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Fractions and Pythagorean Tuning—An Interdisciplinary Study in Secondary Education

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Abstract: Formal education is experiencing a series of reforms that favor the integration of the contents of different areas in the teaching and learning of the different educational stages. The present study examined the use of an interdisciplinary music and mathematics experience in Secondary Education in Galicia (Spain) in the 2016/17 academic year. A descriptive–exploratory design was used, through a Likert questionnaire applied to 197 students with a diagnostic test and a reference test, and a study of multiple cases was carried out in which information was collected through classroom observations. The results show improvements in the understanding of mathematical and musical concepts, and attitudes and procedures so we can argue that the use of interdisciplinary activities have favored the development of teaching–learning opportunities in mathematical and musical training.

Keywords: music; mathematics; compulsory secondary education; learning; didactic activity

1. Introduction

Education has moved, in recent times, towards an interdisciplinary approach with the purpose of preparing students for a world that is increasingly complex and interconnected. One of its aims, according to Majó-Masferrer and Baqueró-Alós [1], is to acquire and apply knowledge, skills, and values in an integrated way, relating the different contents and areas of knowledge to be able to find a response to different purposes. In the case of mathematics, a subject that presents difficulties for students, demotivation, and disinterest, working in an interdisciplinary way, they find it stimulating and it has a greater impact on them. If this is done through music, a greater stimulus is achieved, and they also have a positive impact on each other, not only in cognitive development and skills related to both disciplines, but also in academic results [2].

The connection between music and mathematics has been debated over the centuries. Peralta [3] claims that mathematical geniuses such as Mersenne, Descartes, Euler, and D'Alembert elaborated specific treatises on music, and that composers such as Bach, Mozart, Chopin, and Rossini resorted to mathematics to justify certain qualities, or even to create music (Bartók and the golden ratio, Xenakis' stochastic music, Halffter's Fibonacciana etc.). Certainly, mathematics intervenes in all the qualities of sound: pitch, duration, loudness, and timbre. In all of them we find a common denominator—the fraction—its use being circumscribed by the pitch of the sound, and more specifically, by Pythagoras' musical tuning system. This is because there are few contributions, experiences or teaching materials in school that link them despite the great reciprocity that exists between the two disciplines [4].

The incipient recognition of integrated music and math teaching for infant and primary school students is evident in research by authors such as Fernández-Carrión [2] or Mato-Vázquez, Chao-Fernández and Chao-Fernández [5,6] confirming an improvement in student learning and

attitudes. However, there is no record of studies linking both these disciplines in compulsory secondary education (ESO).

1.1. The Fractions

Fractions are present in the most diverse contexts; they form part of the secondary education curriculum and are essential for learning algebra, geometry, and other areas of higher mathematics [7].

Research on this construct creates a certain interest because many students consider fractions to be among the most complex mathematical concepts tackled in the early school years. The difficulties stem from a lack of conceptual understanding and procedural fluidity due to the limited use of manipulative teaching materials, visual representations and/or assessment of responses before attempting to use a formal algorithm [8]. The students see fractions as meaningless symbols or look at the numerator and denominator as separate numbers instead of understanding them as a unified whole [9]. In addition, these doubts persist when they arrive at secondary school as they have not understood the concept or have learnt it badly.

Perrenoud [10] argues that the reason students do not make adequate progress stems from the lack of teacher competence in teaching math effectively and not being able to apply mathematics to daily life. In this regard, Castro-Rodríguez, Rico and Gómez [11] emphasize that innovative materials and forms of teaching are very important elements as they capture the schoolchildren's attention and allow them to obtain satisfactory results both academically and personally. Therefore, it is necessary for educators to break the monotony in their classrooms, stimulate their students by introducing novel aspects that arouse their interest and present them with approaches that will motivate and provoke a taste for and curiosity about the world of mathematics. Taking this into account, the main objective of our work is to be able to understand fractions significantly and use them in a range of actions, concepts and procedures linked to varied situations, encouraging students to discover them on their own, and as a team, and then to discuss the strengths and weaknesses of each strategy [7].

1.2. Pythagorean Tuning

The pitch of sounds is the quality that allows us to differentiate whether they are high or low. This depends on the frequency of the vibrations from the emitting body, and is measured in hertz (Hz). The most used reference for measuring the absolute value of tones is 440 Hz (a fingerboard frequency equal to the A note located in the 2nd space of the stave in the key of G). To select and locate the particular sounds used by the music out of all the possible sounds, and the method for doing so, constitutes a tuning system, which consists of dividing the octave into a certain number of preset frequency sounds—the musical notes [12].

