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Transformation Based Interpolation with Generalized Representative Values

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***Abstract*— Fuzzy interpolation offers the potential to model problems with sparse rule bases, as opposed to dense rule bases deployed in traditional fuzzy systems. It thus supports the simplification of complex fuzzy models and facilitates inferences when only limited knowledge is available. This paper first introduces the general concept of representative values (RVs), and then uses it to present an interpolative reasoning method which can be used to interpolate fuzzy rules involving arbitrary polygonal fuzzy sets, by means of scale and move transformations. Various interpolation results over different RV implementations are illustrated to show the flexibility and diversity of this method. A realistic application shows that the interpolation-based inference can outperform the conventional inferences.**

I. Introduction

Fuzzy rule interpolation helps reduce the complexity of fuzzy models and supports inference in systems that employ sparse rule sets [8][9]. Despite these significant advantages, earlier work in fuzzy interpolative reasoning does not guarantee the convexity of the derived fuzzy sets [10], which is often a crucial requirement of fuzzy reasoning to attain more easily interpretable practical results. Significant work has been reported in the literature [1][4][10][11] in an effort to eliminate this non-convexity drawback.

However, almost all existing methods lack the flexibility to generate results that meet different application requirements. This paper, based on the initial work carried out by the authors [5][6], introduces a general RV definition (which covers the RV notions previously used, of course) and presents an enhanced interpolation method based on this generalized definition. The enhanced method offers a degree of freedom to provide a variety of unique, normal and convex results.

The rest of the paper is organized as follows: Section II introduces the general representative value definition for arbitrarily complex polygonal fuzzy sets. Section III describes the scale and move transformations used to perform interpolative inference and summarizes the interpolation procedure. Section IV compares the interpolation results obtained by employing different RV definitions. Section V demonstrates the usage of the interpolation in a real world problem. Finally, Section VI concludes the paper.

II. General Representative Value

To facilitate the discussion of the transformation based interpolation method, the *representative value* (RV) of the (polygonal) fuzzy sets involved must be defined first. This value captures important information such as the overall location of a fuzzy set, and will be used as the guide to perform transformations. Consider an arbitrary polygonal fuzzy set with n odd points, $A = (a_0, \ldots, a_{n-1})$, as shown in Fig. 1. It has $\lfloor \frac{n}{2} \rfloor$ *supports* (horizontal intervals between every pair of odd points which have the same membership value) and $2(\lceil \frac{n}{2} \rceil - 1)$ *slopes* (non-horizontal intervals between every pair of consecutive odd points). Note that two top points (of the membership value 1) do not have to be different. Although this figure explicitly assumes that evenly paired odd points are given at each α-cut level, this does not affect the generality of the fuzzy set as artificial odd points can be created to construct evenly paired odd points. Given such an arbitrary polygonal fuzzy set its general RV is defined by

$$Rep(A) = \sum_{i=0}^{n-1} w_i a_i, \tag{1}$$

where w_i is the weight assigned to point a_i.

Specifying the weights is necessary for a given application. The simplest case (which is called the *average RV* hereafter) is that all points take the same weight value, i.e., $w_i = \frac{1}{n}$. Note that [5] uses this RV definition.

An alternative definition named the *weighted average RV* assumes that the weights increase upwards from the bottom support to the top support, to reflect the significance of the fuzzy membership values. For instance, assuming the weights increase upwards from $\frac{1}{2}$ to 1, such an RV is defined by

$$Rep(A) = \frac{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}(a_i + a_{n-1-i})}{\sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} \frac{1+\alpha_i}{2}}. \tag{2}$$

One of the most widely used defuzzification methods – the center of core can also be used to define the *center of core RV*. In this case, the RV is solely determined by those points with a fuzzy membership value of 1:

$$Rep(A) = \frac{1}{2}(a_{\lceil \frac{n}{2} \rceil - 1} + a_{n - \lceil \frac{n}{2} \rceil}). \tag{3}$$

Note that the general RV definition can be simplified if the lengths of the $\lfloor \frac{n}{2} \rfloor$ supports $S_0, \ldots, S_{\lfloor \frac{n}{2} \rfloor - 1}$ (with the indices arranged in ascending order from the bottom to the top) are known. Indeed, as $a_{n-1-i} = a_i + S_i$, $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, the general form of (1) can be re-written as:

$$Rep(A) = \sum_{i=0}^{\lceil \frac{n}{2} \rceil - 1} a_i(w_i + w_{n-1-i}) + C, \tag{4}$$

where $C = S_0 w_{n-1} + \ldots + S_{\lceil \frac{n}{2} \rceil - 1} w_{n - \lceil \frac{n}{2} \rceil}$ is a constant.

