \{z\}$, $A(D^*) = A(D\langle R \rangle) \cup \{z \rightarrow r : r \in V(R), z \rightarrow r \text{ in } D\} \cup \{r \rightarrow z : r \in V(R) - V^*\}$ and $X^* = V(R) \cup \{z\}$. Suppose D has a cycle covering X . Then, by Lemma 5.2, D has an \mathcal{F} -canonical cycle C covering X . Since $|Z| = 1$, C must contain the path $C_t[z^+, z]$ and by the definition of an \mathcal{F} -canonical cycle, the predecessor r of z^+ on C does not belong to V^* . Hence we get a cycle containing X^* in D^* just by replacing the path $rC_t[z^+, z]$ in C by the arc $r \rightarrow z$. Conversely, if C^* is a cycle covering X^* in D^* , then the predecessor r^* of z on C^* does not belong to V^* and hence we can convert C^* to a cycle covering X in D by substituting the path $r^*C_t[z^+, z]$ for the arc $r^* \rightarrow z$. Thus we have shown that D^* has a cycle covering X^* if and only if D has a cycle covering X . Clearly, $|V(D)| > |V(D^*)|$.

Case 2: $|Z| \geq 2$. First construct a new digraph D' as follows: Set $V(D') = V(R) \cup Z \cup \{a', b'\}$, where a' and b' are new vertices, $A(D') = A(D\langle R \rangle) \cup \{z \rightarrow r : z \in Z, r \in V(R), z \rightarrow r \text{ in } D\} \cup \{r \rightarrow z : r \in V(R) - V^*, z \in Z\} \cup \{a' \rightarrow b'\} \cup \{r \rightarrow b' : r \in V(R)\} \cup \{r \rightarrow a' : r \in V(R) - V^*\} \cup \{b' \rightarrow z : z \in Z\}$, $X' = V(R) \cup \{a', b'\}$. Observe that in D' the vertex a' and all vertices of Z are in the colour class V^* (compare with Note 4.3, a', b' play the role of x_α and x_β).

Claim : D contains a cycle covering X if and only if D' has a cycle covering X' .

Proof of claim: Suppose that D has a cycle covering X . Then, by Lemma 5.2, D contains a \mathcal{F} -canonical cycle C covering X . Let C have the form given in (12). Transform C into a cycle C' in D' covering X' as follows: For every $i = 1, 2, \dots, l$, replace the path $C_t[v_{i-1}^+, v_i]T_i$ in C with the path v_iT_i . By the definition of an \mathcal{F} -canonical cycle, the terminal vertex r_i in T_i is not in V^* (as $v_i^+ \in V^*$). Thus, by the definition of D' , $r_{i-1} \rightarrow v_i$ in D' since $v_i \in Z$ (for all $i = 1, 2, \dots, l$). Therefore, after all substitutions above, we obtain a cycle. Since $r_{i-1} \rightarrow a'$ and $b' \rightarrow v_i$ in D' for $i = 1, 2, \dots, l$, the arc $a' \rightarrow b'$ can be inserted into this cycle to form a cycle C' covering X' .

Suppose now that D' has a cycle C' covering X' . Since b' is the only out-neighbour

of a' , the arc $a' \rightarrow b'$ is on C' . Thus, since b' only dominates vertices in Z , $Z' = Z \cap V(C') \neq \emptyset$. Let $z \in Z$ be the successor of b' on C' and let r be the predecessor of a' on C' . By the definition of D' , $r \rightarrow z$ in D' . Set $C'' = C'[z, r]z$.

Let $Z' = \{z_1, z_2, \dots, z_l\}$, where the vertices z_1, \dots, z_l are labeled such that no vertex from $\{z_1, z_2, \dots, z_l\} - \{z_i, z_{i+1}\}$ is on $C_t[z_i, z_{i+1}]$ for all $i = 1, 2, \dots, l$. Let r_i be the predecessor of z_i on C'' . By the definition of D' , $r_i \in V(R) - V^*$. Thus, by Proposition 3.6 (2) and (1), $r_i \rightarrow z_{i-1}^+$ in D (recall that z_{i-1}^+ is the successor of z_{i-1} on C_t). Replace every arc $r_i \rightarrow z_i$ in C'' with the path $r_i C_t[z_{i-1}^+, z_i]$. Clearly, the resulting cycle in D covers X . Note that above we actually proved how to construct the cycle C from C' . This completes the proof of the claim. \diamond

If D' contains fewer vertices than D , then set $D^* = D'$ and $X^* = X'$. Assume now that $|V(D')| \geq |V(D)|$. Observe that then $|Z| \geq 2$ implies that $V(C_t) = S \cup S^-$ and $|V(C_t)| = 4$, thus we must have $|V(D')| = |V(D)|$.