There is no unique way to choose the tuned sounds since to interpret a melody, it is not the absolute frequency of each note that is important but the relationship that exists between them [13]. According to Goldázar [14], there are several different tuning systems: those of just intonation (of Zarlino and Delezenne) and the regular cyclic temperaments (the equal temperament system and that of Holder). The most common are the 12-sound equal-tempered systems, used in practically all Western music, and Pythagorean tuning, the system still used to tune fretless stringed instruments [15] this is the one we will address in our study.

Pythagoras is considered the founder of a philosophical mystical movement Pythagoreanism (in the 4th and 3rd centuries B.C.), which considered the whole Universe as being governed by mathematical relationships, from the movement of celestial bodies to music [16].

According to some scholars, he arrived at his first results concerning the consonance and dissonance of sounds during his travels in Egypt and Babylon [17] although the most widespread theory is that of biographers such as Nicomachus, who refers to Pythagoras as being influenced by his knowledge of means (arithmetic, geometric, and harmonic). He discovered the mathematical proportions of music as he passed through a blacksmith's forge [12,18], where combinations of pleasant sounds were produced from percussion.

To refine and define musical intervals, Pythagoras used a monochord, verifying that the frequency of a musical note was inversely proportional to the length of the percussed string. Using the same string, different musical notes could be obtained if its length were regulated, which also allowed one to measure the intervals. This enabled him to confirm that strings with ratio lengths of 1:2 (the extremes 1 and 2), 2:3 (the harmonic mean of 1 and 2) and 3:4 (the arithmetic mean of 1 and 2) produced pleasant (harmonic) sounds when vibrating. He called them the diapason, diapente and diatessaron (today known as the eighth, fifth, and fourth of the Pythagorean diatonic scale). In his opinion, these were the only perfect notes (tetraktys), formed by the first four natural numbers [3]. To calculate the notes, one splits the interval of linked fifths (the circle of fifths) [3,12].

1.3. The Spanish Educational System in Not-University Levels in Relation to the Type of Centers

Since the Ministry of Education enacted the Organic Law Regulating the Right to Education (LODE) in 1985 [19], there two types of schools depending on their ownership: public and private (which can be secular or religious).

A public school is one that is supported by public funds and supervised by certain Public Administration and teaches compulsory education cycles (from 6 to 16 years old). Teaching must be secular and free. Students are admitted according to certain factors, such as family income, the proximity of the center to the address of the school, and the presence of other brothers or sisters in the center.

Within private schools, there are two types:

- Concerted centers, whose owner is a private individual or legal entity, but supported by public funds under a concert regime, so they are required to adapt to certain parameters established by the government. They are usually governed by religious entities with their own ideology and, although the education is subsidized by the state, the rest of the offered school services must be paid by the parents (dining room, transportation, extracurricular activities...).
- Centers financed exclusively with private funds, managed by a private education company financed exclusively by the parents of the students. They have complete freedom of management and certain curriculum autonomy, within the limits established by the government.

There are numerous studies that analyze whether there are differences in academic performance depending on the ownership of schools, the results being different. Calero, Escardíbul and Choi [20] and Mato-Vázquez, Chao-Fernández and Ferreiro [21] among others, find evidence that the results of students that come from private centers are better than public ones. However Witte [22], Cordero, Crespo and Pedraja [23] conclude that there is no significant difference in relation to the ownership of the center, coinciding with Coleman, Hoffer and Kilgore [24], who claim that the center has little impact on the academic achievements of the students, being other inputs more important in the determination of the educational output, such as personal and family characteristics.

It should be noted that tuning is rarely addressed in Spanish secondary education music classes [5] and that violin strings are tuned at separation intervals of a fifth. There is no consistent research that analyzes the interdisciplinary differences between the two subjects or between the concepts discussed in this study, and none refer to variables such as sex or the type of education center.

2. Materials and Method

The objective of this work is to analyze the benefits of implementing an interdisciplinary music and math experience, in particular, how using numbers allows the specific musical contents to be worked together with the actual mathematics to promote the students' learning of fractions and musical tuning. Our approach was designed and coordinated by the researchers. For the data collection, a descriptive–exploratory design was employed using a Likert questionnaire [25] which was administered to secondary school students in the 2nd year mathematics course, before and after implementing the approach. The technique of observing and analyzing documents was also used.

2.1. Context and Participants

The research was carried out during the 2016–2017 academic year with a convenience sample consisting of 197 students from the 2nd year of ESO (Compulsory Secondary Education) from ten education centers in Galicia (Spain). At first, 12 public schools and 5 concerted schools with easy access for researchers were selected, finally being the accepting sample of 7 public and 3 concerted.

