RANDOM SERIES IN $L^p(X, \Sigma, \mu)$ USING UNCONDITIONAL BASIC SEQUENCES AND l^p STABLE SEQUENCES: A RESULT ON ALMOST SURE ALMOST EVERYWHERE CONVERGENCE

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Abstract. Here we study the almost sure almost everywhere convergence of random series of the form $\sum_{i=1}^{\infty} a_i f_i$ in the Lebesgue spaces $L^p(X, \Sigma, \mu)$, where the a_i's are centered random variables, and the f_i's constitute an unconditional basic sequence or an l^p stable sequence. We show that if one of these series converges in the norm topology almost surely, then it converges almost everywhere almost surely.

1. Introduction

Here, we consider random series of the form:

$$\sum_{i=1}^{\infty} a_i f_i \,, \tag{1.1}$$

where the a_i's are independent centered random variables, and the f_i's are either an unconditional basic sequence, or an l^p stable sequence in a Lebesgue space $L^p(X, \Sigma, \mu)$, where p is not necessarily equal to 2 and μ is a σ-finite measure. The main goal is to show that *if one of these series converges in the norm topology almost surely, then it converges almost everywhere almost surely.* More precesily, in Section 3 we prove:

Theorem 1.1. *a) Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of independent random variables such that there exists a constant $C > 0$ such that $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^p)^2$, $\forall j$, and $\mathbb{E}(a_j) = 0$; let $\{f_j\}_{j\in\mathbb{N}} \subset L^p(X, \Sigma, \mu)$, $1 \leq p < \infty$, be an l^p-stable sequence, and let μ be a σ-finite measure. If $\sum_{i=1}^{\infty} a_i f_i$ converges in $L^p(X, \Sigma, \mu)$ a.s., then $\sum_{i=1}^{\infty} a_i f_i$ converges $[\mu]$ almost everywhere a.s.*

b) Let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of independent random variables such that there exists a constant $C > 0$ such that $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^2)^p$, $\forall j$, and $\mathbb{E}(a_j) = 0$; let $\{f_j\}_{j\in\mathbb{N}} \subset L^p(X, \Sigma, \mu)$ be an unconditional basic sequence, and let μ be a σ-finite measure. If $\sum_{i=1}^{\infty} a_i f_i$ converges in $L^p(X, \Sigma, \mu)$ a.s., then $\sum_{i=1}^{\infty} a_i f_i$ converges $[\mu]$-almost everywhere a.s. If $1 \leq p < 2$, the last assertion remains true with the additional condition: $(\mathbb{E}|a_i|^p)^{\frac{1}{p}} \geq c(\mathbb{E}|a_i|^2)^{\frac{1}{2}}$.

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It is remarkable that the case $p = 2$ in Theorem 1.1 can be easily derived from the results in [2], with the only assumption that the series (1.1) converges in norm a.s. This is a consequence of the fact that $L^2(X, \Sigma, \mu)$ is a Hilbert space, and that the independence of the a_i's makes the $a_i f_i$'s behave as orthogonal elements; i.e., they are orthogonal in $L^2(X \times \Omega) = L^2(\Omega, L^2(X))$. Since unconditional bases are good bases and keep some of the properties of an orthogonal basis, it is reasonable that this result can be extended to the case $p \neq 2$ when $\{f_j\}_{j\in\mathbb{N}}$ is an unconditional basis.

Additionally, as a consequence, we obtain an interesting result similar to the corollary of [6], which can be described as follows:

Corollary 1.1. *Let $f \in Span\{f_j\}_{j\in\mathbb{N}} \subset L^p(X, \Sigma, \mu)$ with μ σ-finite, $\{\theta_j\}_j$ a sequence of i.i.d. r.v.'s taking values in $\{+1, -1\}$ with equal probability, and $\{f_j\}_{j\in\mathbb{N}}$ an unconditional basic (l_p-stable) sequence. If $f = \sum_i a_i f_i$ is the expansion of f in this basis, then the random series $\sum_i \theta_i a_i f_i$ converges a.e. $[\mu]$ a.s.*

Proof. It will become clear that this result is a direct application of Theorem 1.1 below and the definition of an unconditional basic (l^p-stable) sequence. □

Intuitively, given $f \in Span\{f_j\}_j \subset L^p(X, \Sigma, \mu)$, one may expect the series expansion of f in the basis $\{f_j\}_{j\in\mathbb{N}}$ to converge not only in the norm of $L^p(X, \Sigma, \mu)$ but also almost everywhere. It should be pointed out that the exceptional set of zero probability may not necessarily be void for an arbitrary unconditional basic sequence (basis). This follows from the following result in Ergodic Theory: orthonormal bases in a Hilbert space are unconditional bases, but Menchoff [5] showed that if (X, Σ, μ) is $[0, 1]$ with Lebesgue Measure, then there exists an orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of $L^2[0, 1]$ and an $f_0 \in L^2[0, 1]$ such that the sequence $P_k f_0$ of projections of f_0 on the subspaces spanned by $\{f_1, ..., f_k\}$ diverges a.e.

