

CHARACTERIZATION OF GENERALIZED ORLICZ SPACES

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ABSTRACT. The norm in classical Sobolev spaces can be expressed as a difference quotient. This expression can be used to generalize the space to the fractional smoothness case. Because the difference quotient is based on shifting the function, it cannot be used in generalized Orlicz spaces. In its place, we introduce a smoothed difference quotient and show that it can be used to characterize the generalized Orlicz–Sobolev space. Our results are new even in Orlicz spaces and variable exponent spaces.

1. INTRODUCTION

Bourgain, Brézis, and Mironescu [6, 7] studied the limit behavior of the Gagliardo semi-norms

$$\|f\|_{W^{s,p}}^p = \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|^p}{|x - y|^{n+sp}} dx dy, \quad 0 < s < 1,$$

as $s \rightarrow 1$, and established the appropriate scaling factor for comparing the limit with the L^p -norm of the gradient of f . They characterized the Sobolev space $W^{1,p}$ and proved the convergence of certain imaging models of Aubert and Kornprobst [2] to the well-known total variation model of Rudin, Osher, and Fatemi [35]. Our aim in this paper is to extend the characterization to generalized Orlicz spaces defined on open subsets of \mathbb{R}^n .

Generalized Orlicz spaces $L^{\varphi(\cdot)}$ have been studied since the 1940's. A major synthesis of functional analysis in these spaces is given in the monograph of Musielak [31] from 1983, for which reason they have also been called Musielak–Orlicz spaces. These spaces are similar to Orlicz spaces, but defined by a more general function $\varphi(x, t)$ that may vary with the location in space: the norm is defined by means of the integral

$$\int_{\mathbb{R}^n} \varphi(x, |f(x)|) dx,$$

whereas in an Orlicz space, φ would be independent of x , $\varphi(|f(x)|)$. When $\varphi(t) = t^p$, we obtain the Lebesgue spaces, L^p . Generalized Orlicz spaces are motivated by applications to image processing [10, 23], fluid dynamics [37], and differential equations [4, 20]. Recently, harmonic analysis in this setting has been studied e.g. in [15, 25, 29].

We have in mind two principal classes of examples of generalized Orlicz spaces: variable exponent spaces $L^{p(\cdot)}$, where $\varphi(x, t) := t^{p(x)}$ [14, 17], and dual phase spaces, where $\varphi(x, t) := t^p + a(x)t^q$ [3, 4, 5, 8, 9, 11, 12, 13, 34]. It is interesting to note that our general methods give optimal results in these two disparate cases, cf. [25]. Also covered are variants of the variable exponent case such as $t^{p(x)} \log(e + t)$ [20, 30, 32, 33].

It is not difficult to see that a direct generalization of the difference quotient to the non-translation invariant generalized Orlicz case is not possible [27, Section 1]. For instance Besov and Triebel–Lizorkin spaces in this context have been defined using Fourier theoretic approach [1, 18]. Hästö and Ribeiro [27], following [16], adopted a more direct approach with a smoothed difference quotient expressed by means of the sharp averaging operator

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$M_{B(x,r)}^\#$. This is the general approach adopted also in this paper. This paper improves [27] in three major ways:

1. Instead of variable exponent spaces, we consider more general generalized Orlicz spaces;
2. Instead of \mathbb{R}^n , we allow arbitrary open sets $\Omega \subset \mathbb{R}^n$; and
3. In our main result, we relax the technical assumption $f \in L^1(\Omega)$ to its natural form; i.e., $f \in L_{\text{loc}}^1(\Omega)$.

The latter two generalizations have been previously established in the case L^p by Leoni and Spector [28], see also [19, Section 1]. In order to achieve these goals, the methods of the main results (Section 4) are completely different from those in [27] and involve a new bootstrapping scheme.

We introduce some notation to state our main result. We refer to the next section for the precise definition of $L^{\varphi(\cdot)}$ and the assumptions in the theorem. Let $\Omega \subset \mathbb{R}^n$ be an open set, $\varphi \in \Phi_w(\Omega)$, and $\varepsilon > 0$. Let ψ_ε be a set of functions such that

$$(1.1) \quad \psi_\varepsilon \in L^1(0, \infty), \quad \psi_\varepsilon \geq 0, \quad \int_0^\infty \psi_\varepsilon(r) dr = 1,$$

and, for every $\gamma > 0$,

$$(1.2) \quad \lim_{\varepsilon \rightarrow 0^+} \int_\gamma^\infty \psi_\varepsilon(r) dr = 0.$$

We define a weak quasi-semimodular, $\varrho_{\#, \Omega}^\varepsilon(f)$, on $L_{\text{loc}}^1(\Omega)$ by setting

$$\varrho_{\#, \Omega}^\varepsilon(f) := \int_0^\infty \int_{\Omega_r} \varphi\left(x, \frac{1}{r} M_{B(x,r)}^\# f\right) dx \psi_\varepsilon(r) dr$$

for $f \in L_{\text{loc}}^1(\Omega)$, where $\Omega_r := \{x \in \Omega : \text{dist}(x, \partial\Omega) > r\}$ and

$$M_{B(x,r)}^\# f := \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \quad \text{with} \quad f_{B(x,r)} := \int_{B(x,r)} f(y) dy.$$

The associated quasi-norm, $\|f\|_{\#, \Omega}^\varepsilon$, is defined by

$$\|f\|_{\#, \Omega}^\varepsilon := \inf \{ \lambda > 0 \mid \varrho_{\#, \Omega}^\varepsilon(f/\lambda) \leq 1 \}.$$

The following is our main result stated for a Φ -function—it is also possible to state it for weak Φ -functions, see Theorem 4.6. The proof follows from Propositions 4.1 and 4.5.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\varphi \in \Phi(\Omega)$ satisfy Assumptions (A), (aInc), and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). Assume that $f \in L_{\text{loc}}^1(\Omega)$. Then,*

$$\nabla f \in L^{\varphi(\cdot)}(\Omega; \mathbb{R}^n) \quad \Leftrightarrow \quad \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty.$$

In this case,

$$\lim_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) = \varrho_{\varphi(\cdot), \Omega}(c_n |\nabla f|) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon = c_n \|\nabla f\|_{\varphi(\cdot), \Omega},$$

where $c_n := \int_{B(0,1)} |x \cdot e_1| dx$.

Remark 1.2. Note that the previous result is new even in the case of classical Orlicz spaces. In this case, Assumption (A) automatically holds.

Remark 1.3. In the case $\varphi(x, t) = t^{p(x)}$, Assumptions (A0) and (loc) always hold, while Assumptions (A1) and (A2) are equivalent to the local log-Hölder continuity and Nekvinda's decay condition, respectively. Moreover, if $p_- := \inf_{x \in \Omega} p(x) > 1$, then (aInc) holds; and if $p_+ := \sup_{x \in \Omega} p(x) < \infty$, then (aDec) holds.

2. PRELIMINARIES

This section is organized as follows. In Subsection 2.1, we collect some notation used throughout this paper. Then, in Subsection 2.2, we recall the definition of Φ -functions and of some of its generalizations; we recall also the associated Orlicz spaces, norms, and semimodulars. Finally, in Subsection 2.3, we introduce and discuss our main assumptions on the (generalized weak) Φ -functions and relate them with other assumptions in the literature. We conclude by proving two auxiliary results, Lemmas 2.11 and 2.12. The first one can be interpreted as a counterpart in our setting of the weighted power-mean inequality for the function $\varphi(x, t) = t^p$ for some $p \geq 1$; the second one is a Jensen-type inequality in the spirit of [25, Lemma 4.4] and [27, Lemma 2.2].

2.1. NOTATION. We denote by \mathbb{R}^n the n -dimensional real Euclidean space. We write $B(x, r)$ for the open ball in \mathbb{R}^n centered at $x \in \mathbb{R}^n$ and with radius $r > 0$. We use c as a generic positive constant; i.e., a constant whose value may change from appearance to appearance. If $E \subset \mathbb{R}^n$ is a measurable set, then $|E|$ stands for its (Lebesgue) measure and χ_E denotes its characteristic function; we denote by $L^0(E)$ the space of all Lebesgue measurable functions on E .

Let $g, h : D \subset \mathbb{R}^m \rightarrow [0, \infty]$ be two functions. We write $g \lesssim h$ to mean that there exists a positive constant, C , such that $g(z) \leq Ch(z)$ for all $z \in D$. If $g \lesssim h \lesssim g$, we write $g \approx h$. Also, given a sequence $(g_\varepsilon)_\varepsilon$ of non-negative functions in D , the notation $\lim_{\varepsilon \rightarrow 0} g_\varepsilon \approx h$ means that there is a positive constant, C , such that $\frac{1}{C}h \leq \liminf_{\varepsilon \rightarrow 0} g_\varepsilon \leq \limsup_{\varepsilon \rightarrow 0} g_\varepsilon \leq Ch$ in D ; i.e., for the equivalence we do not require the limit to exist, only the upper and lower limits to be within a constant of each other.

Moreover, if $D = [0, \infty)$, we say that g and h are *equivalent*, written $g \simeq h$, if there exists $L \geq 1$ such that for all $t \geq 0$, we have $h(\frac{t}{L}) \leq g(t) \leq h(Lt)$. In this case, L is said to be the *equivalence constant*.

