Lifting properties in operator ranges

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Abstract

Given a bounded positive linear operator A on a Hilbert space $\mathcal{H}$ we consider the semi-Hilbertian space $(\mathcal{H}, \langle\ ,\ \rangle_A)$, where $\langle \xi, \eta \rangle_A = \langle A\xi, \eta \rangle$. On the other hand, we consider the operator range $R(A^{1/2})$ with its canonical Hilbertian structure, denoted by $\mathbf{R}(A^{1/2})$. In this paper we explore the relationship between different types of operators on $(\mathcal{H}, \langle\ ,\ \rangle_A)$ with classical subsets of operators on $\mathbf{R}(A^{1/2})$, like Hermitian, normal, contractions, projections, partial isometries and so on. We extend a theorem by M. G. Krein on symmetrizable operators and a result by M. Mbekhta on reduced minimum modulus.

Introduction

Let $\mathcal{H}$ be a complex Hilbert space and let $A : \mathcal{H} \to \mathcal{H}$ be a positive (semidefinite bounded linear operator) operator. Consider the semi-inner product defined by A, namely, $\langle \xi, \eta \rangle_A := \langle A\xi, \eta \rangle$ for all $\xi, \eta \in \mathcal{H}$. The set of all $T \in L(\mathcal{H})$ which are A-adjointable, i.e., for which there exists $W \in L(\mathcal{H})$ such that $\langle T\xi, \eta \rangle_A = \langle \xi, W\eta \rangle_A$ for all $\xi, \eta \in \mathcal{H}$, is

$$L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : T^* R(A) \subseteq R(A)\}.$$

On the other side, if $\|\xi\|_A = \langle \xi, \xi \rangle_A^{1/2} = \|A^{1/2}\xi\|$, the set of all $\|\ \|_A$-bounded operators in $L(\mathcal{H})$ is

$$L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : T^* R(A^{1/2}) \subseteq R(A^{1/2})\}.$$

These characterizations follow from the well known Douglas' range inclusion theorem [11]. A recent result by S. Hassi, Z. Sebestyén and H. de Snoo [16] implies that $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$. In what follows, any element in $L_{A^{1/2}}(\mathcal{H})$ will be called an A-operator.

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Among the A-operators, the A-symmetrizable operators have been studied since the beginning of operator theory. Recall that $T \in L(\mathcal{H})$ is called A-symmetrizable if $AT = T^*A$, which means that AT is Hermitian or selfadjoint. The book of A.C. Zaanen [26] and the papers by M. G. Krein [17], P. Lax [18], J. Dieudonné [10], B. A. Barnes [4], and Z. Sebestyén, and J. Stochel [24] contain many results, examples and applications of symmetrizable operators. More recently, P. Cojuhari and A. Gheondea [7], S. Hassi et al. [16] have extended the theory to unbounded operators $T : \mathcal{D}(T) \subseteq \mathcal{H} \to \mathcal{K}$, with semi inner products $\langle\ ,\ \rangle_A$ on $\mathcal{H}$ and $\langle\ ,\ \rangle_B$ on $\mathcal{K}$, where B is a positive operator on $\mathcal{K}$.

The semi-inner product $\langle\ ,\ \rangle_A$ induces on the quotient $\mathcal{H}/N(A)$ an inner product which is not complete unless $R(A)$ is closed (here $N(A)$ denotes the nullspace and $R(A)$ the range of A). A canonical construction due to de Branges and Rovnyak [5], [6] shows that the completion of $\mathcal{H}/N(A)$ is isometrically isomorphic to the range $R(A^{1/2})$ of the positive square root of A, with the inner product $(A^{1/2}\xi, A^{1/2}\eta) := \langle P\xi, P\eta\rangle$, where P denotes the orthogonal projection onto the closure of $R(A)$ in $\mathcal{H}$. The Hilbert space $(R(A^{1/2}), (\ ,\ ))$ will be denoted by $\mathbf{R}(A^{1/2})$. The books of T. Ando [1] and D. Sarason [20] and a series of papers of Z. Sebestyén [21], [22], [23], and Z. Sebestyén and J. Stochel [24] are excellent sources for this construction.

This paper is devoted to explore the relationship between A-operators in $L(\mathcal{H})$ and the algebra $L(\mathbf{R}(A^{1/2}))$ of all (bounded linear) operators on $\mathbf{R}(A^{1/2})$. There is a unitary operator U_A from the closure of $R(A)$ in $\mathcal{H}$ onto the space $\mathbf{R}(A^{1/2})$. The conjugation by U_A provides an isometric isomorphism between $L(\overline{R(A)})$ and $L(\mathbf{R}(A^{1/2}))$. However, this isomorphism has no good properties with respect to $\langle\ ,\ \rangle_A$. Our choice is to study the way in which the operator $W_A : \mathcal{H} \to \mathbf{R}(A^{1/2})$ defined by $\xi \mapsto A\xi$, and a certain adjoint of W_A transform A-operators in $L(\mathcal{H})$ into operators in $L(\mathbf{R}(A^{1/2}))$, and conversely.

We describe now the main results of this paper. In 1937 M. G. Krein [17] (and, later and independently, P. Lax [18]) proved the following theorem. Consider an inner product space L with an additional Banach norm $\|\ \|_B$ and let $T : L \to L$ be a linear operator such that $\langle T\xi, \eta\rangle = \langle \xi, T\eta\rangle$ for all $\xi, \eta \in L$. If T is $\|\ \|_B$-bounded then it is also $\|\ \|_L$-bounded. Our extension is the following: if $L = \mathbf{R}(A^{1/2})$ and $T : L \to L$ is linear and it admits a $\langle\ ,\ \rangle$-adjoint $V : L \to L$, then T is $\|\ \|_{\mathcal{H}}$-bounded if it is $\|\ \|_{\mathbf{R}(A^{1/2})}$-bounded.

The second main result is the construction of partially defined homomorphisms $\alpha : L(\mathcal{H}) \to L(\mathbf{R}(A^{1/2}))$, $\beta : L(\mathbf{R}(A^{1/2})) \to L(\mathcal{H})$ such that they basically transport the Hermitian and normal operators, the contractions, the partial isometries and projections, from one side to the other. In a paper by Cojuhari and Gheondea [7], the operator $\alpha(T) \in L(\mathbf{R}(A^{1/2}))$ is called the lifting of T; we follow their terminology.

Finally, we extend to A-operators a result by M. Mbekhta [19] on the reduced minimum modulus of a partial isometry.

The contents of the paper are the following. Section 1 contains basic results on A-operators. There is also a description of the range inclusion theorem of R. G. Douglas [11], which is a key for several results of this paper. Section 2 is devoted to the description of $L(\mathbf{R}(A^{1/2}))$ and to the extension of Krein's theorem. In section 3 we study the correspondence between A-operators and classes of operators in $L(\mathbf{R}(A^{1/2}))$. The final section 4 contains the results on the A-reduced minimum modulus.

1 Preliminaries

Throughout $\mathcal{G}$, $\mathcal{H}$ and $\mathcal{K}$ denote complex Hilbert spaces with inner product $\langle\ ,\ \rangle$. By $L(\mathcal{H}, \mathcal{K})$ we denote the space of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$, and we abbreviate $L(\mathcal{H}) = L(\mathcal{H},\mathcal{H})$. $L(\mathcal{H})^+$ is the cone of positive (semidefinite) operators of $L(\mathcal{H})$, i.e., $L(\mathcal{H})^+ := \{A \in L(\mathcal{H}) : \ \langle A\xi,\xi\rangle \geq 0 \ \forall \xi \in \mathcal{H}\}$. For every $T \in L(\mathcal{H},\mathcal{K})$ its range is denoted by $R(T)$, its nullspace by $N(T)$ and its adjoint by T^*. Given a closed subspace $\mathcal{S}$ of $\mathcal{H}$, $P_{\mathcal{S}}$ denotes the orthogonal projection onto $\mathcal{S}$.

