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Conditions for the spectrum associated with an asymptotically straight leaky wire to contain an interval $[-\alpha^2/4, \infty)$

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1. Introduction

Let Γ be a continuous and piecewise smooth curve extending to infinity in the plane $\mathbf{R}^2$, and consider the spectrum σ associated with the perturbed Laplacian

$$H := -\Delta - \alpha(\mathbf{x})\delta(\mathbf{x} - \Gamma) \tag{1.1}$$

in $\mathbf{R}^2$ where δ is the Dirac delta function and $\alpha(\mathbf{x}) \geq 0$ is a given continuous and bounded function. In this spectral context, Γ is called a leaky wire and the Hamiltonian (1.1) represents the motion of a particle under the influence of a singular attraction (since $\alpha(\mathbf{x}) \geq 0$) along Γ. We refer to the recent extensive survey [5] for the physical motivation of studying this model and details of the influence that the geometry of Γ has on the nature of the spectrum. In the simplest case where α is constant and Γ is a straight line, we have $\sigma = [-\alpha^2/4, \infty)$ [6, (5.1)] but, for much more general curves Γ, it was shown in [6, section 5] that the essential spectrum σ_{ess} is

$$\sigma_{ess} = [-\alpha^2/4, \infty) \tag{1.2}$$

under certain global conditions of Γ which include the idea of asymptotic straightness [6, (3.1) and (3.2)]. An example of this idea [6, Remark 5.6] is that, in terms of the arc length s, the curvature $k(s)$ of Γ satisfies $| k(s) | \leq (\text{const.}) | s |^{-\beta}$ for some $\beta > 5/4$.

There are two aspects to the proof of (1.2) in [6]. One is that

$$\sigma_{ess} \supset [-\alpha^2/4, \infty) \tag{1.3}$$

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and the other is that $(-\infty, -\alpha^2/4)$ is not in σ_{ess}. In this paper we are concerned with (1.3) and, in view of the considerable technicalities in [6], we give a much simpler proof of (1.3) using the singular (or Weyl) sequence method. Further, our approach covers the case of $\alpha(\mathbf{x})$, a bounded and continuous function, tending to a finite limit at infinity. This approach is in the spirit of the early work on the Schrödinger operator in [2], [3] and [8, section 49] and requires conditions imposed only on long disjoint sections of Γ, rather than globally. Our main result is given in Theorem 3.1.

2. Operator realisation

The formal definition (1.1) can be made precise by the procedure indicated in [1, section 4] and [5, section 2] (see also [6] and [7]). Thus we assume that Γ is a piecewise C^1 curve without cusps and that for each compact subset K of $\mathbf{R}^2$ we have $\int_K \alpha(\mathbf{x})\delta(\mathbf{x}-\Gamma)d\mathbf{x} < \infty$. (For simplicity we only consider the case when Γ divides the plane into two regions R_1 and R_2.) In addition we assume that $\alpha(\mathbf{x})$ is non-negative, bounded and continuous on Γ. In this case we have that

$$\int_{\mathbf{R}^2} (1+\alpha(\mathbf{x})) \mid \psi(\mathbf{x}) \mid^2 \delta(\mathbf{x}-\Gamma)d\mathbf{x} \leq c \int_{\mathbf{R}^2} \left(\mid \nabla\psi(\mathbf{x}) \mid^2 + \mid \psi(\mathbf{x}) \mid^2\right) d\mathbf{x}$$

for $\psi(\cdot) \in C_0^\infty(\mathbf{R}^2)$. Therefore we can define the quadratic form

$$q(f,g) := \int_{\mathbf{R}^2} \nabla f(\mathbf{x}) \nabla \bar{g}(\mathbf{x}) d\mathbf{x} - \int_\Gamma \alpha(\mathbf{x}) f(\mathbf{x}) \bar{g}(\mathbf{x}) ds$$

