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Publication date:

2004

Citation for published version (APA):

Huang, Z., & Shen, Q. (2004). *Scale and move transformation-based fuzzy interpolative reasoning: A revisit*. 623-628.

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Scale and Move Transformation-Based Fuzzy Interpolative Reasoning: A Revisit

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Abstract—This paper generalises the previously proposed interpolative reasoning method [5] to cover interpolations involving complex polygon, Gaussian or other bell-shaped fuzzy membership functions. This can be achieved by the generality of the proposed scale and move transformations. The method works by first constructing a new inference rule via manipulating two given adjacent rules, and then by using scale and move transformations to convert the intermediate inference results into the final derived conclusions. This generalised method has two advantages thanks to the elegantly proposed transformations: 1) It can easily handle interpolation of multiple antecedent variables with simple computation; and 2) It guarantees the uniqueness as well as normality and convexity of the resulting interpolated fuzzy sets. Numerical examples are provided to demonstrate the use of this method.

I. INTRODUCTION

The success of fuzzy modelling relies on its ability to approximate any complex system with human-like reasoning. However, the *curse of dimensionality*, which is referred to the exponential growth of the possible rule number as the input variables increase, inevitably deteriorates the transparency of such a fuzzy model. In order to alleviate this problem, there has been much work attempted such as orthogonal based methods [13], similarity based methods [10] and interpolative reasoning methods. As one of the simplification techniques, interpolation offers the potential for sparse fuzzy rule-based modelling. This is in contrast to the classic fuzzy modelling in the sense that it is capable of obtaining the results when the given fuzzy rule base is not complete, whilst fuzzy rules which may be interpolated from their neighbouring rules can be omitted from the rule base.

The first published fuzzy interpolation method [6] highlighted a number of important properties of the approach. However, it cannot guarantee convex results given antecedent values are normal and convex [11][16]. In order to eliminate this drawback and to develop more sensible interpolation techniques, there has been considerable work reported in the literature. For instance, Vas, Kalmar and Kóczy have proposed an algorithm [12] that reduces the problem of non-convex conclusions. Qiao, Mizumoto and Yan [9] have published an improved method which uses similarity transfer reasoning to guarantee the convex results. Hsiao, Chen and Lee [4] have introduced a new interpolative method which exploits the slopes of the fuzzy sets. General fuzzy interpolation and extrapolation techniques [1] and a modified α -cut based method [2] have also been proposed. In addition, Bouchon, Marsala and Rifqi have created an interpolative method by exploiting the concept

of graduality [3]. Yam and Kóczy [14][15] have proposed a fuzzy interpolative method based on Cartesian representation.

Nevertheless, some of the existing methods may include complex computation. It becomes more difficult when they are extended to multiple variables interpolation. Some others may only apply to simple fuzzy membership functions limited to triangular or trapezoidal sets. Others may not be able to obtain unique as well as normal and convex fuzzy (NCF) results. This paper proposes a general interpolative reasoning method which avoids the problems mentioned above. It is a generalised version of the work previously presented in [5]. Intermediate fuzzy rules are constructed by their adjacent rules. These, together with the observations, are converted into the final fuzzy consequences by the proposed scale and move transformations, which ensure unique, normal and convex results in an elegant manner.

The rest of the paper is organised as follows. Section II reviews the previous work [5] which applies to antecedent variables with triangular sets. Section III generalises this method to complex fuzzy sets such as trapezoidal, Gaussian and other bell-shaped membership functions. Section IV extends the idea to multiple variable interpolation. Examples are shown in Section V to demonstrate the usage of this method and to facilitate comparative studies. Finally, Section VI concludes the paper and points out some further work.

II. SINGLE ANTECEDENT VARIABLE WITH TRIANGULAR FUZZY SETS

This section reviews the method proposed in [5] which applies to triangular fuzzy sets, followed by more complex fuzzy membership functions such as trapezoidal and Gaussian in the next section. To facilitate this discussion, the *representative value* of a triangle is defined as the average of all points' x coordinate values (actually it is the x coordinate of the centre of gravity of such a triangle). This can be used to represent the overall location of a given triangular fuzzy set A denoted as (a_0, a_1, a_2) , as shown in Fig. 1. In mathematical formulae the representative value is of the form

$$Rep(A) = \frac{a_0 + a_1 + a_2}{3}. \quad (1)$$

In fuzzy interpolation, the simplest case is commonly used to demonstrate the underlying techniques without losing any generality. That is, given two adjacent rules as follows

If X is A_1 then Y is B_1 ,

If X is A_2 then Y is B_2 ,

which are denoted as $A_1 \Rightarrow B_1, A_2 \Rightarrow B_2$ respectively, together with the observation A^* which is located between fuzzy sets A_1 and A_2 , the interpolation is supposed to achieve the fuzzy result B^* . In another form this simplest case can be represented through the modus ponens interpretation (2), and as illustrated in Fig. 2.