1.1 The semi-Hilbertian space $(\mathcal{H}, \langle\ ,\ \rangle_A)$

Given $A \in L(\mathcal{H})^+$, the functional $\langle\ ,\ \rangle_A : \mathcal{H}\times\mathcal{H} \to \mathbb{C}$, $\langle \xi,\eta\rangle_A := \langle A\xi,\eta\rangle$, defines a Hermitian sesquilinear form which is positive semidefinite, i.e., a semi-inner product on $\mathcal{H}$. So, $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is a semi-Hilbertian space. By $\|\ .\ \|_A$ we denote the seminorm on $\mathcal{H}$ induced by $\langle\ ,\ \rangle_A$, i.e., $\|\xi\|_A = \langle \xi,\xi\rangle_A^{1/2}$. Given a subspace $\mathcal{S}$ of $\mathcal{H}$ its A-orthogonal subspace is the subspace $\mathcal{S}^{\perp_A} = \{\xi \in \mathcal{H} : \ \langle \xi,\eta\rangle_A = 0 \ \forall\ \eta \in \mathcal{S}\}$. Observe that $\langle\ ,\ \rangle_A$ induces a seminorm on a subset of $L(\mathcal{H})$. More precisely, given $T \in L(\mathcal{H})$, if there exists a constant $c > 0$ such that $\|T\omega\|_A \leq c\|\omega\|_A$ for every $\omega \in \mathcal{H}$ then it holds $\|T\|_A := \sup\limits_{\substack{\omega\in\overline{R(A)}\\ \omega\neq 0}} \frac{\|T\omega\|_A}{\|\omega\|_A} < \infty$. Define $L_{A^{1/2}}(\mathcal{H}) := \{T \in L(\mathcal{H}) :$ for some $c > 0$, $\|T\xi\|_A \leq c\|\xi\|_A \ \forall \xi \in \mathcal{H}\}$. $L_{A^{1/2}}(\mathcal{H})$ is a subalgebra of $L(\mathcal{H})$. Note that given $T \in L_{A^{1/2}}(\mathcal{H})$, in general, $T^* \notin L_{A^{1/2}}(\mathcal{H})$.
Given $T \in L(\mathcal{H})$ we say that $W \in L(\mathcal{H})$ is an **A-adjoint** of T if $\langle T\xi,\eta\rangle_A = \langle \xi, W\eta\rangle_A$ for every $\xi,\eta \in \mathcal{H}$, or, which is equivalent, if W satisfies the equation $AX = T^*A$. The operator T is called **A-selfadjoint** if $AT = T^*A$. The existence of an A-adjoint operator is not guaranteed. Observe that a given $T \in L(\mathcal{H})$ may admit none, one or many A-adjoints: in fact, if W is an A-adjoint of T and $AZ = 0$ for some $Z \in L(\mathcal{H})$ then $W + Z$ is also an A-adjoint of T. This kind of equations can be studied applying the next theorem of R. G. Douglas (for its proof see [11] or [13]).

Theorem (Douglas) *Let $B \in L(\mathcal{H},\mathcal{K})$ and $C \in L(\mathcal{G},\mathcal{K})$. The following conditions are equivalent:*

1. *$R(C) \subseteq R(B)$.*
2. *There is a positive number λ such that $CC^* \leq \lambda BB^*$.*
3. *There is $D \in L(\mathcal{G},\mathcal{H})$ such that $BD = C$.*

If one of these conditions holds then there is a unique operator $E \in L(\mathcal{G},\mathcal{H})$ such that $BE = C$ and $R(E) \subseteq \overline{R(B^*)}$. Furthermore, $N(E) = N(C)$. Such E is called the **reduced solution** *or* **Douglas solution** *of $BX = C$.*
The reduced solution of the equation $BX = C$ can be explicitly obtained by means of the Moore-Penrose inverse of B. Recall that given $B \in L(\mathcal{H},\mathcal{K})$ the Moore-Penrose inverse of B, denoted by $B^\dagger$, is defined as the unique linear extension of $\tilde{B}^{-1}$ to $\mathcal{D}(B^\dagger) := R(B) + R(B)^\perp$ with $N(B^\dagger) = R(B)^\perp$, where $\tilde{B}$ is the isomorphism $B|_{N(B)^\perp} : N(B)^\perp \longrightarrow R(B)$. It holds that $B^\dagger$ is the unique solution of the four "Moore-Penrose equations":

$$BXB = B, \quad XBX = X, \quad XB = P_{N(B)^\perp} \ \text{ and } \ BX = P_{\overline{R(B)}}|_{\mathcal{D}(B^\dagger)}.$$

$B^\dagger$ is a bounded operator with closed range if and only if $R(B)$ is closed. the reduced solution of the equation $BX = C$ with $R(C) \subseteq R(B)$, is $B^\dagger C$. the range inclusion guarantees its boundedness. For this and other results concerning different generalized inverses of B and solutions of the equations $BX = C$, see Engl and Nashed [12] and Arias et al. [3].
In what follows, we denote $L_A(\mathcal{H}) := \{T \in L(\mathcal{H}) : \ T$ admits A-adjoint$\}$. The next proposition shows that the notations $L_A(\mathcal{H})$ and $L_{A^{1/2}}(\mathcal{H})$ which look quite different, are consistent.

Proposition 1.1. Let $A \in L(\mathcal{H})^+$. Then:

1. $L_A(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ T^*R(A) \subseteq R(A)\}$.
2. $L_{A^{1/2}}(\mathcal{H}) = \{T \in L(\mathcal{H}) : \ T^*R(A^{1/2}) \subseteq R(A^{1/2})\}$.

Proof. (1) It is a straightforward application of Douglas theorem.
(2) Observe that $T \in L_{A^{1/2}}(\mathcal{H})$ if and only if $T^*AT \leq cA$, and apply Douglas theorem. $\square$

The next result has been proved in a more general context by Hassi, Sebestyén and de Snoo ([16], Theorem 5.1). Here we present a short proof due to J. Antezana, valid for bounded operators, which only uses the so called Jensen operator inequality.

Proposition 1.2. Let $A \in L(\mathcal{H})^+$. Then, $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$.

Proof. Let $T \in L_A(\mathcal{H})$. Without loss of generality it is enough to consider the case where T is a contraction. In this case the map $\phi : L(\mathcal{H}) \to L(\mathcal{H})$ defined by $\phi(E) = T^*ET$ is a contractive positive map. If there is an operator $C \in L(\mathcal{H})$ such that $AC = T^*A$ then

$$T^*A^2T = ACC^*A \leq \|C\|^2A^2.$$

Now, by Jensen's inequality (see [14], [15]), we obtain that $T^*AT \leq (T^*A^2T)^{1/2}$. On the other hand, $(T^*A^2T)^{1/2} \leq \|C\|A$ because $f(t) = t^{1/2}$ is operator monotone. This proves that

$$(T^*A^{1/2})(T^*A^{1/2})^* = T^*AT \leq \|C\|A.$$

Therefore, by Douglas theorem, $T \in L_{A^{1/2}}(\mathcal{H})$. $\square$

Remark 1.3. The same proof, changing $t \to t^{1/2}$ by $t \to t^s$ shows that $L_A(\mathcal{H}) \subseteq L_{A^s}(\mathcal{H})$ for all $s \in (0,1)$. More generally, if $0 < s < s' < 1$ then $L_{A^{s'}}(\mathcal{H}) \subseteq L_{A^s}(\mathcal{H})$. Moreover, $L_{A^{s'}}(\mathcal{H}) = L_{A^s}(\mathcal{H})$ if and only if $R(A)$ is closed.

2 The algebra $L(\mathbf{R}(A^{1/2}))$

Let $A \in L(\mathcal{H})^+$. $R(A^{1/2})$ be equipped with the inner product

$$(A^{1/2}\xi, A^{1/2}\eta) := \langle P\xi, P\eta\rangle \text{ for every } \xi, \eta \in \mathcal{H},$$

where we abbreviate $P_{\overline{R(A)}}$ by P. It can be checked that $\mathbf{R}(A^{1/2}) = (R(A^{1/2}), (\ ,\ ))$ is a Hilbert space. Moreover, $R(A)$ is dense in $\mathbf{R}(A^{1/2})$ and $(A\xi, A\eta) = \langle \xi, \eta\rangle_A$ for every $\xi, \eta \in \mathcal{H}$.
In this section we describe $L(\mathbf{R}(A^{1/2}))$. For this, we consider some operators between $\mathcal{H}$ and $\mathbf{R}(A^{1/2})$, and $\overline{R(A)}$ and $\mathbf{R}(A^{1/2})$, namely,

$$Z_A : \mathcal{H} \to \mathbf{R}(A^{1/2}) \text{ defined by } Z_A\xi = A^{1/2}\xi;$$

$$U_A : \overline{R(A)} \to \mathbf{R}(A^{1/2}) \text{ defined by } U_A\xi = A^{1/2}\xi;$$

$$W_A : \mathcal{H} \to \mathbf{R}(A^{1/2}) \text{ defined by } W_A\xi = A\xi.$$

Following Z. Sebestyén and J. Stochel [24], we use the notations Z_A, U_A and W_A just to distinguish them from $A^{1/2} : \mathcal{H} \to \mathcal{H}$, $A^{1/2}|_{\overline{R(A)}} : \overline{R(A)} \to \mathcal{H}$ and $A : \mathcal{H} \to \mathcal{H}$, respectively. In fact, when taking adjoints, the differences between $A^{1/2}$, Z_A and U_A (respectively, A and W_A) become apparent.

