PONTIFICIA UNIVERSIDAD CATÓLICA DEL PERÚ

FACULTAD DE CIENCIAS E INGENIERÍA

Revisión Teórica de Oscilaciones de Neutrinos

TRABAJO DE INVESTIGACIÓN PARA LA OBTENCIÓN DEL GRADO DE BACHILLERA EN CIENCIAS CON MENCIÓN EN FÍSICA

AUTOR

Alicia Pérez García

ASESOR

Alberto Gago Medina

Lima, enero, 2020

Abstract

Neutrinos play an important role in understanding nature's behavior, from the suggestion of their existence, to the experimental issues found in the way of trying to present evidence for it. The oscillation induced by mass is supported by overwhelming experimental evidence.

This work proposes a theoretical revision of the quantum mechanical description of neutrino oscillations, discussing the inconsistencies of the usual approaches and giving a more precise one. Moreover, the mechanism for oscillations in matter is studied with the purpose of finding the differential equation to solve for the evolution of the neutrino states.

For finding the evolved states, a code for solving the Schrödinger equation numerically was developed. The results were compared with the data from an already existing simulation software for neutrino experiments, allowing the validation of our solutions. Some steps following this work, such as the introduction of Non-Standard physics and predictions at future experiments, are described.

Contents

Chapter 1

Historical Review

To refer to the history of neutrinos is to refer to that of weak interactions, which starts with the discovery of the radioactivity of uranium, and two of its types of products, α and β particles, by Becquerel and Rutherford respectively. In 1914, Chadwick experimentally demonstrated the energy spectrum of electrons emitted in β-decay to be continuous. This fact was controversial, as the beta decay of a nucleus was thought to generate a single particle (an electron) and conservation of energy implies that this particle should have a well-defined energy. In an attempt to solve this problem, conservation of energy was questioned, with N. Bohr suggesting that energy could be conserved only in a statistical sense. However, Pauli proposed the existence of a weakly interacting fermion that was emitted in β-decay [1].

In 1933, Francis Perrin suggested that the new particle, now with the name of neutrino, needed to have a mass smaller than the one of the electron, a velocity close to the speed of light, and spin 1/2. That same year, Fermi formulated a theory on β-decay [2], establishing the process as:

$$n \to p + e^- + \bar{\nu} \tag{1.1}$$

After these contributions, few doubted the existence of the neutrino, but it was only observed in the 1950s by Reines and Cowan, with their measurement of inverse β-decay ($\bar{\nu} + p \to n + e^+$). For this purpose, they used the flux of anti-neutrinos from a nuclear reactor and 1400 liters of liquid scintillators, constituting the first reactor neutrino experiment.

There are three known neutrino flavors: the electron neutrino ν_e, observed by Reines and Cowan, the muon neutrino ν_μ, first observed in accelerator neutrino experiments (beams made from the reaction $\pi^+ \to \mu^+ + \nu_\mu$, and $\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu$), and the tau neutrino ν_τ, whose evidence is only inferred from the τ decay modes. In distinction with the other fermions, neutrinos are only sensible to weak interactions: a tiny fraction from a sample of neutrinos in a medium will interact with matter [3].

Apart from nuclear reactors and accelerators, neutrinos can also come from the Sun, atmospheric reactions or extragalactic sources. The Sun liberates its energy in nuclear fusion reactions taking place in the solar core, in a network of two-particle reactions, of which the most important one is the pp chain:

$$p + p \to {}^2\mathrm{H} + e^+ + \nu_e \tag{1.2}$$

We are also lead to other neutrino production reactions:

$$^{3}\mathrm{He} + {}^{4}\mathrm{He} \to {}^{7}\mathrm{Be} + \gamma \tag{1.3}$$

$$^{7}\mathrm{Be} + e \to {}^{7}\mathrm{Li} + \nu_e \tag{1.4}$$

$$^{7}\mathrm{Be} + p \to {}^{8}\mathrm{B} + \gamma \tag{1.5}$$

$$^{8}\mathrm{B} \to {}^{8}\mathrm{B} + e + \nu_e \tag{1.6}$$

This gives rise to the dominant energy generation mechanism in the Sun [4], and it represents a pure flux of electron neutrinos. However, the measurements made by experiments at the time did not give fluxes as great as the one predicted by the solar standard model for solar neutrinos [5]. This resulted in the so called Solar neutrino problem: the aforementioned experiments measured only electron flavor neutrinos and there was loss of this flux on the neutrinos way to Earth. For example, The SAGE and GALLEX experiments made measurements by making solar neutrinos react with gallium, according to:

$$\nu_e + {}^{71}\mathrm{Ga} \to {}^{71}\mathrm{Ge} + e^- \tag{1.7}$$

While the Kamiokande and Super-Kamiokande experiments used the reaction:

$$\nu_e + e^- \to \nu_e + e^- \tag{1.8}$$

with an energy threshold of 5 MeV in a water Cherenkov counter.

To solve the solar neutrino problem, many proposals were made, from doubting the correctness of the standard solar model, doubting the estimation of the cross section for the reactions in the experiments, to assuming that neutrinos could decay into new particles that would be invisible to detectors. As these scenarios are fairly unlikely, neutrino oscillations, along with its resonances due to the presence of a heavily dense medium, could also offer an explanation about the mechanism that causes the solar neutrino problem. Nevertheless, the difference between the measured and the predicted neutrino fluxes was solved by the SNO experiment. It was able to measure different neutrino flavors by using their reactions with a deuteron:

$$\nu_e + d \rightarrow p + p + e^- \tag{1.9}$$

$$\nu_x + d \rightarrow p + n + \nu_x \tag{1.10}$$

The total neutrino flux it obtained was in agreement with the predicted flux, even if the mechanism for this result is not clear [6].

Having formerly been unwanted background for experiments, atmospheric neutrinos have provided the first indication for neutrino oscillations. The mechanism for muon production in the atmosphere from Cosmic Rays is known and accepted. A charged particle from an extragalactic source arrives at the Earth's atmosphere and interacts with nuclei there, causing both an electromagnetic and a hadronic shower. From the latter, pions are produced and hey then decay, as mentioned in the accelerator experiments case. Thus, giving an expectation of a ratio of 2 to 1 between the number of ν_μ and the number of ν_e. Once again, the measured flux was not in agreement with the theoretical prediction, until Super-Kamiokande measured not only the energy of the neutrinos, but their direction, and, indirectly, the differences in their traveled path. It was found that the neutrino flavor transition was actually dependent on this length [7].

Based on these observations, the most widely accepted theory was that of a periodic change in neutrino flavor, induced by their mass differences. Following the $K_0 - \bar{K}_0$ particle system, Pontecorvo suggested that oscillations happened because the flavor eigenstates did not have a defined mass, but were a quantum superposition of mass states (the eigenstates of the corresponding evolution operator). Both bases were related by a rotation angle, θ given by nature, and this formulation gave rise to the usual approach to neutrino oscillations, which will be described in more detail in the following chapter.

