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LONELY RUNNERS WITH COMPLEX TIME

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Dedicated to Professors Zoltán Daróczy and Imre Kátai
on the occasion of their 75th birtday

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Abstract. In this short note we propose a generalization of the so called lonely runner problem to complex times. We give definitions, prove some basic results and formulate open questions.

1. Introduction

Suppose $n+1$ runners are running along a circular track of unit length with distinct constant speeds $v_0, v_1, \ldots, v_n \in \mathbb{R}$. They all start at the same starting position. We say that runner k is ε-lonely at time t if all other runners are at least ε distance away at time t, the distances being measured on the circle. The lonely runner problem asks how lonely the runners can get, more precisely, what is the supremum value of ε for which each runner gets ε-lonely some time. It is conjectured that for all choices of $v_0, v_1, \ldots, v_n$, this supremum is at least $1/(n+1)$. It is not too hard to verify that for the choice $v_0 = 0$, $v_1 = 1, \ldots, v_n = n$, this value is sharp.

We formulate the conjecture in mathematical form using modulo 1 real numbers for describing positions on the track, placing the starting position in 0, focusing on the loneliness of the 0th runner and assuming that $v_0 = 0 <$ $< v_1 < v_2 < \cdots < v_n$. The last assumption means no restriction since adding or

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subtracting the same amount from everyone's speed does not affect distances. Using the notation $\|x\|$ for the distance of x from the nearest integer, the distance of runners k and l at time t is $\|v_k t - v_l t\|$. The formal statement of the conjecture then says

$$\inf_{0<v_1<\dots<v_n\in\mathbb{R}} \sup_{t\in\mathbb{R}} \min_{k=1,\dots,n} \|v_k t\| = \frac{1}{n+1}\,. \tag{1.1}$$

The purpose of the present paper is to give a possible generalization of the lonely runner problem for complex speeds and times, basically by allowing complex values in 1.1 and defining $\|x\|$ appropriately.

The paper is built up as follows: we review some results about the original conjecture in Section 2, we investigate possible generalizations to complex times and speeds in Section 3, we consider $n = 2$ in Section 4, give computational results in Section 5, then summarize and pose open questions in Section 6.

2. Former results

The lonely runner problem dates back to [2] and independently to [5] where it was formulated as a diophantine approximation question and a view obstruction problem, respectively. These papers both prove the conjecture for $n \leq 3$. The $n = 4$ case was covered in [6]. The $n = 5$ case was proved in [4] and later simplified in [7]. The current largest value for which the conjecture is known to hold is $n = 6$, see [1]. Note that the number of runners in the titles of the cited papers is $n + 1$ rather than n.

A good overview of the theoretical background and connections to other problems is found in [3]. We will need two simple observations. The first one states that we may restrict the investigation to rational or – by rescaling – integer speeds, see e.g. [4].

Lemma 2.1. *The lonely runner conjecture is equivalent to*

$$\inf_{0<v_1<\dots<v_n\in\mathbb{Z}} \sup_{t\in\mathbb{R}} \min_{k=1,\dots,n} \|v_k t\| = \frac{1}{n+1}\,. \tag{2.1}$$

Note that the supremum here is taken above integer speed tuples.

The other observation is an almost trivial one, but we make it explicit.

Lemma 2.2. *Fix $v_1 < v_2 < \dots < v_n \in \mathbb{N}^+$ and $r, s \in \mathbb{Z}$. Then*

$$\sup_{t\in\mathbb{R}} \min_{k=1,\dots,n} \|v_k t\| \geq \frac{r}{s}$$

is eqivalent to

$$\sup_{t\in \frac{1}{m}\mathbb{N}\cap[0,1]} \min_{k=1,\dots,n} \|v_k t\| \geq \frac{r}{s}\,,$$

with $m = lcm(v_1, v_2, \dots, v_n, s)$.

The lemma states that for determining if (r/s)-loneliness is achieved, one is allowed to consider time in discrete steps of $1/m$. This is straightforward since by the definition of m, every runner steps in and out of the "loneliness interval" $[-r/s, r/s]$ at times that are multiples of $1/m$. The intersection with the time interval $[0,1]$ can be taken because motion is periodic with the period of one time unit. We will use this lemma for computational purposes.

3. Generalization to complex values

First note that if we allow complex speeds but time is still real, then this just doubles the dimension of the problem. So we generalize to complex speeds and *complex time*. We let the speeds be arbitrary (distinct) nonzero complex numbers (even if not stated explicitly later), we let the time be complex and we identify positions on the "track" with complex numbers modulo the Gaussian integers, that is taking both coordinates modulo 1.