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rules: if } X \text{ is } A_1, \text{ then } Y \text{ is } B_1 \\ \quad \text{if } X \text{ is } A_2, \text{ then } Y \text{ is } B_2 \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \quad (2)$$

Here, $A_i = (a_{i0}, a_{i1}, a_{i2}), B_i = (b_{i0}, b_{i1}, b_{i2}), i = 1, 2$, and $A^* = (a_0, a_1, a_2), B^* = (b_0, b_1, b_2)$.

The method proposed in [5] begins with constructing a new fuzzy set A' which has the same representative value as A^* . To support the generalisation of the present work, the distance between A_1 and A_2 is herein re-represented by the following (which has the same mathematical interpretation as the distance defined in [5] for triangular fuzzy sets, of course):

$$d(A_1, A_2) = d(Rep(A_1), Rep(A_2)). \quad (3)$$

An interpolative ratio $\lambda_{Rep} (0 \leq \lambda_{Rep} \leq 1)$ is introduced to represent the important impact of A_2 when constructing A' :

$$\begin{aligned} \lambda_{Rep} &= \frac{d(A_1, A^*)}{d(A_1, A_2)} \\ &= \frac{d(Rep(A_1), Rep(A^*))}{d(Rep(A_1), Rep(A_2))}. \end{aligned} \quad (4)$$

That is to say, if $\lambda_{Rep} = 0$, A_2 plays no part in the construction of A' . While if $\lambda_{Rep} = 1$, A_2 plays full weight on A' . Then by using the simplest linear interpolation, a'_0, a'_1 and a'_2 of A' are calculated as follows:

$$a'_0 = (1 - \lambda_{Rep})a_{10} + \lambda_{Rep}a_{20}, \quad (5)$$

$$a'_1 = (1 - \lambda_{Rep})a_{11} + \lambda_{Rep}a_{21}, \quad (6)$$

$$a'_2 = (1 - \lambda_{Rep})a_{12} + \lambda_{Rep}a_{22}, \quad (7)$$

which are collectively abbreviated to

$$A' = (1 - \lambda_{Rep})A_1 + \lambda_{Rep}A_2. \quad (8)$$

Now, A' has the same representative value as A^* , this is because

$$Rep(A') = \frac{a'_0 + a'_1 + a'_2}{3}.$$

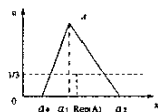


Fig. 1. Representative value of a triangular fuzzy set

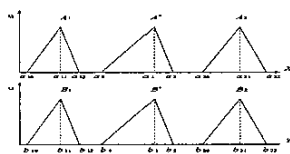


Fig. 2. Interpolation with triangular membership functions

With (5)–(7) and (4),

$$\begin{aligned} Rep(A') &= (1 - \lambda_{Rep}) \frac{a_{10} + a_{11} + a_{12}}{3} + \lambda_{Rep} \frac{a_{20} + a_{21} + a_{22}}{3} \\ &= (1 - \lambda_{Rep})Rep(A_1) + \lambda_{Rep}Rep(A_2) \\ &= Rep(A^*) \end{aligned}$$

Also, it is worth noting that A' is a convex fuzzy set as the following holds given $a_{10} \leq a_{11} \leq a_{12}, a_{20} \leq a_{21} \leq a_{22}$ and $0 \leq \lambda_{Rep} \leq 1$:

$$\begin{aligned} a'_2 - a'_1 &= (1 - \lambda_{Rep})(a_{12} - a_{11}) + \lambda_{Rep}(a_{22} - a_{21}) \geq 0, \\ a'_3 - a'_2 &= (1 - \lambda_{Rep})(a_{12} - a_{11}) + \lambda_{Rep}(a_{22} - a_{21}) \geq 0. \end{aligned}$$

Similarly, the consequent fuzzy set B' can be obtained by

$$b'_0 = (1 - \lambda_{Rep})b_{10} + \lambda_{Rep}b_{20}, \quad (9)$$

$$b'_1 = (1 - \lambda_{Rep})b_{11} + \lambda_{Rep}b_{21}, \quad (10)$$

$$b'_2 = (1 - \lambda_{Rep})b_{12} + \lambda_{Rep}b_{22}, \quad (11)$$

with abbreviated notation:

$$B' = (1 - \lambda_{Rep})B_1 + \lambda_{Rep}B_2. \quad (12)$$

In so doing, the newly derived rule $A' \Rightarrow B'$ involves the use of only normal and convex fuzzy sets.