III. Transformation Based Interpolation

A. Construct the Intermediate Rule

To be concise, the simplest case is herein used to illustrate the underlying techniques for fuzzy interpolation. Given two adjacent rules as follows

$$If\ X\ is\ A_1\ then\ Y\ is\ B_1,$$
$$If\ X\ is\ A_2\ then\ Y\ is\ B_2,$$

which are denoted as $A_1 \Rightarrow B_1$, $A_2 \Rightarrow B_2$ respectively, together with an observation A^* which is located between fuzzy sets A_1 and A_2, an interpolation is performed to achieve the fuzzy result B^*. In another form this simplest case can be represented through the *modus ponens* interpretation (5).

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rules: if } X \text{ is } A_1 \text{, then } Y \text{ is } B_1 \\ \qquad\quad \text{if } X \text{ is } A_2 \text{, then } Y \text{ is } B_2 \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \tag{5}$$

Here, $A_i = (a_{i0}, \ldots, a_{i,n-1})$, $B_i = (b_{i0}, \ldots, b_{i,n-1})$, $i = \{1, 2\}$, and $A^* = (a_0, \ldots, a_{n-1})$, $B^* = (b_0, \ldots, b_{n-1})$.

The transformation based interpolation begins with constructing a new fuzzy set A' which has the same RV as that of A^*. To support this work, the distance between A_1 and A_2 is herein defined by

$$d(A_1, A_2) = d(Rep(A_1), Rep(A_2)), \tag{6}$$

where the actual scheme adopted to compute RVs is fixed for both A_1 and A_2 of course. A ratio λ_{Rep} $(0 \le \lambda_{Rep} \le 1)$ is introduced to represent the important impact of A_2 upon the construction of A' with respect to A_1:

$$\lambda_{Rep} = \frac{d(A_1, A^*)}{d(A_1, A_2)}. \tag{7}$$

That is to say, if $\lambda_{Rep} = 0$, A_2 plays no part in constructing A', while if $\lambda_{Rep} = 1$, A_2 plays a full role in determining A'. Then by using the simplest linear interpolation, the a'_i, $i = \{0, \ldots, n-1\}$, of A' are calculated as follows:

$$a'_i = (1 - \lambda_{Rep}) a_{1i} + \lambda_{Rep} a_{2i}. \tag{8}$$

It can be proved [1] that A' has the same representative value as A^* and that A' is convex and normal. Similarly, the consequent fuzzy set B' can be obtained by B_1, B_2 and λ_{Rep}. In so doing, the newly derived rule $A' \Rightarrow B'$ involves the use of only normal and convex fuzzy sets.

[1]Proofs are omitted, interested readers may contact the authors for more details.

As $A' \Rightarrow B'$ is derived from $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, when A^* is given it is feasible to perform fuzzy reasoning with this new rule without further reference to its originals. The interpolative reasoning problem is therefore changed from (5) to the new *modus ponens* interpretation:

$$\begin{array}{l} \text{observation: } X \text{ is } A^* \\ \text{rule: if } X \text{ is } A' \text{, then } Y \text{ is } B' \\ \hline \text{conclusion: } Y \text{ is } B^*? \end{array} \tag{9}$$

This interpretation retains the same results as (5) in dealing with the extreme cases: If $A^* = A_1$, then from (7) $\lambda_{Rep} = 0$, and according to (8), $A' = A_1$, and similarly $B' = B_1$, so the conclusion $B^* = B_1$. Likewise, if $A^* = A_2$, then $B^* = B_2$.

Other than the extreme cases, *similarity* measures are used to support the application of this new *modus ponens*. In particular, (9) can be interpreted as

$$The\ more\ similar\ X\ to\ A',\ the\ more\ similar\ Y\ to\ B'. \tag{10}$$

Suppose that a certain degree of similarity between A' and A^* is established, it is intuitive to require that the consequent parts B' and B^* attain the same similarity degree. The question is now how to obtain an operator which can capture the similarity degree between A' and A^*, and to allow transforming B' to B^* with the desired degree of similarity. To this end, the following two component transformations are proposed.