Proposition 2.1. The following assertions hold:

1. $Z_A \in L(\mathcal{H}, \mathbf{R}(A^{1/2}))$ and Z_A is onto;
2. $Z_A^* \in L(\mathbf{R}(A^{1/2}), \mathcal{H})$, $Z_A^*(A^{1/2}\eta) = P\eta$;
3. $Z_A^* Z_A = P$ and $Z_A Z_A^* = I_{\mathbf{R}(A^{1/2})}$, in particular Z_A is a coisometry;
4. $U_A \in L(\overline{R(A)}, \mathbf{R}(A^{1/2}))$ is an unitary operator;
5. $Z_A|_{\overline{R(A)}} = U_A$;
6. $W_A \in L(\mathcal{H}, \mathbf{R}(A^{1/2}))$ and $R(W_A) = R(A)$ is dense in $\mathbf{R}(A^{1/2})$;
7. $W_A^* : \mathbf{R}(A^{1/2}) \to \mathcal{H}$, $W_A^*(A^{1/2}\eta) = A^{1/2}\eta$, and $R(W_A^*) = R(A^{1/2})$;
8. $W_A^* W_A = A$ and $Z_A^* W_A = A^{1/2}$.

Proof. Straightforward. □

The next result gives necessary and sufficient conditions for a linear operator $\tilde{T} : R(A^{1/2}) \to R(A^{1/2})$ to be bounded under the norm $\| \;\|_{\mathbf{R}(A^{1/2})}$.

Proposition 2.2. Let $\tilde{T} : R(A^{1/2}) \to R(A^{1/2})$ be a linear operator. Then there exists a unique linear operator $V : \mathcal{H} \to \mathcal{H}$ such that $R(V) \subseteq \overline{R(A)}$ and $A^{1/2}V = \tilde{T}A^{1/2}$. Moreover, $\tilde{T}$ is bounded in $\mathbf{R}(A^{1/2})$ if and only if V is bounded in $\mathcal{H}$. In such case, $V = Z_A^* \tilde{T} Z_A$ and it is the reduced solution of the equation $Z_A X = \tilde{T} Z_A$. Moreover, $\|\tilde{T}\|_{\mathbf{R}(A^{1/2})} = \|V\|$.

Proof. Given $\xi \in \mathcal{H}$ there exists a unique $\eta \in \overline{R(A)}$ such that $\tilde{T}(A^{1/2}\xi) = A^{1/2}\eta$. Define $V : \mathcal{H} \to \mathcal{H}$ by $V\xi = \eta$. It is easy to see that V is linear and $R(V) \subseteq \overline{R(A)}$. Furthermore, $A^{1/2}V = \tilde{T}A^{1/2}$. The uniqueness is straightforward. Now, suppose that $\tilde{T}$ is bounded in $\mathbf{R}(A^{1/2})$. Hence, as $\tilde{T}Z_A = Z_A V$ then, by Douglas theorem, V is bounded. Moreover, since $R(V) \subseteq \overline{R(A)}$ then V is the reduced solution of the equation $\tilde{T}Z_A = Z_A X$ and $V = Z_A^* \tilde{T} Z_A$. Conversely, if V is bounded then there exists $c > 0$ such that $\|V\xi\| \leq c\|\xi\|$ for every $\xi \in \mathcal{H}$. In particular, $\|VP\xi\| \leq c\|P\xi\|$ for every $\xi \in \mathcal{H}$. Now, since $N(A^{1/2}) \subseteq N(\tilde{T}A^{1/2}) = N(V)$, then $VP = V$. Hence, $\|V\xi\| \leq c\|P\xi\|$ for every $\xi \in \mathcal{H}$ or, which is equivalent, $\|\tilde{T}(A^{1/2}\xi)\|_{\mathbf{R}(A^{1/2})} \leq c\|A^{1/2}\xi\|_{\mathbf{R}(A^{1/2})}$ for every $\xi \in \mathcal{H}$. So, $\tilde{T}$ is bounded. On the other hand, since $\tilde{T}Z_A = Z_A V$, $R(V) \subseteq \overline{R(A)}$ and $N(A) \subseteq N(V)$ it holds

$$\begin{aligned}
\|\tilde{T}\|_{\mathbf{R}(A^{1/2})} &= \sup\{ \|\tilde{T}A^{1/2}\xi\|_{\mathbf{R}(A^{1/2})} : \; \|A^{1/2}\xi\|_{\mathbf{R}(A^{1/2})} = 1, \xi \in \mathcal{H}\} \\
&= \sup\{ \|A^{1/2}V\xi\|_{\mathbf{R}(A^{1/2})} : \; \|A^{1/2}\xi\|_{\mathbf{R}(A^{1/2})} = 1, \xi \in \mathcal{H}\} \\
&= \sup\{ \|PV\xi\| : \; \|P\xi\| = 1, \xi \in \mathcal{H}\} \\
&= \sup\{ \|V\xi\| : \; \|\xi\| = 1, \xi \in \mathcal{H}\} \\
&= \|V\|.
\end{aligned}$$

□

In his groundbreaking paper [17], M. G. Krein proved the following theorem. Let $(L, \langle\ ,\ \rangle)$ be an inner product space with Euclidean norm $\|\ \|_L$ such that there exists a (complete) Banach norm $\|\ \|_B$ on L. Let $T : L \to L$ be a linear operator such that $\langle T\xi, \eta\rangle = \langle \xi, T\eta\rangle$ $\forall \xi, \eta \in L$. If T is $\|\ \|_L$-bounded then it is also $\|\ \|_B$-bounded. We prove now that, for the special case $L = R(A^{1/2})$ with the inner product of $\mathcal{H}$ and the Banach norm $\|\ \|_{\mathbf{R}(A^{1/2})}$, the same conclusion holds for a wider class of operators, namely, it holds for all linear operators $T : L \to L$ such that it admits an adjoint $Z : L \to L$ in the sense that $\langle T\xi, \eta\rangle = \langle \xi, Z\eta\rangle$ $\forall\ \xi, \eta \in L$.

Theorem 2.3. Let $\tilde{T} : R(A^{1/2}) \to R(A^{1/2})$ and $Z : R(A^{1/2}) \to R(A^{1/2})$ be linear operators such that $\left\langle \tilde{T}(A^{1/2}\xi), A^{1/2}\eta \right\rangle = \left\langle A^{1/2}\xi, Z(A^{1/2}\eta) \right\rangle$ for every $\xi, \eta \in \mathcal{H}$. If $\tilde{T}$ is bounded in $\mathbf{R}(A^{1/2})$ then $\tilde{T}$ is bounded in $\mathcal{H}$.