Of the 197 students, 102 (51.78%) were women and 95 (48.22%) men. The average age of the students was between 14 and 16 years old. In terms of the education centers they attended, 134 were enrolled at public schools and 63 at state-associated private schools. The sample size was not counted for this study and the total number of participants was equal to total number of students in classrooms where the research was conducted.

2.2. Research Design

Before explaining the research design, it should be emphasized that all the teachers we contacted about the research had to meet certain prerequisites: in addition to teaching mathematics in ESO, they needed to possess a minimum level of musical knowledge, or at least have a positive attitude towards the interdisciplinary teaching of math and music. Only in this way could they demonstrate interest in the study to be conducted in their classrooms with the students. The ten who accepted and went along with our proposal had either an intermediate of elementary level musical qualification, played a musical instrument or had taken a course in music school. In addition, although they did not participate in developing the workshops directly, they all demonstrated a high level of commitment towards their professional development and facilitating the researchers' work in any way they could—so much so that when preparing the contents of the proposal, we could count on most of them participating. They also provided us with the mathematics and music programs prepared for the course in accordance with Decree 86/2015, which sets out the compulsory secondary education curriculum in the Autonomous Community of Galicia [26]. Bearing in mind this legislation, the contents brought together fractions, measurement and geometry (in terms of mathematics), and musical instruments and Pythagorean tuning (in terms of music).

For the practical calculation of the notes, the fifth interval was used in a linked form (the circle of fifths), starting from fractions. This approach was carried out in accordance with the literature: [12]; Anderson [27]; Casals, Carrillo and González-Martín [4] and Jones-Lewis [28], among others.

Likewise, the approach was based on four propositions: instruction, guidance, group work, and individual work. These are mandatory prerequisites because the complexity of both issues is high, and it is necessary for the learners to have a clear explanation of the work, the steps, conditions, benefits, and criteria for assessing their effectiveness and difficulties in executing a strategy. To this end, they were asked questions about each problem which encouraged them to reflect and reorganize everything that had been planned, and they were required to draw up a table containing all the information.

Because of the above-mentioned complication, efforts were made to closely tutor the students to keep them motivated against any possible failures and so that they acquired knowledge themselves. This meant that students expressed their thoughts aloud as they solved the tasks. The researcher collaborated with them, creating space for analysis, discussion, and reflection on the procedures used.

Activities were also encouraged in work groups where students presented their views and explained their processes while interacting with others, thus facilitating socialization, participation, cooperative dialogue, argumentation, and peer-to-peer learning.

Finally, through individual work, the students had to use the strategies autonomously and solve the mathematical problems themselves. The researcher verified what the students had learnt using an assessment.

The process was progressive in terms of ceding learning control to the students gradually until they possessed the ability to fend for themselves and solve the mathematical problems on their own. Although the guidance was basic [29], it was also necessary for the student to continue the presented sequencing since the approach was designed in such a way that some contents could not be acquired

without first having passed through the previous step satisfactorily. It was a matter of creating a symbiosis between the two areas so that the contents could be worked on in the same session thus benefitting both; it is in this way that interdisciplinarity is conceived. Each day the different moments of the process were recorded; for example, when students asked questions that demonstrated the attitudes, behaviors, and achievements acquired in each session.

In addition, the students engaged in self-reporting, which was very enriching as a reflection and regulation strategy to establish the errors or successes, the support received from the researcher, the difficulties they faced in the cooperative work, and as an evaluative modality for the researcher to gain understanding of the metacognitive process of each student.

The intervention program at each education center comprised 6 sessions of 50 min each (2 sessions per week for 3 weeks). All of these were conveniently distributed and guided by researchers during school hours. An activities summary for each session is displayed in Table 1.

Table 1. Distribution of the activities according to the sessions.

	Tuning and Numbers			
S	Initial assessment: music and math	Pythagorean tuning	Proposal on Pythagoras	-
S2	Presentation proposed Pythagoras	Development process tuning system	Fifth Circle Values	Doubts/Proposals for improvement
S3	Selection of musical instrument	Value presentation Fractions	Tuning of string instrument	Neck-fretting development
S4	Present measurements obtained from the instrument neck.	Development of assessment procedure	Conversion of fractional values to metric values	Overlaying metric values to the instrument neck
S5	Overlaying notation musical	Assessment results final product	Final assessment of the educational process	-
S6	Design and construction of the instrument neck	Visit to an education center. Presentation	-	-

2.3. Measurement Tool

To investigate quantitatively the interdisciplinary use of the two areas of knowledge, a questionnaire consisting of 20 items with values ranging from nothing at all (1) to a lot (5) was developed.