B. Scale Transformation for Generalized RVs

Consider applying scale transformation to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ (as shown in Fig. 1) to generate $A' = (a'_0, \ldots, a'_{n-1})$ such that A and A' will have the same RV, and $a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i)$, where s_i are scale rates and $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$. In order to achieve this, $\lfloor \frac{n}{2} \rfloor$ equations $a'_{n-1-i} - a'_i =$

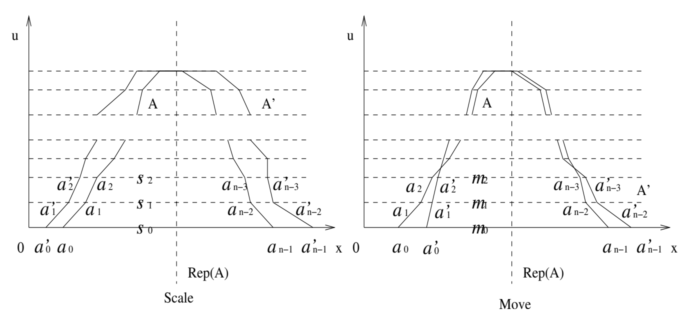

Fig. 1. Scale and move transformations

$s_i(a_{n-1-i} - a_i)$, $i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, are imposed to obtain the supports with desired lengths, and $(\lceil \frac{n}{2} \rceil - 1)$ equations $\frac{a'_{i+1} - a'_i}{a'_{n-1-i} - a'_{n-2-i}} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}}$, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$ are imposed to equalize the ratios between the left $(\lceil \frac{n}{2} \rceil - 1)$ slopes' lengths and the right $(\lceil \frac{n}{2} \rceil - 1)$ slopes' lengths of A' to the ratio counterparts of the original fuzzy set A. The equation $\sum_{i=0}^{n-1} w_i a'_i = \sum_{i=0}^{n-1} w_i a_i$ which ensures the representative values to remain the same before and after the transformation is added to make up of $\lfloor \frac{n}{2} \rfloor + (\lceil \frac{n}{2} \rceil - 1) + 1 = n$ equations. For clarity, these n equations are collectively written as:

$$\begin{cases} a'_{n-1-i} - a'_i = s_i(a_{n-1-i} - a_i) = S_i \\ (i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}) \\ \frac{a'_{i+1} - a'_i}{a'_{n-1-i} - a'_{n-2-i}} = \frac{a_{i+1} - a_i}{a_{n-1-i} - a_{n-2-i}} = R_i \\ (i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}) \\ \sum_{i=0}^{n-1} w_i a'_i = \sum_{i=0}^{n-1} w_i a_i \end{cases} \quad (11)$$

where S_i is the i-th support length of the resultant fuzzy set and R_i is the ratio between the i-th left slope length and the i-th right slope length. Solving these n equations simultaneously results in a unique and convex fuzzy set A' given that the resultant set has the support lengths in a descending order from the bottom to the top. The proof of this is omitted due to space limit. It can also be shown that given a fuzzy set A and the support scale rates s_i, the use of a different RV will not affect the geometrical shape of the resultant fuzzy set. Instead, it only affects the position of the transformed fuzzy set.

However, arbitrarily choosing the i-th support scale rate when the $(i-1)$-th scale rate is fixed may lead the i-th support to becoming wider than the $(i-1)$-th support, i.e., $S_i > S_{i-1}$. To avoid this, the i-th *scale ratio* $\mathbb{S}_i$, which represents the actual increase of the ratios between the i-th supports and the $(i-1)$-th supports, before and after the transformation, normalized over the maximal of such an increase (in the sense that it does not lead to non-convexity), is introduced to restrict s_i with respect to s_{i-1}:

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{1 - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} & (if\ s_i \geq s_{i-1} \geq 0) \\ \dfrac{\frac{s_i(a_{n-i-1} - a_i)}{s_{i-1}(a_{n-i} - a_{i-1})} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{\frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} & (if\ s_{i-1} \geq s_i \geq 0) \end{cases} \quad (12)$$

If $\mathbb{S}_i \in [0, 1]$ (when $s_i \geq s_{i-1} \geq 0$) or $\mathbb{S}_i \in [-1, 0]$ (when $s_{i-1} \geq s_i \geq 0$), then $S_{i-1} \geq S_i$. Again, the proof is omitted. In summary, if given s_i ($i = \{0, \ldots, \lfloor \frac{n}{2} \rfloor - 1$) such that $\mathbb{S}_i \in [0, 1]$ or $\mathbb{S}_i \in [-1, 0]$ (depending on whether $s_i >= s_{i-1}$ or not), $i = \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$, the scale transformation guarantees to produce a normal and convex fuzzy set.