Suppose that $\bar{X} \neq \emptyset$. Then, by Lemma 4.5, the arc $x_\alpha \rightarrow x_\beta$ is on C_t . Note that $x_\beta \notin S^-$, since $S \subset X$ and $X - \{x_\beta\} \Rightarrow x_\beta$. Hence by the remark above ($V(C_t) = S \cup S^-$), we must have $x_\beta \in S$. However, then we would get $|Z| \leq 1$ (by (1), (8) and Definition 3.5 (β)), a contradiction. Thus we must have $\bar{X} = \emptyset$. Apply the construction of D' , X' once more to D' and X' instead of D and X starting, from a new X' -leftshifted X' -singular cycle subgraph of D' (if there is no cycle subgraph covering X' in D' , then by the claim, D has no cycle covering X). Finally, set $D^* = (D')'$, $X^* = (X')'$. Observe that, by construction, $V(D') - X' \neq \emptyset$ and hence $|V(D)| = |V(D')| > |V(D^*)|$. Furthermore, by the arguments above, D has a cycle covering X if and only if D^* has a cycle covering X^* .

This completes the description of our algorithm and the proof of its correctness. \diamond

Clearly, Theorem 5.3 implies our main goal:

Corollary 5.4 *The Hamiltonian cycle problem is solvable in polynomial time for semi-complete multipartite digraphs.* \diamond

6 Constructing leftshifted singular cycle subgraphs

The purpose of this section is to prove Theorem 4.6.

Lemma 6.1 *Suppose $\bar{X} \neq \emptyset$ and that C and C' are disjoint cycles such that the arc $x_\alpha \rightarrow x_\beta$ is on C and C' contains a vertex $z \in \bar{X}$, then there exists a cycle C^* with $V(C^*) = V(C) \cup V(C')$.*

Proof: Since x_β only dominates vertices in \bar{X} , its successor u on C belongs to \bar{X} . By Note 4.3, x_β dominates z and the predecessor z^- of z on C' dominates u , implying that $C[u, x_\beta]C'[z, z^-]$ is the desired cycle. \diamond

Lemma 6.2 *Let $\mathcal{F} = C_1 \cup C_2 \cup \dots \cup C_t$ be an X -singular cycle subgraph and let B be a cycle in $D - V(\mathcal{F})$. Let $A \subseteq \bar{X} - V(\mathcal{F} \cup B)$, $D' = D \langle V(\mathcal{F} \cup B) \cup A \rangle$ and $X' = X \cap V(D')$. Suppose that every cycle $C_i \in \mathcal{F}$ ($1 \leq i \leq t$) has a vertex $q_i \in V(C_i) \cap X$, such that*

$q_i \Rightarrow V(C_{i+1} \cup C_{i+2} \cup \dots \cup C_t \cup B)$. Then, in $O(n^4)$ time, we can find an X -singular cycle subgraph, $\mathcal{F}' = Z_1 \cup Z_2 \cup \dots \cup Z_l \cup B'$ of D' covering X' , where $l \in \{0, 1, \dots, t\}$, B' is a cycle with $V(C_{l+1} \cup C_{l+2} \cup \dots \cup C_t \cup B) \cap X \subseteq V(B')$ and $Z_i = C_i$ for $i = 1, 2, \dots, l$.

Proof: We first describe a procedure **Left-B-growth** which transforms \mathcal{F} into \mathcal{F}' defined above. Then we shall prove the correctness of **Left-B-growth** and determine its complexity. Let \mathcal{G} be a cycle subgraph of D . In the procedure we apply **Singular** (see Note 3.8) to \mathcal{G} in the digraph $D\langle\mathcal{G}\rangle$ in order to find an X -singular subgraph \mathcal{L} of $D\langle\mathcal{G}\rangle$ which covers $X \cap V(D\langle\mathcal{G}\rangle)$. We denote this as follows: $\mathcal{L} := \mathbf{Singular}(\mathcal{G})$. Similarly, we also apply the algorithm **Merge** from Theorem 3.4. Note that only Step 3 involves vertices from A .