Chapter 2

Description of Neutrino Oscillations

2.1 A first approach

To describe the change of a neutrino flavor as it propagates through space, it is necessary to notice first that flavor states are not energy (mass) eigenstates and therefore, are not the Evolution Operator eigenstates, so interactions cannot be understood directly from them. Nevertheless, the energy eigenstates constitute a basis and can be used to write a given flavor as a superposition of them:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i\rangle \tag{2.1}$$

$$|\nu_\alpha(t)\rangle = \sum_i U_{\alpha i} e^{\frac{iHt}{\hbar}} |\nu_i\rangle \tag{2.2}$$

The standard approach will usually apply the evolution operator on the mass eigenstates and obtain the energy for each one of them, which would be equal to the energy-momentum relation $E^2 = p^2 + m^2$. For example, for the particular case of two generation mixing, where U represents a rotation matrix that depends on a parameter from nature, θ:

$$U = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \tag{2.3}$$

Each mass state $|\nu_i\rangle$ has an energy E_i, and the evolved flavor state is:

$$|\nu_\alpha(t)\rangle = e^{-iE_1 t}\cos\theta\,|\nu_1\rangle + e^{-iE_2 t}\sin\theta\,|\nu_2\rangle \tag{2.4}$$

Then, the transition probability from the initial neutrino flavor α into a flavor β is:

$$P_{\nu_\alpha\to\nu_\beta} = \sin^2 2\theta \sin^2((E_2 - E_1)t) \tag{2.5}$$

The energies involved in the probability formula can be expressed using the relativistic approximation:

$$E = \sqrt{p^2 + m^2} \tag{2.6}$$

$$E \simeq p + \frac{m^2}{2p} \tag{2.7}$$

The use of this approximation is completely justified: the current boundaries for neutrino masses indicate that they are less than 1 eV and their energy needs to be higher than 100 keV to be detected. This bears a tiny ratio of $\frac{\Delta m^2}{p^2} \leq 10^{-10}$ [8].

By introducing this into Eq. (2.5) and making $t = \frac{L}{c}$, we obtain the master formula for two generations:

$$P_{\nu_\alpha\to\nu_\beta} = \sin^2 2\theta \sin^2(\frac{\Delta m^2}{4E}L) \tag{2.8}$$

For three generation oscillations, the mixing matrix becomes more complex:

$$U_{PMNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{CP}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{CP}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{CP}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{CP}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{CP}} & c_{23}c_{13} \end{pmatrix} \tag{2.9}$$

where $c_{mn} = cos\theta_{mn}$, $s_{mn} = sin\theta_{mn}$ and δ_{CP} is a CP-violating phase. Obtaining a master formula for three generation mixing becomes considerably more difficult.

Although the steps followed for obtaining the probability formulas work well and are in fact widely used experimentally, notation should be able to describe the space-time and energy-momentum degrees of freedom for a neutrino state, by factorizing the general states as described in [8]: considering that the aforementioned quantities are not certainly known for flavor states, only mass states can be written by defining the complete Hilbert space $\mathscr{H}$ as the product of a space corresponding to the momentum and another one, to the mass of the neutrino.

$$\mathscr{H} := \mathscr{H}_d \otimes \mathscr{H}_m \tag{2.10}$$

Thus, a general neutrino mass state $|\nu_i\rangle \in \mathscr{H}$ will have definite kinematical properties [9] and will be expressed as:

$$|\nu_i\rangle := |\nu_i^m\rangle \otimes |p_i\rangle \tag{2.11}$$

and a flavor state can be written as:

$$|\nu_\alpha\rangle = \sum_i U_{\alpha i} |\nu_i^m\rangle \otimes |p_i\rangle \tag{2.12}$$

The wave function for this initial neutrino flavor state is:

$$|\nu_\alpha(\vec{x})\rangle = \langle\vec{x}|\nu_\alpha\rangle = \sum_i U_{\alpha i} e^{i\vec{p}_i\cdot\vec{x}} |\nu_i^m\rangle \tag{2.13}$$

Now, we can apply the evolution operator to the state, and the neutrino flavor state for a time t and position $\vec{x}$ becomes:

$$|\nu_\alpha(t,\vec{x})\rangle = \sum_i U_{\alpha i} e^{-iE_i t} e^{i\vec{p}_i\cdot\vec{x}} |\nu_i^m\rangle = \sum_i U_{\alpha i} e^{ip_i x} |\nu_i^m\rangle \tag{2.14}$$

Finally, the transition probability of the initial flavor α into a flavor β, defined by $\left|\langle\nu_\beta|\nu_\alpha(t,\vec{x})\rangle\right|^2$, depends on the phase differences between mass states:

$$\Delta\phi_{ik} = \Delta E_{ik}\cdot t - \Delta\vec{p}_{ik}\cdot\vec{x} \tag{2.15}$$

2.2 Approximations and their accuracy

To evaluate Eq. (2.15), some assumptions are usually made, as will be described below. The analysis in the following sections will be based on the work in [10], summarizing and specifying some details.

1. Same momentum: By establishing $p_i = p_k = p$, the space dependence of the phase difference vanishes, and the oscillation probability depends only on time and energy. According to the relativistic approximation, Eq. (2.7),

we obtain:

$$\Delta\phi_{ik} \simeq \frac{\Delta m_{ik}^2}{2p} t \tag{2.16}$$

Now, the distance between source and detector is known much more accurately than the time of propagation in neutrino experiments. This is handled here by making the time to space conversion: as neutrinos are considered to be ultra-relativistic, $L \simeq t$ and

$$\Delta\phi_{ik} \simeq \frac{\Delta m_{ik}^2}{2p} L \tag{2.17}$$

2. Same energy: As neutrinos are created in weak interactions, they have a well-defined flavor at their source. Thus, it is only necessary to examine the behavior of a single energy state: oscillation probabilities can be found by evaluating a linear superposition of mass states with same energy and different momenta [11]. Following this argument, using the same energy for all mass states ($\Delta E = 0$) in Eq. (2.15) is also a common approach. It is also possible to consider $\vec{x}||\vec{p}$, as the distance from source to the detector is much larger than the transverse sizes [10]. By also applying an approximation analogous to Eq. (2.7), the oscillation phase is:

$$\Delta\phi_{ik} = -\Delta p_{ik} \cdot L \simeq \frac{\Delta m_{ik}^2}{2E} L \tag{2.18}$$

Inconsistencies can be found in the approaches, as several contradictions arise. It is first necessary to establish some aspects regarding the observability of oscillations in experiments:

a) The processes of neutrino production and detection are localized and have different coordinates in space, so we must be able to distinguish positions.

b) Oscillation amplitudes must depend on space coordinates.

c) The production and detection of neutrinos also happen at specific instants in time.

d) The description of oscillations is done by using quantum mechanics, which implies that the phenomenon is the result of momentum (space) and energy (time) uncertainties. Therefore, our description must account for them.

Let us first recall Eq. (2.16). The probability will not depend on any spatial coordinates, so same momentum assumption cannot define production and detection regions, contradicting aspect a). On the other hand, a time-only dependence of the phase difference could lead us to believe that detecting neutrinos, for example, at their source would be sufficient to observe oscillations. This contradicts the definition of flavors by the weak charged current and supports aspect b). Of course, the approach solves this by using the time to space conversion, for which it is necessary to have a classical velocity (for a point-like particle) [10].

Assuming that the neutrino mass states all have the same momentum means that it is well-defined -i.e., they are momentum eigenstates and their wave functions in momentum space are delta functions:

$$\psi_i(\vec{p}) = \delta(\vec{p} - \vec{p}_i) \tag{2.19}$$

In the position space, this wave function becomes:

$$\psi_i(\vec{x},t) = \frac{1}{(2\pi)^{\frac{3}{2}}} e^{i\vec{p}\cdot\vec{x} - iE_i t} \tag{2.20}$$

The same momentum assumption then allows us to consider the wave functions as plane waves with definite $\vec{p}_i$ [8]. A group velocity cannot be defined for plane waves, but it is the equivalent to a point-like particle velocity, used for the time to space conversion. Therefore, we arrive at an internal inconsistency of the approach.

Finally, plane waves do not account for energy-momentum spread, which contradicts aspect d). The oscillation phenomenon would not be possible with plane waves.