Proof. By Proposition 2.2, there exist linear operators $V, V_1 : \mathcal{H} \to \mathcal{H}$ such that $\tilde{T}A^{1/2} = A^{1/2}V$, $ZA^{1/2} = A^{1/2}V_1$ and $R(V), R(V_1) \subseteq \overline{R(A)}$. As $\tilde{T}$ is bounded in $\mathbf{R}(A^{1/2})$, then V is bounded. Moreover, for every $\xi, \eta \in \mathcal{H}$ it holds $\langle \xi, AV_1\eta\rangle = \left\langle A^{1/2}\xi, A^{1/2}V_1\eta \right\rangle = \left\langle A^{1/2}\xi, ZA^{1/2}\eta \right\rangle = \left\langle \tilde{T}A^{1/2}\xi, A^{1/2}\eta \right\rangle = \left\langle A^{1/2}V\xi, A^{1/2}\eta \right\rangle = \langle \xi, V^*A\eta\rangle$. Thus, $AV_1 = V^*A$. So $V \in L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$. Therefore, by Proposition 1.1, there exists $c > 0$ such that $V^*AV \leq cA$, or which is the same $\|A^{1/2}V\xi\| \leq c\|A^{1/2}\xi\|$ for every $\xi \in \mathcal{H}$. Thus, $\|\tilde{T}(A^{1/2}\xi)\| = \|A^{1/2}V\xi\| \leq c\|A^{1/2}\xi\|$ for every $\xi \in \mathcal{H}$. Therefore $\tilde{T}$ is bounded in $\mathcal{H}$. $\square$

3 Relationship among A-operators and operators of $L(\mathbf{R}(A^{1/2}))$

In this section we study the problem of relating classes of A-operators with similar classes of operators on $\mathbf{R}(A^{1/2})$. For this, note that if one needs to work with $T \in L(\mathcal{H}, \mathcal{K})$ and there are positive operators $A \in L(\mathcal{H})^+$, $B \in L(\mathcal{K})^+$ inducing semi-inner products $\langle\ ,\ \rangle_A$ on $\mathcal{H}$ and $\langle\ ,\ \rangle_B$ on $\mathcal{K}$, respectively, then T is AB-adjointable, in the sense that there exists $W \in L(\mathcal{K}, \mathcal{H})$ such that $\langle T\xi, \eta\rangle_B = \langle \xi, W\eta\rangle_A$ $\forall \xi \in \mathcal{H}$, $\eta \in \mathcal{K}$, if and only if the equation $AX = T^*B$ admits a solution; by Douglas theorem, this is equivalent to $R(T^*B) \subseteq R(A)$. However, if $R(T^*B) \not\subseteq R(A)$, the definition of AB-adjoint of T can be extended as follows:

Definition 3.1. Given $T \in L(\mathcal{H}, \mathcal{K})$ its AB-**adjoint** is the operator $\boldsymbol{T^\sharp}$ defined by

$$\mathcal{D}(T^\sharp) = \{\xi \in \mathcal{K} :\ \exists \eta \in \overline{R(A)} \text{ such that } \langle T\nu, \xi\rangle_B = \langle \nu, \eta\rangle_A\ \ \forall \nu \in \mathcal{H}\};$$

and $T^\sharp\xi = \eta$ for each $\xi \in \mathcal{D}(T^\sharp)$.

Proposition 3.2. Let $A \in L(\mathcal{H})^+$, $B \in L(\mathcal{K})^+$ and $T \in L(\mathcal{H}, \mathcal{K})$. The next assertions hold:

1. $T^\sharp$ is a well defined linear operator.
2. If $R(T^*B) \subseteq R(A)$ then $T^\sharp$ is the reduced solution of the equation $AX = T^*B$, i.e. $T^\sharp = A^\dagger T^*B$.

Proof. 1. If given $\xi \in \mathcal{D}(T^\sharp)$ there exist $\eta_1, \eta_2 \in \overline{R(A)}$ such that $\langle \nu, \eta_1\rangle_A = \langle T\nu, \xi\rangle_B = \langle \nu, \eta_2\rangle_A$ for every $\nu \in \mathcal{H}$ then $\langle A\nu, \eta_1 - \eta_2\rangle = 0$ for every $\nu \in \mathcal{H}$. So, $A(\eta_1 - \eta_2) = 0$. Therefore, $\eta_1 = \eta_2$ because $\eta_1, \eta_2 \in \overline{R(A)}$. Thus $T^\sharp$ is well defined.

2. It is a straightforward application of Douglas theorem. $\square$

Observe that if $T \in L_A(\mathcal{H})$ then $T^\sharp$ denotes the reduced solution of the equation $AX = T^*A$. We work with the next classes of A-operators.

Definition 3.3. Let $T \in L(\mathcal{H})$.

1. $T \in L_A(\mathcal{H})$ is an A**-normal** operator if $T^\sharp T = TT^\sharp$.
2. T is an A**-contraction** if $\|T\xi\|_A \leq \|\xi\|_A$ for every $\xi \in \mathcal{H}$.
3. T is called an A**-isometry** if $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \mathcal{H}$.
4. $T \in L_A(\mathcal{H})$ is an A**-unitary** operator if T and $T^\sharp$ are A-isometries.
5. $T \in L_A(\mathcal{H})$ is called an A**-partial isometry** if $T^\sharp T$ is a projection.

In [2] the above classes of operators are studied. The definition of A-partial isometry can be extended for $T \notin L_A(\mathcal{H})$ (see [2]). However, in that case, the A-partial isometries are not A-operators, in general. For more results on A-contractions, see [25] and the references therein.
We denote by $L^{sa}(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is selfadjoint}\}$, $\mathcal{N}(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is normal}\}$, $\mathcal{P}(\mathcal{H}) := \{Q \in L^{sa}(\mathcal{H}) : Q \text{ is projection}\}$, $\mathcal{C}(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is a contraction}\}$, $\mathcal{I}(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is an isometry}\}$, $\mathcal{U}(\mathcal{H}) := \{U \in L(\mathcal{H}) : U \text{ is unitary}\}$ and $\mathcal{J}(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is a partial isometry}\}$. We shall denote, $L^{sa}_A(\mathcal{H}) := \{T \in L(\mathcal{H}) : T \text{ is } A\text{-selfadjoint}\}$ and similarly $\mathcal{N}_A(\mathcal{H})$, $\mathcal{P}_A(\mathcal{H})$, $\mathcal{C}_\mathcal{A}(\mathcal{H})$, $\mathcal{I}_\mathcal{A}(\mathcal{H})$, $\mathcal{U}_\mathcal{A}(\mathcal{H})$ and $\mathcal{J}_A(\mathcal{H})$.

Remark 3.4. The definition 3.3 can be easily adapted to the case $T \in L(\mathcal{H}, \mathcal{K})$ where $A \in L(\mathcal{H})^+$, $B \in L(\mathcal{K})^+$ induce semi-inner products on $\mathcal{H}$ and $\mathcal{K}$, respectively. In this case, the contractions (resp. isometries, unitaries, partial isometries, normal operators) respect to these semi-inner products will be called AB**-contractions** (resp. AB**-isometries,** AB**-unitaries**, AB**-partial isometries**, AB**-normal operators**).

Observe that the standard way of transfer selfadjoints operators, isometries, projections, unitary operators and partial isometries of $L(\mathcal{H})$ to similar classes of operator of $L(\mathcal{K})$, if $\mathcal{H}$ and $\mathcal{K}$ are isomorphic as Hilbert spaces, is by mean the application $T \rightarrow UTU^*$ where $U : \mathcal{H} \rightarrow \mathcal{K}$ is an unitary operator. Nevertheless, note that there is not unitary transformation between $(\mathcal{H}, \langle\ ,\ \rangle_A)$ and $\mathbf{R}(A^{1/2})$; indeed, $(\mathcal{H}, \langle\ ,\ \rangle_A)$ is not a Hilbert space. However, there exists an AI-unitary operator between them which will play the role of U, namely, W_A. Therefore, we shall transfer A-operators to operators of $L(\mathbf{R}(A^{1/2}))$ by means of $W_A T W_A^\sharp$.

Proposition 3.5. The next assertions hold

1. $W_A^\sharp = W_A^\dagger$.
2. $W_A \in L(\mathcal{H}, \mathbf{R}(A^{1/2}))$ is an AI-unitary operator.