Procedure Left-B-growth

Step 1. For j from t down to 1 do the following: $C := \mathbf{Singular}(B \cup C_j)$, if C is a cycle then $B := C$ (otherwise, B is not changed). Let $I = \{i_1, \dots, i_k\}$ be the set of subscripts of the cycles in \mathcal{F} which have been merged with a current B in the loop above. If $I = \emptyset$ then set $\mu := t + 1$, $\mathcal{H} := \mathcal{F} \cup B$ and go to Step 3. Set $\mu := \min I$. If $|I| = t$, then return $\mathcal{F}' := B$.

Step 2. While there is $j \in \{\mu + 1, \mu + 2, \dots, t\} - I$ do the following: $B := \mathbf{Merge}(B \cup C_j)$, $I := I \cup \{j\}$. If $\mu = 1$ then return $\mathcal{F}' := B$.

After the execution of Step 2 we have obtained the cycle subgraph $\mathcal{H} = C_1 \cup C_2 \cup \dots \cup C_{\mu-1} \cup B$. (Below we prove that every application of **Merge** to a subgraph $B \cup C_j$ in Step 2 results in a new cycle B).

Step 3. Apply only the first iteration of **Singular** to \mathcal{H} in D' (recall Note 3.8). If the output of the iteration is \mathcal{H} itself, then return \mathcal{H} . Otherwise, **Singular** has replaced two or more cycles from \mathcal{H} by one new cycle C . If $V(B) \cap X \not\subseteq V(C)$, i.e. B was not involved in this merging, then let $B := \mathbf{Merge}(B \cup C)$ (as we shall prove later, B will, in fact, be a cycle after this step). Otherwise let $B := C$. Add the subscripts of those cycles C_j ($j < \mu$), merged into C above, to the set I , $\mu := \min I$ and go to Step 2.

First note that the call **Singular** $(B \cup C_j)$ may release some vertices from \bar{X} which can be used to merge other cycles C_p, C_q (for some $1 \leq p < q \leq t$). This is why we may need to repeat Step 2 and Step 3.

Now we prove the correctness of **Left-B-growth**. Clearly, the procedure is correct if every call involving **Merge** in Steps 2 and 3 provides us with a new cycle B . Below we show that this is indeed so.

Assume that just after execution of Step 1 $|I| < t$, i.e. there exists an index $j \in \{1, 2, \dots, t\} - I$. This means that in the $(t - j + 1)$ 'th execution of Step 1 **Left-B-growth** has tried to merge C_j and the cycle $B = B_j$, existing at that moment of time, but has not succeeded because $C_j \cup B_j$ is X -singular in $D\langle C_j \cup B_j \rangle$ (see Note 3.8). Thus, by Definition 3.5, either $C_j \rightsquigarrow B_j$ or $B_j \rightsquigarrow C_j$. However, $B_j \rightsquigarrow C_j$ is impossible since before the algorithm starts, the vertex q_j strongly dominates $V(B_j) \subseteq$

$V(C_{j+1}) \cup \dots \cup V(C_t) \cup V(B)$. Thus $C_j \simeq B_j$. If $C_j \Rightarrow B_j$, then clearly $V(C_j) \Rightarrow w_j$ for all $w_j \in V(B_j)$. Otherwise let $b \rightarrow c$ be an arc from B_j to C_j . By Theorem 3.7 and Proposition 3.6, $V(C_j) \Rightarrow b^+$ and $b^+ \in X$. Thus we have shown that B_j has an X -vertex w_j such that $V(C_j) \Rightarrow w_j$. Observe that the vertex w_j will remain in the current B till the end of the execution of the algorithm (see Note 3.8).

Now we prove that every application of **Merge** to the subgraph $B \cup C_j$ in Step 2 results in a new cycle B . By Theorem 3.4, it is sufficient to show that one of the alternatives (A)-(C) hold for the cycles B, C_j . We have proved the existence of a vertex $w_j \in X \cap B$ strongly dominated by C_j . On the other hand, B contains the vertex q_μ which strongly dominates C_j . Therefore alternative (B) holds and, by Theorem 3.4, **Merge** will merge B and C_j into one cycle.