The same energy approach results in a phase difference that does not account for the time dependence of oscillations, and contradicts aspect c). Furthermore, it could be argued that same energy for the mass states can be a reality for certain Lorentz frames. Let us follow the logic in [12], and apply first order corrections to the relation $p_i = E_i = E$ by adding a term depending on a parameter ξ that depends on production conditions:

$$p_i = E_i - \xi \frac{m_i^2}{2E} \tag{2.21}$$

Then, Eq. (2.7) gives:

$$E_i \simeq E + (1-\xi)\frac{m_i^2}{2E} \tag{2.22}$$

Mass eigenstates with the same energy can then be obtained if we considered a boosted reference frame at a velocity v, say for the case where a decaying particle is not at rest, we can arrive at a parameter $\xi' = \xi'(\xi, v) = 1$. Nevertheless, this frame does not coincide with the laboratory frame, meaning it is not useful for calculations and predictions, and would depend on the energy of the decaying particle. In realistic conditions, particle beams are not monochromatic, and the non-existence of one single value for the energy means that the boost needed to arrive at the desired reference frame is not unique.

2.3 A more accurate treatment

As established in the previous section, plane waves cannot describe localized neutrino production and detection, or a single localized particle for that matter. In general, the latter is usually described by a wave packet: a superposition of plane waves. Additionally, a wave packet has a momentum spread σ_p around a central momentum $\vec{p}_0$, as the uncertainty relation implies that, if we have a localized state in space, then we can know its momentum only with an uncertainty $\sigma_p \gtrsim \frac{1}{\sigma_x}$ [1]. Just by examining the definition of wave packet, we see that it is in agreement with the aspects for oscillation observability, thus validating the use of wave packets for describing the states. Let us take a closer look a this formalism.

Using the wave packet approach, a particle of mass m_i is represented in the coordinate space by a wave function of the form:

$$\Psi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} f_{\vec{p}_0} \exp(i(\vec{p}\cdot\vec{x} - E_i(p)t)) \tag{2.23}$$

where $f_{\vec{p}_0}$ is the corresponding momentum distribution with a peak at $\vec{p}_0$, momentum spread σ_p and, of course, energy given by the dispersion relation, Eq. (2.6). Then, the evolved state for the neutrino flavor in Eq. (2.14) becomes:

$$|\nu_\alpha(t,\vec{x})\rangle = \sum_i U_{\alpha i}\Psi_i(t,\vec{x})\,|\nu_i\rangle \tag{2.24}$$

and any other flavor states $|\nu_\beta\rangle$ must also be described by a wave packet [10]. We now have to find the oscillation phase using the properties of a wave packet rather than any of the previously discussed approaches.

The individual waves that compose a wave packet each have a value for their respective momentum that is in fact close to the central value $\vec{p}_0$, determined by the production and detection processes. We can then, in general, expand the energy as:

$$E(\vec{p}_0+\Delta\vec{p}) = E(\vec{p}_0) + \frac{dE}{dp}\Delta p \tag{2.25}$$

Our state is a wave packet composed of a main wave and modulations to it, which have a velocity:

$$v_g = \frac{dE}{dp}\Big|_{p_0} \tag{2.26}$$

Proceeding as in [10], we can make a similar expansion for ΔE, considering that neutrinos are relativistic (the difference in energy of the mass states is small in comparison to the average values E), and by also taking into account the dependence of the energy on the mass of the neutrino state from Eq. (2.6):

$$\Delta E_{ik} = v_g \Delta p_{ik} + \frac{1}{2E}\Delta m^2_{ik} \tag{2.27}$$

Let us notice that this newly defined group velocity can be used for establishing relations of the form $x = vt$, as it is interpreted as the velocity of the localized particle described by our wave packet. Thus, allowing us to make use of time to space conversion $t = L/v_g$ with no contradictions. If we introduce Eq. (2.27) into the phase difference found at first, Eq. (2.15), we can obtain two possible new expressions for the latter:

$$\Delta\phi_{ik} = \Delta E_{ik} \cdot t - \Delta\vec{p}_{ik} \cdot \vec{x} = \Delta E_{ik} \cdot t - \Delta p_{ik} L \tag{2.28}$$

$$\Delta\phi_{ik} = (v_g t - L)\Delta p_{ik} + \frac{1}{2} E \Delta m_{ik}^2 \tag{2.29}$$

$$\Delta\phi_{ik} = \frac{1}{v_g}(v_g t - L)\Delta E_{ik} + \frac{L}{2p}\Delta m_{ik}^2 \tag{2.30}$$

Both Eq. (2.29) and Eq. (2.30) can have their first term on the right hand side vanish only for the case of the center of the wave packet, where $v_g t = L$. However, other positions cannot have a distance to the center of over σ_x to it:

$$\left| v_g t - L \right| \leq \sigma_x \tag{2.31}$$

and we can establish a condition for each expression so their first term vanishes:

$$\sigma_x \left| \Delta p_{ik} \right| \ll 1 \tag{2.32}$$

$$\frac{\sigma_x}{v_g} \left| \Delta E_{ik} \right| \ll 1 \tag{2.33}$$

Let us also recall that Heisenberg's uncertainty indicates that $\sigma_x \sim \frac{1}{\sigma_p}$, so our condition in Eq. (2.32) becomes:

$$\left| \Delta p_{ik} \right| \ll \sigma_p \tag{2.34}$$

From $E^2 = p^2 + m^2$, we can obtain $\sigma_E = \frac{p}{E}\sigma_p = v_g\sigma_p$. Then, the uncertainty principle also implies $\sigma_x \sim \frac{v_g}{\sigma_E}$, and the condition in Eq. (2.33) becomes:

$$|\Delta E_{ik}| \ll \sigma_E \tag{2.35}$$

If the relations in Eq. (2.34) and (2.35) hold for the mass states, the phase differences found with the same momentum and same energy assumptions are recovered.

2.4 Uncertainties and Coherence

It is not a coincidence that our conditions for oscillations are related to quantum uncertainties. In fact, they are the reason for the entire phenomenon to be observable, apart from just the use of quantum mechanics in the description.

Let us take a look at a pion decay, $\pi^+ \to \mu^+ \nu_\mu$, and follow the standard procedure of 4-momentum conservation ($P_\pi = P_\mu + P_\nu$). If we apply this for the case in which the pion is not at rest, so that we can express the mass of the neutrino in terms of the masses and momenta of the pion and muon. If the experiment to be performed were able to precisely measure the momenta, it would be able to determine the neutrino mass squared m_ν^2 and distinguish it from the other neutrino masses, as described below, following [13].

If we consider that the experiment will detect only a specific neutrino flavor, the rate of the events of a particular neutrino mass state will be proportional to its probability to trigger detection and the probability for a pion to decay in the muon channel. Therefore, the results will depend on how a neutrino flavor state is a superposition of mass states, but not vary with spatial coordinates: there will be no oscillation pattern.

This can be explained as follows: if the experiment can determine E and p with independent errors, the dispersion relation can be used to find an uncertainty for the neutrino mass squared:

$$\sigma_{m^2} = \sqrt{[2E\sigma_E]^2 + [2p\sigma_p]^2} \tag{2.36}$$

For a mass state to be identified, it is necessary that $\Delta m^2 > \sigma_{m^2}$. For this to hold, we see from Eq. (2.36) that we need:

$$2p\sigma_p < \Delta m^2 \tag{2.37}$$

and by using the uncertainty principle, we arrive at:

$$\sigma_x > \frac{2p}{\Delta m^2} \tag{2.38}$$

Before continuing, we need to define a relevant quantity: the oscillation length. Eq. (2.14) can be expressed using Eq. (2.7) as:

$$|\nu_\alpha(t,\vec{x})\rangle = \sum_i U_{\alpha i} e^{-i\frac{m_i^2}{2p}t} |\nu_i^m\rangle \tag{2.39}$$

The probability of measuring a state given by Eq.(2.13) is then [13]:

$$P_{\nu_\alpha \to \nu_\beta} = \sum_i U_{\alpha i}^2 U_{\beta i}^2 + \sum_{i \neq j} U_{\alpha i} U_{\beta i}^* U_{\alpha j}^* U_{\beta j} \cos(x\frac{\Delta m^2}{2p}) \tag{2.40}$$

As the periodicity of the cosine is given by 2π, we can write the term as

$cos(2\pi\frac{x}{l_{osc}})$, where

$$l_{osc} = 2\pi\frac{2p}{\Delta m^2} \tag{2.41}$$

is the oscillation length for our probability. Except for a factor of 2π, this is precisely the condition stated in Eq. (2.38).