Let $\varphi : [0, \infty) \rightarrow [0, \infty]$ be an increasing function. We denote by $\varphi^{-1} : [0, \infty] \rightarrow [0, \infty]$ the left-continuous generalized inverse of φ ; that is, for all $s \in [0, \infty]$,

$$(2.1) \quad \varphi^{-1}(s) := \inf\{t \geq 0 \mid \varphi(t) \geq s\}.$$

Let $\Omega \subset \mathbb{R}^n$ be open, let $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$, and let $B \subset \mathbb{R}^n$. Then, $\varphi_B^+ : [0, \infty) \rightarrow [0, \infty]$ and $\varphi_B^- : [0, \infty) \rightarrow [0, \infty]$ are the functions defined for all $t \in [0, \infty)$ by

$$\varphi_B^+(t) := \sup_{x \in \Omega \cap B} \varphi(x, t) \quad \text{and} \quad \varphi_B^-(t) := \inf_{x \in \Omega \cap B} \varphi(x, t).$$

Note that if $\varphi(x, \cdot)$ is increasing for every $x \in \Omega$, then so are $\varphi_B^+(\cdot)$ and $\varphi_B^-(\cdot)$; thus, these functions admit a left-continuous generalized inverse in the sense of (2.1).

2.2. Φ -FUNCTIONS AND GENERALIZED ORLICZ SPACES. We start this subsection by introducing the notion of almost increasing and almost decreasing functions, after which we recall the definition of Φ -functions and of some of its generalizations.

Definition 2.1. We say that a function $g : D \subset \mathbb{R} \rightarrow [0, \infty]$ is *almost increasing* if $g(t_1) \leq cg(t_2)$ for every $t_1 \leq t_2$ in D and some c . We say that c is the *monotonicity constant* of g . *Almost decreasing* is defined analogously.

Definition 2.2. Let $\varphi : [0, \infty) \rightarrow [0, \infty]$ be an increasing function satisfying $\varphi(0) = \lim_{t \rightarrow 0^+} \varphi(t) = 0$ and $\lim_{t \rightarrow \infty} \varphi(t) = \infty$. We say that φ is a

- (i) *weak Φ -function* if $t \mapsto \frac{\varphi(t)}{t}$ is almost increasing.
- (ii) *Φ -function* if it is left-continuous and convex.

We denote by Φ_w the set of all weak Φ -functions and by Φ the set of all Φ -functions.

Remark 2.3. If $\varphi \in \Phi$, then, by convexity and because $\varphi(0) = 0$, for $0 < t_1 < t_2$, $\varphi(t_1) = \varphi(\frac{t_1}{t_2}t_2 + (1 - \frac{t_1}{t_2})0) \leq \frac{t_1}{t_2}\varphi(t_2)$; thus, $t \mapsto \frac{\varphi(t)}{t}$ is increasing. Hence, $\Phi \subset \Phi_w$. Conversely, if $\varphi \in \Phi_w$, then there exists $\psi \in \Phi$ such that $\varphi \simeq \psi$ [22, Proposition 2.3].

Definition 2.4. Let $\Omega \subset \mathbb{R}^n$ be an open set, and let $\varphi : \Omega \times [0, \infty) \rightarrow [0, \infty]$ be a function such that $\varphi(\cdot, t) \in L^0(\Omega)$ for every $t \in [0, \infty)$. We say that φ is a

- (i) *generalized weak Φ -function* on Ω if $\varphi(x, \cdot) \in \Phi_w$ uniformly in $x \in \Omega$; i.e., the monotonicity constant is independent of x .
- (ii) *generalized Φ -function* on Ω if $\varphi(x, \cdot) \in \Phi$ for every $x \in \Omega$.

We denote by $\Phi_w(\Omega)$ and $\Phi(\Omega)$ the sets of generalized weak Φ -functions and generalized Φ -functions, respectively.

By this definition, it is clear that properties of (weak) Φ -functions carry over to generalized (weak) Φ -functions point-wise uniformly. In particular, this holds for Remark 2.3. Similarly, $\varphi \simeq \psi$ means that $\varphi(x, \cdot) \simeq \psi(x, \cdot)$ with constant uniform in x , etc.

Next, we recall the definition of the generalized Orlicz space, quasi-norm, and quasi-semimodular associated with a generalized weak Φ -function.

Definition 2.5. Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\varphi \in \Phi_w(\Omega)$, and consider the *weak quasi-semimodular*, $\varrho_{\varphi(\cdot), \Omega}$, on $L^0(\Omega)$ defined by

$$\varrho_{\varphi(\cdot), \Omega}(f) := \int_{\Omega} \varphi(x, |f(x)|) dx$$

for all $f \in L^0(\Omega)$. The *generalized Orlicz space*, $L^{\varphi(\cdot)}(\Omega)$, is given by

$$L^{\varphi(\cdot)}(\Omega) := \{f \in L^0(\Omega) \mid \varrho_{\varphi(\cdot), \Omega}(\lambda f) < \infty \text{ for some } \lambda > 0\}.$$

We endow $L^{\varphi(\cdot)}(\Omega)$ with the quasi-norm

$$\|f\|_{\varphi(\cdot), \Omega} := \inf \{\lambda > 0 \mid \varrho_{\varphi(\cdot), \Omega}(f/\lambda) \leq 1\}.$$

If, in the Definition 2.5, $\varphi \in \Phi(\Omega)$, then $\varrho_{\varphi(\cdot), \Omega}(\cdot)$ defines a semimodular on $L^0(\Omega)$ and $\|\cdot\|_{\varphi(\cdot), \Omega}$ a norm on $L^{\varphi(\cdot)}(\Omega)$ (see [17]).

Remark 2.6. If $\varphi, \psi \in \Phi_w(\Omega)$ and $\varphi \simeq \psi$, then $L^{\varphi(\cdot)} = L^{\psi(\cdot)}$ with equivalent quasi-norms.

2.3. MAIN ASSUMPTIONS. We begin by introducing our main assumptions on the generalized (weak) Φ -functions on Ω . The first three assumptions, (Δ_2) , **(aInc)**, and **(aDec)**, extend three known properties for Φ -functions to generalized Φ -functions point-wise uniformly. The fourth assumption, **(A)**, relates the behavior of generalized Φ -functions at different values of the variable in Ω . The last assumption, **(loc)**, implies that simple functions belong to the generalized Orlicz space. These last two assumptions hold trivially for Orlicz spaces.

Let $\Omega \subset \mathbb{R}^n$ be open and $\varphi \in \Phi_w(\Omega)$. We denote by φ^{-1} the generalized inverse of φ with respect to the second variable (see (2.1)).

- (Δ_2) φ is *doubling*; i.e., there exists $A > 0$ such that $\varphi(x, 2t) \leq A\varphi(x, t)$ for a.e. $x \in \Omega$ and for all $t \geq 0$.
- (aInc) There exists a constant $\varphi_\uparrow > 1$ such that for a.e. $x \in \Omega$, the map $s \mapsto s^{-\varphi_\uparrow}\varphi(x, s)$ is almost increasing with monotonicity constant c_\uparrow independent of x .
- (aDec) There exists a constant $\varphi_\downarrow > 1$ such that for a.e. $x \in \Omega$, the map $s \mapsto s^{-\varphi_\downarrow}\varphi(x, s)$ is almost decreasing with monotonicity constant c_\downarrow independent of x .
- (A) There exist $\beta, \sigma > 0$ for which:
- (A0) $\varphi(x, \beta\sigma) \leq 1 \leq \varphi(x, \sigma)$ for all $x \in \Omega$;
- (A1) $\varphi(x, \beta t) \leq \varphi(y, t)$ for every ball $B \subset \Omega$, $x, y \in B$, and $t \in [\sigma, \varphi^{-1}(y, \frac{1}{|B|})]$;
- (A2) there exists $h \in L^1(\Omega) \cap L^\infty(\Omega)$ such that for a.e. $x, y \in \Omega$ and for all $t \in [0, \sigma]$, we have $\varphi(x, \beta t) \leq \varphi(y, t) + h(x) + h(y)$.
- (loc) There exists $t_0 > 0$ such that $\varphi(\cdot, t_0) \in L^1_{\text{loc}}(\Omega)$.

The notation $(\text{aInc})_1$ is used for a version of (aInc) with $\varphi_\uparrow \geq 1$; i.e., equality included. Note that, for any weak Φ -function, $(\text{aInc})_1$ holds for $\varphi_\uparrow = 1$.

Remark 2.7. Let us collect several observations regarding the assumptions above.

1. Each of the previous conditions is invariant under equivalence of (weak) Φ -functions; i.e., if $\varphi \simeq \psi$, then φ satisfies a condition if and only if ψ satisfies it.
2. For doubling (weak) Φ -functions, \simeq and \approx are equivalent.
3. (aDec) and (Δ_2) are equivalent [24, Lemma 2.6].
4. (A0) implies (loc) (choose $t_0 := \beta\sigma$).
5. If (aInc) and (aDec) hold, then $\varphi^\uparrow \leq \varphi^\downarrow$.
6. Finally, note that if φ satisfies (aDec) and (loc), then $\varphi(\cdot, t) \in L^1_{\text{loc}}(\Omega)$ for every $t \geq 0$.

It follows directly from the definition of the left-inverse that $\varphi \simeq \psi$ implies $\varphi^{-1} \approx \psi^{-1}$. Furthermore, if $\varphi \simeq \psi$, then $\varphi_B^- \simeq \psi_B^-$ and so $(\varphi_B^-)^{-1} \approx (\psi_B^-)^{-1}$.

The next proposition shows that Assumption (A1) is equivalent to its counterpart in [25].