In the following, in Section 2 we give some definitions and auxiliary results, and in Section 3 we prove Theorem 1.1, from which Corollary 1.1 follows.

2. Auxiliary results and definitions

Here we will be considering two measure spaces: a probability space, say $(\Omega, \mathcal{F}, \mathbf{P})$, and another measure space (X, Σ, μ), with μ σ-finite. As usual, we define the Lebesgue spaces $L^p(X, \Sigma, \mu)$. We talk about properties that hold *almost everywhere* $[\mu]$ *almost surely.* This must be understood without ambiguity meaning that such a property holds for a measurable set defined in $X \times \Omega$ with the complete measure $\mu \times \mathbf{P}$ [2]. The main target is to deal with certain random elements [7] in $L^p(X, \Sigma, \mu)$, but some results remain true in a general separable Banach space with arbitrary norm $\|.\|$. In this case we will denote it by just $(E, \|.\|)$, and in order to make things work we must consider in E, $\mathcal{B}(E)$ [7], the Borel σ-algebra generated by the open sets of $(E, \|.\|)$. Then, a random element is a measurable map $X : \Omega \longrightarrow E$, where sometimes $E = L^p(X, \Sigma, \mu)$.

In a Banach space E, a sequence $\{f_j\}_{j\in\mathbb{N}} \subset E$ is called a (Schauder) basis if $\forall x \in E$ there exists a unique sequence $\{a_j\}_{j\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ such that $x = \sum_{i\in\mathbb{N}} a_i f_i$, where this must be understood as a limit in the norm topology. A Schauder basis $\{f_j\}_{j\in\mathbb{N}}$ is called [3] an *unconditional* basis if $\forall a \in \mathbb{R}^{\mathbb{N}}$ such that $\sum_{i\in\mathbb{N}} a_i f_i$ converges, then $\sum_{i\in\mathbb{N}} \theta_i a_i f_i$ converges, provided that $\theta_j = \mp 1$. A sequence $\{f_j\}_{j\in\mathbb{N}}$ is called a (unconditional) basic sequence, if it is a (unconditional) basis of a closed subspace of E.

Now, we need some results from probability theory. The following can be found, for example, in [4]:

Theorem 2.1 (Generalized Kolmogorov inequality)**.** *Let $X_1, X_2, \ldots$ be independent r.v.'s with $\mathbb{E}X_i = 0\ \forall\, i$, and let $p \geq 1$, $\delta > 0$. Then*

$$\mathbf{P}\left(\bigvee_{j=1}^{n}\left|\sum_{i=1}^{j} X_i\right| > \delta\right) \leq \frac{1}{\delta^p}\mathbb{E}\left|\sum_{i=1}^{n} X_i\right|^p .$$

We will also need the following inequality [2]:

Lemma 2.1. *Let $0 < \lambda < 1$, $X \in L^2(\Omega, \mathcal{F}, \mathbf{P})$ and $X > 0$ a.s. Then*

$$\mathbf{P}\left(X \geq \lambda\mathbb{E}(X)\right) \geq (1-\lambda)^2\frac{(\mathbb{E}(X))^2}{\mathbb{E}X^2}.$$

l^p-stability is defined as an equivalence of norms:

Definition 1 ([1])**.** Let $(E, \|.\|)$ be a Banach space. Then $\{f_j\}_{j\in\mathbb{N}} \subset E$ is an l^p-stable sequence ($p \in [1, \infty)$) if there exist positive constants c_p and K_p, such that:

$$c_p\,\|a\|_{l^p} \leq \left\|\sum_i a_i f_i\right\| \leq K_p\,\|a\|_{l^p}\,, \quad \forall\, a \in l_0\,.$$

In the following μ will be a σ-finite measure. We are also interested in unconditional basic sequences, and the following result from [9] will be very important in the sequel:

Theorem 2.2. *Let $\{f_j\}_{j\in\mathbb{N}}$ be a basic sequence in $L^p(X, \Sigma, \mu)$ $(1 \leq p \leq \infty)$. Then it is unconditional if and only if there exist positive constants A_p, B_p such that:*

$$A_p\left\|\left(\sum_j |a_j f_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(X)} \leq \left\|\sum_j a_j f_j\right\|_{L^p(X)} \leq B_p\left\|\left(\sum_j |a_j f_j|^2\right)^{\frac{1}{2}}\right\|_{L^p(X)}$$

$\forall\ \sum_j a_j f_j\ \in L^p(X, \Sigma, \mu)$.