with the domain $W^{1,2}(\mathbf{R}^2)$ which gives rise to the selfadjoint operator H from (1.1) by the same construction as described in [5, section 2]. The same operator can be constructed from the essentially self-adjoint operator $\tilde{H}$ defined by

$$\tilde{H}\psi\ (\mathbf{x}) = -\Delta\psi(\mathbf{x}) \quad (\mathbf{x} \in \mathbf{R}^2 \backslash \Gamma) \tag{2.1}$$

with domain $D(\tilde{H})$ consisting of functions $\psi \in W^{2,2}(\mathbf{R}^2\backslash\Gamma)$ which are continuous at Γ and with the normal derivatives having a jump in the sense that

$$\frac{\partial\psi}{\partial n_1}(\mathbf{x}) + \frac{\partial\psi}{\partial n_2}(\mathbf{x}) = -\alpha(\mathbf{x})\psi(\mathbf{x}) \quad (\mathbf{x} \in \Gamma). \tag{2.2}$$

Here n_1 and n_2 denote the normals directed away from Γ on the two sides of Γ. This operator reproduces the form q on the core $C_0^\infty(\mathbf{R}^2)$ of q (see [5, Section 2] and [1, Remark 4.1]). Thus H is the closure of $\tilde{H}$ and, in particular, $D(\tilde{H}) \subset D(H)$.

A real number λ is in σ_{ess} if and only if there is a sequence f_m in $D(H)$ such that

$$\| f_m \| = 1, \quad f_m \rightharpoonup 0 \text{ (weak convergence)}$$

and

$$\| (H-\lambda I) f_m \| \to 0 \tag{2.3}$$

as $m \to \infty$ [4, p.415]. Such a sequence is called a singular (or Weyl) sequence. In section 3, our choice of f_m will lie in $D(\tilde{H})$ so that (2.3) becomes simply

$$\int_{R_1} | (\Delta + \lambda I) f_m |^2 \, d\mathbf{x} + \int_{R_2} | (\Delta + \lambda I) f_m |^2 \, d\mathbf{x} \to 0 \tag{2.4}$$

by (2.1), subject to f_m satisfying the normal derivative condition (2.2).

3. The singular sequence

Our idea of asymptotic straightness is simply that Γ should lie close to arbitrarily long disjoint line segments as Γ recedes to infinity. The segments can be located without restriction in $\mathbf{R}^2$ but, purely for convenience in the proof which follows, we take them to lie along the $x-$axis. Thus we assume that there are disjoint intervals $I_m = (c_m - a_m, c_m + a_m)$ on the $x-$axis with $c_m \to \infty$ and $a_m \to \infty$ and, for x in each I_m, Γ has the equation $y = \gamma(x)$ with

$$\gamma(x) \to 0 \tag{3.1}$$

as $x \to \infty$ through the I_m. As in section 2, we take it that $\alpha(\mathbf{x})$ is non-negative, bounded and continuous on Γ.

Theorem 3.1. *Let $\gamma(x)$ have continuous derivatives up to order 3 in each I_m and, in addition to (3.1), let*

$$\gamma^{(r)}(x) \to 0 \quad (r = 1, 2, 3) \tag{3.2}$$

as $x \to \infty$ through the I_m. For $x \in I_m$, we write $\alpha(x) := \alpha(x, \gamma(x))$ and assume that $\alpha(x)$ has a continuous second derivative. As $x \to \infty$ through the I_m, let $\alpha(x)$ tend to a finite limit $\alpha_0 \ (> 0)$ with $\alpha^{(r)}(x) \to 0 \ (r = 1, 2)$. Then $\sigma_{ess} \supset [-\alpha_0^2/4, \infty)$.