The process of elaboration consisted, first, of a bibliographic search and compilation concerning the relationship between music and mathematics, as well as an exhaustive review of the existing instruments regarding the benefits of working the two areas together; this resulted in an initial questionnaire consisting of 27 items. Next, attending Hernández, Fernández, and Baptista [30] a group of experts—math and music professors and researchers—carried out an initial selection consisting of 23 items, which was reviewed by judges. Subsequently, appropriate modifications were made resulting in the pilot questionnaire, which was administered to a reduced sample group of students from 2nd year of ESO different from those who would finally participate in the final test (of 8 people).

The reliability and validity analyses eliminated 3 items that had a very low homogeneity index (IHc < 40) resulting in the final questionnaire consisting of 20 items, which was administered to a definitive sample of 197 students.

Two tests were used: a diagnostic (pre-test) on the first day, before starting the work sessions, with a Cronbach's Alpha (internal consistency) of 0.8054, and another reference test (post-test) on the last day, once the interdisciplinary mathematical-musical strategies had been applied; this had a Cronbach's Alpha of 0.9206, indicating high reliability in each of the tests.

The instrument consisted of 5 categories (C) sharing the same core idea with associated questions:

Category 1 (C1) Mathematics (M) and school music (SM) (items 1 and 4) refers to the difficulties of these subjects with respect to other subjects in the curriculum, and to fractions and tuning with respect to other contents (Cronbach's Alpha of 0.9112).

Category 2 (C2) Special method used in the classroom (items 5 to 8) focuses on finding out whether both subjects and/or their contents are worked interdisciplinarily: fractions-tuning (Cronbach's Alpha of 0.8791).

Category 3 (C3) Concepts. Mathematics/music (items 9 and 12) examines the fundamental musical and mathematical knowledge, the theoretical-practical corpus of what tuning is and the possibility of applying fractions to real situations; specifically fractions and tuning (Cronbach's Alpha of 0.8675).

Category 4 (C4) Attitudes. Mathematics/music (items 13 to 16) refers to the students' motivation towards the learning of fractions and musical tuning, as well as their confidence, interest and predisposition towards the activity (Cronbach's Alpha of 0.9213).

Category 5 (C5) Procedures. Mathematics/music (items 17 and 20) investigates how students act in class, the implementation of the researchers' working script and its indications, the individual autonomy in resolving doubts and the joint intervention to expand knowledge (Cronbach's Alpha of 0.8099).

2.4. Observations

In line with Simons [31], the observation has allowed us to capture an image of reality, providing us with a detailed description of the phenomenon, which we can then analyze and interpret. From this perspective, the six classes in which the approach was carried out were video recorded, to analyze the most significant facts related to the experiences of the investigated. In order not to distract them and favor the normal development of the session, the location of the researcher was on one side in the classroom and in silence, without intervening.

Subsequently, three members of the research team analyzed the teaching surveys using observational rubrics, with Insufficient (1), Adequate (2), Good (3) and Excellent (4) values for each of the students. Finally, after the dump and exposure of the data, the conclusions were drawn up.

3. Analysis and Results

To address the set objectives, statistical analyses were performed: describing the items and categories of the pre-test and post-test, the percentages and measurements of the central tendencies (the mean) and of the dispersion (the standard deviation).

The non-parametric Mann–Whitney statistical test was used for the two independent samples (differentiated by sex and by education center). For the contrast tests, the absence of a normal distribution for the considered variables was taken into account.

3.1. Pre-Test and Post-Test Percentage Distribution of the Questionnaire Items

In the description of the results in Table 2, which refers to the percentage distribution of the 197 students on the value scales of each item in the questionnaire, it should be noted that the pre-test values manifested the students' absence of knowledge regarding musical tuning. Ninety percent of the responses were concentrated in value 1 (Nothing at all), while no data were collected in values 4 and 5. Something similar occurred, although to a lesser extent, when they were asked about interdisciplinary work (Category 2), where the nothing at all values were the highest.

It can also be seen that students consider fractions difficult and that they possess insufficient knowledge although a high percentage expressed an interest in learning about them.

In contrast, the items for which the highest values were recorded were those belonging to Category 5, which expressed the students' confidence in their abilities and in group work; however, this was less for individual work.

Once the experience had been carried out, the post-test results reflected an overall improvement in all the items, especially those in Category 2, whose pre-test values had been the worst. In fact, those items in C2, C3, and C4 that had previously obtained 0 responses in values 4 and 5, afterwards presented estimable results.