The most challenging part is defining loneliness. We do this by defining $\|z\|$ on the complex plane. There are several ways to proceed. First consider the situation when $\|z\|$ is the ∞-distance from the nearest Gaussian integers, that is

$$\|z\|_\infty = \|x + yi\|_\infty = \max(\|x\|, \|y\|) \tag{3.1}$$

This definition seems promising first but we claim that it is uninteresting for $n = 2$ in most cases.

Lemma 3.1. *Let* v_1 *and* v_2 *be two complex speeds that are not collinear as real vectors. Let* $0 < \varepsilon < 1/2$. *Then* ε*-loneliness is achieved for some time* $t \in \mathbb{C}$, *that is* $\|v_k t\|_\infty \geq \varepsilon$ *for* $k = 1, 2$.

Proof. Let $T_k = \{t \in \mathbb{C} \mid \|v_k t\|_\infty \geq \varepsilon\}$ for $k = 1, 2$. Then T_k contains the infinite strip between the lines $\{(\varepsilon+yi)/v_k \mid y \in \mathbb{R}\}$ and $\{(1-\varepsilon+yi)/v_k \mid y \in \mathbb{R}\}$. Since these two strips are not parallel by the choice of v_1 and v_2, they have nonempty intersection. ∎

Since in dimension 2 any two different norms are equivalent, we get similar results if we use the 1-distance or 2-distance from the nearest Gaussian integer, making the amount of loneliness essentially independent of the chosen speeds.

For obtaining interesting results we therefore use the following definition for $\|z\|$. In the rest of the paper, this notation is only used in the now defined meaning. The original distance function defined on the reals (distance from the nearest integer) wiil be denoted by $\|x\|_{\mathbb{R}}$.

Definition 3.1. Let $z = x + yi \in \mathbb{C}$. We define $\|z\|$ as the *minimum* of $\|x\|_{\mathbb{R}}$ and $\|y\|_{\mathbb{R}}$.

Informally, a runner is considered lonely if the others are far away in both the real and imaginary coordinates. The points that are at least at ε distance from the Gaussian integers form a set $S(\varepsilon)$, which is a union of squares with side length $1 - 2\varepsilon$, centered at the points of the plane with half-integer coordinates.

For convenience we also define the maximal loneliness function on n-tuples of complex speeds. In the original conjecture, the infimum of this quantity is taken over all speed tuples. For brevity, a speed tuple $v_1, \ldots, v_n$ is denoted by V, and if there is no danger of confusion, V will also stand for the set of speeds.

Definition 3.2. Let $v_1, v_2, \ldots, v_n \in \mathbb{C} \setminus \{0\}$ be distinct complex speeds. The maximal loneliness of this speed sequence is defined as

$$\operatorname{maxlon}(V) = \sup_{t \in \mathbb{C}} \max_{k=1,\ldots,n} \|v_k t\| .$$

A third definition is convenient for speaking about the time instants when a specific runner is farther away than a prescribed value.

Definition 3.3. Let v be a complex speed and $\varepsilon > 0$ a real number. The away-time for runner with speed v with distance ε is defined as

$$\operatorname{away}(v, \varepsilon) = \{t \in \mathbb{C} \mid \|vt\| \geq \varepsilon\} .$$

The away-time is a union of squares whose centers form a translated lattice. We have $\operatorname{away}(1, \varepsilon) = S(\varepsilon)$, here the centers of the squares are at points with half-integer coordinates. If the speed is v, the away-time is $S(\varepsilon)/v = \{t/v \mid t \in$ $\in S(\varepsilon)\}$.

We also have

$$\operatorname{maxlon}(V) \geq \varepsilon \text{ if and only if } \bigcap_{k=1,\ldots,n} \operatorname{away}(v_k, \varepsilon) \neq \emptyset .$$

We illustrate this fact in Figure 1.