Conversely, if two convex sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ which have the same RV are given, the scale rate of the bottom support, s_0, and the scale ratio of the i-th support, $\mathbb{S}_i$ ($\mathbb{S}_i$, $i = \{1, \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$) can be calculated by:

$$s_0 = \frac{a'_{n-1} - a'_0}{a_{n-1} - a_0} \quad (13)$$

$$\mathbb{S}_i = \begin{cases} \dfrac{\frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{1 - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} \in [0, 1] \\ (if\ \frac{a'_{n-i-1} - a'_i}{a_{n-i-1} - a_i} \geq \frac{a'_{n-i} - a'_{i-1}}{a_{n-i} - a_{i-1}} \geq 0) \\ \\ \dfrac{\frac{a'_{n-i-1} - a'_i}{a'_{n-i} - a'_{i-1}} - \frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}}{\frac{a_{n-i-1} - a_i}{a_{n-i} - a_{i-1}}} \in [-1, 0] \\ (if\ \frac{a'_{n-i} - a'_{i-1}}{a_{n-i} - a_{i-1}} \geq \frac{a'_{n-i-1} - a'_i}{a_{n-i-1} - a_i} \geq 0) \end{cases} \quad (14)$$

Since A and A' are both convex, $\mathbb{S}_i$ must be within the range as given in (14). Again the proof is omitted.

C. Move Transformation for Generalized RVs

Now, consider the move transformation (also shown in Fig. 1) applied to an arbitrary polygonal fuzzy membership function $A = (a_0, \ldots, a_{n-1})$ to generate $A' = (a'_0, \ldots, a'_{n-1})$, such that A and A' have the same RV and the same lengths of supports, and $a'_i = a_i + l_i$, $i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$. In order to achieve this, the move transformation is decomposed to $(\lceil \frac{n}{2} \rceil - 1)$ sub-moves. The i-th sub-move ($i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\}$) moves the i-th support (indexed from bottom to top beginning with 0) to a desired place. This operator moves all the odd points on and above the i-th support, whilst unalter those points under this support. To measure the degree of the i-th sub-move, the first possible maximal move distance (in the sense that the corresponding sub-move does not lead to the above part of the fuzzy set becoming non-convexity) should be worked out first. To simplify the description of the sub-move procedure, only the move on the right side (from a_i's point of view) is considered in the discussion hereafter. The left direction simply mirrors this operation.

If the i-th point is supposed to move to the right direction, the maximal position $a_i^{(i)*}$ can be calculated as follows when $\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j}) > 0$:

$$a_i^{(i)*} = \frac{\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1} a_j(w_j + w_{n-1-j}) - A}{\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j})} \quad (15)$$

where $A = \sum_{\substack{w_k + w_{n-1-k} < 0 \\ i < k < \lceil \frac{n}{2} \rceil}} [(S_{k-1} - S_k) \sum_{m=k}^{\lceil \frac{n}{2} \rceil - 1}(w_m + w_{n-1-m})]$ and S_k is the length of the k-th support (either before or after move transformation as they are the same). If however $\sum_{j=i}^{\lceil \frac{n}{2} \rceil - 1}(w_j + w_{n-1-j}) < 0$, the maximal position $a_i^{(i)*}$ is calculated similarly to (15) except that the condition $w_k + w_{n-1-k} < 0$ in term A is changed to $w_k + w_{n-1-k} > 0$. Once again, the proofs are omitted here. It can be shown that the other extreme moving points $a_j^{(i)*}$ ($j = \{i+1, \ldots, \lceil \frac{n}{2} \rceil - 1\}$) which are on the left side of the fuzzy set in the i-th sub-move can be computed by:

$$a_j^{(i)*} = \begin{cases} a_{j-1}^{(i)*} & if\ w_j + w_{n-1-j} > 0 \\ a_{j-1}^{(i)*} + S_{j-1} - S_j & if\ w_j + w_{n-1-j} < 0 \end{cases} \quad (16)$$

Also, it can be seen that all the extreme points determine a normal and convex fuzzy set $A^{(i)*}$ (as illustrated in Fig. 2) which must have at least a vertical slope between any two consecutive α-cuts above the i-th support. This fuzzy set will have the same RV as $A^{(i-1)}$ with respect to the move transformation. That is:

$$\sum_{j=0}^{\lceil \frac{n}{2} \rceil - 1} a_j^{(i)*}(w_j + w_{n-1-j}) = \sum_{j=0}^{\lceil \frac{n}{2} \rceil - 1} a_j^{(i-1)}(w_j + w_{n-1-j}) \quad (17)$$