Table 2. Pre-test and post-test percentages of the questionnaire items.

	I	Pre-Test					Pos-Test				
		1	2	3	4	5	1	2	3	4	5
C1	1. This year mathematics is easier than other subjects	14.7	28.9	35.0	17.3	4.1	6.2	3.9	62.8	15.4	11.7
	2. This year music is easier than other subjects	18.8	35.5	25.4	14.2	6.1	12.6	22.4	24.9	23.8	16.3
	3. Fractions are easier than other mathematical concepts	9.1	9.6	34.0	31.0	16.3	7.4	6.7	49.7	23.4	12.8
	4. Musical tuning is easier than other music concepts	31.0	21.3	29.9	11.2	6.6	9.7	14.3	42.0	22.6	11.4
C2	5. The teacher relates mathematics with other subjects	64.2	27.9	7.9	0	0	8.4	21.7	24.3	26.5	17.1
	6. The teacher relates music with other subjects	75.4	18.8	5.8	0	0	7.2	9.5	16.3	45.4	21.6
	7. The teacher relates fractions with other subject contents	43.7	50.5	5.8	0	0	3.2	5.6	18.7	39.4	32.1
	8. The teacher relates musical tuning with other subject contents	85.4	12.5	2.1	0	0	3.6	4.9	20.8	33.4	37.3
C3	9. My knowledge about fractions is enough	32.3	30.3	15.4	18.3	3.7	12.1	19.7	39.4	21.3	7.5
	10. I can organize the musical notes on the diatonic scale	25.4	23.9	36.5	9.6	4.6	5.6	13.7	36.9	29.2	14.6
	11. I can calculate fractions to apply Pythagorean tuning	26.4	18.3	38.6	12.2	4.6	8.6	19.7	32.2	24.2	15.3
	12. I can sort the position of each fret on a violin	98.8	1.2	0	0	0	11.7	27.9	30.5	22.8	7.1
C4	13. I'm interested in learning fractions	8.1	19.8	40.6	20.8	10.7	6.4	12.8	20.6	40.7	19.5
	14. I'm interested in learning tuning	92.9	4.8	2.3	0	0	5.1	15.7	25.4	33.0	20.8
	15. I have confidence in my abilities to work with fractions	33.5	29.9	26.9	5.6	4.1	14.7	20.1	48.2	10.1	6.9
	16. I have confidence in my abilities to work with tuning	98.8	1.2	0	0	0	20.8	9.6	26.9	20.4	22.3
C5	17. I am looking for information related to Pythagoras and its relationship with music	90.1	9.5	0.4	0	0	2.9	6.1	18.7	33.7	38.6
	18. I perform the tasks indicated by the teacher	10.7	18.3	41.6	20.8	8.6	9.1	17.4	42.3	21.5	9.7
	19. I can work in a group	23.9	29.4	27.4	14.2	5.1	5.6	14.2	34.	33.0	20.8
	20. I can perform the tasks indicated by the teacher individually	9.2	11.4	24.3	38.6	19.5	7.1	8.3	24.7	38.8	21.1

3.2. Mean and Standard Deviations for the Pre-Test and Post-Test by Education Centers

Table 3 shows the mean (M) and standard deviation (SD) of the pre-test and post-test items in relation to the type of education center. The original scores were very low, both in the public and associated education centers, with a significant improvement being observed following the start of the teaching experience.

Table 3. Mean and standard deviations for the pre-test and post-test by education centers.