Lemma 2.8. *Let $\varphi \in \Phi_w(\Omega)$, $\sigma > 0$, and $\beta \in (0, 1)$. Then, φ satisfies (A1) for (σ, β) if and only if for every ball $B \subset \Omega$, and for all finite $t \in [\sigma, (\varphi_B^-)^{-1}(1/|B|)]$, we have*

$$(2.2) \quad \varphi_B^+(\beta t) \leq \varphi_B^-(t).$$

Proof. Because $(\varphi_B^-)^{-1}(\frac{1}{|B|}) \geq \varphi^{-1}(y, \frac{1}{|B|})$, it follows that if φ satisfies (2.2), then it also satisfies (A1).

Conversely, assume that φ satisfies (A1). We first consider the case when $t \in [\sigma, (\varphi_B^-)^{-1}(\frac{1}{|B|})]$. In this case, we can find $(y_i)_{i \in \mathbb{N}} \subset B$ such that $t \in [\sigma, \varphi^{-1}(y_i, \frac{1}{|B|})]$ for all $i \in \mathbb{N}$ and $\varphi_B^-(t) = \lim_{i \rightarrow \infty} \varphi(y_i, t)$. Then, by (A1), we have

$$\varphi(x, \beta t) \leq \varphi(y_i, t)$$

for a.e. $x \in B$ and for all $i \in \mathbb{N}$. Taking the supremum over $x \in B$ and then letting $i \rightarrow \infty$ in the previous estimate, we obtain $\varphi_B^+(\beta t) \leq \varphi_B^-(t)$. Finally, assume that $t = (\varphi_B^-)^{-1}(\frac{1}{|B|}) < \infty$, and let $t' \in [\sigma, t)$. By the previous case,

$$(2.3) \quad \varphi(x, \beta t') \leq \varphi_B^+(\beta t') \leq \varphi_B^-(t') \leq \varphi_B^-(t)$$

for a.e. $x \in B$, where in the last inequality we used the fact that φ_B^- is increasing. Taking the limit $t' \rightarrow t$ in (2.3), the left-continuity of $\varphi(x, \cdot)$ yields $\varphi(x, \beta t) \leq \varphi_B^-(t)$ for a.e. $x \in B$. Hence, taking the supremum over $x \in B$, we conclude that φ satisfies (2.2). \square

The next lemma shows that left-inverse commutes with infimum, even when the function is not continuous.

Lemma 2.9. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $B \subset \Omega$, and let $\varphi \in \Phi_w(\Omega)$. Then, $(\varphi_B^-)^{-1} = (\varphi^{-1})_B^+$.*

Proof. Fix $s \in [0, \infty]$. For every $x \in B$, we have

$$(\varphi_B^-)^{-1}(s) = \inf \{t \geq 0 \mid \varphi_B^-(t) \geq s\} \geq \inf \{t \geq 0 \mid \varphi(x, t) \geq s\} = \varphi^{-1}(x, s).$$

Thus, taking the supremum over $x \in B$,

$$(\varphi_B^-)^{-1}(s) \geq (\varphi^{-1})_B^+(s).$$

To prove the converse inequality, we may assume that $\bar{t} := (\varphi^{-1})_B^+(s) < \infty$ without loss of generality. Fix $\varepsilon > 0$. By definition of \bar{t} and because $\varphi(x, \cdot)$ is increasing for every $x \in B$, we have $\varphi(x, \bar{t} + \varepsilon) \geq s$ for all $x \in B$. Hence, $(\varphi_B^-)^{-1}(s) \leq \bar{t} + \varepsilon$. Letting $\varepsilon \rightarrow 0$, we conclude the desired inequality. \square

The following lemma allows us to relate Assumption (A1) with its counterpart in [21].

Lemma 2.10. *Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\varphi \in \Phi(\Omega)$ is doubling and satisfies (A0). Then, (A1) is equivalent to the following condition:*

(A1') $\varphi^{-1}(x, s) \lesssim \varphi^{-1}(y, s)$ for every ball $B \subset \Omega$, $x, y \in B$, and $s \in [1, \frac{1}{|B|}]$.

Proof. Because φ belongs to $\Phi(\Omega)$ and is doubling, it is a bijection with respect to the second variable from $[0, \infty)$ to $[0, \infty)$. Applying φ^{-1} to (A0) and (A1), we find that

$$(2.4) \quad (\text{A0}) \quad \Leftrightarrow \quad \beta\sigma \leq \varphi^{-1}(x, 1) \leq \sigma \text{ for a.e. } x \in \Omega;$$

$$(\text{A1}) \quad \Leftrightarrow \quad \beta\varphi^{-1}(x, s) \leq \varphi^{-1}(y, s) \text{ for a.e. } x, y \in B \text{ and for all } s \in \left[\varphi(x, \sigma), \frac{1}{|B|}\right].$$

If $s \in [1, \varphi(x, \sigma)]$, then $\beta\varphi^{-1}(x, 1) \leq \beta\varphi^{-1}(x, s) \leq \beta\sigma$ because $\varphi^{-1}(x, \cdot)$ is increasing. Thus, using (2.4), $\beta\varphi^{-1}(x, s) \leq \varphi^{-1}(y, s)$ holds for all such s . \square

As mentioned at the beginning of Section 2, the following lemma can be interpreted as a counterpart in our setting of the weighted power-mean inequality for the function $\varphi(x, t) = t^p$ for some $p \geq 1$.

Lemma 2.11. *Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\varphi \in \Phi_w(\Omega)$ satisfies (aDec). Then, for all $\delta > 0$ and $a, b \geq 0$ and for a.e. $x \in \Omega$, we have*

$$(2.5) \quad \varphi(x, a + b) \leq \varphi(x, (1 + \delta)a) + \frac{1}{c_\downarrow} \left(1 + \frac{1}{\delta}\right)^{\varphi_\downarrow} \varphi(x, b)$$

and

$$(2.6) \quad \varphi(x, a + b) \leq \frac{1}{c_\downarrow} \left[(1 + \delta)^{\varphi_\downarrow} \varphi(x, a) + \left(1 + \frac{1}{\delta}\right)^{\varphi_\downarrow} \varphi(x, b) \right].$$

Proof. If $b \leq \delta a$, then the monotonicity of $\varphi(x, \cdot)$ yields

$$\varphi(x, a + b) \leq \varphi(x, (1 + \delta)a).$$

If $a < \delta^{-1}b$, then the monotonicity of $\varphi(x, \cdot)$ and (aDec) yield

$$\varphi(x, a + b) \leq \varphi(x, (1 + \delta^{-1})b) \leq \frac{1}{c_\downarrow} (1 + \delta^{-1})^{\varphi_\downarrow} \varphi(x, b).$$

Thus, (2.5) holds. Further, (2.6) follows from (2.5) by (aDec). \square

The following lemma is a variant of [25, Lemma 4.4] without the assumption $\rho_{\varphi(\cdot)}(f\chi_{\{|f|>\sigma\}}) < 1$ and correspondingly weaker conclusion (see also [27, Lemma 2.2]).

Lemma 2.12. *Let $\Omega \subset \mathbb{R}^n$ be open and assume that $\varphi \in \Phi_w(\Omega)$ satisfies Assumption (A). Then, there exists $\beta' > 0$ such that, for every ball $B \subset \Omega$, $f \in L^{\varphi(\cdot)}(\Omega)$, and a.e. $x \in B$, we have*

$$\varphi\left(x, \beta' \min\left\{(\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right), \int_B |f(y)| dy\right\}\right) \leq \int_B \varphi(y, |f(y)|) dy + h(x) + \int_B h(y) dy,$$

where h is the function provided by (A2).

Proof. Without loss of generality, we may assume that $f \geq 0$. By Remarks 2.3 and 2.7, we may also assume that $\varphi \in \Phi(\Omega)$. Then, $t \mapsto \frac{\varphi(x,t)}{t}$ is increasing for every $x \in \Omega$.

Fix a ball $B \subset \Omega$, and denote $\alpha := (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)$. Let (σ, β) be given by Assumption (A), and set $f_1 := f\chi_{\{f>\sigma\}}$, $f_2 := f - f_1$, and $F_i := \int_B f_i dy$ for $i \in \{1, 2\}$. Because $\varphi(x, \cdot)$ is convex and increasing,

$$(2.7) \quad \varphi\left(x, \frac{\beta}{4} \min\left\{\alpha, \int_B f dy\right\}\right) \leq \varphi\left(x, \frac{\beta}{2} \min\{\alpha, F_1\}\right) + \varphi(x, \beta F_2).$$

We start by estimating the first term on the right-hand side of (2.7). Suppose first that $\frac{1}{2} \min\{\alpha, F_1\} > \sigma$. Then, by definition of α , $\frac{1}{2} \min\{\alpha, F_1\} \in [\sigma, (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)]$. Thus, Lemma 2.8 and the monotonicity of $\varphi(x, \cdot)$ and φ_B^- yield $\varphi\left(x, \frac{\beta}{2} \min\{\alpha, F_1\}\right) \leq \varphi_B^-\left(\frac{1}{2} F_1\right)$. Using now [25, Lemma 4.3], we obtain

$$\varphi\left(x, \frac{\beta}{2} \min\{\alpha, F_1\}\right) \leq \varphi_B^-\left(\frac{1}{2} F_1\right) \leq \int_B \varphi_B^-(f_1(y)) dy \leq \int_B \varphi(y, f_1(y)) dy.$$

