Note on alternating directed cycles

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Abstract

The problem of the existence of an alternating simple dicycle in a 2-arc-coloured digraph is considered. This is a generalization of the alternating cycle problem in 2-edge-coloured graphs (proved to be polynomial time solvable) and the even dicycle problem (the complexity is not known yet). We prove that the alternating dicycle problem is $\mathcal{NP}$-complete. Let $f(n)$ ($g(n)$, resp.) be the minimum integer such that if every monochromatic indegree and outdegree in a strongly connected 2-arc-coloured digraph (any 2-arc-coloured digraph, resp.) D is at least $f(n)$ ($g(n)$, resp.), then D has an alternating simple dicycle. We show that $f(n) = \Theta(\log n)$ and $g(n) = \Theta(\log n)$.

Keywords: Alternating cycles, even cycles, edge-coloured directed graphs.

1 Introduction, terminology and notation

We shall assume that the reader is familiar with the standard terminology on graphs and digraphs and refer the reader to [4]. We consider digraphs without loops and multiple arcs. The arcs of digraphs are coloured with two colours: colour 1 and colour 2. By a *cycle* in a digraph (in a graph) we mean a directed simple cycle (a simple cycle). A cycle C is *alternating* if any consecutive arcs (edges) of C have distinct colours.

The problem of the existence of an alternating cycle in a 2-arc-coloured digraph (*the ADC problem*) generalizes the following two problems: the existence of an alternating cycle in a 2-edge-coloured graph (this problem is polynomial time solvable, cf. [2]; the faster of two polynomial algorithms described in [2] follows from a nice characterization [7] of 2-edge-coloured graphs containing an alternating cycle) and the existence of an even length cycle in a digraph (the complexity is not known yet, cf. [9, 10]).

To see that the ADC problem generalizes the even cycle problem, replace every arc (x, y) of a digraph D by two vertex disjoint alternating paths of length three, one starting from colour 1 and the other - from colour 2. Clearly, the obtained 2-edge-coloured digraph has an alternating cycle if and only if D has a cycle of even length.

We prove that the ADC problem is $\mathcal{NP}$-complete by providing a transformation from the well-known 3-SAT to the ADC problem.

To indicate that an arc (x, y) has colour $i \in \{1, 2\}$ we shall write $(x, y)_i$. For a vertex v in a 2-arc-coloured digraph D, $d_i^+(v)$ ($d_i^-(v)$) denotes the number of arcs of colour i leaving

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(entering) v, $i = 1, 2$; $\delta_{mon}(v) = \min\{d_i^+(v), d_i^-(v) : \ i = 1, 2\}$. The following parameter is of importance to us:

$$\delta_{mon}(D) = \min\{\delta_{mon}(v) : \ v \in V(D)\}.$$

We study a function $f(n)$ ($g(n)$, resp.), the minimum integer such that if $\delta_{mon}(D) \geq f(n)$ ($\delta_{mon}(D) \geq g(n)$, resp.), for a strongly connected digraph (any digraph, resp.) D with n vertices, then D has an alternating cycle. We show that $f(n) = \Theta(\log n)$ and $g(n) = \Theta(\log n)$.

By contrast with that, the corresponding function $f(n)$ for the even cycle problem does not exceed three (see [9]). Using Theorem 3.2 in [8], one can show that the corresponding function $g(n)$ for the even cycle problem equals $\Theta(\log n)$. By Theorem 3.2 in [8], there exists a digraph H_n with n vertices and minimum outdegree at least $\frac{1}{2}\log n$ [1] not containing even cycles. Let H'_n be the digraph obtained from H_n by reorienting all arcs. Take vertex disjoint copies of H_n and H'_n and add all arcs from H'_n to H_n. The obtained digraph and the upper bound in Theorem 3.2 of [8] provide the estimate $\Theta(\log n)$.

2 $\mathcal{NP}$-completeness

Theorem 2.1 *The ADC problem is $\mathcal{NP}$-complete.*

Proof: To show that the ADC problem is $\mathcal{NP}$-hard, we transform the well-known problem 3-SAT ([6], p. 46) to the ADC problem. Let $U = \{u_1, ..., u_k\}$ be a set of variables, let $C = \{c_1, ..., c_m\}$ be a set of clauses such that every c_i has three literals, and let v_{il} be the lth literal in the clause c_i.

We construct a 2-arc-coloured digraph D which has an alternating cycle if and only if C is satisfiable. The vertex set of D consists of two disjoint sets X and Y, where $X = \{x_i : \ i = 1, 2, ..., m+2\}$ and $Y = \{y_{j0}, y_{j,t+1}, y^r_{j1}, y^r_{j2}, ..., y^r_{jt} : \ r = 1, 2; \ j = 1, 2, ..., k\}$ ($t = 6m$).