	CP				CC			
	Pre-Test		Pos-Test		Pre-Test		Pos-Test	
I	M	D	M	D	M	D	M	D
1. This year mathematics is easier than other subjects	1.71	1.09	3.64	1.12	1.57	0.95	3.73	1.10
2. This year music is easier than other subjects	2.81	1.01	3.46	1.09	2.89	1.16	3.89	1.07
3. Fractions are easier than other mathematical concepts	1.90	1.12	4.17	1.10	2.28	1.11	4.11	1.08
4. Musical tuning is easier than other music concepts	1.34	1.08	3.65	1.07	1.62	1.13	3.74	1.09
5. The teacher relates mathematics with other subjects	0.82	0.879	4.32	1.10	0.92	0.91	4.56	1.01
6. The teacher relates music with other subjects	0.67	1.14	3.99	1.23	0.43	1.15	4.12	1.12
7. The teacher relates fractions with other subject contents	1.05	1.12	4.54	1.21	0.99	1.15	4.63	1.09
8. The teacher relates musical tuning with other subject contents	0.12	1.14	3.23	1.01	0.24	1.15	3.54	1.02
9. My knowledge about fractions is enough	2.08	1.16	4.27	1.14	2.15	1.18	4.19	1.14
10. I can organize the musical notes on the diatonic scale	3.35	1.09	4.09	1.00	3.70	1.11	4.18	1.02
11. I can calculate fractions to apply Pythagorean tuning	1.03	1.14	4.14	1.02	1.34	1.15	3.96	1.04
12. I can sort the position of each fret on a violin	0.88	1.12	3.04	1.09	0.97	1.14	3.58	1.07
13. I'm interested in learning fractions	2.06	1.08	4.62	1.04	2.38	1.05	4.98	1.03
14. I'm interested in learning tuning	1.49	1.13	4.03	1.01	1.55	1.15	3.73	1.02
15. I have confidence in my abilities to work with fractions	2.15	1.06	3.97	1.03	2.21	1.05	3.88	1.01
16. I have confidence in my abilities to work with tuning	1.13	1.10	2.36	0.99	1.15	1.08	2.41	1.00
17. I am looking for information related to Pythagoras and its relationship with music	0.48	1.11	3.49	1.04	0.54	1.01	3.63	1.04
18. I perform the tasks indicated by the teacher	3.94	1.09	4.81	1.01	4.09	1.04	4.14	1.02
19. I can work in a group	2.78	1.15	4.25	0.97	2.97	0.990	4.82	0.98
20. I can perform the tasks indicated by the teacher individually	3.38	1.08	3.62	1.02	3.26	1.00	3.41	1.01

Subcategory (S) Items (I) Pre-test (Pr) Post-test (Pt) (MT) Public Mean (MP) Associated Mean (MC) (D) Standard Deviation.

It should be emphasized that the initial average in items 10, 18, and 20 was already high, since students already considered themselves able to order notes on the diatonic scale, as well as to perform the tasks given by the teacher, and to work individually; this was the case both in the public and associated education centers.

The items referring to the interdisciplinarity of the two subjects, or the contents relating to fractions and tuning (5, 6, 8, 11, and 14) are those in which a greater difference is perceived from that of the initial situation. It should be noted that the response variability was high for all the items.

Table 4 shows the mean (M) and standard deviation (SD) for the categories in the pre-test and post-test. The results indicate that Category 2 had the lowest values, and this was the one that experienced the greatest rise, followed by Category 4, which referred to the students' attitudes regarding the two areas of knowledge.

Table 4. Mean and standard deviations for the categories in the pre-test and post-test.

	Categories	Pre-Test		Pos-Test	
		M	D	M	D
C1	Mathematics (M) and school music (SM)	2.01	1.08	3.8	1.08
C2	Special method used in the classroom	0.65	1.10	4.12	1.07
C3	Concepts. Mathematics / music	1.9	1.09	3.9	1.10
C4	Attitudes. Mathematics / music	1.8	1.08	3.75	1.05
C5	Procedures. Mathematics / music	2.7	1.06	4.2	1.01

3.3. Analysis by Category for the Sex and Center-Type Variables

Regarding sex, the Mann–Whitney U test showed the absence of significant pre-test differences between female and male students in all categories. For the post-test, the differences were significant in two cases: "Mathematics and school music" (Sig. < 0.001), and "Attitudes. Mathematics/music" (0.029). Conversely, for "Special method used in classroom", "Concepts. Mathematics/music" and "Procedures. Mathematics/music" which obtained values above 0.05, no differences were observed (Table 5).

Table 5. Contrast statistics. The Mann–Whitney U test for sex by category.

		C1	C2	C3	C4	C5
Pre-test	Mann–Whitney U	241,901.000	154,249.500	196,217.500	110,343.000	210,765.500
	Wilcoxon W	345,096.000	360,120.500	3,403,128.500	346,628.000	306,351.500
	Z	−4.951	−1.434	−1.200	−1.194	−1.186
	Sig.	0.098	0.421	0.296	0.729	0.656
Pos-test	Mann–Whitney U	161,901.000	178,249.500	178,387.500	173,333.000	185,350.500
	Wilcoxon W	363,196.000	350,240.500	350,378.500	374,628.000	357,341.500
	Z	−3.951	−1.224	−1.210	−2.188	−0.068
	Sig. < 0.001	0.000	0.221	0.226	0.029	0.946

Regarding the education center type, the results indicate in both the pre-test and the post-test, the absence of significant differences in all the established categories; thus, we can accept the null hypothesis (Table 6).

Table 6. Contrast statistics. The Mann–Whitney U test for the center type by category.