Proof. (1) First, let us prove that $\mathcal{D}(W_A^\sharp) = R(A)$. Let $\xi = A^{1/2}\eta \in \mathcal{D}(W_A^\sharp)$. Then, there exists $\phi \in \overline{R(A)}$ such that $(W_A\psi, A^{1/2}\eta) = \langle \psi, \phi \rangle_A$, for every $\psi \in \mathcal{H}$; or which is the same, $\left\langle A^{1/2}\psi, P\eta \right\rangle = \left\langle A^{1/2}\psi, A^{1/2}\phi \right\rangle$ for every $\psi \in \mathcal{H}$. Therefore, $P\eta = A^{1/2}\phi$ and so $\xi = A^{1/2}\eta = A\phi \in R(A)$. On the other hand, let $A\eta \in R(A)$. Then for every $\xi \in \mathcal{H}$, $(W_A\xi, A\eta) = \langle \xi, P\eta \rangle_A$, i.e., $A\eta \in \mathcal{D}(W_A^\sharp)$ and $W_A^\sharp A\eta = P\eta$. Hence, $\mathcal{D}(W_A^\sharp) = R(A)$. Moreover, as $W_A^\sharp A\eta = P\eta$, we get that $W_A^\sharp = W_A^\dagger$.
(2) First, as $\|W_A\xi\|_{\mathbf{R}(A^{1/2})} = \|A^{1/2}\xi\| = \|\xi\|_A$ for every $\xi \in \mathcal{H}$, then $W_A \in L(\mathcal{H}, \mathbf{R}(A^{1/2}))$ is an AI-isometry. On the other hand, $\|W_A^\sharp(A\xi)\|_A = \|P\xi\|_A = \|A^{1/2}\xi\| = \|A\xi\|_{\mathbf{R}(A^{1/2})}$. Thus, $W_A^\sharp$ is an IA-isometry and so $W_A \in L(\mathcal{H}, \mathbf{R}(A^{1/2}))$ is an AI-unitary operator.

$\square$

Observe that the conjugation $W_A T W_A^\sharp$ is not bounded for every $T \in L(\mathcal{H})$ and that $W_A^\sharp \tilde{T} W_A$ is not defined for every $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$. Thus, this sort of conjugation by means of the AI-unitary W_A is not as perfect as it is in the case of isomorphic Hilbert spaces. The study of these conjugations is equivalent to determine conditions for the commutativity of the following diagram:

$$\begin{tikzcd}
\mathcal{H} \arrow[r, "T"] \arrow[d, "W_A"'] & \mathcal{H} \arrow[d, "W_A"] \\
\mathbf{R}(A^{1/2}) \arrow[r, "\tilde{T}"] & \mathbf{R}(A^{1/2})
\end{tikzcd}$$

More precisely, we study two different lifting problems:

1. given $T \in L(\mathcal{H})$ under which conditions there exists $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ such that $W_A T = \tilde{T} W_A$;

2. given $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ under which conditions there exists $T \in L(\mathcal{H})$ such that $W_A T = \tilde{T} W_A$.

The next result is due to Barnes [4] if A is injective. The general case, but with an unnecessary extra hypothesis, can be found in [9]. We present a proof based on Douglas theorem.

Proposition 3.6. Consider $T \in L(\mathcal{H})$. Then, there exists $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ such that $\tilde{T} W_A = W_A T$ if and only if $T \in L_{A^{1/2}}(\mathcal{H})$. In such case $\tilde{T}$ is unique.

Proof. If $T \in L_{A^{1/2}}(\mathcal{H})$ then $T^* R(A^{1/2}) \subseteq R(A^{1/2})$. By Douglas theorem, equation $W_A^* X = T^* W_A^*$ has solution $\tilde{S} \in L(\mathbf{R}(A^{1/2}))$, because $R(T^* W_A^*) = T^* R(A^{1/2}) \subseteq R(A^{1/2}) = R(W_A^*)$; take $\tilde{T} = \tilde{S}^*$. Conversely, if $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ satisfies $W_A T = \tilde{T} W_A$ then $T^* W_A^* = W_A^* \tilde{T}^*$ and, as before, $T^* R(A^{1/2}) \subseteq R(A^{1/2})$. Observe that if there exists such $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$, automatically $\tilde{T}^* \in L(\mathbf{R}(A^{1/2}))$ and so $R(\tilde{T}^*) \subseteq R(A^{1/2}) \subseteq \overline{R(A)}$. This means that $\tilde{T}^*$ is the reduced solution of the equation $T^* W_A^* = W_A^* \tilde{T}^*$, and, as such, it is unique. $\square$

Remark 3.7. Cojuhari and Gheondea [7] proved a similar result under more general conditions on A. See also the paper by Hassi et al. [16]. Basically, they suppose that operators $T : \mathcal{H} \to \mathcal{K}$, $V : \mathcal{K} \to \mathcal{H}$ satisfy $BT = V^* A$, where $A \in L(\mathcal{H})^+$ and $B \in L(\mathcal{K})^+$ and they prove the existence of unique $\tilde{T} : \mathbf{R}(A^{1/2}) \to \mathbf{R}(B^{1/2})$ and $\tilde{V} : \mathbf{R}(B^{1/2}) \to \mathbf{R}(A^{1/2})$ such that $W_B T = \tilde{T} W_A$, $W_A V = \tilde{V} W_B$ and $\tilde{T}^* = \tilde{V}$.

In the previous proposition, we studied under which conditions an operator $T \in L(\mathcal{H})$ comes from some $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ in the sense that $W_A T = \tilde{T} W_A$. The next lemma goes in the reverse direction, namely, given $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ under which conditions there exists some $T \in L(\mathcal{H})$ such that $\tilde{T} W_A = W_A T$.

Proposition 3.8. Given $\tilde{T} \in L(\mathbf{R}(A^{1/2}))$ there exists $T \in L(\mathcal{H})$ such that $W_A T = \tilde{T} W_A$ if and only if $R(\tilde{T} W_A) \subseteq R(W_A) = R(A)$. In such case, there exists a unique $T \in L_{A^{1/2}}(\mathcal{H})$ such that $R(T) \subseteq \overline{R(A)}$.

Proof. The first part is a straightforward consequence of Douglas theorem. Moreover, if $R(\tilde{T} W_A) \subseteq R(W_A)$ then the reduced solution T of the equation $W_A X = \tilde{T} W_A$ verifies that $R(T) \subseteq \overline{R(W_A^*)} = \overline{R(A)}$. On the other hand, $R(T^* A^{1/2}) = R(T^* W_A^*) = R(W_A^* \tilde{T}^*) \subseteq R(A^{1/2})$. So, $T \in L_{A^{1/2}}(\mathcal{H})$. $\square$

Define $\tilde{L}(\mathbf{R}(A^{1/2})) := \{\tilde{T} \in L(\mathbf{R}(A^{1/2})) : R(\tilde{T}W_A) \subseteq R(A)\}$. $\tilde{L}(\mathbf{R}(A^{1/2}))$ is a non closed subalgebra of $L(\mathbf{R}(A^{1/2}))$. Moreover, observe that $\tilde{T} \in \tilde{L}(\mathbf{R}(A^{1/2}))$ does not imply $\tilde{T}^* \in \tilde{L}(\mathbf{R}(A^{1/2}))$, in general. In fact, $\tilde{T}$ and $\tilde{T}^* \in \tilde{L}(\mathbf{R}(A^{1/2}))$ if and only if $R(A)$ reduces $\tilde{T}$. In the sequel, we denote $\widetilde{L}^{sa}(\mathbf{R}(A^{1/2})) = L^{sa}(\mathbf{R}(A^{1/2})) \cap \tilde{L}(\mathbf{R}(A^{1/2}))$. Similarly we define $\widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$, $\widetilde{\mathcal{C}}(\mathbf{R}(A^{1/2}))$ and $\widetilde{\mathcal{I}}(\mathbf{R}(A^{1/2}))$. On the other hand, we denote by $\widetilde{\mathcal{N}}(\mathbf{R}(A^{1/2})) = \{\tilde{T} \in \tilde{L}(\mathbf{R}(A^{1/2})) \cap \mathcal{N}(\mathbf{R}(A^{1/2})) :$ $R(A)$ reduces $\tilde{T}\}$. Analogously we define $\widetilde{\mathcal{U}}(\mathbf{R}(A^{1/2}))$ and $\widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2}))$.

By Propositions 3.6 and 3.8, the next mappings are well defined:

$$\alpha : L_{A^{1/2}}(\mathcal{H}) \longrightarrow \tilde{L}(\mathbf{R}(A^{1/2})),\, T \longmapsto \tilde{T}$$

where $\tilde{T}W_A\xi = W_AT\xi$ for all $\xi \in \mathcal{H}$, and

$$\beta : \tilde{L}(\mathbf{R}(A^{1/2})) \longrightarrow L_{A^{1/2}}(\mathcal{H}),\, \tilde{T} \longmapsto T$$

where $\tilde{T}W_A\xi = W_AT\xi$ for all $\xi \in \mathcal{H}$ and $R(T) \subseteq \overline{R(A)}$.