We can then interpret the disappearance of oscillations with precise momentum measurements. As the pion momentum is more accurately defined, its position will be more undetermined and the neutrino production will be delocalized [13]. When the pion momentum is measured with enough precision such that σ_{m^2} is less than all Δm^2, the uncertainty in the coordinate of the production point exceeds the oscillation length and oscillations are averaged.

Even though this entire argument was made by taking into account only the production of a neutrino, it can be similarly done for neutrino detection. Thus, we have arrived at a condition for the observability of oscillations: we must not be able to distinguish what mass state has been produced or detected. This condition is satisfied only if neutrino production and detection have their spatial coordinates well-defined (they have small values for σ_x).

It could be argued that this violates energy-momentum conservation. However, given that neutrinos (and their processes) have spatial and time coordinates with their respective uncertainties, so do their energy and momentum values. Then, the states that represent our particles are not exactly eigenstates of energy and momentum, but this does not imply the 4-momentum is not conserved [10].

Another formulation for the coherence conditions can be done in configuration space and arrive at conditions that are in agreement with the discussion so far.

For that, we focus on the fluctuations of the oscillation phase, which must be small as not to average the probability over the oscillation phase:

$$|\delta\phi| = |\Delta E \cdot \delta t - \Delta\vec{p} \cdot \delta\vec{x}| \ll 1 \tag{2.42}$$

If we assume that in no frame will the terms in Eq. (2.42) cancel (or rather if we are not looking to get Lorentz invariant conditions), both terms need to be small on their own. Furthermore, the fluctuations in, for example, production position and time are, at most, equal to the uncertainties of those quantities, $\delta t \lesssim \sigma_t$ and $|\delta\vec{x}| \lesssim \sigma_x$. All these relations finally arrive at [10]:

$$|\Delta E| \ll \sigma_E \tag{2.43}$$

$$|\Delta p| \ll \sigma_p \tag{2.44}$$

These are the exact same conditions assumed for evaluating the oscillation phase in the wave packet approach, and support the role of the quantum uncertainties in observing oscillations.

While we have established that coherence in production and detection is needed for neutrino oscillations to be observed, this condition is not sufficient by itself. For a given momentum, the waves for each mass state will travel at different speeds v_g, as they depend on the respective mass, resulting in the separation of their centers. When the waves do not overlap, the mass states cannot interfere and produce oscillations [13] and so, coherence is lost.

However, the waves will maintain coherence while travelling some distance in space, called the coherence length. Its value can be deduced logically from the conditions already mentioned. Let us consider an average group velocity for the mass states, v_g, and the average distance traveled by the different waves, $l = v_g t$. The separation between mass states after a time t will be given by $\Delta l = \Delta v_g t$, and for there to be interference of the states, this separation must be less than the wave spatial spread σ_x, or at most, equal to it after a time $t_{coh} = \frac{l_{coh}}{v_g}$. As a result of these equations, the coherence length is found to be:

$$l_{coh} \simeq \frac{v_g}{\left|\Delta v_g\right|}\sigma_x \tag{2.45}$$

Considering ultra relativistic neutrinos, we can use $p \sim E$ and $v_g \simeq 1$. Also, by considering an average energy, $\Delta v_g \simeq \frac{\Delta p}{E} \simeq \frac{\Delta m^2}{2E^2}$, and the coherence length is:

$$l_{coh} \simeq \frac{2E^2}{|\Delta m^2|}\sigma_x \tag{2.46}$$

Neutrino oscillations can be observed only for distances traveled that satisfy $L \ll l_{coh}$, which is in fact very large [10]. If this condition is not met, the detection process will be able to distinguish each mass state wave as they arrive at different times. However, depending on the characteristics of the detection, coherence could be restored.

If the detection process takes longer than the time it takes all the wave packets to arrive to the detection point, there may still be a coherent event. Hence, the coherence length must have a dependence on the time resolution of the detector [14]. From $l = v_g t$, we can obtain $\sigma_x = v_g \sigma_t = v_g/\sigma_E$, interpreted as an effective length of the wave packet by taking into account the uncertainties of both production and detection [10]. Then, for example, if the energy measurement in the experiment became extremely accurate, the effective length of the wave packet would be infinite, and so would the coherence length: the propagation of the neutrino mass states would not cause the loss of coherence.

2.5 Summarizing the Coherence Conditions

Neutrino oscillations are observable only if the neutrino flavor state is a coherent superposition the mass states at all times: production, propagation and detection of neutrinos are coherent. There is, however, a possible contradiction among the conditions this sets.

Going back to the example of a very accurate measurement of energy in the detection process, it would not allow the condition in Eq. (2.43) to be satisfied. Furthermore, it would imply an infinitely large uncertainty in time, so the instant for the detection is completely undefined, contradicting the argument against the same energy approach.

Let us try and evaluate how both coherence conditions could work together. They can be expressed as:

$$\Delta E \sim \frac{\Delta m^2}{2E} \ll \sigma_E \tag{2.47}$$

$$\frac{\Delta m^2}{2E^2} L \ll \frac{v_g}{\sigma_E} \tag{2.48}$$

and would represent limits for σ_E [10]:

$$\frac{\Delta m^2}{2E} \ll \sigma_E \ll \frac{2E^2}{\Delta m^2 L} \tag{2.49}$$

The stated relation between the limits will work as long as Δm^2 is small (this explains why neutrinos oscillate, while charged leptons do not). Moreover, two conditions can be obtained:

$$2\pi \frac{L}{l_{osc}} \ll \frac{2E^2}{\Delta m^2} \tag{2.50}$$

$$(\frac{L}{l_{osc}}) \ll \frac{E}{2\pi\sigma_E} \tag{2.51}$$

The first one ensures the compatibility of the conditions, while the second one represents a fulfillment condition. Experimentally, σ_E is actually the energy resolution of the detector, which is larger. This establishes conditions to be considered in experiment design, such as the maximum number of observable oscillations, $\frac{l_{coh}}{l_{osc}}$, and the baseline.

2.6 Closing Remarks on the formalism

The conditions for neutrino oscillations to occur and be observable discussed so far have a dependency on the production and detection processes. For example, in Eq. (2.23), we find a dependence on a momentum distribution, which will be different for the neutrino states in production and detection. On the other hand, a flavor state needs to be normalized and for that to be ensured, they need to satisfy:

$$\int \frac{d^3p}{(2\pi)^3} \left| f^P_{\vec{p}_0} \right|^2 \left| f^D_{\vec{p}_0} \right|^2 = 1 \tag{2.52}$$

So, for probabilities to even make sense, the production and detection processes need to be taken into account. However, oscillations are a phenomenon in nature, and not induced from experimental conditions, ergo the oscillation probabilities found with the formalism have to be universal.