It remains to prove that every call $B := \mathbf{Merge}(B \cup C)$ in Step 3 results in a cycle B . Since $V(C) \cap V(B) = \emptyset$ and since any subset of cycles of an X -singular cycle subgraph is X -singular (as always with respect to the corresponding induced subgraphs), C must contain some vertices from \bar{X} , i.e. vertices that were left out in a previous merging using **Singular** (recall that \mathcal{F} is X -singular and we use only one iteration of **Singular** in Step 3). Thus, since C contains a vertex q_l (for some $l \leq \mu - 1$) which strongly dominates B and a vertex $u \in \bar{X}$, we see (by Note 4.3) that alternative (C) of Theorem 3.4 holds for the pair C, B after deleting possible arcs from u to B . Hence, by Theorem 3.4, **Merge** will indeed return a cycle as desired.

It is easy to check that, by Theorem 3.7, the complexity of our procedure is as claimed. Indeed, the performance of Step 1 takes $O(n^4)$ time by Theorem 3.7 and the fact that $t < n$. In Steps 2 and 3, each call of **Merge** or the first iteration of **Singular** either results in fewer cycles to be considered or halts the procedure. Again, $t < n$ and Theorems 3.4 and 3.7 imply $O(n^4)$ time for the total performance of Steps 2 and 3. \diamond

We shall now prove Theorem 4.6, by showing that the following algorithm **Leftshift** returns the desired cycle subgraph. In the algorithm we use procedure **Left-B-growth** from the previous lemma and the following procedure **Right-B-growth**. It is easy to check that the complexity of **Right-B-growth** is $O(n^5)$.

Procedure **Right-B-growth**

Input: A digraph D , a cycle B and a cycle subgraph \mathcal{R} of $D - V(B)$.

Output: A cycle B' and a cycle subgraph \mathcal{R}' such that $X \cap V(B \cup \mathcal{R}) \subseteq V(B' \cup \mathcal{R}')$ and $V(B) \subseteq V(B')$. Furthermore, there is no 1-path-cycle subgraph $P' \cup C_1 \cup \dots \cup C_l$ of $D - V(B')$ covering $X \cap V(\mathcal{R}')$ with the property that P' is an (a, b) -path and there exists some $q \in V(B')$ such that $q \rightarrow a$ and $b \rightarrow q^+$.

Step 1. Using Lemma 3.2, for every $q \in V(B)$, try to construct a 1-path-cycle subgraph $P \cup C_1 \cup \dots \cup C_s$ of $D - V(B)$ covering $X \cap V(\mathcal{R})$ such that the path P terminates at a vertex dominating q^+ and starts at a vertex dominated by q . If such a subgraph exists, then set $B := B[q^+, q]Pq^+$ and $\mathcal{R} := C_1 \cup \dots \cup C_s$ and go to the beginning of Step 1.

Step 2. Return $B' := B$ and $\mathcal{R}' := \mathcal{R}$.