If a literal v_{il} is a variable, u_j, then let $par(i,l) = 1$ and $ind(i,l) = j$; and if v_{il} is the negation of a variable u_j, then let $par(i,l) = 2$ and $ind(i,l) = j$. Let $y(v_{il}) = y^{par(i,l)}_{j,q}$, where $j = ind(i,l)$ and $q = 6(i-1) + 2l$.

The arc set of D is $A(D) = (\cup_{j=1}^k \cup_{r=1}^2 P^r_j) \cup (\cup_{i=1}^m \cup_{p=1}^3 Q_{ip}) \cup B$, where the sets in $A(D)$ are defined as follows:

$$\begin{aligned} B &= \{(x_{m+1}, x_{m+2})_1, (x_{m+2}, y_{1,0})_2, (y_{k,t+1}, x_1)_2\} \cup \{(y_{p,t+1}, y_{p+1,0})_2 : \ 1 \leq p \leq k-1\}; \\ P^r_j &= \{(y_{j0}, y^r_{j1})_1, (y^r_{j1}, y^r_{j2})_2, (y^r_{j2}, y^r_{j3})_1, (y^r_{j3}, y^r_{j4})_2, ..., (y^r_{j,t-1}, y^r_{jt})_2, (y^r_{jt}, y_{j,t+1})_1\}; \\ Q_{ip} &= \{(x_i, y(v_{ip}))_1, (y(v_{ip}), x_{i+1})_2\}. \end{aligned}$$

Suppose now that C is satisfiable and consider a truth assignment α for U that satisfies all the clauses in C. Then, for every $i = 1, 2, ..., m$, there exists an l_i such that v_{i,l_i} is true under α. It is easy to check that D has the following alternating cycle:

$$\begin{gathered} (x_1, y(v_{1l_1}), x_2, y(v_{2l_2}), x_3, ..., x_m, y(v_{ml_m}), x_{m+1}, \\ x_{m+2}, y_{1,0}, y^{r(1)}_{1,1}, y^{r(1)}_{1,2}, ..., y^{r(1)}_{1,t}, y^{r(2)}_{2,1}, ..., \\ y^{r(2)}_{2,t}, y_{2,t+1}, y_{3,0}, ..., y_{k,0}, y^{r(k)}_{k,1}, ..., y^{r(k)}_{k,t}, y_{k,t+1}, x_1), \end{gathered} \tag{1}$$

where $r(j) = 2$ if u_j is true under α and $r(j) = 1$, otherwise.

[1] All logarithms in this paper are of basis 2.

Now suppose that D has an alternating cycle. We prove that C is satisfiable. Because of the above correspondence between a truth assignment for U and an alternating cycle in D of the form (1), to show that C is satisfiable it suffices to prove that every alternating cycle in D is of the form (1).

We first prove that every alternating cycle in D contains the arc $(y_{k,t+1}, x_1)_2$. Assume that this is not true, i.e. the digraph $D' = D - (y_{k,t+1}, x_1)$ has an alternating cycle. The vertex x_1 cannot belong to an alternating cycle in D' as its indegree in D' is zero. Assuming that x_2 is in an alternating cycle we easily conclude that x_1 must be one of the predecessors of x_2 in such a cycle. Thus x_2 is not in an alternating cycle. Similar arguments show that no vertex in X is in an alternating cycle. However, the subgraph of D induced by Y has no alternating cycle.

Let F be an alternating cycle in D. We have proved that F has x_1. It is easy to check that F thus contains either Q_{11} or Q_{12} or Q_{13}. In any case F contains x_2. Thus F has either Q_{21} or Q_{22} or Q_{23}. Repeating this argument we conclude that, for every $i = 1, 2, ..., m$, F contains either Q_{i1} or Q_{i2} or Q_{i3}. Now we see that y_{10} is also in F. Therefore, for every $j = 1, 2, ..., k$, either P_j^1 or P_j^2 is in F. Thus we have proved that F is of the form (1). □

We do not know what is the complexity of the ADC problem restricted to tournaments.

Problem 2.2 *Does there exist a polynomial algorithm to check whether a 2-arc-coloured tournament has an alternating cycle?*

3 Functions $f(n)$ and $g(n)$

As $f(n) \leq g(n)$ we shall only prove a lower bound for $f(n)$ in Theorem 3.4 and an upper bound for $g(n)$ in Theorem 3.6.

Let $S(k)$ be the set of all sequences whose elements are from the set $\{1, 2\}$ such that neither 1 nor 2 appears more that k times in a sequence. We assume that the sequence without elements (i.e. the empty sequence) is in $S(k)$. We start with three technical lemmas.