		C1	C2	C3	C4	C5
Pre-test	Mann–Whitney U	178,452.500	191,142.600	107,560.500	114,567.000	192,340.500
	Wilcoxon W	33,121.800	321,340.500	335,621.500	351,232.000	315,232.500
	Z	−0.245	−0.231	−0.654	−0.234	−0.067
	Sig.	0.540	0.341	0.266	0.387	0.832
Post-test	Mann–Whitney U	183,482.500	180,642.500	179,380.500	173,873.000	178,950.500
	Wilcoxon W	355,473.500	340,200.500	360,323.500	384,529.000	327,387.500
	Z	−0.371	−0.321	−1.010	−0.843	−0.098
	Sig.	0.710	0.221	0.226	0.309	0.946

3.4. Observation

The observations have provided us with a meticulous portrait of reality from a natural perspective, allowing us to analyze and interpret things openly, guiding us exclusively by the descriptions of the rubrics, avoiding the evaluation of aspects not contemplated in them.

We consider students the key element in all classroom programming and the central focus in the teaching–learning process. Throughout the course of the activities, we collected information about students' opinions and/or feelings, motivations, and attitudes (Table 7). In this regard, the responsibility and interest shown in carrying out the tasks must be influenced in terms of the researchers' explanations.

Table 7. Observation rubric. Opinions and/or feelings, motivations, and attitudes of the students.

	1	2	3	4
1. Follow the teacher's instructions	0.5	7.1	37.6	54.8
2. Actively participate in collaborative work	0	12.7	36.6	50.7
3. Is properly attentive in class	0	11.7	40.1	48.2
4. It strives in the realization and timely delivery of activities related to the experience	0.5	12.3	38	49.2
5. Express your opinions in a reasoned way	1.1	13.2	39	46.7
6. Avoid talking about topics outside the activity	1.1	12.7	38.5	47.7
7. Ask timely and important questions about the topic	0	14.2	40.9	44.9
8. Organize and collect class information in the notebook	1.1	14.7	41.1	43.1
9. Listen carefully to what your classmates say	0	13.7	38.5	47.7
10. Show interest in innovation in class	0	10.6	37.6	51.8
11. Share your knowledge with the group	0	12.3	38.5	49.2
12. Take initiatives, proposing activities and ideas related to the topic	1.1	14.7	39.3	44.9
13. Accept criticism and take them into account	1.1	11.7	39	48.2
14. Accept other ideas even if they are different from yours	0.5	12.3	40.6	46.6
15. Knows how to make constructive criticism, without offending or disturbing colleagues	0.5	16.2	42.2	41.1
16. Help partners	0	12.3	37.5	50.2
17. Show positive attitudes before math and music	0.5	10.6	37	51.9
18. Shows interest in interdisciplinary activities	0	7.1	40.9	52
19. He gets excited about playful work	0	0	0	100

It should be noted that the explanation about what they were going to do had an impact on them, perhaps because of the novelty of working with two subjects that in principle, were not related. However, their curiosity made them want to participate from the start. This gave the researchers some level of assurance that the students liked class innovation. The students also showed enthusiasm for practical and dynamic work, involvement in the activity, participation, pleasure, collaboration, and respect and empathy for the researchers. They expressed exceptional surprise at the possibilities of fractions.

We should also refer to the moments of dialogue and reflection about what was done in class. Something that was not usual. We can point to how students improved their personal and social skills and attitudes regarding math and music. With the work on tuning, we found a link was established, a fully didactic bridge, between the study of one subject and its impact on the other. It was motivating for the students; they cooperated as a team, respected each other and participated fully.

To achieve a full development of skills it is necessary to work them in a relaxed and playful environment and, above all, in a group. In this case, we found that the group atmosphere was wonderful, everyone formed a good team.

We observed that the students were not afraid of making mistakes and were open to criticism and suggestions from both peers and researchers. They were also open to integrating this innovative activity. They developed their artistic ability and understood that there are other avenues of learning beyond what is in a textbook. Having practiced fractions and tuning, students remembered the tasks performed with pleasure and positively appreciated the influence that interdisciplinarity can have on content assimilation. In short, as shown in Table 7, interdisciplinary work has been motivating, attractive, interesting, and productive for students.

4. Discussion and Conclusions

This study explored the relationship between mathematics and music in the school context, focusing specifically on musical tuning and fractions. The main purpose was to find out whether a proper interdisciplinary strategy led to changes in the pre-test and post-test responses given by the students. At the same time, we were interested in underlining the importance of implementing activities in the classroom that provoke in students a liking for knowledge in certain subjects and contents that for a variety of reasons, arouse less interest than others in the curriculum. The results of our research are consistent with studies by authors that show the benefit of combining these subjects to strengthen the pedagogical practices of both [4]. They have also made it possible to analyze the potentialities and difficulties of this sample in terms of its categories and subcategories.