Next, suppose that $\frac{1}{2} \min\{\alpha, F_1\} \leq \sigma$. Then, $\min\left\{\frac{\alpha}{2\sigma}, \frac{F_1}{2\sigma}\right\} \leq 1$, which, together with the convexity of $\varphi(x, \cdot)$ and the condition $\varphi(x, 0) = 0$, yields

$$\varphi\left(x, \frac{\beta}{2} \min\{\alpha, F_1\}\right) = \varphi\left(x, \beta\sigma \min\left\{\frac{\alpha}{2\sigma}, \frac{F_1}{2\sigma}\right\}\right) \leq \varphi\left(x, \beta\sigma \min\left\{\frac{\alpha}{2\sigma}, \frac{F_1}{2\sigma}\right\}\right) \leq \varphi\left(x, \beta\sigma\right) \frac{F_1}{2\sigma}.$$

For the same reason, if $f_1(y) \geq \sigma$, then $\frac{f_1(y)}{\sigma} \varphi(y, \sigma) \leq \varphi(y, f_1(y))$. If $f_1(y) = 0$, this inequality also holds, so it holds for all $y \in B$. Then, using these estimates, the inequalities $\varphi(\cdot, \beta\sigma) \leq 1 \leq \varphi(\cdot, \sigma)$ given by (A0), and the definition of F_1 , we conclude that

$$\begin{aligned} \varphi\left(x, \frac{\beta}{2} \min\{\alpha, F_1\}\right) &\leq \varphi\left(x, \beta\sigma\right) \frac{F_1}{2\sigma} \leq \frac{1}{2\sigma} \int_B f_1(y) dy \leq \frac{1}{2} \int_B \frac{f_1(y)}{\sigma} \varphi(y, \sigma) dy \\ &\leq \frac{1}{2} \int_B \varphi(y, f_1(y)) dy. \end{aligned}$$

To estimate the second term on the right-hand side of (2.7), we invoke the convexity and monotonicity of $\varphi(x, \cdot)$ and Assumption (A2) to obtain

$$\varphi\left(x, \beta F_2\right) \leq \int_B \varphi\left(x, \beta f_2(y)\right) dy \leq \int_B \varphi\left(y, f_2(y)\right) dy + h(x) + \int_B h(y) dy.$$

Recalling that $\alpha = (\varphi_B^-)^{-1}\left(\frac{1}{|B|}\right)$ and $\varphi(x, 0) = 0$, the claim follows. \square

3. AUXILIARY RESULTS

Lemma 3.1. *Let Ω and U be two open subsets of \mathbb{R}^n such that $U \subset\subset \Omega$. Let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (aDec) and (loc), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying*

(1.1) and (1.2). Then,

$$(3.1) \quad \lim_{\varepsilon \rightarrow 0^+} \varrho_{\#,U}^\varepsilon(f) \approx \varrho_{\varphi(\cdot),U}(c_n |\nabla f|)$$

for all $f \in C^\infty(\Omega)$, where $c_n := \int_{B(0,1)} |x \cdot e_1| dx$. If, in addition, $\varphi(x, \cdot)$ is continuous for all $x \in U$, then (3.1) holds with equality.

Remark 3.2. Note that $\varphi(x, \cdot)$ is continuous for all $x \in \Omega$ if $\varphi \in \Phi(\Omega)$ satisfies (aDec).

Proof. Let $f \in C^\infty(\Omega)$. We start by treating the case in which $\varphi(x, \cdot)$ is continuous for all $x \in U$. We claim that

$$(3.2) \quad \lim_{\varepsilon \rightarrow 0^+} \varrho_{\#,U}^\varepsilon(f) = \varrho_{\varphi(\cdot),U}(c_n |\nabla f|).$$

By the Taylor expansion formula, for any $x \in \Omega$ and $y \in \Omega$ such that $|y - x| < \text{dist}(x, \partial\Omega)$, we have

$$f(y) = f(x) + \nabla f(x) \cdot (y - x) + R(x, y),$$

where $R(x, y) = o(|x - y|)$ as $y \rightarrow x$. Denote $h(x, r) := \frac{2}{r} \int_{B(x,r)} |R(x, y)| dy$, for $r > 0$ and $x \in \Omega_r$, and $c_n = \int_{B(0,1)} |x \cdot e_1| dx$. As proved in [27, Lemma 3.1], we have the point-wise estimate

$$c_n r |\nabla f(x)| - r h(x, r) \leq M_{B(x,r)}^\# f \leq c_n r |\nabla f(x)| + r h(x, r).$$

Consequently, we can define a function $\alpha : \Omega \times \mathbb{R}^+ \rightarrow [-1, 1]$ such that $\frac{1}{r} M_{B(x,r)}^\# f = c_n |\nabla f(x)| + \alpha(x, r) h(x, r)$ when $x \in \Omega_r$. Then,

$$(3.3) \quad \varrho_{\#,U}^\varepsilon(f) = \int_0^\infty \int_{U_r} \varphi(x, c_n |\nabla f(x)| + \alpha(x, r) h(x, r)) dx \psi_\varepsilon(r) dr.$$

Next, we prove that $\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#,U}^\varepsilon(f) \leq \varrho_{\varphi(\cdot),U}(c_n |\nabla f|)$. Fix $\delta > 0$. Because $c_n |\nabla f(x)| + \alpha(x, r) h(x, r) \leq c_n |\nabla f(x)| + h(x, r)$, the monotonicity of φ and (2.5) yield

$$\varphi(x, c_n |\nabla f(x)| + \alpha(x, r) h(x, r)) \leq \varphi(x, (1 + \delta) c_n |\nabla f(x)|) + \frac{1}{c_\downarrow} \left(1 + \frac{1}{\delta}\right)^{\varphi_\downarrow} \varphi(x, h(x, r)).$$

Hence, invoking (3.3) and (1.1), we obtain

$$(3.4) \quad \begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#,U}^\varepsilon(f) &\leq \varrho_{\varphi(\cdot),U}((1 + \delta) c_n |\nabla f|) \\ &+ \frac{1}{c_\downarrow} \left(1 + \frac{1}{\delta}\right)^{\varphi_\downarrow} \limsup_{\varepsilon \rightarrow 0^+} \int_0^\infty \int_{U_r} \varphi(x, h(x, r)) dx \psi_\varepsilon(r) dr. \end{aligned}$$

We claim that

$$(3.5) \quad \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \int_{U_r} \varphi(x, h(x, r)) dx \psi_\varepsilon(r) dr = 0,$$

from which the estimate on the upper limit follows by dominated convergence as $\delta \rightarrow 0^+$ in (3.4) taking also into account the continuity of $\varphi(x, \cdot)$.

To prove (3.5), we start by observing that because U is bounded, the set U_r is empty for all $r > 0$ sufficiently large. Thus, there exists $r_0 > 0$ for which we have

$$\int_0^\infty \int_{U_r} \varphi(x, h(x, r)) dx \psi_\varepsilon(r) dr = \int_0^{r_0} \int_{U_r} \varphi(x, h(x, r)) dx \psi_\varepsilon(r) dr.$$

Moreover, since $f \in C^2$, $|R(x, y)| \leq \|f\|_{W^{2,\infty}(U)} |x - y|^2$ for all $r \in (0, r_0)$, $x \in U_r$, and $y \in B(x, r)$. Set $C := \|f\|_{W^{2,\infty}(U)}$. Then, for all $r \in (0, r_0)$ and $x \in U_r$, $h(x, r) \leq 2Cr$. Fix $0 < \gamma < \min\{1, r_0\}$. We have that $\varphi(x, h(x, r)) \leq \varphi(x, 2Cr_0)$ whenever $r \in (\gamma, r_0)$; moreover, denoting by c the monotonicity constant of $t \mapsto \frac{\varphi(x,t)}{t}$, we have also

$\varphi(x, h(x, r)) \leq c\gamma\varphi(x, 2C)$ whenever $r \in (0, \gamma]$. Using, in addition, the condition $\varphi \geq 0$ and (1.1), it follows that

$$\int_0^{r_0} \int_{U_r} \varphi(x, h(x, r)) dx \psi_\varepsilon(r) dr \leq c\gamma \int_U \varphi(x, 2C) dx + \int_U \varphi(x, 2Cr_0) dx \int_\gamma^\infty \psi_\varepsilon(r) dr.$$

In view of (1.2) and Assumption (loc), together with Remark 2.7, letting $\varepsilon \rightarrow 0^+$ in this estimate first, and then $\gamma \rightarrow 0^+$, we obtain (3.5).