Lemma 3.1 $|S(k)| = \binom{2(k+1)}{k+1} - 1$.

Proof: Clearly, $|S(k)| = \sum_{i=0}^{k} \sum_{j=0}^{k} \binom{i+j}{i}$. Using the well-known identity $\sum_{i=0}^{m} \binom{n+i}{n} = \binom{n+m+1}{n+1}$, we obtain

$$|S(k)| = \sum_{i=0}^{k} \binom{i+k+1}{i+1} = \sum_{i=0}^{k} \binom{i+k+1}{k} = (\sum_{t=0}^{k+1} \binom{k+t}{k}) - 1 = \binom{2(k+1)}{k+1} - 1.$$

□

Lemma 3.2 *For every $k \geq 1$,*

$$\binom{2k}{k} < \frac{1}{\sqrt{\pi}} \frac{4^k}{\sqrt{k}}. \tag{2}$$

Proof: Using the well-known inequality (see, e.g., [5], p. 54)

$$\sqrt{2\pi} n^{n+1/2} e^{-n} e^{(12n+1)^{-1}} < n! < \sqrt{2\pi} n^{n+1/2} e^{-n} e^{(12n)^{-1}},$$

we obtain

$$
\begin{aligned}
\binom{2k}{k} &= \frac{(2k)!}{k!k!} \\
&< \frac{\sqrt{2\pi}(2k)^{2k+1/2}e^{-2k}e^{(24k)^{-1}}}{(\sqrt{2\pi}k^{k+1/2}e^{-k}e^{(12k+1)^{-1}})^2} \\
&= \frac{2^{2k}k^{2k}\sqrt{2k}e^{-2k}}{\sqrt{2\pi}k^{2k}ke^{-2k}} \times e^{\frac{1}{24k}-\frac{2}{12k+1}} \\
&= \frac{4^k}{\sqrt{\pi}\sqrt{k}} \times e^{\frac{1-36k}{(24k)(12k+1)}}
\end{aligned}
$$

As $(1-36k)/((24k)(12k+1)) < 0$ when $k \geq 1$, we arrive at (2). □

Let $d(n) = \lfloor \frac{1}{4}\log n + \frac{1}{8}\log\log n - a \rfloor$, where $a = \frac{5-\log\pi}{8}$ (≤ 0.5).

Lemma 3.3 $\binom{2(2d(n)+1)}{2d(n)+1} < n$, *for all* $n \geq 24$.

Proof: Let $s = \binom{2(2d(n)+1)}{2d(n)+1}$ and assume that $s \geq n$, for the sake of contradiction. Let $\phi(x) = x - \log(2x+1)/8$. By the definition of s and the inequality (2), we obtain that

$$\frac{1}{4}\log s < a - \frac{1}{8} + \phi(d(n)). \tag{3}$$

Since $d(n) \leq d(s) \leq \frac{1}{4}\log s + \frac{1}{8}\log\log s - a$ and the function $\phi(x)$ is monotonically increasing for $x \geq 0$, we obtain the following from (3).

$$\frac{1}{4}\log s < a - \frac{1}{8} + \phi(\frac{1}{4}\log s + \frac{1}{8}\log\log s - a). \tag{4}$$

By observing that $\frac{\log\log s}{8} - \frac{1}{8} = \frac{\log\log\sqrt{s}}{8}$, we obtain the following from (4).

$$\frac{1}{8}\log(\log\sqrt{s} + \frac{1}{4}\log\log s - 2a + 1) < \frac{1}{8}\log\log\sqrt{s} \tag{5}$$

Thus, $\frac{1}{4}\log\log s - 2a + 1 < 0$, a contradiction when $d(n) \geq 1$ (i.e. $n \geq 24$). Therefore, $s < n$, when $n \geq 24$. □

Theorem 3.4 *For every integer* $n \geq 24$, *there exists a 2-arc-coloured strongly connected digraph* G_n *with* n *vertices and* $\delta_{mon}(G_n) \geq d(n)$ *not containing an alternating cycle.*

Proof: Let the vertex set of a digraph D_n be $S(2d(n))$ and let two vertices of D_n be connected if and only if one of them is a prefix of the other one. Moreover, if $x = (x_1, x_2, ..., x_p)$ and $y = (y_1, y_2, ..., y_q)$ are vertices of D_n, and x is a prefix of y (namely, $x_i = y_i$ for every $i = 1, 2, ..., p$), then the arc $a(x, y)$ between x and y has colour y_{p+1} and $a(x, y)$ is oriented from x to y if and only if $|\{j : j \geq p+1 \text{ and } y_j = y_{p+1}\}| \leq d(n)$.