Of course, measurable quantities do need to take into account the experimental aspects of production, propagation and detection, and they can be represented by the probability of the complete process. If this probability can be expressed as factors of flux (production), interaction probability (detection) and the experiment-independent probability, then we could be able to obtain universal probabilities [10]. This type of factorization will only be possible if all three aspects are independent of each other. Propagation and detection of the neutrino states will not be dependent of each other. They will also not depend on the production process if the latter generates neutrinos with the same kinematics, which happens if the mass of different states does not affect the momentum. This can all be reduced to having the production process not be able to discriminate between mass states, i.e, production must be coherent. On that note, propagation and detection must also not distinguish masses, so the already established conditions ensure independence for experiments as well.

Finally, universality also implies that the probabilities do not depend on the used frame of reference. The phase difference that defines the probability explicitly includes the relation $\frac{L}{E}$ and, it can be shown that the relation remains invariant. Following the treatment found in [15], let us assume an inertial reference frame $\mathscr{O}'$ moving with velocity v in the x direction with respect to another reference frame $\mathscr{O}$, and recall the Lorentz transformations for space and time:

$$\Delta x' = \gamma(\Delta x - vt) \tag{2.53}$$

$$\Delta t' = \gamma(-v\Delta x + \Delta t) \tag{2.54}$$

where the parameter $\gamma = 1/(1-v^2)$. The distance between the neutrino source and the detector measured in $\mathscr{O}'$ is:

$$L' = \frac{L}{\gamma} \tag{2.55}$$

and the transformations for momentum and energy are:

$$p' = \gamma(p - vE) \tag{2.56}$$

$$E' = \gamma(-vp + E) \tag{2.57}$$

In the ultra-relativistic limit,

$$E' = p' = \gamma(1-v)E \tag{2.58}$$

Evidently, L' and E' will not result in probabilities equal to the ones using L and E. However, for obtaining our phase difference, we have used the approximation of $L = t$. The parameter L does not represent the distance between the source and detector, but rather the distance traveled by the neutrino. On one hand, we have for $\mathscr{O}$, $\Delta x = \Delta t$, and for $\mathscr{O}'$:

$$\Delta x' = \Delta t' = \gamma(1-v)\Delta x \tag{2.59}$$

Then, the correct transformation for L is:

$$L' = \gamma(1-v)L \tag{2.60}$$

The relation L'/E' can now be seen to be equal to L/E [15]. That being the case, probabilities now can be considered as universal.

Chapter 3

Oscillations in Matter

So far, we have established the acceptable formalism for neutrino oscillations and found arguments to support the more simple approaches, as the necessary conditions are usually satisfied. This has been done by only taking into account the properties of the particles themselves, without any additional potentials from the medium they are in -i.e., we have considered neutrino oscillations in vacuum.

According to the Standard Model, neutrinos are have no mass and are left-handed, which allows them to only interact through weak force. On the other hand, we have so far assumed that the reason for the oscillation phenomenon to occur is that neutrinos are not actually massless, giving us a hint for new physics, which will be studied in future work. For the purpose of this work, we will consider only the standard interactions via weak force, through either charged or neutral currents. Even though their interaction rate is relatively low, oscillations can be affected when considering neutrinos in a medium, in the presence of nucleons and electrons. The introduction of the interactions in the theory to be used will be done by obtaining them from electroweak theory, to

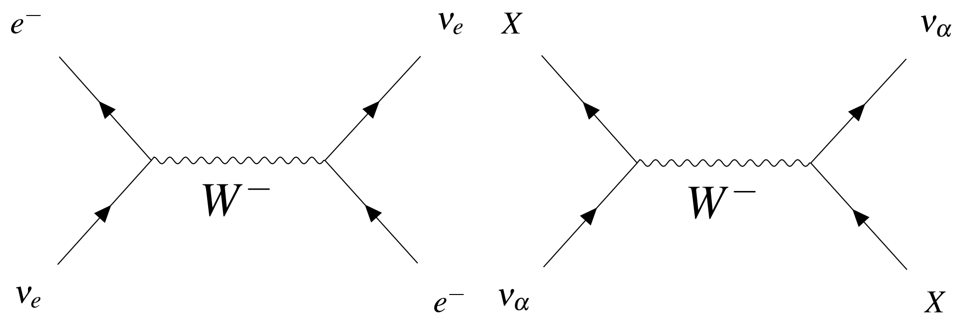

Figure 3.1: Feynman diagrams of Coherent Forward Scattering processes through charged (left) and neutral (right) current. X represents and electron or a nucleon, and ν_α, any flavor of neutrino.

ensure a correct understanding of the phenomenon, and following the procedure in [1].

Specifically, the processes that will affect neutrinos in a medium are coherent forward elastic scattering, where the other interacting particle will recoil as a whole (Fig. (3.1)), and incoherent scattering. However, the latter is a small fraction of the total scattering events, as the neutrino mean free path considering in normal matter can be demonstrated to be about 0.1 light years [1]. Thus, we can neglect this process.

3.1 Evolution Equation

The number of events of coherent interactions will not be vanishingly small and they will affect the neutrino flavor states evolution, as they add an effective potential term to the energy. Let us recall the Schrödinger equation:

$$i\frac{d}{dx}\nu = \mathscr{H}\nu \tag{3.1}$$

For our case, $\mathscr{H}$ is the total Hamiltonian, and includes matter effects as:

$$\mathscr{H} = \mathscr{H}_0 + \mathscr{H}_I \tag{3.2}$$

where $\mathscr{H}_I |\nu_\alpha\rangle = V_\alpha |\nu_\alpha\rangle$. The expression for V_α can be found from the formulation of the interaction in electroweak theory. For the case of the charged current shown in Fig. (3.1), the effective Hamiltonian is:

$$\mathscr{H}^{CC}(x) = \frac{G_F}{\sqrt{2}}[\overline{\nu_e}(x)\gamma^\rho(1-\gamma^5)e(x)][\overline{e}(x)\gamma_\rho(1-\gamma^5)\nu_e(x)] \tag{3.3}$$

After being transformed conveniently and averaged over the electron background:

$$\overline{\mathscr{H}}^{CC}(x) = V_{CC}\overline{\nu_{e,L}}(x)\gamma^0\nu_{e,L}(x) \tag{3.4}$$

where $V_{CC} = \sqrt{2}G_F N_e$. Similarly, the effective Hamiltonian for the neutral current interaction is:

$$\mathscr{H}^{NC}(x) = \frac{G_F}{\sqrt{2}}\sum_\alpha[\overline{\nu_\alpha}(x)\gamma^\rho(1-\gamma^5)\nu_\alpha(x)]\sum_f[\overline{f}(x)\gamma_\rho(g_\nu^f - g_A^f\gamma^5)f(x)] \tag{3.5}$$

By comparing with our previous equations, we arrive at the potential for neutral current $V_{NC}^f = \sqrt{2}G_F N^f g_V^f$ for each particle represented by f [1], where g_V^f are real dimensionless parameters for the couplings of the Z boson to fermions [16]. The sums over neutrino flavors α and particles f appear because the interaction is not limited to only electrons and electron neutrinos. However, this also implies that the potentials produced by interactions with protons and electrons will cancel out, as their values for g_V^f are the same, but with opposite signs. Therefore, the potential will be reduced to that of the interaction between neutrinos and neutrons, and we arrive at:

$$V_{NC} = -\frac{1}{2}\sqrt{2}G_F N_n \tag{3.6}$$

Finally, the potentials can be summarized as [1]:

$$\overline{\mathscr{H}}_{eff}(x) = \sum_{\alpha=e,\mu,\tau} V_\alpha \overline{\nu_{\alpha,L}}(x)\gamma^0 \nu_{\alpha,L}(x) \tag{3.7}$$

$$V_\alpha = V_{CC}\delta_{\alpha e} + V_{NC} \tag{3.8}$$

We can now express the total Hamiltonian in matrix form, taking into account that the eigenstates of $\mathscr{H}_0$ are the neutrino mass states, while the eigenstates of $\mathscr{H}_I$ are the flavor states. On top of that, we will express the $\mathscr{H}_0$ eigenvalues as the relativistic approximation for the energy, $E = p + \frac{m^2}{2E}$. Our total $\mathscr{H}$ then will be, in the flavor basis:

$$\mathscr{H} = U \begin{pmatrix} p + \frac{m_1^2}{2E} & 0 & 0 \\ 0 & p + \frac{m_2^2}{2E} & 0 \\ 0 & 0 & p + \frac{m_3^2}{2E} \end{pmatrix} U^\dagger + \begin{pmatrix} V_{CC} + V_{NC} & 0 & 0 \\ 0 & V_{NC} & 0 \\ 0 & 0 & V_{NC} \end{pmatrix} \tag{3.9}$$