As $A' \Rightarrow B'$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, it is feasible to perform fuzzy reasoning with this new rule without further reference to its originals. The interpolative reasoning problem is therefore changed from (2) to the new modus ponens interpretation:

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rule: if } X \text{ is } A', \text{ then } Y \text{ is } B' \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \quad (13)$$

This interpretation retains the same results as (2) in dealing with the extreme cases: If $A^* = A_1$, then from (4) $\lambda_{Rep} = 0$, and according to (8) and (12), $A' = A_1$ and $B' = B_1$, so the conclusion $B^* = B_1$. Similarly, if $A^* = A_2$, then $B^* = B_2$.

Other than the extreme cases, *similarity* measures are used to support the application of this new modus ponens as done in [9]. In particular, (13) can be interpreted as

$$\text{The more similar } X \text{ to } A', \text{ the more similar } Y \text{ to } B'. \quad (14)$$

Suppose that a certain degree of similarity between A' and A^* is established, it is reasonable to require that the consequent parts B' and B^* attain the same similarity degree. The question is now how to obtain an operator which will allow transforming B' to B^* with the desired degree of similarity. To this end, the following two component transformations are first introduced:

Scale Transformation Given a *scale rate* $s (s \geq 0)$, in order to transform the current fuzzy support $(a_2 - a_0)$ into a new support $(s * (a_2 - a_0))$ while keeping the representative value and the ratio of left-support $(a'_1 - a'_0)$ to right-support $(a'_2 - a'_1)$ of the transformed fuzzy set the same as those of its original, that is, $Rep(A') = Rep(A)$ and $\frac{a'_1 - a'_0}{a'_2 - a'_1} = \frac{a_1 - a_0}{a_2 - a_1}$, the new a'_0, a'_1 and a'_2 must satisfy (see Fig. 3. A):

$$a'_0 = \frac{a_0(1 + 2s) + a_1(1 - s) + a_2(1 - s)}{3}, \quad (15)$$

$$a'_1 = \frac{a_0(1 - s) + a_1(1 + 2s) + a_2(1 - s)}{3}, \quad (16)$$

$$a'_2 = \frac{a_0(1 - s) + a_1(1 - s) + a_2(1 + 2s)}{3}. \quad (17)$$

This is obvious. In fact, to satisfy the conditions imposed over the transformation, the linear equations below must hold simultaneously,

$$\begin{cases} \frac{a'_0 + a'_1 + a'_2}{3} = \frac{a_0 + a_1 + a_2}{3} \\ \frac{a'_1 - a'_0}{a'_2 - a'_1} = \frac{a_1 - a_0}{a_2 - a_1} \\ a'_2 - a'_0 = s(a_2 - a_0) \end{cases}$$

Solving these equations leads to the solutions as given in (15)–(17). The scale transformation guarantees that the transformed fuzzy sets are convex as the following holds given $a_0 \leq a_1 \leq a_2$ and $s \geq 0$:

$$\begin{cases} a'_1 - a'_0 = s(a_1 - a_0) \geq 0 \\ a'_2 - a'_1 = s(a_2 - a_1) \geq 0 \end{cases}$$