Proposition 3.9. The following properties of α and β hold:

1. α is the homomorphism $\alpha(T) = \overline{W_ATW_A^\sharp}$; α is injective if and only if A is injective.
2. β is the homomorphism $\beta(\tilde{T}) = W_A^\sharp\tilde{T}W_A$; β is always injective.
3. $\|\alpha(T)\|_{\mathbf{R}(A^{1/2})} = \|T\|_A$ and $\|\beta(\tilde{T})\|_A = \|\tilde{T}\|_{\mathbf{R}(A^{1/2})}$.
4. The compositions $\alpha\beta$ and $\beta\alpha$ can be explicitly computed as
   $\alpha\beta : \tilde{L}(\mathbf{R}(A^{1/2})) \longrightarrow \tilde{L}(\mathbf{R}(A^{1/2})),\ \alpha\beta(\tilde{T}) = \tilde{T}$ and
   $\beta\alpha : L_{A^{1/2}}(\mathcal{H}) \longrightarrow L_{A^{1/2}}(\mathcal{H}),\ \beta\alpha(T) = PTP.$

Proof. (1) As $W_A^\sharp = W_A^\dagger$ then $\alpha(T) = \overline{W_ATW_A^\sharp}$. The linearity of $\alpha(T)$ is trivial. If $T, T_1 \in L_{A^{1/2}}(\mathcal{H})$ then $W_ATT_1 = \tilde{T}W_AT_1 = \tilde{T}\tilde{T}_1W_A$. So $\alpha(TT_1) = \alpha(T)\alpha(T_1)$. Thus α is an homomorphism. Now, note that if $T \in L_{A^{1/2}}(\mathcal{H})$ then $PTP \in L_{A^{1/2}}(\mathcal{H})$. Therefore, if A is not injective there exists $T \in L_{A^{1/2}}(\mathcal{H})$ such that $T \neq PTP$ and it holds $\alpha(T) = \alpha(PTP)$. So α is not injective. Let $T, T_1 \in L_{A^{1/2}}(\mathcal{H})$ such that $\overline{W_ATW_A^\sharp} = \overline{W_AT_1W_A^\sharp}$. Then, it holds $PTP = PT_1P$ and so we obtain that $T = T_1$ because A is injective; hence α is injective.

(2) As $W_A^\sharp = W_A^\dagger$, it is clear that $\beta(\tilde{T}) = W_A^\sharp\tilde{T}W_A$. The linearity of β is immediate. In addition, if $\tilde{T}, \tilde{T}_1 \in \tilde{L}(\mathbf{R}(A^{1/2}))$ then $\tilde{T}\tilde{T}_1W_A = \tilde{T}W_AT_1 = W_ATT_1$. Furthermore $R(TT_1) \subseteq \overline{R(A)}$. Thus $\beta(\tilde{T}\tilde{T}_1) = \beta(\tilde{T})\beta(\tilde{T}_1)$. So, β is an homomorphism. Now, if $\beta(\tilde{T}) = \beta(\tilde{T}_1)$ then $\tilde{T}W_A\xi = \tilde{T}_1W_A\xi$ for all $\xi \in \mathcal{H}$. Now, as $R(W_A)$ is dense in $\mathbf{R}(A^{1/2})$, then $\tilde{T} = \tilde{T}_1$. Thus β is injective.

(3) If $W_AT = \tilde{T}W_A$ then it is sufficient to show that $\|T\|_A = \|\tilde{T}\|_{\mathbf{R}(A^{1/2})}$. Now,

$$\begin{aligned}\|T\|_A &= \sup_{0\neq\xi\in\overline{R(A)}} \frac{\|T\xi\|_A}{\|\xi\|_A} = \sup_{0\neq\xi\in\overline{R(A)}} \frac{\|W_AT\xi\|_{\mathbf{R}(A^{1/2})}}{\|\xi\|_A} \\ &= \sup_{0\neq\xi\in\overline{R(A)}} \frac{\|\tilde{T}W_A\xi\|_{\mathbf{R}(A^{1/2})}}{\|A\xi\|_{\mathbf{R}(A^{1/2})}} = \|\tilde{T}\|_{\mathbf{R}(A^{1/2})}\end{aligned}$$

(4) It is straightforward. $\square$

The next result and, later, item (1) of Proposition 3.13, show a relationship between the adjoint operation in $L(\mathbf{R}(A^{1/2}))$ and the $^\sharp$ operation in $L_{A^{1/2}}(\mathcal{H})$. This result for partially defined positive operators is due to Cojuhari and Gheondea ([7], Theorem 3.1). Here, we present a shorter proof for the case $A \in L(\mathcal{H})^+$.

Proposition 3.10. Suppose that $T, W \in L(\mathcal{H})$ satisfies that $AW = T^*A$. Then, $T, W \in L_A(\mathcal{H})$ and

$$\tilde{W} = \tilde{T}^*$$

In other words, $\alpha(W) = \alpha(T)^*$.

Proof. Indeed, for every $\xi, \eta \in \mathcal{H}$ it holds

$$\begin{aligned}(\tilde{T}(A\xi), A\eta) &= (W_A T\xi, A\eta) = \left\langle A^{1/2}T\xi, A^{1/2}\eta \right\rangle = \langle AT\xi, \eta\rangle \\ &= \langle W^* A\xi, \eta\rangle = \langle A\xi, W\eta\rangle = (A\xi, AW\eta) \\ &= (A\xi, \tilde{W}(A\eta)).\end{aligned}$$

Therefore, $\alpha(W) = \alpha(T)^*$. □

The next theorem which is the main result of this section relates, by means of α, the classes of A-operators defined above with similar classes in $L(\mathbf{R}(A^{1/2}))$.

Theorem 3.11. Let $A \in L(\mathcal{H})^+$.Then, the following equalities hold:

1. $\alpha(L_A^{sa}(\mathcal{H})) = \widetilde{L}^{sa}(\mathbf{R}(A^{1/2}))$,
2. $\alpha(\mathcal{N}_A(\mathcal{H})) = \widetilde{\mathcal{N}}(\mathbf{R}(A^{1/2}))$,
3. $\alpha(\mathcal{P}_A(\mathcal{H})) = \widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$,
4. $\alpha(\mathcal{C}_A(\mathcal{H})) = \widetilde{\mathcal{C}}(\mathbf{R}(A^{1/2}))$,
5. $\alpha(\mathcal{I}_A(\mathcal{H})) = \widetilde{\mathcal{I}}(\mathbf{R}(A^{1/2}))$,
6. $\alpha(\mathcal{U}_A(\mathcal{H})) = \widetilde{\mathcal{U}}(\mathbf{R}(A^{1/2}))$,
7. $\alpha(\mathcal{J}_A(\mathcal{H})) = \widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2}))$.

Remark 3.12. Observe that $L_A^{sa}(\mathcal{H})$, $\mathcal{N}_A(\mathcal{H})$, $\mathcal{P}_A(\mathcal{H})$, $\mathcal{U}_A(\mathcal{H})$ and $\mathcal{J}_A(\mathcal{H})$ are subsets of $L_A(\mathcal{H})$, a fortiori of $L_{A^{1/2}}(\mathcal{H})$. However, $\mathcal{C}_A(\mathcal{H})$ and $\mathcal{I}_A(\mathcal{H})$ are not contained in $L_A(\mathcal{H})$, in general, but they are subsets of $L_{A^{1/2}}(\mathcal{H})$.

For the proof of Theorem 3.11 we shall need the following result in which we determine the images by β of certain subsets of $\tilde{L}(\mathbf{R}(A^{1/2}))$.

Proposition 3.13. Let $A \in L(\mathcal{H})^+$. The next assertions hold:

1. If $\tilde{T} \in \tilde{L}(\mathbf{R}(A^{1/2}))$ and $R(A)$ reduces $\tilde{T}$ then $\beta(\tilde{T}^*) = \beta(\tilde{T})^\sharp$.
2. $\beta(\widetilde{L}^{sa}(\mathbf{R}(A^{1/2}))) \subseteq L_A^{sa}(\mathcal{H})$,
3. $\beta(\widetilde{\mathcal{N}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{N}_A(\mathcal{H})$,
4. $\beta(\widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{P}_A(\mathcal{H})$,

5. $\beta(\widetilde{\mathcal{C}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{C}_A(\mathcal{H})$,

6. $\beta(\widetilde{\mathcal{I}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{I}_A(\mathcal{H})$,

7. $\beta(\widetilde{\mathcal{U}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{U}_A(\mathcal{H})$.