The terms with V_{NC} in $\mathscr{H}_I$ will only add a global phase to any flavor state we will be evaluating, and so will the term $p + \frac{m_1^2}{2E}$ from $\mathscr{H}_0$. Then, we can take the terms out and finally obtain:

$$\mathscr{H} = \frac{1}{2E}(U\mathscr{M}^2 U^\dagger + A) \tag{3.10}$$

where U is the PMNS matrix and $\mathscr{M}^2$ and A are defined as [1]:

$$\mathscr{H} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Delta m_{21}^2 & 0 \\ 0 & 0 & \Delta m_{31}^2 \end{pmatrix} \tag{3.11}$$

$$A = \begin{pmatrix} A_{CC} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{3.12}$$

with $A_{CC} \equiv 2EV_{CC} = 2\sqrt{2}EG_F N_e$.

For the case of two generation oscillations, for example, the new term in the Hamiltonian will modify the sole mixing angle considered in vacuum oscillations, so we can substitute the variables as:

$$\nu_1, \nu_2 \to \nu_{1,m}, \nu_{2,m} \tag{3.13}$$

$$\frac{m_1^2}{2E}, \frac{m_2^2}{2E} \to \frac{m_{1,M}^2}{2E}, \frac{m_{2,M}^2}{2E} \tag{3.14}$$

$$\theta \to \theta_m \tag{3.15}$$

The new variables are defined as:

$$\Delta m_M^2 = \sqrt{(\Delta m^2 \cos(2\theta) - Acc)^2 + (\Delta m^2 \sin(2\theta))^2} \tag{3.16}$$

$$\sin(2\theta_M) = \frac{\Delta m^2 \sin(2\theta)}{\Delta m_M^2} \tag{3.17}$$

and the oscillation probability will now depend on $\sin^2(2\theta_m)$ [1].

3.2 Mikheyev-Smirnov-Wolfenstein Effect

We have established that neutrino flavor conversion is a product of coherent mixtures of mass eigenstates, which will have different phases when evolving, and this relative phase will produce interference. We have also established that, when propagating in normal media, the Hamiltonian for the neutrino flavors will depend on the effective potential due only to charged current interactions. Thus, adding a term to our phase difference $\Delta\phi_{matter} = (V_\alpha - V_\beta)t$ and a dependence on the matter density and neutrino energy, according to Eq. (3.12).

Analogous to the oscillation length defined in Eq. (2.41), we can define a refraction length, l_0 from the matter term in the phase difference, equal to the distance in which the term becomes 2π [17]:

$$l_o = \frac{2\pi}{V_\alpha - V_\beta} \tag{3.18}$$

As seen from Eq. (3.17), our oscillation parameter has a dependence on the ratio of the oscillation and refraction lengths [17]:

$$x \equiv \frac{l_{osc}}{l_o} = \frac{2EV}{\Delta m^2} \tag{3.19}$$

and it has a resonance for $\frac{l_{osc}}{l_o} = \cos 2\theta$, making flavor mixing maximal. With this new parameter x we can find a resonance density:

$$N_e^R = \frac{\Delta m^2 \cos 2\theta}{2\sqrt{2} E G_F} \tag{3.20}$$

The effective mixing angle then gives $\theta_m = \pi/4$ and the effective mass squared difference Δm^2 reaches a minimum. At this resonance, the initial flavor can completely disappear [1]. This is the Mikheyev, Smirnov, Wolfenstein (MSW) Effect and it evidences the difference between oscillations in vacuum and matter as follows.

For media with constant matter density, enhanced transitions are produced. As the new parameters for the probabilities will still be fixed with a certain value of N_e, their oscillatory behavior observed in vacuum remains [17]. However, the effective potential for normal matter is positive and the resonance density in Eq. (3.20) can only happen if $\theta < \pi/4$. Then, the probability is not symmetric if we change θ for $\pi/2 - \theta$, as it is in vacuum [1].

When neutrinos propagate in a medium whose density changes on their way, N_e depends also on the spatial coordinate, and so will the new mixing angle $\theta_m(x)$. This leads to having the eigenstates of the Hamiltonian also change with the propagation. Considering adiabatic interactions, where the rate of change of the effective mixing angle in space is small, allows us to neglect these changes in eigenstates and oscillations still occur. Nevertheless, probabilities will differ: we now need to take into account the dependence of our parameters on the coordinate for the differential equation. Oscillations will then be produced by both adiabatic conversion and change of phase [17]. The MSW Effect can then also be interpreted as the adiabatic neutrino flavor transition in a medium of varying density (following this reasoning, it has been used to explain the oscillation of massless neutrinos).

A particular case of varying density is that in which it can be approximated by having the mass states propagate in tiny regions (slabs) of constant density, picking up the corresponding phase, which takes into account its width Δx. This translates into having to apply the evolution operator for each slab succesively to the initial neutrino state. This approach can lead us to clearly notice the effect of parametric resonance in the oscillations, which occurs if the matter effective potential varies periodically along the neutrino path [1].

Chapter 4

Numerical Calculation of Oscillation Probabilities

After solving the Schrödinger equation for the evolution of neutrino flavor states, the probabilities for flavor transitions can be obtained. For this work, a code for solving the equation numerically has been developed. It takes into account matter effects and a medium with constant density. An already existent package for neutrino experiments (GLoBES) has also been used, for comparing the solutions.

4.1 Developed Code

The code for finding the numerical solution for the Schrödinger equation from Chapter 3 was developed using Python. The neutrino flavor states were defined as vectors with two or three complex components each, and the effective Hamiltonians, as 2x2 or 3x3 matrices with complex elements, for oscillations in two or three generations respectively. The values used for the parameters θ_{ik}

and $\delta_{\rm CP}$ for the PMNS matrix, and Δm^2_{ik} for the Hamiltonian term, $\mathscr{H}_0$, were the latest best-fit updates in NU-fit [18]. The electron density in matter was found with a function describing the density profile of the Earth, as used for the evaluation of ultrahigh-energy neutrino interactions [19]. The fraction of electrons in normal matter is taken to be ~ 0.494.

The method for solving the initial value problem was implicit, as our "vector components" are not independent. A function (solve_ivp) that contains Backward Differentiation Formulas (BDF), already implemented in the SciPy library [20], was used. Several relative and absolute tolerances were tried out as conditions for convergence of the solution, with different orders of magnitude (the relative error manages the number of correct digits in the result). The results for different values of tolerance and initial flavor states are discussed in section 4.3.