Move Transformation Given a moving distance l , in order to transform the current fuzzy set from the starting location a_0 to a new starting position $a_0 + l$ while keeping the representative value and the length of support ($a_2 - a_0$) the same, i.e., $Rep(A') = Rep(A)$ and $a'_2 - a'_0 = a_2 - a_0$, the new a'_0 , a'_1 and a'_2 must be (as shown in Fig. 3. B):

$$a'_0 = a_0 + l, \quad (18)$$

$$a'_1 = a_1 - 2l, \quad (19)$$

$$a'_2 = a_2 + l, \quad (20)$$

where $0 \leq l \leq l_{max} = (a_1 - a_0)/3$. If $l > l_{max}$, the transformation generates the non-convex fuzzy sets. For instance, consider the extreme case where A is transformed to A'' where the left slope of A'' becomes vertical (i.e. $a'_0 = a'_1$) as shown in Fig. 3. B. Here, $l = l_{max}$. Any further increase in l will lead to the resulting transformed fuzzy set being a non-NCF set. To avoid this, the *move ratio* M is introduced:

$$M = \frac{l}{(a_1 - a_0)/3}. \quad (21)$$

If move ratio $M \in [0, 1]$, then $l \leq l_{max}$ holds. This ensures that the transformed fuzzy set A' will be normal and convex if A is itself an NCF set. Note that the move transformation has two possible moving directions, the above discusses the left-direction case (from the viewpoint of a_1) with $l > 0$, the right direction with $l < 0$ should hold by symmetry:

$$M = \frac{l}{(a_2 - a_1)/3} \in [-1, 0]. \quad (22)$$

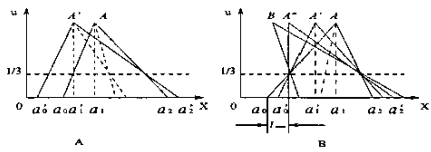


Fig. 3. Triangle scale and move transformations

III. SINGLE ANTECEDENT VARIABLE WITH COMPLEX FUZZY SETS

It is potentially very useful to extend this interpolative reasoning method to apply to more complex fuzzy membership functions. Consider a trapezoid fuzzy set A , denoted as (a_0, a_1, a_2, a_3) as shown in Fig. 4, the *support*, *left support*, *right support* and *top support* of A are defined as $a_3 - a_0$, $a_1 - a_0$, $a_3 - a_2$ and $a_2 - a_1$ respectively. The representative value of this fuzzy set is defined as:

$$Rep(A) = \frac{1}{3} \left(a_0 + \frac{a_1 + a_2}{2} + a_3 \right). \quad (23)$$

There may be alternative definitions (such as $Rep(A) =$

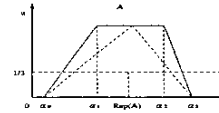


Fig. 4. Trapezoid representative value

$\frac{a_0 + a_1 + a_2 + a_3}{3}$) which are still able to represent the overall location of such a trapezoid. However, the adoption of (23) ensures the compatibility to the triangular case. This is covered by the situation where a_1 and a_2 in a trapezoid are collapsed into a_1 , thereby degenerating into a triangle. The representative value definitions for trapezoid (23) and triangle (1) remain the same. It is worth noting that these definitions do not affect the uniqueness, normality and convexity properties of resultant fuzzy sets.

The calculation of the intermediate fuzzy rule $A' \Rightarrow B'$ follows a similar process as applying to triangular membership functions except that A' and B' here are trapezoids. It is straightforward to verify the extreme cases (such as if $A^* = A_1$ then $B^* = B_1$) in the same way as applying to triangles. To adapt the proposed method to be suitable for trapezoidal fuzzy sets, attention is drawn to the two transformations.

Scale Transformation Given two *scale rates* s_s and s_t ($s_s \geq 0$ and $s_t \geq 0$) for support scale and top support scale respectively, in order to transform the current fuzzy support ($a_3 - a_0$) to the new support ($s_s * (a_3 - a_0)$), and the top support ($a_2 - a_1$) to the new top support ($s_t * (a_2 - a_1)$), while keeping the representative value and the ratio of left support ($a'_1 - a'_0$) to right support ($a'_3 - a'_2$) of the transformed fuzzy set the same as those of its original, that is, $Rep(A') = Rep(A)$ and $\frac{a'_1 - a'_0}{a'_3 - a'_2} = \frac{a_1 - a_0}{a_3 - a_2}$, the new a'_0 , a'_1 , a'_2 and a'_3 must satisfy (as illustrated in Fig. 5. A):