Finally, we prove that $\liminf_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \geq \varrho_{\varphi(\cdot), U}(c_n |\nabla f|)$. Fix $\delta > 0$, and denote $a' := c_n |\nabla f(x)|$ and $b' := \alpha(x, r)h(x, r)$. If $r > 0$ and $x \in U_r$ are such that $\alpha(x, r) < 0$, then applying (2.5) with $a := \frac{a'+b'}{1+\delta}$ and $b := -\frac{b'}{1+\delta}$ gives

$$(3.6) \quad \begin{aligned} \varphi(x, c_n |\nabla f(x)| + \alpha(x, r)h(x, r)) &= \varphi(x, (1+\delta)a) \geq \varphi(x, a+b) - \frac{1}{c_\downarrow} (1 + \frac{1}{\delta})^{\varphi_\downarrow} \varphi(x, b) \\ &\geq \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) - \frac{1}{c_\downarrow} (1 + \frac{1}{\delta})^{\varphi_\downarrow} \varphi(x, h(x, r)), \end{aligned}$$

where we used the monotonicity of φ and the estimate $0 \leq -\frac{b'}{1+\delta} \leq h(x, r)$. If $r > 0$ and $x \in U_r$ are such that $\alpha(x, r) \geq 0$, then the monotonicity of φ yields

$$(3.7) \quad \varphi(x, c_n |\nabla f(x)| + \alpha(x, r)h(x, r)) \geq \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}).$$

Thus, by (3.6) and (3.7), for every $r > 0$ and $x \in U_r$, we have

$$\varphi(x, c_n |\nabla f(x)| + \alpha(x, r)h(x, r)) \geq \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) - \frac{1}{c_\downarrow} (1 + \frac{1}{\delta})^{\varphi_\downarrow} \varphi(x, h(x, r)).$$

Consequently, by (3.3), it follows that

$$(3.8) \quad \varrho_{\#, U}^\varepsilon(f) \geq \int_0^\infty \int_{U_r} \left[\varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) - \frac{1}{c_\downarrow} (1 + \frac{1}{\delta})^{\varphi_\downarrow} \varphi(x, h(x, r)) \right] dx \psi_\varepsilon(r) dr.$$

Fix $0 < \gamma < 1$ and $r_0 > 0$ such that $U_{r_0} \neq \emptyset$. By (1.1) and (1.2), there exists $\varepsilon_0 = \varepsilon_0(\gamma, r_0)$ such that, for all $\varepsilon \in (0, \varepsilon_0)$, we have

$$\int_0^{r_0} \psi_\varepsilon(r) dr \geq 1 - \gamma.$$

Thus, we have also, for $0 < \varepsilon < \varepsilon_0$,

$$\begin{aligned} \int_0^\infty \int_{U_r} \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) dx \psi_\varepsilon(r) dr &\geq \int_0^{r_0} \int_{U_{r_0}} \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) dx \psi_\varepsilon(r) dr \\ &\geq (1 - \gamma) \int_{U_{r_0}} \varphi(x, \frac{c_n |\nabla f(x)|}{1+\delta}) dx. \end{aligned}$$

This estimate, (3.8), and (3.5) yield

$$\liminf_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \geq (1 - \gamma) \varrho_{\varphi(\cdot), U_{r_0}}(\frac{c_n |\nabla f|}{1+\delta}).$$

The conclusion then follows by using the continuity of $\varphi(x, \cdot)$ and by letting $\delta \rightarrow 0^+$, $\gamma \rightarrow 0^+$, and $r_0 \rightarrow 0^+$ in this order. This completes the proof of (3.2) under the continuity assumption.

Suppose now that φ is a general weak Φ -function. By Remarks 2.3 and 2.7, there exists $\psi \in \Phi(\Omega)$ satisfying the same assumptions as φ and such that $\psi \approx \varphi$. Then, there is $C > 0$ such that $\frac{1}{C}\varphi \leq \psi \leq C\varphi$. Hence, by the first part of the proof,

$$\begin{aligned} \frac{1}{C^2} \varrho_{\varphi(\cdot), U}(c_n |\nabla f|) &\leq \frac{1}{C} \varrho_{\psi(\cdot), U}(c_n |\nabla f|) = \frac{1}{C} \liminf_{\varepsilon \rightarrow 0^+} \varrho_{\#, U, \psi}^\varepsilon(f) \leq \liminf_{\varepsilon \rightarrow 0^+} \varrho_{\#, U, \varphi}^\varepsilon(f) \\ &\leq \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U, \varphi}^\varepsilon(f) \leq C \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U, \psi}^\varepsilon(f) = C \varrho_{\psi(\cdot), U}(c_n |\nabla f|) \leq C^2 \varrho_{\varphi(\cdot), U}(c_n |\nabla f|). \square \end{aligned}$$

Next, we derive an auxiliary upper bound which holds for all functions $f \in L^1_{\text{loc}}(\Omega)$ with $|\nabla f| \in L^{\varphi(\cdot)}(\Omega)$.

Lemma 3.3. *Let Ω be an open set of \mathbb{R}^n , let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (A) and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). If $f \in L^1_{\text{loc}}(\Omega)$ and $|\nabla f| \in L^{\varphi(\cdot)}(\Omega)$, then*

$$\varrho_{\#, \Omega}^\varepsilon(f) \leq c \max \left\{ \|\nabla f\|_{\varphi(\cdot), \Omega}^{\varphi^\uparrow}, \|\nabla f\|_{\varphi(\cdot), \Omega}^{\varphi^\downarrow} \right\}.$$

Proof. By Remarks 2.3, 2.6, and 2.7 we may assume without loss of generality that $\varphi \in \Phi(\Omega)$. Then, in view of Assumptions (A0) and (aInc)₁, we have $L^{\varphi(\cdot)}(B) \subset L^1(B)$ for every bounded set $B \subset \Omega$ by [22, Lemma 4.4].

Let $f \in L^1_{\text{loc}}(\Omega)$ with $|\nabla f| \in L^{\varphi(\cdot)}(\Omega)$ be such that $\|\nabla f\|_{\varphi(\cdot), \Omega} \leq 1$. Fix $r > 0$ and $x \in \Omega_r$. By the Poincaré inequality in L^1 , we have

$$(3.9) \quad \frac{1}{r} M_{B(x,r)}^\# f = \frac{1}{r} \int_{B(x,r)} |f(y) - f_{B(x,r)}| dy \leq c \int_{B(x,r)} |\nabla f(y)| dy.$$

On the other hand, in view of Lemma 2.8, we may invoke [26, Lemma 4.4] (with $\gamma = 1$) that gives the existence of a constant, $\beta' \in (0, 1)$, depending only on the constants in (A), such that

$$(3.10) \quad \varphi \left(x, \beta' \int_{B(x,r)} |\nabla f(y)| dy \right) \leq \int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy + h(x) + \int_{B(x,r)} h(y) dy.$$

Using the monotonicity of φ , (aDec), (3.9), and (3.10), we obtain

$$(3.11) \quad \begin{aligned} \varrho_{\#, \Omega}^\varepsilon(f) &\lesssim \int_0^\infty \int_{\Omega_r} \varphi \left(x, \beta' \int_{B(x,r)} |\nabla f(y)| dy \right) dx \psi_\varepsilon(r) dr \\ &\leq \int_0^\infty \int_{\Omega_r} \left(\int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy + h(x) + \int_{B(x,r)} h(y) dy \right) dx \psi_\varepsilon(r) dr. \end{aligned}$$

Next, by changing the order of integration, we observe that

$$\begin{aligned} \int_{\Omega_r} \int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy dx &= \int_{\Omega_r} \int_{\Omega} \varphi(y, |\nabla f(y)|) \frac{\chi_{B(y,r)}(x)}{|B(x,r)|} dy dx \\ &\leq \int_{\Omega} \varphi(y, |\nabla f(y)|) dy = \varrho_{\varphi(\cdot), \Omega}(|\nabla f|) \leq 1. \end{aligned}$$

Similarly,

$$\int_{\Omega_r} \int_{B(x,r)} h(y) dy dx \leq \int_{\Omega} h(y) dy.$$

Consequently, because $h \in L^1(\Omega)$ and $\int_0^\infty \psi_\varepsilon(r) dr = 1$, from (3.11) we conclude that

$$(3.12) \quad \varrho_{\#, \Omega}^\varepsilon(f) \leq c \text{ for all } f \in L^1_{\text{loc}}(\Omega) \text{ with } |\nabla f| \in L^{\varphi(\cdot)}(\Omega) \text{ and } \|\nabla f\|_{\varphi(\cdot), \Omega} \leq 1.$$

By considering the cases $\lambda \leq 1$ and $\lambda > 1$ and use Assumptions (aInc)₁ and (aDec), respectively, we find that $\varrho_{\#, \Omega}^\varepsilon(f) \leq c \varrho_{\#, \Omega}^\varepsilon(\frac{f}{\lambda}) \max\{\lambda^{\varphi^\uparrow}, \lambda^{\varphi^\downarrow}\}$ for all $\lambda > 0$ and $f \in L^1_{\text{loc}}(\Omega)$ with $|\nabla f| \in L^{\varphi(\cdot)}(\Omega)$. Using this estimate with $\lambda := \|\nabla f\|_{\varphi(\cdot), \Omega} + \delta$ for $\delta > 0$ and then invoking (3.12), it follows that

$$\begin{aligned} \varrho_{\#, \Omega}^\varepsilon(f) &\leq c \varrho_{\#, \Omega}^\varepsilon \left(\frac{f}{\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta} \right) \max \left\{ (\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)^{\varphi^\uparrow}, (\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)^{\varphi^\downarrow} \right\} \\ &\leq c \max \left\{ (\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)^{\varphi^\uparrow}, (\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)^{\varphi^\downarrow} \right\}. \end{aligned}$$

Letting $\delta \rightarrow 0^+$, we conclude the proof of Lemma 3.3. \square

4. MAIN RESULTS

In this section we prove our main result, which provides a characterization of generalized Orlicz spaces. Theorem 1.1 is an immediate consequence of Propositions 4.1, 4.3 and 4.5 below.

Proposition 4.1. *Let $\Omega \subset \mathbb{R}^n$ be open, let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (A), (aInc), and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). Assume that $f \in L^1_{\text{loc}}(\Omega)$ with $|\nabla f| \in L^{\varphi(\cdot)}(\Omega)$. Then,*

$$(4.1) \quad \lim_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) \approx \varrho_{\varphi(\cdot), \Omega}(c_n |\nabla f|) < \infty$$

where $c_n = \int_{B(0,1)} |x \cdot e_1| dx$. If, in addition, $\varphi(x, \cdot)$ is continuous for all $x \in \Omega$, then (4.1) holds with equality.