From (15), the first maximal move distance can be calculated. However, the i-th sub-move should not only consider above non-convexity, but also ensure the avoidance of below non-convexity (i.e., the part of the fuzzy set below i-th support). Otherwise it may still lead to non-convexity as illustrated in Fig. 2. For this, the second maximal move distance is calculated as $(a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)})$. It is intuitive to select the minimal of these two maximal move distances to act as the actual maximal move distance for use to avoid either above or below non-convexity. The move ratio $\mathbb{M}_i$, which is used to measure the degree of such a sub-move, is thus calculated by:

$$\mathbb{M}_i = \begin{cases} \frac{l_i-(a_i^{(i-1)}-a_i)}{min\{a_i^{(i)*}-a_i^{(i-1)}, a_{n-i}^{(i-1)}-a_{n-1-i}^{(i-1)}\}} \ (if\ l_i \geq (a_i^{(i-1)} - a_i)) \\ \frac{l_i-(a_i^{(i-1)}-a_i)}{min\{a_i^{(i-1)}-a_i^{(i)*}, a_i^{(i-1)}-a_{i-1}^{(i-1)}\}} \ (if\ l_i \leq (a_i^{(i-1)} - a_i)) \end{cases} \tag{18}$$

where the notation $a_i^{(i-1)}$ represents a_i's new position after the $(i-1)$-th sub-move. Initially, $a_i^{(-1)} = a_i$.

If $\mathbb{M}_i \in [0,1]$ when $l_i \geq (a_i^{(i-1)} - a_i)$, or $\mathbb{M}_i \in [-1,0]$ when $l_i \leq (a_i^{(i-1)} - a_i)$, the sub-move is carried out as follows. The odd points under the i-th support are not changed: $a_j^{(i)} = a_j^{(i-1)} (j = \{0,\ldots,i-1,n-i,\ldots,n-1\})$ while the other points $a_i^{(i-1)}, a_{i+1}^{(i-1)}, \ldots, a_{n-1-i}^{(i-1)}$ are being moved. At the beginning, when $i = 0$, all odd points are moved of course. If moving to the right side from the viewpoint of $a_i^{(i-1)}$, i.e., $\mathbb{M}_i \in [0,1]$, the moving distances of $a_j^{(i-1)}$ $(j = \{i, i+1, \ldots, \lceil \frac{n}{2} \rceil - 1\})$ which are on the left side of the fuzzy set $A^{(i-1)}$ are calculated by multiplying $\mathbb{M}'_i$ with the distances between the extreme positions $a_j^{(i)*}$ and themselves. In so doing, $a_j^{(i-1)}$ will move the same proportion of distances to their respective extreme positions. That is:

$$a_j^{(i)} = a_j^{(i-1)} + \mathbb{M}'_i(a_j^{(i)*} - a_j^{(i-1)}), \tag{19}$$

where

$$\mathbb{M}'_i = \mathbb{M}_i \frac{min\{a_i^{(i)*} - a_i^{(i-1)}, a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}\}}{a_i^{(i)*} - a_i^{(i-1)}}. \tag{20}$$

This represents the *applied move ratio* for the i-th sub-move. If $\mathbb{M}_i \in [0,1], \mathbb{M}'_i \in [0, \mathbb{M}_i]$. The adoption of applied move ratio $\mathbb{M}'_i$ avoids the potential below non-convexity. Such a move strategy leads to a fuzzy set $A^{(i)} = \{a_0^{(i)}, \ldots, a_{n-1}^{(i)}\}$ which is convex, has the same RV as A, and has the new point $a_i^{(i)}$ on the desired position, i.e., $a_{j+1}^{(i)} - a_j^{(i)} \geq 0$ $(j = \{0, \ldots, n-2\})$, $Rep(A^{(i)}) = Rep(A)$, and $a_i^{(i)} = a_i + l_i$. These properties have been proved but details are omitted here.

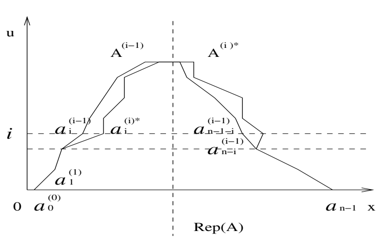

Fig. 2. The extreme move positions in the i-th sub-move

In summary, if given move ratios $\mathbb{M}_i \in [-1,1]$, $(i = \{0, \ldots, \lceil \frac{n}{2} \rceil - 2\})$, the $(\lceil \frac{n}{2} \rceil - 1)$ sub-moves transform a given normal and convex set $A = (a_0, \ldots, a_{n-1})$ to a new normal and convex set $A' = (a'_0, \ldots, a'_{n-1})$ with the same lengths of supports and the same RV.