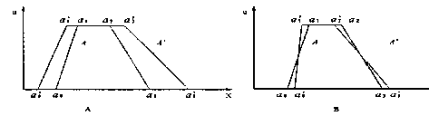


Fig. 5. Trapezoid scale and move transformations

$$a'_0 = A - \frac{C(2a_1 + a_3 - 2a_0 - a_2) - D(a_1 + a_2 - a_0 - a_3)}{B}, \quad (24)$$

$$a'_1 = A - \frac{C(a_0 + a_3 - a_1 - a_2) - D(5a_0 + a_2 - 5a_1 - a_3)}{B}, \quad (25)$$

$$a'_2 = A - \frac{C(a_0 + a_3 - a_1 - a_2) - D(a_1 + 5a_3 - a_0 - 5a_2)}{B}, \quad (26)$$

$$a'_3 = A - \frac{C(a_0 + 2a_2 - a_1 - 2a_3) - D(a_1 + a_2 - a_0 - a_3)}{B}, \quad (27)$$

where $A = \frac{2a_0 + a_1 + a_2 + 2a_3}{6}$, $B = 6(a_1 + a_3 - a_0 - a_2)$, $C = 2s_s(a_3 - a_0)$ and $D = s_t(a_2 - a_1)$.

These results can be achieved by solving the conditions below imposed over the transformation,

$$\begin{cases} \frac{1}{3}(a'_0 + \frac{a'_1 + a'_2}{2} + a'_3) = \frac{1}{3}(a_0 + \frac{a_1 + a_2}{2} + a_3) \\ \frac{a'_1 - a'_0}{a'_3 - a'_2} = \frac{a_1 - a_0}{a_3 - a_2} \\ a'_3 - a'_0 = s_s(a_3 - a_0) \\ a'_2 - a'_1 = s_t(a_2 - a_1) \end{cases}$$

Note that the scale transformation guarantees that the transformed fuzzy sets are convex, given that the support of the desired fuzzy set is longer than the top support. This is can be shown by

$$\begin{cases} a'_1 - a'_0 = \frac{(a_1 - a_0)(\text{sup}(A') - \text{top}(A'))}{a_1 + a_3 - a_0 - a_2} \geq 0 \\ a'_2 - a'_1 = s_t(a_2 - a_1) \geq 0 \\ a'_3 - a'_2 = \frac{(a_3 - a_2)(\text{sup}(A') - \text{top}(A'))}{a_1 + a_3 - a_0 - a_2} \geq 0 \end{cases}$$

where $\text{sup}(A')$ and $\text{top}(A')$ are the length of support and that of top support of transformed fuzzy set A' , respectively. However, arbitrarily choosing s_t when s_s is fixed may lead the top support of the resultant fuzzy set to becoming wider than the support. To avoid this, the *scale ratio* \mathbb{S}_t , which represents the actual increase of the ratios between the top supports and the supports, before and after the transformation, normalised over the maximal possible such increase (in the sense that it does not lead to non-convexity), is introduced to restrict s_t with respect to s_s :

$$\mathbb{S}_t = \begin{cases} \frac{\frac{s_t(a_2 - a_1)}{s_s(a_3 - a_0)} - \frac{a_2 - a_1}{a_3 - a_0}}{1 - \frac{a_2 - a_1}{a_3 - a_0}} & \text{if } s_t \geq s_s \geq 0 \\ \frac{\frac{s_t(a_2 - a_1)}{s_s(a_3 - a_0)} - \frac{a_2 - a_1}{a_3 - a_0}}{\frac{a_2 - a_1}{a_3 - a_0}} & \text{if } s_s \geq s_t \geq 0 \end{cases} \quad (28)$$

Thus if $\mathbb{S}_t \in [0, 1]$ (when $s_t \geq s_s > 0$) or $\mathbb{S}_t \in [-1, 0]$ (when $s_s \geq s_t > 0$), $s_s(a_3 - a_0) \geq s_t(a_2 - a_1)$, i.e., $\text{sup}(A') \geq \text{top}(A')$. This constraint along with the scale transformation ensures that the resultant fuzzy set A' to be a unique, normal and convex fuzzy set.