Proof. We start with the upper bound. Let $U \subset\subset \Omega$. Using the same arguments as in [22, Theorems 6.5 and 6.6], we find $g_\nu \in C^\infty(\Omega)$, $\nu \in \mathbb{N}$, such that $(\nabla g_\nu)_{n \in \mathbb{N}}$ converges to ∇f in $L^{\varphi(\cdot)}(U; \mathbb{R}^n)$. For $r > 0$ and $x \in U_r$, we have $M_{B(x,r)}^\# f \leq M_{B(x,r)}^\#(f - g_\nu) + M_{B(x,r)}^\# g_\nu$ by the triangle inequality. Combining this estimate with (2.5) applied to $a = \frac{1}{r} M_{B(x,r)}^\# g_\nu$ and $b = \frac{1}{r} M_{B(x,r)}^\#(f - g_\nu)$, we obtain

$$\varrho_{\#, U}^\varepsilon(f) \leq \varrho_{\#, U}^\varepsilon((1 + \delta)g_\nu) + c_\delta \varrho_{\#, U}^\varepsilon(f - g_\nu)$$

for all $\delta > 0$, where we also used the monotonicity of $\varphi(x, \cdot)$. Invoking now Lemmas 3.1 and 3.3, we conclude that

$$(4.2) \quad \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \leq \varrho_{\varphi(\cdot), U}(c_n(1 + \delta)|\nabla g_\nu|) + c_\delta \max \left\{ \|\nabla(f - g_\nu)\|_{\varphi(\cdot), U}^{\varphi^\uparrow}, \|\nabla(f - g_\nu)\|_{\varphi(\cdot), U}^{\varphi^\downarrow} \right\}.$$

Because $\|\nabla(f - g_\nu)\|_{\varphi(\cdot), U} \rightarrow 0$, it follows from (aDec) that $\lim_{\nu \rightarrow \infty} \varrho_{\varphi(\cdot), U}(c_n(1 + \delta)|\nabla(f - g_\nu)|) = 0$. Thus, letting $\nu \rightarrow \infty$ in (4.2) and using (2.5) again, we obtain

$$(4.3) \quad \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \leq \varrho_{\varphi(\cdot), U}(c_n(1 + \delta)^2|\nabla f|) \leq \varrho_{\varphi(\cdot), \Omega}(c_n(1 + \delta)^2|\nabla f|).$$

By (aDec), $\varphi(x, c_n(1 + \delta)^2|\nabla f(x)|) \leq c\varphi(x, c_n|\nabla f(x)|)$. This proves that $\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \leq c\varrho_{\varphi(\cdot), U}(c_n|\nabla f|)$ for a general φ . If, in addition, $\varphi(x, \cdot)$ is continuous, we use $c\varphi(x, c_n|\nabla f(x)|)$ as a majorant and let $\delta \rightarrow 0$ in (4.3). Then, Lebesgue's dominated convergence theorem yields

$$\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \leq \varrho_{\varphi(\cdot), \Omega}(c_n|\nabla f|).$$

We claim that

$$(4.4) \quad \limsup_{U \nearrow \Omega} \limsup_{\varepsilon \rightarrow 0^+} \int_0^\infty \int_{\Omega_r \setminus U_r} \varphi(x, \frac{1}{r} M_{B(x,r)}^\# f) dx \psi_\varepsilon(r) dr = 0,$$

which, together with the bound on $\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f)$ established above, concludes the proof of the upper bound.

To prove (4.4), we observe that replacing f by $\tilde{f} := \|\nabla f\|_{\varphi(\cdot), \Omega}^{-1} f$ if necessary, we may assume that $\|\nabla f\|_{\varphi(\cdot), \Omega} \leq 1$. Indeed, by (aDec), if (4.4) holds for \tilde{f} , then it also holds for f . Arguing as in Lemma 3.3, we find that

$$(4.5) \quad \begin{aligned} & \int_0^\infty \int_{\Omega_r \setminus U_r} \varphi(x, \frac{1}{r} M_{B(x,r)}^\# f) dx \psi_\varepsilon(r) dr \\ & \leq c \int_0^\infty \int_{\Omega_r \setminus U_r} \left(\int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy + h(x) + \int_{B(x,r)} h(y) dy \right) dx \psi_\varepsilon(r) dr. \end{aligned}$$

Let $\gamma > 0$. If $r \leq \frac{\gamma}{2}$, then $y \in \Omega \setminus U_\gamma$ whenever $x \in \Omega_r \setminus U_r$. Therefore, we may estimate the right-hand side of (4.5) as follows:

$$(4.6) \quad \begin{aligned} & \int_0^\infty \int_{\Omega_r \setminus U_r} \left(\int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy + h(x) + \int_{B(x,r)} h(y) dy \right) dx \psi_\varepsilon(r) dr \\ & \leq \int_0^\infty \int_{\Omega_r \setminus U_r} \left(\int_{\Omega \setminus U_\gamma} (\varphi(y, |\nabla f(y)|) + h(y)) \frac{\chi_{B(y,r)}(x)}{|B(0,r)|} dy + h(x) \right) dx \psi_\varepsilon(r) dr \\ & \quad + \int_{\frac{\gamma}{2}}^\infty \int_{\Omega_r} \left(\int_{B(x,r)} \varphi(y, |\nabla f(y)|) dy + h(x) + \int_{B(x,r)} h(y) dy \right) dx \psi_\varepsilon(r) dr. \end{aligned}$$

Changing the order of integration as in Lemma 3.3, from (4.5) and (4.6), we obtain

$$\begin{aligned} & \int_0^\infty \int_{\Omega_r \setminus U_r} \varphi\left(x, \frac{1}{r} M_{B(x,r)}^\# f\right) dx \psi_\varepsilon(r) dr \\ & \lesssim \varrho_{\varphi(\cdot), \Omega \setminus U_\gamma}(|\nabla f|) + 2\|h\|_{L^1(\Omega \setminus U_\gamma)} + \left(\varrho_{\varphi(\cdot), \Omega}(|\nabla f|) + 2\|h\|_{L^1(\Omega)} \right) \int_{\frac{\gamma}{2}}^\infty \psi_\varepsilon(r) dr. \end{aligned}$$

Then, (4.4) follows by (1.2) as $\varepsilon \rightarrow 0$, $\gamma \rightarrow 0$, and $U \rightarrow \Omega$, in this order.

It remains to prove the lower bound. The proof of this estimate is similar to the proof of the upper bound once we show that for all $U \subset \subset \Omega$, we have

$$(4.7) \quad \varrho_{\#, U}^\varepsilon(f) \geq \varrho_{\#, U}^\varepsilon\left(\frac{1}{1+\delta} g_\nu\right) - c_\delta \varrho_{\#, U}^\varepsilon(f - g_\nu),$$

where $c_\delta = \frac{1}{c_\downarrow} \left(1 + \frac{1}{\delta}\right)^{\varphi_\downarrow}$. Fix $r > 0$ and $x \in U_r$, and set $a' := \frac{1}{r} M_{B(x,r)}^\# g_\nu$ and $b' := \frac{1}{r} M_{B(x,r)}^\#(f - g_\nu)$. Note that $\frac{1}{r} M_{B(x,r)}^\# f \geq |a' - b'|$. We consider two cases:

- If $a' - b' \geq 0$, then using (2.5) with $a = \frac{a' - b'}{1+\delta}$ and $b = \frac{b'}{1+\delta}$, we obtain

$$\varphi(x, a' - b') \geq \varphi\left(x, \frac{a'}{1+\delta}\right) - c_\delta \varphi\left(x, \frac{b'}{1+\delta}\right) \geq \varphi\left(x, \frac{a'}{1+\delta}\right) - c_\delta \varphi(x, b'),$$

where in the last inequality we used the fact that $\varphi(x, \cdot)$ is increasing and $\frac{1}{1+\delta} \leq 1$.

- If $a' - b' \leq 0$, then

$$\varphi\left(x, \frac{a'}{1+\delta}\right) \leq \varphi\left(x, \frac{b'}{1+\delta}\right) \leq c_\delta \varphi(x, b')$$

because $\varphi(x, \cdot)$ is increasing, $\frac{1}{1+\delta} \leq 1$, and $c_\delta \geq 1$. Thus,

$$0 \geq \varphi\left(x, \frac{a'}{1+\delta}\right) - c_\delta \varphi(x, b').$$

Splitting the inner integral defining $\varrho_{\#, U}^\varepsilon(f)$ according to these two cases, we conclude that (4.7) holds. \square

Remark 4.2. In the proof of the previous proposition the assumption (aInc) was used only for the density of smooth functions in the Sobolev space. Presumably, (aInc) is not really needed for this, but it was used in the cited reference.

Proposition 4.3. *Under the assumptions of Proposition 4.1, we have*

$$(4.8) \quad \lim_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon \approx c_n \|\nabla f\|_{\varphi(\cdot), \Omega}.$$

If, in addition, $\varphi \in \Phi(\Omega)$, then (4.8) holds with equality.

Proof. Suppose first that $\varphi \in \Phi(\Omega)$. Since φ is also doubling, $\varphi(x, \cdot)$ is continuous; thus, the modular inequality (4.1) holds with equality. Fixing $\delta > 0$ and applying this equality to the function

$$g_\delta := \frac{f}{c_n(1+\delta)(\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)},$$

we obtain

$$\lim_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(g_\delta) = \varrho_{\varphi(\cdot), \Omega}(c_n |\nabla g_\delta|) \leq \frac{1}{1+\delta} \int_{\Omega} \varphi\left(x, \frac{|\nabla f|}{\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta}\right) dx < 1,$$

where we used convexity in the last estimate. Thus, $\varrho_{\#, \Omega}^\varepsilon(g_\delta) \leq 1$ for all sufficiently small $\varepsilon > 0$. Consequently, also $\|g_\delta\|_{\#, \Omega}^\varepsilon \leq 1$ for all sufficiently small $\varepsilon > 0$; that is, $\|f\|_{\#, \Omega}^\varepsilon \leq c_n(1+\delta)(\|\nabla f\|_{\varphi(\cdot), \Omega} + \delta)$. Letting $\varepsilon \rightarrow 0$ first and then $\delta \rightarrow 0$, we obtain $\limsup_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon \leq c_n \|\nabla f\|_{\varphi(\cdot), \Omega}$.