Algorithm Leftshift

- Step 1. Let $\mathcal{F}' = C'_1 \cup \dots \cup C'_l$ be an X -singular subgraph of D which covers X . If $l = 1$ then return $\mathcal{F} := C'_1$; halt. Set $\mathcal{A} := \emptyset$, $B := C'_1$, $\mathcal{R} = C'_2 \cup \dots \cup C'_l$, $a := 0$ and let p_1 be any vertex in $V_{\chi_{\mathcal{R}}(C'_1)} \cap V(C'_1)$.
- Step 2. Call **Right-B-growth** with input $D - V(\mathcal{A})$, B and \mathcal{R} . Let the output be B' and \mathcal{R}' . Set $B := B'$, $\mathcal{R} := \mathcal{R}'$. If $\mathcal{A} = \emptyset$ and $\mathcal{R} = \emptyset$, then return $\mathcal{F} := B$; halt.
- Step 3. If $\bar{X} \neq \emptyset$, the arc $x_\alpha \rightarrow x_\beta$ is on B and $\mathcal{R} \neq \emptyset$, then successively merge each cycle of \mathcal{R} into B , using **Merge**. Let B^* denote the cycle obtained this way (Note that $V(B^*) = V(B) \cup V(\mathcal{R})$). Let $B := B^*$ and $\mathcal{R} := \emptyset$.
- Step 4. If $\mathcal{A} = \emptyset$, then go to Step 6. Otherwise ($\mathcal{A} = Z_1 \cup \dots \cup Z_a$), call **Left-B-growth** for $\mathcal{A} \cup B$ with $q_i = p_i$, $i = 1, 2, \dots, a$ (recall that q_i is defined in Lemma 6.2) and $A = V(D) - (V(\mathcal{A} \cup B \cup \mathcal{R}))$, to convert it into an X -singular cycle subgraph \mathcal{F}^* of $D \setminus V(\mathcal{A} \cup B)$ covering $X \cap V(\mathcal{A} \cup B)$, such that $\mathcal{F}^* = Z_1 \cup \dots \cup Z_e \cup B'$ for some $e \in \{0, 1, \dots, a\}$ (see Lemma 6.2). If $\mathcal{R} = \emptyset$, then return $\mathcal{F} := \mathcal{F}^*$; halt.
- Step 5. If $e < a$ then set $B := B'$, $\mathcal{A} := Z_1 \cup \dots \cup Z_e$, $a := e$ and go to Step 2.
- Step 6. Use **Singular** to obtain an X -singular cycle subgraph $\mathcal{H} = H_1 \cup \dots \cup H_h$ (of $D - V(\mathcal{A} \cup B)$) from \mathcal{R} . If $h \geq 2$, then let p_{a+2} be any vertex of $V_{\chi_{\mathcal{R}}(H_1)} \cap V(H_1)$. (Here $\chi_{\mathcal{R}}(H_1)$ is defined with respect to \mathcal{H} .)
- Step 7. Let $Z_{a+1} := B$, $\mathcal{A} := Z_1 \cup \dots \cup Z_{a+1}$, $B := H_1$, $\mathcal{R} := H_2 \cup \dots \cup H_h$ and let $a := a + 1$. If $\mathcal{R} = \emptyset$, then go to Step 4, otherwise go to Step 2.

Below we prove the correctness of **Leftshift** and determine its time complexity.

To prove the correctness of the algorithm it is enough to prove that, in every performance of Step 4, p_i is a proper choice for q_i in Lemma 6.2 for all $i = 1, \dots, a$ (that is we can actually apply the algorithm **Left-B-growth** as we claim), and that the output of the algorithm, $\mathcal{F} = C_1 \cup \dots \cup C_t$, is X -leftshifted. The first part follows from the following claim.

Claim: The following properties hold during the execution of **Leftshift**:

- (i) If $V(B) \cap \bar{X}$ becomes non-empty (in Step 2 or Step 4) this will lead to the execution of Step 3 and hence render $\mathcal{R} = \emptyset$.
- (ii) At all times during the execution of **Leftshift** we have $V(\mathcal{A}) \subseteq X$.
- (iii) The vertex p_i is a proper choice for q_i in Lemma 6.2 for all $i = 1, \dots, a$.

Proof of the claim: Suppose $V(B) \cap \bar{X}$ becomes non-empty during the algorithm. If this happens (for the first time) in Step 4, then Step 2 will be executed just after this (via Step 5). Otherwise $V(B) \cap \bar{X}$ becomes non-empty in Step 2. Now consider that execution of Step 2 when $V(B) \cap \bar{X}$ becomes non-empty for the first time, or

has just become non-empty in Step 4. Note that at this point we have $V(\mathcal{A}) \subseteq X$, since vertices are only added to \mathcal{A} in Step 7 and after Step 5 $\mathcal{A} \subset X$, by Lemma 4.4. Furthermore, we claim that the arc $x_\alpha \rightarrow x_\beta$ will be an arc of B before the current execution of Step 2 finishes. Indeed, By Lemma 4.5, $x_\alpha \rightarrow x_\beta$ was an arc of C_l after Step 1 and hence is currently on some cycle C' in \mathcal{R} ; B has a vertex of \bar{X} , thus, the proof of Lemma 6.1 shows that **Right-B-growth** will merge B and C' . Lemma 6.1 implies that when $x_\alpha \rightarrow x_\beta$ has become an arc of B , then Step 2 will either terminate with $\mathcal{R} = \emptyset$, or with $V(\mathcal{R}) \subseteq X$. In the last case Step 3 will be executed and since B contains a vertex p that strongly dominates every vertex in \mathcal{R} (one of the p_i 's defined in an earlier execution of Step 7; compare with proposition 3.6) and B also contains the vertex x_α which is strongly dominated by every vertex in \mathcal{R} , it follows from Theorem 3.4 (alternative (C) holds for B and every cycle of \mathcal{R}) that after execution of Step 3 we have $\mathcal{R} = \emptyset$. Observe that (i) and (ii) follow easily from this, since after \mathcal{R} becomes empty \mathcal{A} will not be redefined anymore.