Move Transformation Given a moving distance l , in order to transform the current fuzzy set from the starting location a_0 to a new starting position $a_0 + l$ while keeping the representative value, the lengths of the support ($a_3 - a_0$) and the top support ($a_2 - a_1$) to remain the same, i.e., $\text{Rep}(A') = \text{Rep}(A)$, $a'_3 - a'_0 = a_3 - a_0$ and $a'_2 - a'_1 = a_2 - a_1$, the new a'_0 , a'_1 , a'_2 and a'_3 must be (as shown in Fig. 5. B):

$$a'_0 = a_0 + l, \quad (29)$$

$$a'_1 = a_1 - 2l, \quad (30)$$

$$a'_2 = a_2 - 2l, \quad (31)$$

$$a'_3 = a_3 + l, \quad (32)$$

where $0 \leq l \leq l_{max} = (a_1 - a_0)/3$. If $l > l_{max}$, the transformation generates the non-convex fuzzy sets. Likewise, the *move ratio* \mathbb{M} is introduced to avoid non-convexity:

$$\mathbb{M} = \frac{l}{(a_1 - a_0)/3}. \quad (33)$$

If move ratio $\mathbb{M} \in [0, 1]$, then $l \leq l_{max}$ holds. Similar to the triangle move transformation, there is an opposite move direction with $l \leq 0$. In this case the condition

$$\mathbb{M} = \frac{l}{(a_3 - a_2)/3} \in [-1, 0] \quad (34)$$

is imposed to ensure the convexity of transformed fuzzy sets.

It is easy to see that trapezoidal transformations actually cover the triangular ones. That is, triangular interpolation is a specific case of the trapezoidal one. Specially, if $a_1 = a_2$ the trapezoid becomes a triangle. Substituting $a_1 = a_2$ and $s_t = 0$ in trapezoidal transformation formulae (24)-(27) and (29)-(32) generate the same results as triangular transformation formulae (15)-(17) and (18)-(20).

There are two specific cases worth noting when applying scale transformation. The first is that if A^* is a singleton while A' is an NCF set (triangle or trapezoid), the scale transformation from A' to A^* is 0. This case can be easily handled by setting the result B^* to a singleton whose value interpolates between $\text{Rep}(B_1)$ and $\text{Rep}(B_2)$ in the same way as A^* does between $\text{Rep}(A_1)$ and $\text{Rep}(A_2)$. The second case (which only exists if both antecedents A_1 and A_2 are singletons) is that if A^* is an NCF set (triangle or trapezoid) while A' is a singleton, the scale transformation from A' to A^* is infinite. Since infinity cannot be used to generate the resulting fuzzy set, a modified strategy is created for this. The ratio between the support (and top support for trapezoid) length of fuzzy set A^* and the distance of $\text{Rep}(A_1)$ and $\text{Rep}(A_2)$ is calculated in order to compute the support (and top support) length of fuzzy set B^* by equalising the corresponding ratio. Note that the fuzzy set obtained by the scale transformation from a singleton is an isosceles triangle (or trapezoid).

Any complex polygonal fuzzy sets are readily covered by the proposed method following an analogous procedure to the extension from triangular to trapezoidal. One open issue is to determine the representative value for given complex polygons although this does not apply to symmetrical fuzzy sets, since in which case the symmetrical x value is naturally taken. For computational simplification, the average of all points' x coordinate values is calculated as the representative value for more complex polygons. Moreover, it is readily extendable to Gaussian and other bell-shaped membership functions. For instance, consider the simplest case where two rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ and the observation A^* all involve the use of Gaussian fuzzy sets of the form (Fig. 6):

$$p(x) = e^{-\frac{(x-c)^2}{2\sigma^2}}, \quad (35)$$

where c and σ are the mean and standard deviation respectively. The construction of the intermediate rule is slightly different from the polygonal fuzzy membership function case in the sense that the standard deviations are used to interpolate. Since the Gaussian shape is symmetrical, c is chosen to be the representative value of such a fuzzy set. In so doing, the antecedent value A' of the intermediate rule has the same representative value as that of observation A^* . That means only scale transformation from A^* to A' as depicted in Fig. 6 is needed to conduct interpolation. Heuristics can be employed to represent the scale rate s in terms of the standard deviation

σ . One of the simplest definitions is to calculate the ratio of two fuzzy sets' σ values when considering transformation from one to the other.

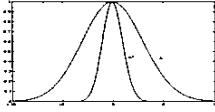


Fig. 6. Gaussian scale transformation

On top of the *scale* and *move* transformations, an *integrated transformation*, denoted $T(A, A')$, between two fuzzy sets A and A' can be introduced such that A' is the derived NCF set of A by applying both transformation components. Obviously, two integrated transformations are said to be identical if and only if both of their *scale rate*, *scale ratios* (for polygonal fuzzy sets more complex than triangular) and *move ratios* are equal.