In the converse case, where two convex fuzzy sets $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ of the same representative value are given, the move ratio as $\mathbb{M}_i$, $i = \{0, 1, \ldots, \lceil \frac{n}{2} \rceil - 2\}$, are computed by:

$$\mathbb{M}_i = \begin{cases} \frac{a'_i - a_i^{(i-1)}}{min\{a_i^{(i)*}-a_i^{(i-1)}, a_{n-i}^{(i-1)}-a_{n-1-i}^{(i-1)}\}} \ (if\ a'_i \geq a_i^{(i-1)}) \\ \frac{a'_i - a_i^{(i-1)}}{min\{a_i^{(i-1)}-a_i^{(i)*}, a_i^{(i-1)}-a_{i-1}^{(i-1)}\}} \ (if\ a'_i \leq a_i^{(i-1)}) \end{cases} \tag{21}$$

where $a_i^{(i-1)}$ is the a_i's new position after the $(i-1)$-th sub-move. Initially, when $i = 0$, $a_i^{(-1)} = a_i$. This (bottom) sub-move will not lead to any below non-convexity as there are no odd points underneath, whilst the other sub-moves need to consider situations where non-convexity may arise both above and underneath. When $i = 0$, $a_{n-i}^{(i-1)} - a_{n-1-i}^{(i-1)}$ and $a_i^{(i-1)} - a_{i-1}^{(i-1)}$ are not defined. In order to keep the expression the same for (21), both of them take value 1 to represent the bottom case.

Since $A = (a_0, \ldots, a_{n-1})$ and $A' = (a'_0, \ldots, a'_{n-1})$ are both convex, the ranges of $\mathbb{M}_i$ (i.e., $\mathbb{M}_i \in [0,1]$ when $a'_i \geq a_i^{(i-1)}$ or $\mathbb{M}_i \in [-1,0]$ when $a'_i \leq a_i^{(i-1)}$), $i = \{0, 1, \ldots, \lceil \frac{n}{2} \rceil - 2\}$, are obvious and hence no proof is needed.

D. Algorithm Outline

As indicated earlier, it is intuitive to maintain the similarity degree between the consequent parts $B' = (b'_0, \ldots, b'_{n-1})$ and $B^* = (b^*_0, \ldots, b^*_{n-1})$ to be the same as that between the antecedent parts $A' = (a'_0, \ldots, a'_{n-1})$ and $A^* = (a^*_0, \ldots, a^*_{n-1})$, in performing interpolative reasoning. The proposed scale and move transformations can be used to entail this by the following algorithm:

1) Calculate scale rates s_i $(i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support from A' to A^* by $s_i = \frac{a^*_{n-1-i} - a^*_i}{a'_{n-1-i} - a'_i}$.
2) Calculate scale rate s_0 of the bottom support (or just get from the first step) and scale ratios $\mathbb{S}_i$ $(i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support from A' to A^* by (13) and (14).
3) Apply scale transformation to A' with scale rates s_i calculated in the first step to obtain A''.
4) Assign scale rate s'_0 of the bottom support of B' to the value of s_0 (i.e., $s'_0 = s_0$), with the scale ratios $\mathbb{S}'_i$, $(i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\})$ of the i-th support of B' calculated as per (14) under the condition that they equal to $\mathbb{S}_i$ ($i = \{1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$) as calculated in step 2:

$$s'_i = \begin{cases} s_i & (i = 0) \\ \frac{s'_{i-1}(s_i - s_{i-1})(\frac{b'_{n-i} - b'_{i-1}}{b'_{n-i-1} - b'_i} - 1)}{s_{i-1}(\frac{a'_{n-i} - a'_{i-1}}{a'_{n-i-1} - a'_i} - 1)} + s'_{i-1} & (s_i \geq s_{i-1} \geq 0) \\ \frac{s'_{i-1} s_i}{s_{i-1}} & (s_{i-1} \geq s_i \geq 0) \end{cases} \tag{22}$$