It should be noted that the teachers became involved in the experience, thus allowing the researchers to break from the usual work model, ceding the classrooms, students, and material, allowing them to freely give their opinion on the educational activities before them. This made it possible to carry out an in-depth analysis of the entire process.

Regarding the aim of integrating fractions and musical tuning through activities common to the contents of both, the experience was appealing to the students and helpful to both the researchers and the teachers for making deductions. As demonstrated by Fernández-Carrión [2], when mathematics and music are mutually supported in an interdisciplinary way, similarities are established with the lived experience, are brought closer to daily life, are more useful, practical, dynamic, and, above all, the teaching–learning is motivating [32].

The results of the analyses carried out using the various instruments have shown that the students' perception of these areas of knowledge underwent considerable change, both in terms of their opinions and about the tasks carried out.

On this point, the students' percentages greatly improved when it came to fretting, by noticing the usefulness of fractions, and by working collaboratively. Specifically, the focus on integrating the mathematical-musical contents helped balance the students who knew nothing about the subject and those who had some idea, making the educational practice valuable to all and favoring the learning process.

We should state that on the first day, because of a lack of practice, we witnessed some difficulties in identifying any successes; there was a certain slowness in organizing or difficulties in resolving issues. However, the satisfaction the students felt in discovering the usefulness of what they were doing in class exceeded all expectations.

Focusing on and comparing the data from the questionnaires, before and after the applied teaching approach, we can say that the use of interdisciplinary strategies has generated significant differences in the learning of fractions and musical tuning. Based on the results, we believe that students need experiences that allow them to work collaboratively, and that these have an impact on them. Students need to see meaning in what they do, feel at ease, and assess and comment on their own successes or failures [27,33–35]. In this sense, the self-reporting provided valuable information by making it clear that you can work interdisciplinarily in the math class.

Likewise, musical tuning contributed to the students' teaching–learning process and the acquisition of content because it made it possible to work on different curriculum contents thus putting all their skills into practice [18]. Consequently, students were able to acquire long-term knowledge in an enjoyable way as music produces a more rapid and significant learning process in our brains [36].

Regarding the education center variable, we can state that there are no significant differences in the assessment of the students, either in the diagnostic test or in the reference test. In relation to sex, there is only a significant difference in the post-test in Categories 1 Mathematics and school music, and 4 Attitudes of students in the classroom. Mathematics and music.

We must stress that going into the field made it possible for us to observe the willingness to work, the communication between students, and their relationship with the instructors, as well as the materials on which they relied.

These observations allowed us to see that the students were working with determination and were making efforts to improve every day. Proof of this was evident in the final result when making the instrument neck, the experience of which surprised and captivated the other members of the school community (both students and teachers) from other classrooms who were very interested, and who asked to visit us on the last day to observe the fruit of the work done. The questionnaire responses show a positive change across all the formulated questions; even regarding attitudes in the classroom, there was an implicit effective commitment following the five weeks spent with the students.

Likewise, the results are consistent with Majó-Masferrer and Baqueró-Alós [1], in what corresponds to acquiring and relating knowledge of the two areas. Along the same lines, and in line with Mato-Vázquez, Chao-Fernández, Chao-Fernández [6], it is perceived that the motivation and interest in mathematics has been increasing when working in an interdisciplinary way with music.

Procedurally, this approach is very useful. The combined group-learning collaboration while creating an interesting opportunity—at the personal, group, and social level—also generated profound effects at the pedagogical, organizational, and didactic levels.

One of the reflections this research offers is that of the students' individual work. Individually, they are less able to ask questions and intervene than when they work together to build knowledge. This leads one to consider again a certain lack of student autonomy and the advantages of group work.

In this investigation we find a limitation that, although it did not condition the student body, allows us to point out that in future investigations there is a schedule of the time needed to take specific notes, record the actions and behaviors observed, record the grade, after each observation.

The study encourages us to continue working to implement new interdisciplinary experiences between these and other areas of knowledge in a contextualized manner, using teaching materials that have a real-life application for the students. Likewise, it reinforces the desire to show alternatives to traditional teaching–learning methods that allow both teachers and students to swap the usual classroom methodology for more constructive educational practices. Although the results show the potential for learning interdisciplinarity among 2nd year ESO students, we cannot generalize them to any school or teacher. The potential lack of interest and/or knowledge of a teacher is an obstacle to the general use of this approach in class.

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