As indicated earlier in (14), it is intuitive to maintain the similarity degree between the consequent parts B' and B^* to be the same as that between the antecedent parts A' and A^* , in performing interpolative reasoning. Now that the integrated transformation allows the similarity degree between two fuzzy sets to be measured by the *scale rate*, *scale ratios* (for fuzzy polygonal sets more complex than triangular) and *move ratios*, the desired conclusion B^* can be obtained by satisfying the following (as shown in Fig. 7 for an interpolation involving triangular fuzzy sets):

$$T(B', B^*) = T(A', A^*). \tag{36}$$

That is, the parameters of *scale rate*, *scale ratios* and *move*

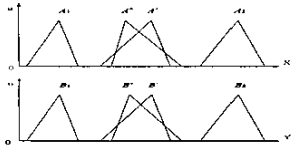


Fig. 7. Proposed interpolative method

ratios calculated from A' to A^* are used to compute B^* from B' . Clearly, B^* will then retain the same similarity degree as that between the antecedent parts A' and A^* .

IV. MULTIPLE ANTECEDENT VARIABLES INTERPOLATION

The interpolations described in Sections II and III concern two adjacent rules with each involving one antecedent variable. This is readily extendable to rules with multiple antecedent attributes. Of course, the attributes appearing in both rules must be the same to make sense for interpolation.

Without losing generality, suppose that two adjacent rules R_i and R_j are represented by

if X_1 *is* A_{1i} *and* ... *and* X_m *is* A_{mi} *then* Y *is* B_i ,
if X_1 *is* A_{1j} *and* ... *and* X_m *is* A_{mj} *then* Y *is* B_j .

Thus, when a vector of observations $(A_1^*, \dots, A_k^*, \dots, A_m^*)$ is given, by direct analogy to the one variable case, the values

A_{ki} and A_{kj} of X_k , $k = 1, 2, \dots, m$, are used to obtain the new NCF set A'_k :

$$A'_k = (1 - \lambda_k)A_{ki} + \lambda_k A_{kj}, \tag{37}$$

where

$$\lambda_k = \frac{d(\text{Rep}(A_{ki}), \text{Rep}(A_k^*))}{d(\text{Rep}(A_{ki}), \text{Rep}(A_{kj}))}.$$

Clearly, the representative value of A'_k will remain the same as that of the k -th observation A_k^* .

The resulting A'_k and the given A_k^* are used to compute the scale rate s_k , scale ratios S_{ki} (for polygonal fuzzy sets more complex than triangular), $i = 1, \dots, p$, and move ratios M_{kj} , $j = 1, \dots, q$, in the same way as the one variable case. Note that p and q are positive integers and they vary with respect to the complexity of the fuzzy sets involved. The combined scale rate s_c , scale ratios S_{ci} , $i = 1, \dots, p$, and move ratios M_{cj} , $j = 1, \dots, q$, over the m conditional attributes are calculated as the arithmetic averages of s_k , S_{ki} and M_{kj} , $k = 1, 2, \dots, m$:

$$s_c = \frac{1}{m} \sum_{k=1}^m s_k, \quad S_{ci} = \frac{1}{m} \sum_{k=1}^m S_{ki}, \quad M_{cj} = \frac{1}{m} \sum_{k=1}^m M_{kj}. \tag{38}$$

Note that, other than using arithmetic average, different mechanisms such as the medium value operator may be employed for this purpose. However, the average helps to capture the intuition that when no particular information regarding which variable has a more dominating influence upon the conclusion, all the variables are treated equally. If such information is available, it may be better to use a weighted average operator.

Regarding the consequent, by analogy to (12), B' can be computed by

$$B' = (1 - \lambda_a)B_i + \lambda_a B_j. \tag{39}$$

Here, λ_a is deemed to be the average of λ_k , $k = 1, 2, \dots, m$, to mirror the approach taken above

$$\lambda_a = \frac{1}{m} \sum_{k=1}^m \lambda_k. \tag{40}$$

As the *combined scale rate* s_c , *combined scale ratios* S_{ci} (for fuzzy polygonal sets more complex than triangular), $i = 1, \dots, p$, and *move ratios* M_{cj} reflect the similarity degree between the observation vector and the values of the given rules, the fuzzy set B^* of the conclusion can then be estimated by transforming B' via the application of the same s_c , S_{ci} and M_{cj} .