8. $\beta(\widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2}))) \subseteq \mathcal{J}_A(\mathcal{H})$.

Proof. (1) For every $\xi, \eta \in \mathcal{H}$ it holds $\left\langle A\beta(\tilde{T}^*)\xi, \eta \right\rangle = (W_A\beta(\tilde{T}^*)\xi, W_A\eta) = (\tilde{T}^* W_A\xi, W_A\eta) = (W_A\xi, \tilde{T}W_A\eta) = (W_A\xi, W_A\beta(\tilde{T})\eta) = \left\langle \xi, A\beta(\tilde{T})\eta \right\rangle$. Therefore, $A\beta(\tilde{T}^*) = \beta(\tilde{T})^*A$. Furthermore $R(\beta(\tilde{T}^*)) \subseteq \overline{R(A)}$. Hence, $\beta(\tilde{T}^*) = \beta(\tilde{T})^\sharp$.
(2) It is consequence of item (1).
(3) Let $\tilde{T} \in \widetilde{\mathcal{N}}(\mathbf{R}(A^{1/2}))$ and $T = \beta(\tilde{T})$. Hence, $TT^\sharp = \beta(\tilde{T})\beta(\tilde{T}^*) = \beta(\tilde{T}\tilde{T}^*) = \beta(\tilde{T}^*\tilde{T}) = \beta(\tilde{T}^*)\beta(\tilde{T}) = T^\sharp T$. Therefore $\beta(\tilde{T}) \in \mathcal{N}_A(\mathcal{H})$.
(4) Let $\tilde{P} \in \widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$. Since β is a homomorphism, $\beta(\tilde{T})$ is idempotent. Furthermore, by (2), $\beta(\tilde{P})$ is A-selfadjoint. Thus, $\beta(\tilde{P}) \in \mathcal{P}_A(\mathcal{H})$.
(5) Let $\tilde{T} \in \widetilde{\mathcal{C}}(\mathbf{R}(A^{1/2}))$ and $T = \beta(\tilde{T})$. Then, for every $\xi \in \mathcal{H}$ it holds $\|T\xi\|_A = \|AT\xi\|_{\mathbf{R}(A^{1/2})} = \|\tilde{T}(A\xi)\|_{\mathbf{R}(A^{1/2})} \leq \|A\xi\|_{\mathbf{R}(A^{1/2})} = \|\xi\|_A$. Therefore, T is an A-contraction.
The proofs of items (6) and (7) are similar to the above one.
(8) If $\tilde{T} \in \widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2}))$ then $\tilde{T}^*\tilde{T} \in \widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$ and $\beta(\tilde{T}^*\tilde{T}) = T^\sharp T \in \mathcal{P}_A(\mathcal{H})$ by item (4). So $T = \beta(\tilde{T})$ is an A-partial isometry.

□

By Proposition 3.13 and since $\alpha\beta = id$, the next inclusions hold: $\widetilde{L}^{sa}(\mathbf{R}(A^{1/2})) \subseteq \alpha(L^{sa}_A(\mathcal{H}))$, $\widetilde{\mathcal{N}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{N}_A(\mathcal{H}))$, $\widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{P}_A(\mathcal{H}))$, $\widetilde{\mathcal{C}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{C}_A(\mathcal{H}))$, $\widetilde{\mathcal{I}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{I}_A(\mathcal{H}))$, $\widetilde{\mathcal{U}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{U}_A(\mathcal{H}))$, and $\widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2})) \subseteq \alpha(\mathcal{J}_A(\mathcal{H}))$. Hence, in order to finish the proof of Theorem 3.11 it only remains to show the reverse inclusions:

Proof of Theorem 3.11
(1) This equality is a particular case of Proposition 3.10.
(2) The equality follows since α is a homomorphism.
(3) Let $Q \in \mathcal{P}_A(\mathcal{H})$. By (1), $\alpha(Q) = \tilde{Q} \in \widetilde{L}^{sa}(\mathbf{R}(A^{1/2}))$. Furthermore $\tilde{Q}$ is idempotent because α is a homomorphism. So, $\tilde{Q} \in \mathcal{P}(\mathbf{R}(A^{1/2}))$.
(4) Let $T \in \mathcal{C}_A(\mathcal{H})$ and $\tilde{T} = \alpha(T)$. Then, for every $\xi \in \mathcal{H}$ it holds $\|\tilde{T}(A\xi)\|_{\mathbf{R}(A^{1/2})} = \|AT\xi\|_{\mathbf{R}(A^{1/2})} = \|T\xi\|_A \leq \|\xi\|_A = \|A\xi\|_{\mathbf{R}(A^{1/2})}$. Hence, as $R(A)$ is dense in $\mathbf{R}(A^{1/2})$, we get that $\tilde{T}$ is a contraction.
The proofs of items (5) and (6) can be done following the same lines that in item (4).
(7) Let $T \in \mathcal{J}_A(\mathcal{H})$ then $T^\sharp T$ is a projection. So $\alpha(T^\sharp T) = \tilde{T}^*\tilde{T} \in \widetilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$. Then $\tilde{T} \in \widetilde{\mathcal{J}}(\mathbf{R}(A^{1/2}))$. □

Remark 3.14. A closed subspace $\mathcal{S}$ of $\mathcal{H}$ and a positive semidefinite operator A are called **compatible** if there exists a (bounded linear) projection Q onto $\mathcal{S}$ which is A-selfadjoint. In [9], the compatibility of a pair $(A, \mathcal{S})$ is related to the existence in the operator range $\mathbf{R}(A^{1/2})$ of a convenient orthogonal projection. More precisely, given a closed subspace $\mathcal{S}$ of $\mathcal{H}$ the pair $(A, \mathcal{S})$ is compatible if and only if $P_{\overline{A(\mathcal{S})}'} \in \tilde{L}(\mathbf{R}(A^{1/2}))$ where $\overline{A(\mathcal{S})}'$ denotes the closure of $A(\mathcal{S})$ in $\mathbf{R}(A^{1/2})$. As a consequence, in general, $\tilde{\mathcal{P}}(\mathbf{R}(A^{1/2})) \neq \mathcal{P}(\mathbf{R}(A^{1/2}))$. In fact, consider $B \in L(\mathcal{H})^+$

with non closed range and $A \in L(\mathcal{H} \oplus \mathcal{H})^+$ defined by $A = \begin{pmatrix} B & B^{1/2} \\ B^{1/2} & I \end{pmatrix}$. Now, by Theorem 2.9, [8], the pair $(A, \overline{R(B)} \oplus \{0\})$ is not compatible. Therefore, if $\mathcal{W} = \overline{A(\overline{R(B)} \oplus \{0\})'}$ then $P_{\mathcal{W}} \notin \tilde{\mathcal{P}}(\mathbf{R}(A^{1/2}))$.

4 The A-reduced minimum modulus

In this section we introduce the concept of A-reduced minimum modulus of an operator. This is a natural generalization of the reduced minimum modulus: recall that the *reduced minimum modulus* of an operator $T \in L(\mathcal{H})$ is defined as

$$\gamma(T) = \inf \left\{ \|T\xi\| : \ \xi \in N(T)^{\perp} \text{ and } \|\xi\| = 1 \right\}.$$

Definition 4.1. Let $A \in L(\mathcal{H})^+$ and $T \in L(\mathcal{H})$. The **A-reduced minimum modulus** of T is

$$\gamma_A(T) = \inf \left\{ \|T\xi\|_A : \ \xi \in N(A^{1/2}T)^{\perp_A} \text{ and } \|\xi\|_A = 1 \right\}.$$

Note that if $T \in L_A(\mathcal{H})$ then $\gamma_A(T) = \inf \left\{ \|T\xi\|_A : \ \xi \in \overline{R(T^\sharp T)} \text{ and } \|\xi\|_A = 1 \right\}$. From now on, given $T \in L_{A^{1/2}}(\mathcal{H})$ we denote by $T^\diamond$ the reduced solution of the equation $A^{1/2}X = T^*A^{1/2}$, namely, $T^\diamond = (A^{1/2})^\dagger T^* A^{1/2}$.