5) Apply scale transformation to B' using s'_i ($i = \{0, 1 \ldots, \lfloor \frac{n}{2} \rfloor - 1\}$) as calculated in step 4 to obtain $B'' = (b''_0, \ldots, b''_{n-1})$.
6) Decompose the move transformation to $(\lceil \frac{n}{2} \rceil - 1)$ sub-moves. For $i = 0, 1, \ldots, \lceil \frac{n}{2} \rceil - 2$,
   a) Calculate the i-th sub-move ratio $\mathbb{M}_i$ from $A^{(i-1)}$ to A^* by (21), where $A^{(i-1)}$ is the fuzzy set obtained after the $(i-1)$-th sub-move with initialization $A^{(-1)} = A''$.
   b) Apply move transformation to $A^{(i-1)}$ using $\mathbb{M}_i$ to obtain $A^{(i)} = \{a_0^{(i)}, a_1^{(i)}, \ldots, a_n^{(i)}\}$.
   c) Apply move transformation to $B^{(i-1)}$ using $\mathbb{M}_i$ to obtain $B^{(i)} = \{b_0^{(i)}, b_1^{(i)}, \ldots, b_n^{(i)}\}$.
7) Return $A^{(\lceil \frac{n}{2} \rceil - 2)} = A^*$ and $B^{(\lceil \frac{n}{2} \rceil - 2)}$, which is the required resultant fuzzy set B^*, once the *for* loop of step 6 terminates.

Note that the interpolation of two rules involving multiple antecedent variables is easily extendable by averaging the scale rate, scale ratios and move ratios [5].

The fuzzy interpolation technique is required to give prompt response when it is used to handle time critical applications. Therefore, the complexity of time is an important issue for the present method. With respect to n (the largest number of odd points for any fuzzy sets involved), the transformation-based interpolation needs $O(n^2)$ computation time mainly owing to step 6 in subsection III-D. This is acceptable given that n is not significantly large in most cases, and that high-speed processors are more and more popularly used.

IV. Impact of RV Definition

The example discussed in this section concerns the interpolation between two adjacent rules $A_1 \Rightarrow B_1$ and $A_2 \Rightarrow B_2$, each involving the use of hexagonal fuzzy sets. Interpolations are carried out using three different RV representations (namely, average RV, weighted average RV and center of core RV), resulting in three unique, normal and convex fuzzy sets respectively when given an observation $A^* = (6, 6.5, 7, 9, 10, 10.5)$. All the attribute values of the given two rules and the results (B^*) are shown in Table I and Fig. 3. It is interesting to note that these three results have almost the same geometrical shape although their positions are slightly different. This is because all the calculations involved are the same except the calculation of the RV. This empirically shows that although different RVs may be chosen for use given a specific problem, their influence on the final interpolative outcomes is not significant. This helps ensure the stability of the inference method developed.

V. An Realistic Application

This section shows the usage of the interpolation-based inference in a real-world prediction problem, based on the comp-activ database [2]. This database consists of a collection of computer activity measures such as the number of system read calls per second. The task is to predict the portion of time that CPUs run in user mode from all measured activities. The data set includes 8192 cases, with each having 22 continuous numeric attributes. The whole data set is divided into a training set and a test set. The training set has approximately $2/3$ of the whole data (5462) and test set takes the rest (2730). Consider there may exist redundant or less relevant information in the initial 22 attributes, a process of attribute selection is carried out to choose the most informative ones. For simplicity, the correlation-based feature subset selection [3] is used for this, resulting 11 selected attributes.

The well-known fuzzy ID3 training scheme [7] is adopted here to form the fuzzy rules. Again, for simplicity, the triangular fuzzy sets are used and they are assumed to be evenly distributed over each attribute domain. Fuzzy ID3 trainings with different configurations (in terms of the number of fuzzy sets and the minimal leaf objects) are carried out and the relative squared error (relative to the simple average predictor) are given in Fig. 4. This reveals a general trend in that the more fuzzy sets used in the training, the better performance the resulting rules have. However, the number of rules may

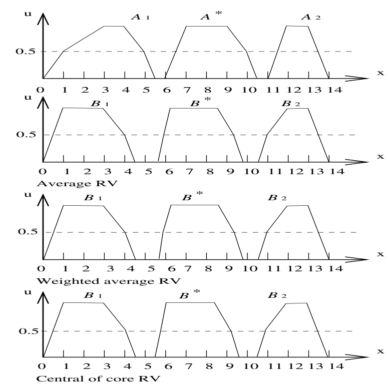

Fig. 3. Interpolation with different RV definitions

TABLE I
Interpolation results with $A^* = (6, 6.5, 7, 9, 10, 10.5)$

A_1	$(0, 1, 3, 4, 5, 5.5)$
A_2	$(11, 11.5, 12, 13, 13.5, 14)$
B_1	$(0, 0.5, 1, 3, 4, 4.5)$
B_2	$(10.5, 11, 12, 13, 13.5, 14)$
B^*(average)	$(5.64, 5.98, 6.29, 8.63, 9.46, 9.93)$
B^*(w_average)	$(5.61, 5.95, 6.26, 8.59, 9.42, 9.89)$
B^*(core center)	$(5.47, 5.79, 6.08, 8.42, 9.23, 9.70)$