V. ILLUSTRATIVE EXAMPLES

In this section, two examples concerning trapezoidal fuzzy membership functions are illustrated to demonstrate the use of the proposed method (denoted as HS method) and facilitate the comparative studies with the KH method [6].

Example 1. This concerns one antecedent variable. Given the observation $A^* = (6, 6, 9, 10)$, two adjacent rules $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$, and the antecedent values, the results are presented in Table I and Fig. 8.A. In this case, the KH method generated a non-convex fuzzy set. However, the HS method resulted in an NCF conclusion, which still maintained the property of the left vertical slope.

TABLE I
RESULTS FOR EXAMPLE 1, WITH $A^* = (6, 6, 9, 10)$

Attribute Values	Results	
	Method	B^*
$A_1 = (0, 4, 5, 6)$	KH HS	(5.45, 4.25, 7.5, 8.5) (5.23, 5.23, 7.61, 8.32)
$A_2 = (11, 12, 13, 14)$		
$B_1 = (0, 2, 3, 4)$		
$B_2 = (10, 11, 12, 13)$		

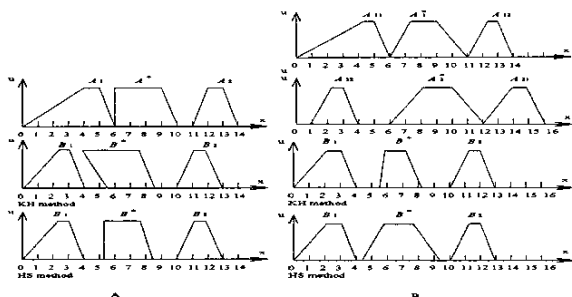


Fig. 8. The reasoning results of Example 1 and 2

Example 2. This example concerns an interpolation of multiple antecedent variables with trapezoidal membership functions. Two rules $A_{11} \wedge A_{21} \Rightarrow B_1$, $A_{12} \wedge A_{22} \Rightarrow B_2$ and the observations $A_1^* = (6, 7, 9, 11)$, $A_2^* = (6, 8, 10, 12)$ are given to determine the result B^* . Table II and Fig. 8.B summarise the results. In this case, the parameter λ_1 for the first variable is 0.54 and λ_2 for the second is 0.44, the average 0.49 is used to calculate the intermediate rule result B' . Then, the average of two support scale rates (1.14 and 1.69) and that of two top support ratios (0.22 and 0.07) are computed, equalling 1.41 and 0.15. They are used as the combined support scale rate and top support scale ratio, respectively. These together with the combined move ratio, which is the average (0.35) of the two move ratios (0.53 and 0.18), are employed to transfer B' to achieve the final result B^* . Both the KH method and the HS method resulted in an NCF set in this example. Interestingly, the resultant fuzzy set of the present work reflects better shapes of the original observations than that obtained by the KH method. More investigations into the problem underlying reasons for this is currently being carried out.

TABLE II
RESULTS FOR EXAMPLE 2, WITH $A_1^* = (6, 7, 9, 11)$ AND
 $A_2^* = (6, 8, 10, 12)$

Attribute Values	Results	
	Method	B^*
$A_{11} = (0, 4, 5, 6)$	KH HS	(5.45, 5.94, 7.13, 8.31) (4.37, 5.55, 7.48, 9.33)
$A_{12} = (12, 14, 15, 16)$		
$A_{21} = (11, 12, 13, 14)$		
$A_{22} = (1, 2, 3, 4)$		
$B_1 = (0, 2, 3, 4)$		
$B_2 = (10, 11, 12, 13)$		

VI. CONCLUSIONS

This paper has proposed a generalised, scale and move transformation-based [5], interpolative reasoning method. The method works by first constructing a new intermediate rule via manipulating two adjacent rules (and the given observations of course), and then converting the intermediate inference result into the final derived conclusion, using the scale and move transformations. This approach not only inherits the common advantages of fuzzy interpolative reasoning: allowing inferences to be performed with simple and sparse rule bases, but also has other two advantages: 1) It can easily handle multiple antecedent variable interpolation with simple computation; and 2) It guarantees the uniqueness as well as normality and convexity of the resultant fuzzy sets.

There is still work needed to improve this method. In particular, the present work only uses two rules to conduct interpolation. The interpolation involved with more rules may be utilised in fuzzy modelling. An extension of the proposed method to cope with such a problem is worth investigating. In addition, this work does not look into the extrapolation problem. Further effort to estimate this issue seems necessary.

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