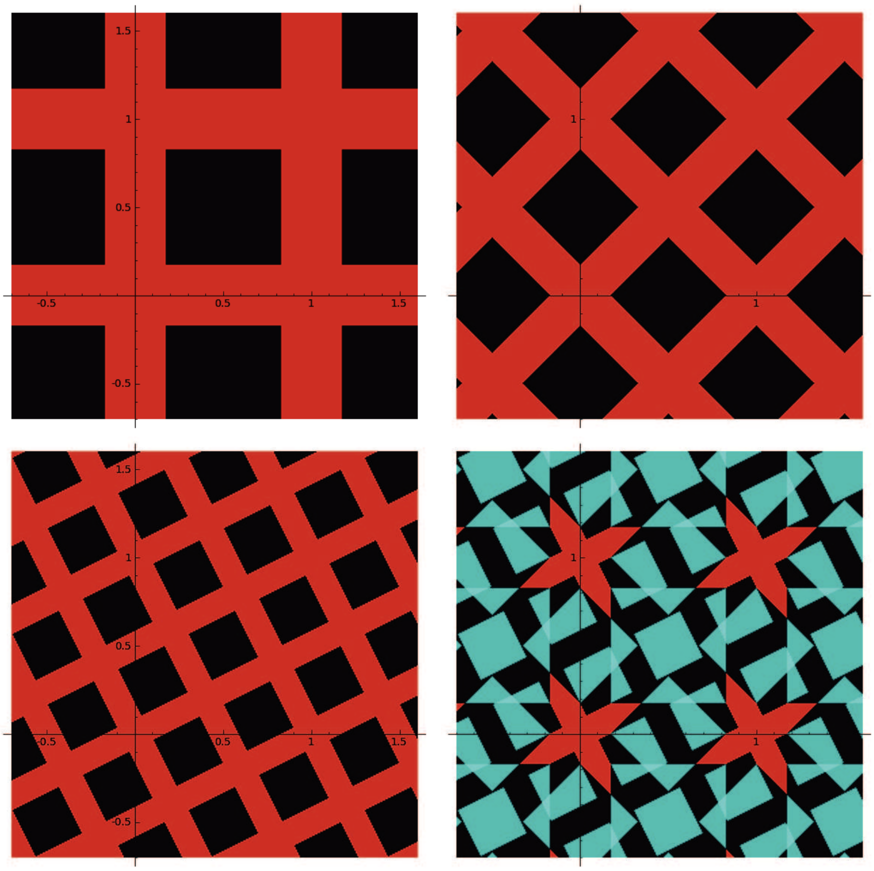

Figure 1. The figure shows the sets away$(v, 0.17)$ for $v = 1$, $v = 1 + i$ and $v = 1 + 2i$. The final picture shows the three sets overlayed, or otherwise expressed, the sum of their characteristic functions. The lightest areas indicate the non-empty intersection.

We finally state an analogue of a result for the original lonely runner problem.

Theorem 3.1. *For a given n, the infimum of the maximum loneliness value is attained at a speed tuple of Gaussian integer speeds, that is*

$$\inf_{V \subseteq \mathbb{C}\setminus\{0\}, |V|=n} maxlon(V) = \min_{V \subseteq \mathbb{Z}[i]\setminus\{0\}, |V|=n} maxlon(V)\,.$$

Therefore from now on we only consider speeds in the Gaussian integers. The proof is essentially identical to the classical case, we refer to e.g.[4].

4. Three complex runners ($n = 2$)

In this section we analyze the case $n = 2$. We prove that the maximal loneliness is always at least $1/4$. First we prove that this value is sharp.

Theorem 4.1.

$$maxlon(1, 1 + i) = \frac{1}{4}.$$

Proof. The away-times for runner 1 and runner 2 with ε are the sets $S = S(\varepsilon)$ and $S/(1 + i)$, respectively, see Figure 2. Since time is doubly periodic with periods 1 and i, we only consider the intersection of the away-times in $[0, 1]^2$. It is visible in the figure, and an easy but technical calculation shows that $S \cap S/(1 + i) \cap [0, 1]^2 = \{1/4 + i/2, 1/2 + i/4, 1/2 + 3i/4, 3/4 + i/2\}$ when $\varepsilon = 1/4$, and this intersection is empty when $\varepsilon > 1/4$. We leave the details to the reader. ■

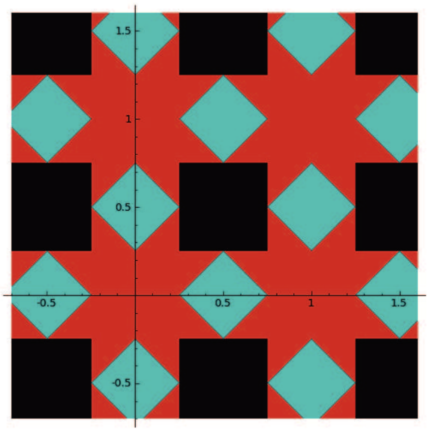

Figure 2. The figure shows the sets away$(v, 1/4)$ for $v = 1$ and $v = 1 + i$. It is visible that the intersection in $[0, 1]^2$ consists of 4 points. Also, it is plausible from the picture that increasing ε (shrinking the squares) will make the intersection empty.

Theorem 4.2. *For every speed pair $v_1, v_2 \in \mathbb{C}$, we have*

$$\mathit{maxlon}(v_1, v_2) \geq \frac{1}{4}\,.$$

Proof. We can make a few assumptions. If we rescale time or speeds by a constant, the the intersection of away-times is only multiplied by the scaling constant. Therefore, we assume $v_1 = 1$. Also, by symmetry we assume $|v| \leq 1$. It remains to prove that for all v with $|v| \leq 1$, there exists a time $t \in \mathbb{C}$ such that $\|t\| \geq 1/4$ and $\|vt\| \geq 1/4$. Also, by rotational and mirror symmetry, we may assume that $0 \leq \arg(v) \leq \pi/4$. We distinguish two cases.