4.2 General Long Baseline Experiment Simulator - GLoBES

GLoBES is a simulation software package for short and long baseline neutrino oscillation experiments [21]. It allows the description of experiments through a newly-defined language, and processing data for obtaining oscillation parameters. While the main purpose of GLoBES is the calculation of χ^2 and its projections onto certain subspaces of parameters, we will focus on its feature of the calculating the oscillation probabilities. For this, the software takes a set of input data, which can include the neutrino flavors for evaluating the probabilities, matter density and certain experiments profiles. The details on the

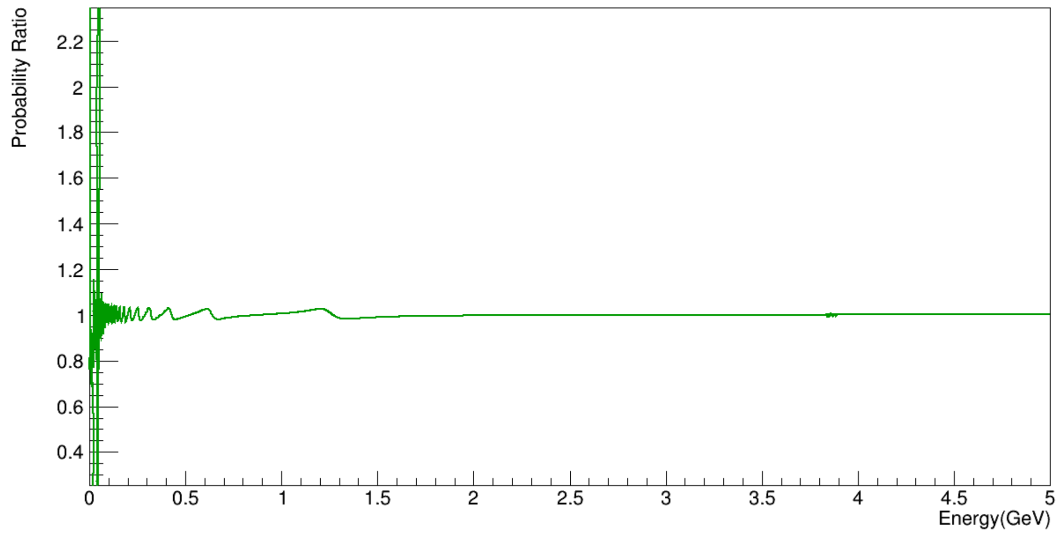

Figure 4.1: Probability ratio for $\nu_\mu \to \nu_e$ process in vacuum.

functions for the calculation of probabilities can be found in [22].

The results from the package were used to validate the developed code. By working with the same initial states and energy values, curves that could be compared were obtained.

4.3 Results

For each case, we studied the behavior of the curves obtained with different tolerances with respect to the results of GLoBES. The values used for the tolerance were 10^{-2} and 10^{-3}, as lesser relative errors resulted in irrelevant data (the values resulted in zero or infinity). In every case, there were no major differences between data obtained with each of the relative errors. Therefore, the graphs shown correspond to the ratio of the probabilities given by our code and GLoBES, using 10^{-3} as tolerance.

The future for this work is the evaluation of experiments from the oscillation

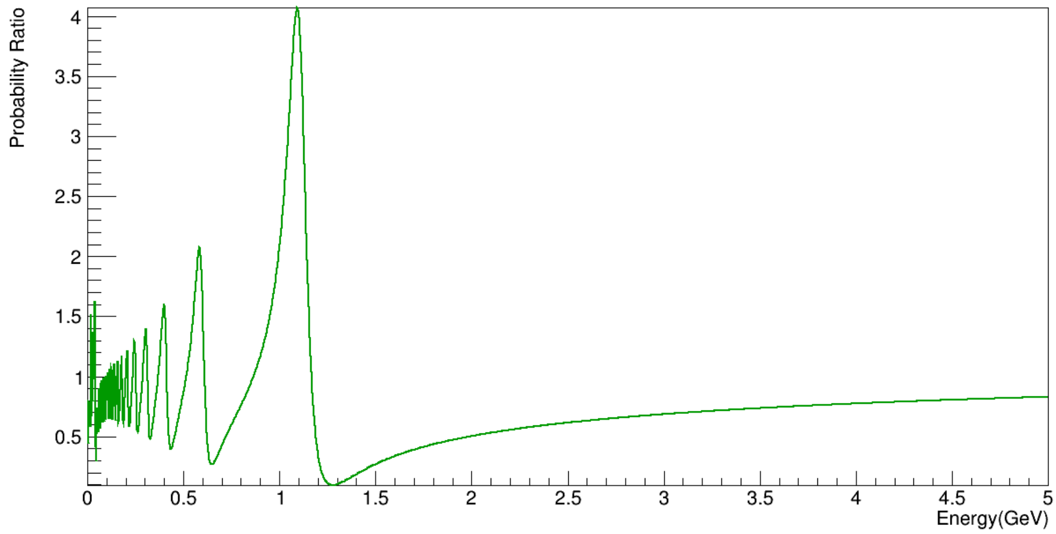

Figure 4.2: Probability ratio for $\nu_\mu \to \nu_e$ process in matter.

probabilities. As neutrino beams are composed of muon neutrinos resulting from the pion decay and are mostly detected as ν_e, we show the relevant case of the transition $\nu_\mu \to \nu_e$ in vacuum and matter, in Fig. (4.1) and (4.2).

The curves for oscillations, in general, describe a similar behavior, as they follow the same pattern. Probabilities fit particularly well for energies in the order of GeV in vacuum. However, they do differ considerably in matter at energies of ~ 1 GeV. This could be product of the chosen method of integration, as it could make the solutions follow a different function.

Experiments also use atmospheric neutrinos to evaluate oscillations, so the process $\nu_e \to \nu_e$ is relevant for the study of the developed code. The results are shown in Fig. (4.3) and (4.4). In contrast to the probability of appearance of ν_e with initial flavor ν_μ, the curves for the survival of ν_e fit the data from GLoBES both in vacuum and matter very well.

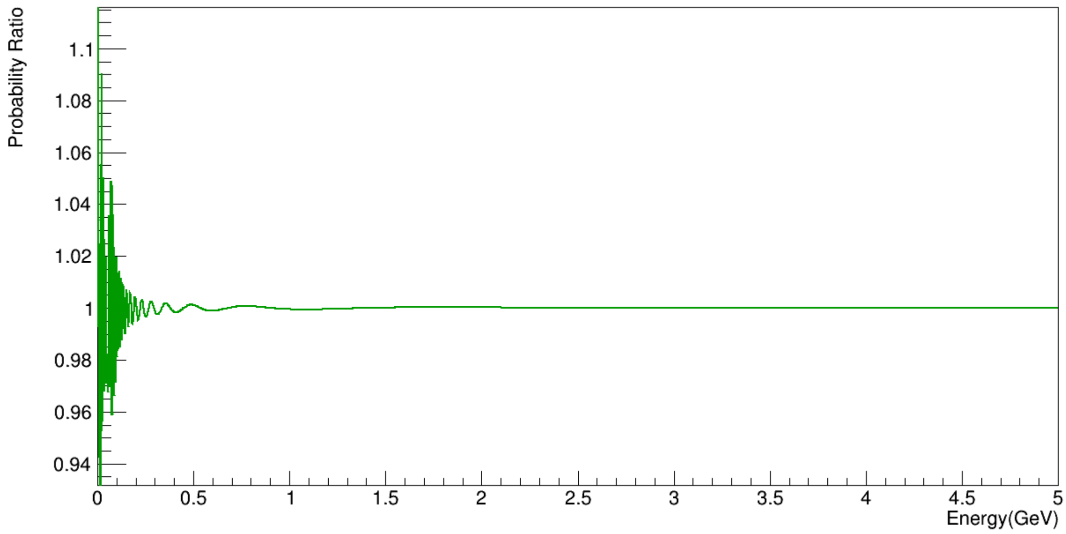

Figure 4.3: Probability ratio for $\nu_e \rightarrow \nu_e$ process in vacuum.