Next, we prove the opposite inequality. Fix $\delta > 0$, let $\varepsilon_j \rightarrow 0$ as $j \rightarrow \infty$ be such that $\liminf_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon = \lim_{j \rightarrow \infty} \|f\|_{\#, \Omega}^{\varepsilon_j}$, and set

$$g_\delta := \frac{f}{\lim_{j \rightarrow \infty} \|f\|_{\#, \Omega}^{\varepsilon_j} + \delta}.$$

Because $\lim_{j \rightarrow \infty} \|g_\delta\|_{\#, \Omega}^{\varepsilon_j} < 1$, we conclude that $\|g_\delta\|_{\#, \Omega}^{\varepsilon_j} \leq 1$ for all sufficiently large $j \in \mathbb{N}$. For all such $j \in \mathbb{N}$, $\varrho_{\#, \Omega}^{\varepsilon_j}(g_\delta) \leq 1$ by the definition of the norm. Letting $j \rightarrow \infty$, the previous proposition yields $\varrho_{\varphi(\cdot), \Omega}(c_n |\nabla g_\delta|) \leq 1$; so, $\|c_n |\nabla g_\delta|\|_{\varphi(\cdot), \Omega} \leq 1$. Therefore, $\|c_n |\nabla f|\|_{\varphi(\cdot), \Omega} \leq \lim_{j \rightarrow \infty} \|f\|_{\#, \Omega}^{\varepsilon_j} + \delta = \liminf_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon + \delta$. Letting $\delta \rightarrow 0$, we obtain the desired inequality.

This completes the proof in the case $\varphi \in \Phi(\Omega)$. If $\varphi \in \Phi_w(\Omega)$, we find $\psi \in \Phi(\Omega)$ with $\varphi \simeq \psi$ (Remarks 2.3 and 2.7). Then, $\|\nabla f\|_{\varphi(\cdot)} \approx \|\nabla f\|_{\psi(\cdot)}$ and similarly for the $\#$ -norm; so, the claim follows from the first part. \square

The next lemma shows that the condition $\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty$ implies that f is locally in a Sobolev space. The estimate for the norm obtained in this way is not uniform, however. Nevertheless, this information is used later to prove a uniform estimate.

Lemma 4.4. *Let $\Omega \subset \mathbb{R}^n$ be open, let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (A), (aInc), and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). Assume that $f \in L_{\text{loc}}^1(\Omega)$ and*

$$\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty.$$

Then, there is a constant $c > 0$ such that, for every $U \subset\subset \Omega$, $\int_U \varphi_U^-(|\nabla f(x)|) dx \leq c$. Moreover, $\nabla f \in L_{\text{loc}}^{\varphi_U^+}(\Omega; \mathbb{R}^n)$.

Proof. Let $U \subset\subset \Omega$ and note that $\varphi_U^- \in \Phi_w$ (the only nontrivial condition is the limit at infinity, which follows from (A0)). By [22, Lemma 2.2], there exists $\xi \in \Phi$ with $\xi \simeq \varphi_U^-$ for which the equivalent constant depends only on the monotonicity constant of $t \mapsto \frac{\varphi(x,t)}{t}$. Then, by (aDec), we can find $c > 0$, independent of U , such that $c^{-1} \varphi_U^- \leq \xi \leq c \varphi_U^-$.

Fix $\delta > 0$, and let G_δ be a standard mollifier; that is, $G_\delta(x) = \delta^{-n} G(x/\delta)$, where $G \in C_0^\infty(B(0,1))$ is a non-negative function, radially symmetric, and $\int_{B(0,1)} G(x) dx = 1$. Let $\tau > 0$ be such that $U \subset\subset \Omega_\tau$. Then, for $0 < \delta < \tau$, $G_\delta * f \in C^\infty(\Omega_\tau)$ and, by Lemma 3.1 with $\varphi = \xi$, $U := U_\delta$, and $\Omega := \Omega_\tau$,

$$(4.9) \quad \lim_{\varepsilon \rightarrow 0^+} \int_0^\infty \int_{U_{\delta+r}} \xi\left(\frac{1}{r} M_{B(x,r)}^\#(G_\delta * f)\right) dx \psi_\varepsilon(r) dr = \varrho_{\xi, U_\delta}(c_n |\nabla(G_\delta * f)|) \\ \geq \varrho_{\xi, U_\tau}(c_n |\nabla(G_\delta * f)|).$$

On the other hand, by the triangle inequality and a change on the order of integration, we obtain, for all $r > 0$ and $x \in U_r$,

$$(4.10) \quad \begin{aligned} M_{B(x,r)}^\#(G_\delta * f) &= \int_{B(x,r)} |(G_\delta * f)(y) - (G_\delta * f)_{B(x,r)}| dy \\ &= \int_{B(x,r)} |(G_\delta * f)(y) - (G_\delta * f_{B(\cdot,r)})(x)| dy \leq (G_\delta * M_{B(\cdot,r)}^\# f)(x). \end{aligned}$$

Hence, the monotonicity of ξ yields

$$\xi\left(\frac{1}{r}M_{B(x,r)}^\#(G_\delta * f)\right) \leq \xi\left((G_\delta * \frac{1}{r}M_{B(\cdot,r)}^\# f)(x)\right).$$

For $x \in \Omega_r$, define $g_r(x) := \frac{1}{r}M_{B(x,r)}^\# f$. Since $G_\delta dx$ is a probability measure and ξ is convex,

$$\xi\left(\frac{1}{r}M_{B(x,r)}^\#(G_\delta * f)\right) \leq \xi((G_\delta * g_r)(x)) \leq (G_\delta * \xi(g_r))(x).$$

Integrating $G_\delta * \xi(g_r)$ over $U_{r+\delta}$ and changing variables and the order of integration, we obtain

$$\begin{aligned} \int_{U_{\delta+r}} \int_{B(x,\delta)} G_\delta(x-z)\xi(g_r(z)) dz dx &= \int_{B(0,\delta)} G_\delta(w) \int_{U_{\delta+r}} \xi(g_r(x-w)) dx dw \\ &\leq \int_{U_r} \xi(g_r(y)) dy. \end{aligned}$$

Combining the previous two estimates, we conclude that

$$\int_{U_{\delta+r}} \xi\left(\frac{1}{r}M_{B(x,r)}^\#(G_\delta * f)\right) dx \leq \int_{U_r} \xi(g_r(x)) dx \leq c \int_{U_r} \varphi\left(x, \frac{1}{r}M_{B(x,r)}^\# f\right) dx,$$

where in the last inequality we used the definition of g_r and the estimate $\xi(t) \leq c\varphi(x, t)$ for $x \in U$. Consequently,

$$(4.11) \quad \begin{aligned} \limsup_{\varepsilon \rightarrow 0^+} \int_0^\infty \int_{U_{\delta+r}} \xi\left(\frac{1}{r}M_{B(x,r)}^\#(G_\delta * f)\right) dx \psi_\varepsilon(r) dr &\leq c \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#,U}^\varepsilon(f) \\ &\leq c \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#,\Omega}^\varepsilon(f) < \infty, \end{aligned}$$

which, together with (4.9), shows that $(\nabla(G_\delta * f))_\delta$ is bounded in $L^\xi(U_\tau; \mathbb{R}^n)$.

By (A0), (aInc), and (aDec), $L^\xi(U_\tau; \mathbb{R}^n)$ is reflexive (see [22, Lemma 2.4 and Proposition 4.6]). Since $(\nabla(G_\delta * f))_\delta$ is a bounded sequence, it has a subsequence which converges weakly in $L^\xi(U_\tau; \mathbb{R}^n)$ to a function $l \in L^\xi(U_\tau; \mathbb{R}^n)$. Using the fact that $G_\delta * f$ converges to f in $L^1(U)$, it follows from the definition of weak derivative that $l = \nabla f$. By weak semi-continuity [17, Theorem 2.2.8], (4.9), and (4.11), we obtain that

$$\begin{aligned} \int_{U_\tau} \varphi_U^-(c_n |\nabla f(x)|) dx &\leq c \rho_{\xi,U_\tau}(c_n |\nabla f|) \leq c \liminf_{\delta \rightarrow 0^+} \rho_{\xi,U_\tau}(c_n |\nabla(G_\delta * f)|) \\ &\leq c \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#,\Omega}^\varepsilon(f) < \infty. \end{aligned}$$

Letting $\tau \rightarrow 0$, we obtain $\int_U \varphi_U^-(|\nabla f(x)|) dx \leq c$ by monotone convergence and (aDec).