D_n is strongly connected since the arc between a pair of vertices $x = (x_1, x_2, ..., x_p)$ and $y = (x_1, x_2, ..., x_p, x_{p+1})$ is oriented from x to y, and the arc between the empty sequence $\emptyset$ and a vertex v of D_n which is a sequence with $4d(n)$ elements is oriented from v to $\emptyset$.

Let $x = (x_1, x_2, ..., x_p)$ be a vertex of D_n. It is easy to see that $d_1^+(x) \geq d(n)$. Indeed, if x contains at most $d(n)$ elements equal one, then $(x, x^r)_1$ is in D_n, where $r = 1, 2, ..., d(n)$ and x is a prefix of x^r followed by r ones. If x contains $t > d(n)$ elements equal one, then $(x, y)_1$ is in D_n, where y is obtained from x by either adding at most $2d(n) - t$ ones or deleting more than $d(n)$ rightmost ones, together with 2's between them, from x.

Analogously, one can show that $d_1^-(x) \geq d(n)$. By symmetry, $\delta_{mon}(D_n) \geq d(n)$.

Now we prove that D_n contains no alternating cycle. Assume that D_n contains an alternating cycle C. The empty sequence $\emptyset$ is not in C as $\emptyset$ is adjacent with the vertices of the form $(i, ...)$ by arcs of colour $i \in \{1, 2\}$, but the vertices of the form $(1, ...)$ are not adjacent with the vertices of the form $(2, ...)$. Analogously, one can prove that the vertices (1) and (2) are not in C. In general, after proving that C has no vertex with p elements, we can show that C has no vertex with $p+1$ elements.

By Lemma 3.1, D_n has $b(n) = \binom{2(2d(n)+1)}{2d(n)+1} - 1$ vertices. By Lemma 3.3, $b(n) < n$. Now we append $n - b(n)$ vertices along with arcs to D_n to obtain a digraph G_n with $\delta_{mon}(G_n) \geq d(n)$: Take a vertex $x \in D_n$ with $4d(n)$ elements. We add $n - b(n)$ copies of x to D_n such that every copy has the same out- and in-neighbours of each colour as x. The vertex x and its copies form an independent set of vertices.

The construction of G_n implies that $\delta_{mon}(G_n) \geq d(n)$, G_n is strongly connected and G_n has no alternating cycle, by the same reason as D_n. □

In order to prove an upper bound for $g(n)$ we need a result on hypergraph colouring. First we give some definitions. A *hypergraph* is a pair $H = (V, E)$, where V is a finite set whose elements are called vertices and E is a family of subsets of V called edges. A hypergraph is *k-uniform* if each of its edges has size k. We say that H is *2-colourable* if there is a 2-colouring of V such that no edge is monochromatic. The following result was proved by J. Beck [3].

Proposition 3.5 *There exists an absolute constant c such that any k-uniform hypergraph with at most $ck^{1/3}2^k$ edges is 2-colourable.*

Now we are ready to prove an upper bound for $g(n)$.

Theorem 3.6 *Let D=(V,A) be a 2-arc-coloured digraph on $|V| = n$ vertices. If $d_i^+(v) \geq \log n - 1/3 \log\log n + O(1)$ for every $i = 1, 2$ and $v \in V$, then D contains an alternating cycle.*

Proof: Without loss of generality assume that $d_i^+(v) = k$ for all $v \in V$ (k will be defined later), otherwise simply remove extra arcs. For each vertex $v \in V$ and each colour $i = 1, 2$, let

$$B_v^i = \{u \in V : \ (v, u) \ is \ an \ arc \ of \ colour \ i\}.$$

The size of each of the sets B_v^i is equal to k, thus they form a k-uniform hypergraph H with n vertices and $2n$ edges. Let $k = \log n - 1/3 \log\log n + b$, where b is a constant. Then it is easy to see that by choosing b large enough we get that $ck^{1/3}2^k > 2n$. By Proposition 3.5, our hypergraph H is 2-colourable. By taking a 2-colouring of H we get a partition $V = X \cup Y$ such that B_v^i intersects both X and Y for every $i = 1, 2$ and $v \in V$. Let D_1 be a subdigraph of D which contains only arcs of colour 1 from X to Y and arcs of colour 2 from Y to X. The outdegree of every vertex in D_1 is positive, since all sets B_v^i intersect both X and Y. Therefore D_1 contains a cycle, which is alternating by the construction of D_1. □

Remark. As pointed out to us by N. Alon a similar approach was used in [1].

It would be interesting to find better bounds for the functions $f(n)$ and $g(n)$ as well as to investigate these functions for tournaments.

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