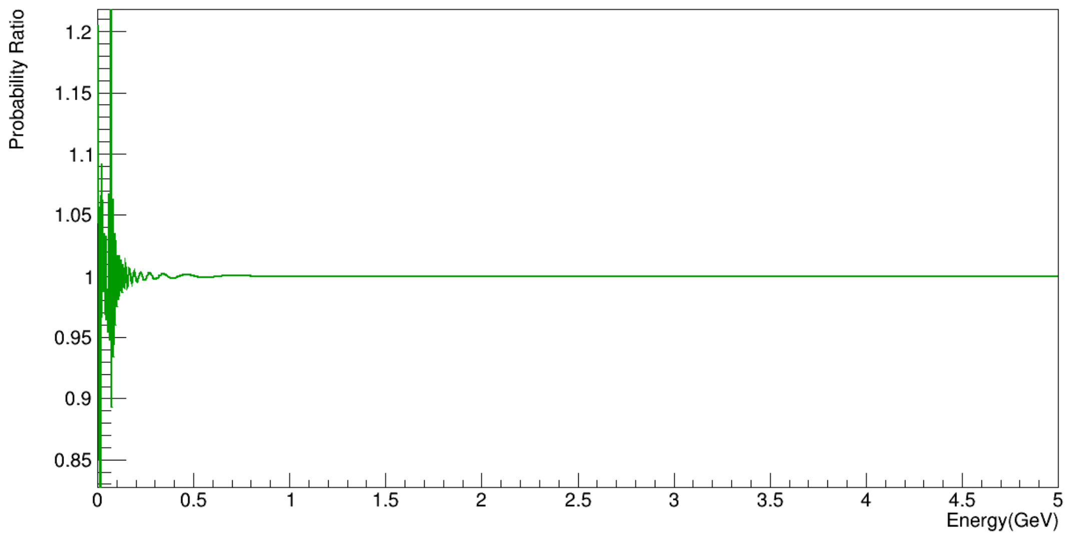

Figure 4.4: Probability ratio for $\nu_e \rightarrow \nu_e$ process in matter.

4.4 Future work

A further study of the numerical method is required for finding the reason behind the difference in the values for ν_e appearance in matter. Our results for the oscillations probability need to be compared also with analytical solutions to the Schrödinger equation. The long-term goal of the project in progress is the introduction of new physics into the neutrino oscillations equations, which can produce a distortion in the prediction of nature's parameters with respect to experimental data, specifically obtained at DUNE (Deep Underground Neutrino Experiment). Thus, our immediate next step is to compare our results with the formula used in DUNE [23].

Afterwards, a review on non-standard physics will be necessary, to allow us to add new effects to the known Hamiltonian. Non-standard neutrino interactions will be added using an effective theory. Changes in the oscillation pattern will be analyzed, along with their correlation with energy, and matter and neutrino properties.

Bibliography

[1] C. GIUNTI, W. CHUNG, 2007. "Fundamentals of neutrino physics and astrophysics". *Fundamentals of Neutrino Physics and Astrophysics, by Carlo Giunti and W. Kim Chung. ISBN 978-0-19-850871-7 (HB). Published by Oxford University Press, Oxford, Great Britain, 2007.* .

[2] V. NESVIZHEVSKY, J. VILLAIN, 2017. "The discovery of the neutron and its consequences (19301940)". *Comptes Rendus Physique 18*(9):592 . ISSN 1631-0705. Science in the making: The Comptes rendus de lAcadmie des sciences throughout history, URL `http://www.sciencedirect.com/science/article/pii/S1631070517300993`.

[3] H.-K. QUANG, X.-Y. PHAM, 1998. Elementary particles and their interactions: concepts and phenomena. ISBN 9783642083495. URL

[4] M. ALTMANN, ET AL., 2001. "Solar neutrinos". *Rep. Prog. Phys 64*:97.

[5] A. DAR, G. SHAVIV, 1999. "The Solar neutrino problem: An Update". *Phys. Rept. 311*:115. `astro-ph/9808098`.

[6] C. GRUPEN, 2005. Astroparticle Physics. Springer. URL `https://www.xarg.org/ref/a/B001089JM6/`.

[7] Y. FUKUDA, ET AL. (Super-Kamiokande), 1998. "Evidence for oscillation of atmospheric neutrinos". *Phys. Rev. Lett. 81*:1562. hep-ex/9807003.

[8] D. KRUPPKE, 2007. On Theories of Neutrino Oscillations: A Summary and Characterisation of the Problematic Aspects. Tesis Doctoral, Bielefeld U. URL http://www2.physik.uni-bielefeld.de/fileadmin/user_upload /theory_e6/Diploma$_T$*heses/dipl_kruppke.pdf*.

[9] C. GIUNTI, C. W. KIM, 2001. "Quantum mechanics of neutrino oscillations". *Found. Phys. Lett. 14*(3):213. hep-ph/0011074.

[10] E. AKHMEDOV, 2019. "Quantum mechanics aspects and subtleties of neutrino oscillations". En "International Conference on History of the Neutrino: 1930-2018 Paris, France, September 5-7, 2018", 1901.05232.

[11] Y. GROSSMAN, H. J. LIPKIN, 1997. "Flavor oscillations from a spatially localized source: A Simple general treatment". *Phys. Rev. D55*:2760. hep-ph/9607201.

[12] C. GIUNTI, 2001. "Energy and momentum of oscillating neutrinos". *Mod. Phys. Lett. A16*:2363. hep-ph/0104148.

[13] B. KAYSER, 1981. "On the Quantum Mechanics of Neutrino Oscillation". *Phys. Rev. D24*:110.

[14] K. KIERS, N. WEISS, 1998. "Neutrino oscillations in a model with a source and detector". *Phys. Rev. D57*:3091. hep-ph/9710289.

[15] C. GIUNTI, 2004. "Lorentz invariance of neutrino oscillations". *American Journal of Physics 72*(5):699. https://doi.org/10.1119/1.1643372, URL https://doi.org/10.1119/1.1643372.

[16] M. TANABASHI, ET AL. (Particle Data Group), 2018. "Review of particle physics". *Phys. Rev. D 98*:030001. URL https://link.aps.org/doi/10.1103/PhysRevD.98.030001.

[17] A. YU. SMIRNOV, 2003. "The MSW effect and solar neutrinos". En "Neutrino telescopes. Proceedings, 10th International Workshop, Venice, Italy, March 11-14, 2003. Vol. 1+2", págs. 23–43. hep-ph/0305106.

[18] I. ESTEBAN, ET AL., 2019. "Global analysis of three-flavour neutrino oscillations: synergies and tensions in the determination of θ_{23}, δ_{CP}, and the mass ordering". *JHEP 01*:106. 1811.05487.

[19] R. GANDHI, ET AL., 1996. "Ultrahigh-energy neutrino interactions". *Astropart. Phys. 5*:81. hep-ph/9512364.

[20] E. JONES, ET AL., 2001–. "SciPy: Open source scientific tools for Python". [Online; accessed], URL http://www.scipy.org/.

[21] P. HUBER, ET AL., 2007. "New features in the simulation of neutrino oscillation experiments with GLoBES 3.0: General Long Baseline Experiment Simulator". *Comput. Phys. Commun. 177*:432. hep-ph/0701187.

[22] P. HUBER, ET AL., 2005. "Simulation of long-baseline neutrino oscillation experiments with GLoBES (General Long Baseline Experiment Simulator)". *Comput. Phys. Commun. 167*:195. hep-ph/0407333.

[23] R. ACCIARRI, ET AL. (DUNE), 2015. "Long-Baseline Neutrino Facility (LBNF) and Deep Underground Neutrino Experiment (DUNE)" `1512.06148`.