Finally, we observe that Assumptions (aInc) and (A0) imply that $t^{\varphi^\dagger} \leq c(\varphi_U^-(t) + 1)$ with $c > 0$ independent of U . Hence,

$$\int_U |\nabla f(x)|^{\varphi^\dagger} dx \leq c \int_U \varphi_U^-(|\nabla f(x)|) + 1 dx \leq c + c|U|,$$

which yields $\nabla f \in L_{\text{loc}}^{\varphi^\dagger}(\Omega; \mathbb{R}^n)$. \square

We now remove the local-condition from the previous lemma:

Proposition 4.5. *Let $\Omega \subset \mathbb{R}^n$ be open, let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (A), (aInc), and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). Assume that $f \in L^1_{\text{loc}}(\Omega)$ and*

$$\limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty.$$

Then, $\nabla f \in L^{\varphi(\cdot)}(\Omega; \mathbb{R}^n)$.

Proof. Let $U \subset\subset V \subset\subset \Omega$ and $\delta \in (0, \frac{1}{6})$ be such that $U^\delta \subset V \subset \Omega_\delta$. Let G_δ and g_r be as in the previous lemma. By Remarks 2.3 and 2.7, there exists $\xi \in \Phi(\Omega)$ equivalent to φ which satisfies the same assumptions. In Lemma 4.4, we proved that $\int_U \xi_U^- (|\nabla f|) dx \leq c$ for every $U \subset\subset \Omega$. In view of (aDec), by scaling the function f , if necessary, we may assume that

$$(4.12) \quad \int_B \xi_B^- (|\nabla f(x)|) dx \leq 3^{-n} \text{ for every ball } B \subset\subset \Omega \text{ with } |B| \leq 1.$$

Let $r \in (0, \delta]$ and $x \in U_r$. By the Poincaré and triangle inequalities,

$$\frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) \leq c \int_{B(x,r)} |\nabla(G_\delta * f)(y)| dy \leq c \int_{B(x,r)} \int_{B(y,\delta)} G_\delta(y-z) |\nabla f(z)| dz dy.$$

Note that $G_\delta \leq c \frac{\chi_{B(0,\delta)}}{|B(0,\delta)|}$; hence, by [25, Lemma 4.3] and (4.12), together with the previous estimate, we obtain

$$\begin{aligned} \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) &\leq c \int_{B(x,r)} \int_{B(y,\delta)} |\nabla f(z)| dz dy \\ &\leq c \int_{B(x,r)} (\xi_{B(y,\delta)}^-)^{-1} \left(\int_{B(y,\delta)} \xi_{B(y,\delta)}^- (|\nabla f(z)|) dz \right) dy \\ &\leq c \int_{B(x,r)} (\xi_{B(y,\delta)}^-)^{-1} \left(\frac{1}{3^n |B(y,\delta)|} \right) dy. \end{aligned}$$

Observe that $B(y, \delta) \subset B(x, 3\delta)$ for every $y \in B(x, r)$, and $|B(y, \delta)| = |B(x, \delta)|$ and $|B(x, 3\delta)| = 3^n |B(x, \delta)|$. By Lemmas 2.9 and 2.10,

$$(\xi_{B(y,\delta)}^-)^{-1} \left(\frac{1}{3^n |B(y,\delta)|} \right) \approx (\xi_{B(x,3\delta)}^-)^{-1} \left(\frac{1}{3^n |B(x,\delta)|} \right) \approx (\xi_{B(x,\delta)}^-)^{-1} \left(\frac{1}{3^n |B(x,\delta)|} \right)$$

when $y \in B(x, r)$. Then, because $(\xi_{B(x,\delta)}^-)^{-1}$ is increasing, we obtain

$$\frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) \leq c (\xi_{B(x,\delta)}^-)^{-1} (3^{-n} |B(x, \delta)|^{-1}) \leq c (\xi_{B(x,\delta)}^-)^{-1} (|B(x, \delta)|^{-1}).$$

On the other hand, by (4.10), we have also $\frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) \leq (G_\delta * g_r)(x) \leq c \int_{B(x,\delta)} |g_r| dy$. Thus,

$$\frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) \leq c \min \left\{ \int_{B(x,\delta)} |g_r| dy, (\xi_{B(x,\delta)}^-)^{-1} \left(\frac{1}{|B(x,\delta)|} \right) \right\}.$$

Then, by the monotonicity of $\xi(x, \cdot)$ and Lemma 2.12, invoking (aDec) if necessary, it follows that

$$\xi \left(x, \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) \right) \leq c \left(\int_{B(x,\delta)} (\xi(y, g_r(y)) + h(y)) dy + h(x) \right).$$

Integrating this estimate over $x \in U_r$, changing the order of integration as in Lemma 3.3, and using the inclusion $(U_r)^\delta \subset V_r$, we obtain

$$\begin{aligned} \int_{U_r} \xi\left(x, \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f)\right) dx &\leq c \left(\int_{(U_r)^\delta} \xi(y, g_r(y)) + h(y) dy + \int_{U_r} h(x) dx \right) \\ &\leq c \left(\int_{V_r} \xi\left(y, \frac{1}{r} M_{B(y,r)}^\# f\right) dy + 2\|h\|_{L^1(\Omega)} \right) \end{aligned}$$

for $r \leq \delta$.

Next, we consider $r \in (\delta, \infty)$. Fix $x \in U_r$, and recall that $\nabla f \in L^{\varphi^\uparrow}(V; \mathbb{R}^n)$ by Lemma 4.4. By the Poincaré and Hölder inequalities, we have

$$\begin{aligned} \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f) &\leq c \int_{B(x,r)} |\nabla(G_\delta * f)(y)| dy \leq c \int_{B(x,r)} \|G_\delta\|_{(\varphi^\uparrow)', B(0,\delta)} \|\nabla f\|_{\varphi^\uparrow, B(y,\delta)} dy \\ &\leq c_\delta \|\nabla f\|_{\varphi^\uparrow, U^\delta} \leq c_\delta \|\nabla f\|_{\varphi^\uparrow, V}. \end{aligned}$$

Hence,

$$\int_{U_r} \xi\left(x, \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f)\right) dx \leq \int_{U_r} \xi(x, c_\delta \|\nabla f\|_{\varphi^\uparrow, V}) dx$$

for $r > \delta$.

Combining the two cases, $r \leq \delta$ and $r > \delta$, we obtain

$$\begin{aligned} \int_0^\infty \int_{U_r} \xi\left(x, \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f)\right) dx \psi_\varepsilon(r) dr \\ \leq c \int_0^\delta \left(\int_{V_r} \xi\left(x, \frac{1}{r} M_{B(x,r)}^\# f\right) dx + 2\|h\|_{L^1(\Omega)} \right) \psi_\varepsilon(r) dr + \int_\delta^\infty \int_{U_r} \xi(x, c_\delta \|\nabla f\|_{\varphi^\uparrow, V}) dx \psi_\varepsilon(r) dr. \end{aligned}$$

The second term on the right-hand side of the previous inequality tends to zero as $\varepsilon \rightarrow 0$ by (loc) and (1.2). Consequently,

$$\limsup_{\varepsilon \rightarrow 0} \int_0^\infty \int_{U_r} \xi\left(x, \frac{1}{r} M_{B(x,r)}^\#(G_\delta * f)\right) dx \psi_\varepsilon(r) dr \leq c \limsup_{\varepsilon \rightarrow 0} \rho_{\#, V}^\varepsilon(f) + c < \infty$$

by hypothesis, where we also used (1.1) and the fact that $\xi \approx \varphi$.

Thus, arguing as in Lemma 4.4, we conclude that $\nabla f \in L^{\xi^{(\cdot)}}(U; \mathbb{R}^n)$. Hence, also $\nabla f \in L^{\varphi^{(\cdot)}}(U; \mathbb{R}^n)$ (see Remark 2.6).

Finally, we appeal to Proposition 4.1 to conclude that

$$\varrho_{\varphi^{(\cdot)}, U}(c_n |\nabla f|) \leq c \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, U}^\varepsilon(f) \leq c \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty.$$

Because the upper bound in the last estimate is independent of U , we conclude the proof of Proposition 4.5 by monotone convergence as $U \nearrow \Omega$. \square

Combining the propositions of this section, we arrive at the following theorem.

Theorem 4.6. *Let $\Omega \subset \mathbb{R}^n$ be an open set, let $\varphi \in \Phi_w(\Omega)$ satisfy Assumptions (A), (aInc), and (aDec), and let $(\psi_\varepsilon)_\varepsilon$ be a family of functions satisfying (1.1) and (1.2). Assume that $f \in L_{\text{loc}}^1(\Omega)$. Then,*

$$\nabla f \in L^{\varphi^{(\cdot)}}(\Omega; \mathbb{R}^n) \quad \Leftrightarrow \quad \limsup_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) < \infty.$$

In this case,

$$\lim_{\varepsilon \rightarrow 0^+} \varrho_{\#, \Omega}^\varepsilon(f) \approx \varrho_{\varphi^{(\cdot)}, \Omega}(c_n |\nabla f|) \quad \text{and} \quad \lim_{\varepsilon \rightarrow 0^+} \|f\|_{\#, \Omega}^\varepsilon \approx c_n \|\nabla f\|_{\varphi^{(\cdot)}, \Omega},$$

where $c_n := \int_{B(0,1)} |x \cdot e_1| dx$.

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ADDENDUM

During the reviewing process of this manuscript, the authors became aware that Sibe Yang, Dachun Yang, and Wen Yuan had completed a related manuscript [36]. Their manuscript is dated of 28 January 2018, over one year after our work was posted on arXiv (see arXiv:1612.04566).

Their work is based on the abstract assumption that the maximal operator is bounded, whereas we assume (A) (which implies the maximal inequality [25]). On the other hand, our results are more general in the sense that we cover arbitrary open sets Ω , instead of the entire Euclidean space \mathbb{R}^n , and we assume only local integrability, whereas they assume global integrability.

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