To prove (iii), we first observe that at all executions of Step 4, except the last one we have $V(\mathcal{A} \cup B) \subseteq X$. This implies that no vertex was lost when any earlier execution of Step 4 was performed. Secondly, no p_i is defined after the (unique) execution of Step 3. Thus it follows from the choice of p_{a+2} in Step 7 and the definition of p_1 that (iii) holds (note that when some cycles of the current \mathcal{A} are merged into B in Step 4 only the vertex p_i with the smallest index among these will survive as a p_i that can later be referred to). This completes the proof of the claim.

We now prove that the output of the algorithm, $\mathcal{F} = C_1 \cup \dots \cup C_t$, is X -leftshifted. Suppose that \mathcal{F} is not X -leftshifted. Then there exist $1 \leq i < j \leq t$ and a 1-path-cycle subgraph $\mathcal{F}' = P \cup M_1 \dots \cup M_k$ such that $X \cap V(C_j) \subseteq V(\mathcal{F}) \cap V(\mathcal{F}') \subseteq V(C_j)$, P is a (u, v) -path and there exists $w \in V(C_i)$ such that $w \rightarrow u$ and $v \rightarrow w^+$.

Consider that performance of Step 7, where C_i enters \mathcal{A} . Just before this step of the algorithm $B = C_i$. Consider also the last performance of Step 2 before the above-mentioned performance of Step 7. Clearly, $B' = B = C_i$ is the output of **Right-B-growth** in this performance of Step 2. Let $k < i$. By Lemma 4.4 and the fact that every subset of the cycles of a singular cycle subgraph is singular, $V(C_k) \subseteq X$. Thus, by Note 3.8 and part (ii) of the claim above, the cycle C_k has not been changed since it entered \mathcal{A} . Moreover, C_k was constructed before C_i . Hence, after the above-mentioned performance of Step 2 $\mathcal{A} = C_1 \cup \dots \cup C_{i-1}$. Thus, the vertices of \mathcal{F}' were available to "enlarge" B in the above-mentioned performance of **Right-B-growth**, contradicting the fact that **Right-B-growth** did not change B .

It is easy to see that the complexity of the algorithm is $O((n_2 + 1)n^5)$, where n_2 is the number of executions of Step 2. To bound n_2 , we introduce a potential function $L = a(\mathcal{A}) + 2|X \cap V(\mathcal{R})|$, where $a(\mathcal{A})$ denotes the number of cycles in \mathcal{A} . We can return to Step 2 from either Step 5 or Step 7. If we return from Step 5, then $a(\mathcal{A})$ decreases and $2|X \cap V(\mathcal{R})|$ remains unchanged. If we return from Step 7, then $a(\mathcal{A})$ increases by one, but $2|X \cap V(\mathcal{R})|$ decreases by at least two. Thus, every time we return to Step 2, the function L decreases. Moreover, $0 \leq L \leq 3n$. Hence, $n_2 = O(n)$. Thus, the complexity of the algorithm is $O(n^6)$. \diamond

7 Open problems and conjectures

The existence of a polynomial algorithm for the Hamiltonian cycle problem in semicomplete multipartite digraphs indicates the possible existence of a "nice" mathematical characterization of Hamiltonian semicomplete multipartite digraphs. So far we have not managed to find one.

We believe that some more general problems for semicomplete multipartite digraphs D are also polynomial time solvable:

Conjecture 7.1 *There are polynomial algorithms for the following problems:*

- 1) *Given set $X \subseteq V(D)$ find a cycle covering X .*
- 2) *Construct a longest cycle in D .*
- 3) *Given set $X \subseteq V(D)$ find a cycle containing the maximum number of vertices from X .*

Conjecture 7.1.1 seems quite difficult, even in the case of semicomplete bipartite digraphs it is still open (see [5]). However, the case of extended semicomplete digraphs is solved in [3]. Conjecture 7.1.2 is settled for both semicomplete bipartite digraphs and extended semicomplete digraphs (cf. [16]).

Another interesting problem is to characterize weakly Hamiltonian-connected semicomplete multipartite digraphs, i.e. semicomplete multipartite digraphs having a Hamiltonian path between every pair of vertices. The problem was settled for bipartite semicomplete and extended semicomplete digraphs in [6] and [2], respectively.

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