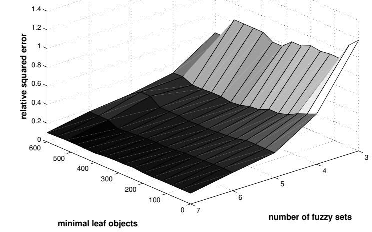

Fig. 4. Relative squared error

become very large at the same time. For instance, with the number of the minimal leaf objects 0, the resulting rule base size increases from 55 to 477 if the number of fuzzy sets increases from 3 to 7. In order to provide a platform to compare the interpolation-based inference with the well known Mamdani inference, both the rule base size and the prediction error have to be considered. For this, a particular resultant rule base, which has 47 rules and an error rate of 13.29% is chosen (where the number of fuzzy sets is 6 and the number of minimal leaf objects is 480). Note that in this rule base, 4 among the 2730 test data are not fired by any of the 47 rules. That is, the obtained rule base is in fact a sparse rule base.

Now the interpolation-based inference is tested over this rule base and the test data. The first step is to compute the intermediate rule. This is not as straightforward as the way discussed in subsection III-A. The distance between a rule and a data object (with crisp or fuzzy values taken by the attributes) is defined as the average of the distances between their individual values (or sets) with regard to each attribute (see (6)). The nearest n ($n \geq 2$) rules are then selected and be used to construct the intermediate rule. In particular, for each input attribute, the weights of the fuzzy sets within the nearest n rules regarding that attribute are computed inversely to the distances between the sets and the test object's value of the corresponding attribute. By using these weights, the calculated intermediate fuzzy set on this attribute may not have the same representative value as the test object's value. A δ is therefore introduced for each input attribute to measure how much move (in proportion to the whole domain space) is needed to move the intermediate fuzzy set to the desired position so that it will have the same representative value as the test object. Note that a positive/negative δ indicates right/left move. The averaged weights and the δs corresponding to all the input attributes are used to determine the intermediate output fuzzy set. In so doing, an intermediate rule which has the same representative value as the test object's value on each input attribute is determined.

The second step is carried out in the same way as described in subsection III-D. Consider that the test data may not be accurate due to measurement noise, a fuzzification for each test data is hence introduced prior to performing interpolation. That is, a vector of crisp values is fuzzified to a vector of fuzzy sets for each test data. The fuzzification of the crisp value on each attribute leads to an isosceles triangular set which has a certain support length. Different versions of such a length (including 0, $1/8$ and $1/4$, with 0 indicating the fuzzification is not performed) are used, and the results with respect to the average, the weighted average, and the center of core RVs are shown in Table II. Note that the error is calculated as the average of the errors in interpolating two or three nearest rules.

These results clearly show that the interpolation-based inference generally obtains better performance than the error rate 13.29% (produced by Mamdani inference). The other advantage is that all the test data are fired by the interpolation-based inference. It is worth noting that the fuzzification of the test data with different support lengths does not significantly affect the prediction error. This ensures the stability of the interpolation-based inference. In particular, if the average RV is used, the results are exactly the same across different support lengths. This is because the value of the average RV over a fuzzy set is exactly the same as the fuzzified crisp value created from the defuzzification method used (center of gravity) over the same fuzzy set.

TABLE II
RELATIVE SQUARED ERROR OF THE INTERPOLATION-BASED INFERENCE

	portion =0	portion =1/8	portion =1/4
average	6.92%	6.92%	6.92%
w_average	6.22%	6.25%	6.28%
core center	8.05%	7.58%	7.20%

VI. CONCLUSIONS

This paper has proposed a generalized, scale and move transformation-based, interpolative reasoning method. It first introduces the general definition of representative values (RVs) and then uses this notion to develop the interpolative reasoning method. The work allows interpolating fuzzy rules involving arbitrary polygonal fuzzy sets, by means of scale and move transformations. The main advantage of this method is its flexibility and diversity: it can choose different RVs to obtain suitable results for different application requirements. The paper has also presented a realistic application of the proposed method.

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