Case 1: v is small. We first give an intuitive description of this case. If v is small, then $S = S(1/4)$ and S/v always intersect, since the latter one consists of so large squares that do not fit in the holes between the squares in S. We make this remark quantitative below. Suppose $v = v_x + iv_y$ with $v_x + v_y \leq 1$. Consider the following problem: let $S = S(1/4)$. How large can a square be on the plane without intersecting S? What if we prescribe the slope α of one of the sides of the square? The best that one can achieve is to place the larger square between 4 neighboring smaller cells, with the sides of the large square touching the vertices of the smaller squares. One obtains that the maximal ratio of the large and the small squares is $\sqrt{1 + \sin(2\alpha)}$. One can verify that the set of v for which this reasoning shows a non-empty intersection of the away-sets is exactly those v for which $v_x + v_y \leq 1$.

Case 2. We have to cover the case of $v = v_x + v_y$ with $|v| \leq 1$, $v_x + v_y \geq 1$ and $0 \leq v_y \leq v_x$. Denote the set of such values by U. We will show that there exists times for each $v \in U$ with loneliness at least $1/4$. Let t_1, t_2 and t_3 denote the times $1/2 + i/2$, $1/2 + i/4$ and $3/4 + i/4$, respectively. These times are in S, the away-time of runner 1. Let T be the square in S centered at $1/2 + i/2$. If $vt_j \in T$ for some j, then the intersection of the away-times contains t_j, so we have proved the statement for v. But one can verify symbolically – which we did by computer – and it is visible in Figure 3 that $U \subseteq T/t_1 \cup T/t_2 \cup T/t_3$.

This finishes both cases of the proof. ■

Corollary 4.1. *From Theorems* 4.1 *and* 4.2 *it follows that*

$$\inf_{|V|=2} \mathit{maxlon}(V) = \frac{1}{4}\,.$$

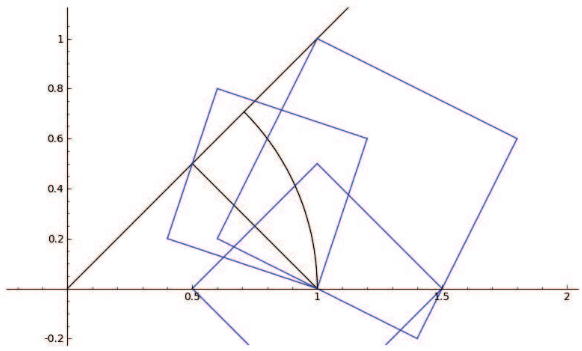

Figure 3. This figure illustrates the proof of case 2 in theorem 4.2. The set U is the half-banana-shaped region bordered by the circular arc and the two darker straight lines. The three squares (one of which is only partially visible) are T/t_1, T/t_2 and T/t_3.

5. Computational results for $n = 3$

Lemma 2.2 about discretizing time steps cannot be carried over easily to the complex setting. Although it is true that if the intersection of the away-times is non-empty, then there is a point in the intersection whose coordinates are rational with denominators bounded above a priori by some calculable constant depending only on V and the rational ε , but this bound is usually large.

Nevertheless, we can discretize time in small steps (1/100, for example), and calculate the maximal loneliness over the discrete time instants. This gives a lower bound very fast. To verify that a lower bound is the exact value of maximal loneliness, symbolic calculations or linear program solvers can be used, which are slower but give exact results.

In the table below we list the maximal loneliness for some speed tuples (mainly triplets) of Gaussian integers with small coordinates. The calculations were carried out on a single desktop machine running a sage server.

Numerical evidence is not satisfactory at the moment to formulate conjectures.

Speed tuple	Maximal loneliness
1, 1+i	1/4
1, 2	1/3
1, 2, 3	1/4
1, 2, 1+i	1/5
1, 2, i	1/3
1, 1+i, 1+2i	1/5
1, 2, 3, 4	1/5
1, 1+i, 1+2i, 1+3i	1/6

Table 1. Table with maximal loneliness values for a few speed tuples.

6. Summary

We considered a possible generalization of the lonely runner problem to complex valued speeds and times. We examined the $n = 2$ case mathematically and gave computational results for $n = 3$.

We pose the following questions:

- Find an efficient method for finding the maximal loneliness for given speeds.
- Determine the exact value of $\inf_V \operatorname{maxlon}(V)$ for $n = 3$ and possibly for $n = 4$, with the aid of a computer if needed (possible candidate values are $1/5$ and $1/6$).
- Based on numerical calculations, formulate a conjecture for all values of n.
- Explore the connections and implications (if any) between the original conjecture and the generalization.

Our further work will also focus on these questions, primarily on the $n = 3$ case.

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