Proposition 4.2. Let $A \in L(\mathcal{H})^+$. If $T \in L_{A^{1/2}}(\mathcal{H})$ then $\gamma_A(T) \leq \gamma(C)$ for every solution C of the equation $A^{1/2}X = T^*A^{1/2}$. In particular, $\gamma_A(T) \leq \gamma(T^\diamond)$.

Proof. Let $T \in L_{A^{1/2}}(\mathcal{H})$ and $C \in L(\mathcal{H})$ such that $A^{1/2}C = T^*A^{1/2}$. If $\xi \in N(A^{1/2}T)^{\perp_A}$ then $\eta = A^{1/2}\xi \in A^{-1/2}(\overline{R(T^*A^{1/2})})$. So $\|\xi\|_A = \|\eta\|$ and $\|T\xi\|_A^2 = \|C^*\eta\|^2$. On the other hand, as $R(C) \subseteq A^{-1/2}(R(T^*A^{1/2}))$, it holds that $N(C^*)^\perp = \overline{R(C)} \subseteq \overline{A^{-1/2}(R(T^*A^{1/2}))} \subseteq A^{-1/2}(\overline{R(T^*A^{1/2})})$. Therefore,

$$\begin{aligned}
\gamma_A(T) &= \inf\{\|T\xi\|_A : \ \xi \in N(A^{1/2}T)^{\perp_A} \text{ and } \|\xi\|_A = 1\} \\
&= \inf\{\|C^*\eta\| : \ \eta \in A^{-1/2}(\overline{R(T^*A^{1/2})}) \text{ and } \|\eta\| = 1\} \\
&\leq \inf\{\|C^*\eta\| : \ \eta \in N(C^*)^\perp \text{ and } \|\eta\| = 1\} \\
&= \gamma(C^*) = \gamma(C).
\end{aligned}$$

□

Proposition 4.3. Let $A \in L(\mathcal{H})^+$, $T \in L_A(\mathcal{H})$ and C be a solution of the equation $A^{1/2}X = T^*A^{1/2}$. If $A^{1/2}\overline{R(T^\sharp T)} \subseteq \overline{R(C)}$ then $\gamma_A(T) = \gamma(C)$.

Proof. Let $C \in L(\mathcal{H})$ be a solution of the equation $A^{1/2}X = T^*A^{1/2}$. Then, by Proposition 4.2, it holds that $\gamma_A(T) \leq \gamma(C)$. Now, as $T \in L_A(\mathcal{H})$, if $\xi \in \overline{R(T^\sharp T)}$ then $\eta = A^{1/2}\xi \in A^{1/2}\overline{R(T^\sharp T)}$. Then,

$$\begin{aligned}
\gamma_A(T) &= \inf\{\|T\xi\|_A : \ \xi \in \overline{R(T^\sharp T)} \text{ and } \|\xi\|_A = 1\} \\
&= \inf\{\|C^*\eta\| : \ \eta \in A^{1/2}\overline{R(T^\sharp T)} \text{ and } \|\eta\| = 1\} \\
&\geq \inf\{\|C^*\eta\| : \ \eta \in N(C^*)^\perp \text{ and } \|\eta\| = 1\} \\
&= \gamma(C^*) = \gamma(C).
\end{aligned}$$

Therefore, $\gamma_A(T) = \gamma(C)$. □

Lemma 4.4. Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then $T^\diamond A^{1/2} = A^{1/2}T^\sharp$.

Proof. As $L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$ there exists $T^\diamond$. Now, $A^{1/2}T^\diamond A^{1/2} = T^*A$. On the other hand, $A^{1/2}A^{1/2}T^\sharp = AT^\sharp = T^*A$. Then, $T^\diamond A^{1/2}$ and $A^{1/2}T^\sharp$ are both reduced solutions of the equation $A^{1/2}X = T^*A$. Therefore $T^\diamond A^{1/2} = A^{1/2}T^\sharp$. $\square$

The next result shows that the A-reduced minimum modulus of an operator $T \in L_A(\mathcal{H})$ coincides with the classical reduced minimum modulus of $T^\diamond$.

Corollary 4.5. Let $A \in L(\mathcal{H})^+$ and $T \in L_A(\mathcal{H})$. Then
(1) $\gamma_A(T) = \gamma(T^\diamond)$.
(2) $\gamma_A(T) = \gamma_A(T^\sharp)$.

Proof. 1. By Proposition 4.3, it is sufficient to show that $A^{1/2}\overline{R(T^\sharp T)} \subseteq \overline{R(T^\diamond)}$. Now, by Lemma 4.4, it holds that $A^{1/2}R(T^\sharp T) = R(T^\diamond A^{1/2}T) \subseteq R(T^\diamond)$. Hence, $\overline{A^{1/2}\overline{R(T^\sharp T)}} = \overline{A^{1/2}R(T^\sharp T)} \subseteq \overline{R(T^\diamond)}$. So, $A^{1/2}\overline{R(T^\sharp T)} \subseteq \overline{R(T^\diamond)}$ and the assertion follows.

2. As $T \in L_A(\mathcal{H})$ then $T^\sharp \in L_A(\mathcal{H})$. By the above item, it is sufficient to show that $(T^\diamond)^*$ is the reduced solution of the equation $A^{1/2}X = (T^\sharp)^*A^{1/2}$. Now, by Lemma 4.4, it holds that $A^{1/2}(T^\diamond)^* = (T^\sharp)^*A^{1/2}$. On the other hand, $R((T^\diamond)^*) \subseteq \overline{R((T^\diamond)^*)} = N(T^\diamond)^\perp = N(T^*A^{1/2})^\perp = \overline{R(A^{1/2}T)} \subseteq \overline{R(A^{1/2})}$ and so the assertion follows.

$\square$

We finish this section extending the following theorem due to Mbekhta [19]:

Theorem (Mbekhta) *If $T \in L(\mathcal{H})$ is a contraction, then the following conditions are equivalent:*

1. *T is a non-zero partial isometry.*
2. *$\gamma(T) = 1$.*

Theorem 4.6. Let $A \in L(\mathcal{H})^+$. If $T \in L_A(\mathcal{H})$ is an A-contraction, then the following conditions are equivalent:

1. T is an A-partial isometry such that $T^\sharp$ is non-zero.
2. $\gamma_A(T) = 1$.

Proof. Let T be an A-partial isometry such that $T^\sharp$ is non-zero. Then $T^\sharp T$ is non-zero and $\|T\xi\|_A = \|\xi\|_A$ for every $\xi \in \overline{R(T^\sharp T)}$. Therefore, by definition, $\gamma_A(T) = 1$. Conversely, since $T \in L_A(\mathcal{H}) \subseteq L_{A^{1/2}}(\mathcal{H})$ then $(T^\diamond)^* = \overline{A^{1/2}T(A^{1/2})^\dagger} \in L(\mathcal{H})$. Thus, $T^\diamond(T^\diamond)^* = \overline{(A^{1/2})^\dagger T^*AT(A^{1/2})^\dagger} \leq \overline{(A^{1/2})^\dagger A(A^{1/2})^\dagger} = P_{\overline{R(A)}} \leq I$. Therefore $(T^\diamond)^*$ is a contraction. On the other hand, by Corollary4.5, $1 = \gamma_A(T) = \gamma(T^\diamond) = \gamma((T^\diamond)^*)$. Then, by Mbekhta's theorem, $(T^\diamond)^*$ is a non-zero partial isometry. Moreover, as $(T^\diamond)^*$ is non-zero, $T^\sharp$ is non-zero. Now,

$$\begin{aligned}
(T^\sharp T)^2 &= A^\dagger T^*ATA^\dagger T^*AT \\
&= (A^{1/2})^\dagger(A^{1/2})^\dagger T^*A^{1/2}A^{1/2}T(A^{1/2})^\dagger(A^{1/2})^\dagger T^*A^{1/2}A^{1/2}T \\
&= (A^{1/2})^\dagger T^\diamond(T^\diamond)^*|_{\mathcal{D}((A^{1/2})^\dagger)}T^\diamond A^{1/2}T = (A^{1/2})^\dagger T^\diamond A^{1/2}T \\
&= T^\sharp T.
\end{aligned}$$

Then T is an A-partial isometry. $\square$

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