Proof. In the square $S_m = (c_m - a_m, c_m + a_m) \times (-a_m, a_m)$ we define

$$f_m(\mathbf{x}) = b_m h_m(x - c_m) h_m(y) \exp\{-\beta(x) \mid y - \gamma(x) \mid + i\nu x\} \tag{3.3}$$

where b_m is the normalisation factor making $\| f_m \| = 1$, $\nu \geq 0$ and h_m is as usual a $\mathbf{C}^{(2)}(-\infty, \infty)$ function such that

$$h_m(t) = 1 \quad (| t | \leq a_m - 1) = 0 \quad (| t | \geq a_m)$$

and with derivatives independent of m. Finally, $\beta(x) \ (\geq 0)$ is chosen so that f_m satisfies (2.2), and we deal with this choice now.

When f_m is substituted into the left-hand side of (2.2), the net result comes only from the modulus term in (3.3). Let $\theta(x) = \tan^{-1} \gamma(x)$ ($| \theta(x) | \leq \pi/2$). Then a simple calculation shows that (2.2) holds if

$$2\beta(\cos\theta + \gamma' \sin\theta) = \alpha,$$

giving

$$\beta = \frac{1}{2}\alpha\cos\theta = \frac{1}{2}\alpha(1 + \gamma'^2)^{-1/2}. \tag{3.4}$$

Then, as $x \to \infty$ through the I_m, we have from (3.2)

$$\beta(x) \to \frac{1}{2}\alpha_0, \quad \beta^{(r)}(x) \to 0 \quad (r = 1, 2). \tag{3.5}$$

It follows now from (3.1) and (3.3) that

$$1 = b_m^2\{1 + o(1)\} \int_{-a_m}^{a_m} dt \int_{-a_m}^{a_m} \exp\{-2\beta(t + c_m) \mid y \mid\} dy$$

and hence, by (3.2) and (3.5),

$$b_m \sim (4a_m/\alpha_0)^{-1/2} \quad (m \to \infty). \tag{3.6}$$

The weak convergence condition $f_m \rightharpoonup 0$ is easily verified from (3.3) and (3.6). Then, to apply (2.4), we consider Δf_m for $\mathbf{x}$ in S_m and $\mathbf{x} \notin \Gamma$. The situation is similar on the two sides of Γ, and we concentrate on $y > \gamma(x)$. Then, by (3.3),

$$\begin{aligned} \Delta f_m = \quad & \{\beta^2 + (\beta\gamma' + \beta'\gamma - y\beta' + i\nu)^2 \\ & + \beta\gamma'' + 2\beta'\gamma' + \beta''\gamma - y\beta''\} f_m + E_m, \end{aligned} \tag{3.7}$$

where E_m denotes terms containing derivatives of h_m. Now $h_m'(t)$ and $h_m''(t)$ are only non-zero when $\mid a_m \mid -1 < t <\mid a_m \mid$, and it follows from (3.6) that $\parallel E_m \parallel = o(1) \quad (m \to \infty)$.

Finally, by (3.1), (3.2) and (3.5), we have from (3.7)

$$\int_{R_1} \mid (\Delta + \lambda I) f_m \mid^2 d\mathbf{x} + \int_{R_2} \mid (\Delta + \lambda I) f_m \mid^2 d\mathbf{x} = (\alpha_0^2/4 - \nu^2 + \lambda)^2 \parallel f_m \parallel^2 + o(1),$$

and hence (2.4) is satisfied with $\lambda = -\alpha_0^2/4 + \nu^2$. Since $\nu \geq 0$ is arbitrary, this proves that $\sigma_{ess} \supset [-\alpha_0^2/4, \infty)$ as required. □

The proof includes the case of constant $\alpha(\mathbf{x})$ $(= \alpha > 0)$ in the I_m, and then $\sigma_{ess} \supset [-\alpha^2/4, \infty)$. We also note that the theorem remains true when $\alpha_0 = 0$. We simply choose an f_m like (3.3) but with $\beta = 0$ and supported on a large square which does not intersect Γ. We omit the familiar details which are as in [2] for example.

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