This result characterizes unconditional basic sequences in terms of equivalency of norms or as a "Littlewood-Paley like" inequality. We will use this equivalence without referring to it, but it will become clear from the context. Moreover, with this, we can prove our first result: a kind of analogue of a result in the work of Paley and Zygmund.

Proposition 2.1. *a) Let $\{f_j\}_{j\in\mathbb{N}} \subset E$ be an l^p-stable sequence, $0 < \lambda < 1$, and let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of independent random variables such that there exists a constant $C > 0$ such that $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^p)^2$, $\forall\, j$. Then equation* (2.1) *holds.*

b) Let $\{f_j\}_{j\in\mathbb{N}} \subset L^p(X, \Sigma, \mu)$ $(\infty > p \geq 2)$ be a basic unconditional sequence, $0 < \lambda < 1$; and let $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of independent random variables such that there exists a constant $C > 0$ such that $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^2)^p$, $\forall\, j$. Then equation (2.1) *holds. If $1 \leq p < 2$, the last assertion remains true with the additional condition: $(\mathbb{E}|a_i|^p)^{\frac{1}{p}} \geq c(\mathbb{E}|a_i|^2)^{\frac{1}{2}}$.*

$$\mathbf{P}\left(\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^p > \lambda\mathbb{E}\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^p\right) \geq (1-\lambda)^2 k, \tag{2.1}$$

where k is a positive constant independent of n.

Remark. The hypothesis $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^p)^2 \;\forall\, j$ is a regularity condition in order to control the values of the a_i's. Similar conditions can be found in [2] dealing, for example, with random Fourier series. This condition prevents a_j from being small with a large probability and from being large with a small probability. For example, if the a_j's are $\mathcal{N}(0,{\sigma_j}^2)$, then it is known that $(\mathbb{E}|a_j|^p)^{\frac{1}{p}} = c(p)(\mathbb{E}|a_j|^{2p})^{\frac{1}{2p}}$.

Part a). First, by Lemma 2.1 we have:

$$\mathbf{P}\left(\left\|\sum_{j=1}^n a_j\, f_j\right\|^p > \lambda\mathbb{E}\left\|\sum_{j=1}^n a_j\, f_j\right\|^p\right) \geq (1-\lambda)^2\frac{\left(\mathbb{E}\left\|\sum\limits_{j=1}^n a_j\, f_j\right\|^p\right)^2}{\mathbb{E}\left\|\sum\limits_{j=1}^n a_j\, f_j\right\|^{2p}}. \tag{2.2}$$

On the other hand,
(2.3)

$$\begin{aligned}
\mathbb{E}\left\|\sum_{j=1}^n a_j\, f_j\right\|^{2p} &\leq K_p^{2p}\mathbb{E}\left(\sum_{j=1}^n |a_j|^p\right)^2 = K_p^{2p}\mathbb{E}\left(\sum_{j=1}^n |a_j|^p \sum_{i=1}^n |a_i|^p\right)\\
&= K_p^{2p}\left(\sum_{j=1}^n \mathbb{E}|a_j|^{2p} + \sum_{i}^n\sum_{j\neq i}^n \mathbb{E}|a_j|^p\mathbb{E}|a_i|^p\right)\\
&\leq K_p^{2p}\left(\sum_{j=1}^n C(\mathbb{E}|a_j|^p)^2 + \left(\sum_{i=1}^n \mathbb{E}|a_i|^p\right)^2\right) \leq K_p^{2p}(C+1)\left(\sum_{j=1}^n \mathbb{E}|a_j|^p\right)^2.
\end{aligned}$$

Clearly, from (2.3):

$$\frac{\left(\mathbb{E}\left\|\sum\limits_{j=1}^n a_j\, f_j\right\|^p\right)^2}{\mathbb{E}\left\|\sum\limits_{j=1}^n a_j\, f_j\right\|^{2p}} \geq \frac{c_p^{2p}\left(\sum_{j=1}^n \mathbb{E}|a_j|^p\right)^2}{K_p^{2p}(C+1)\left(\sum_{j=1}^n \mathbb{E}|a_j|^p\right)^2}.$$

This, together with equation (2.2) implies the desired result.

Part b). To bound $\mathbb{E}\left\|\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{1}{2}}\right\|^p$ we must consider two separate cases: first $\infty > p \geq 2$ and then $1 \leq p \leq 2$. The rest of the proof is valid for all $\infty > p \geq 1$. If $p \geq 2$, then,

(2.4)

$$\begin{aligned}
\mathbb{E}\left\|\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{1}{2}}\right\|^p &= \int\limits_\Omega\int\limits_X \left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} d\mu d\mathbf{P} = \int\limits_X\int\limits_\Omega \left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} d\mathbf{P} d\mu\\
&= \int\limits_X \mathbb{E}\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} d\mu.
\end{aligned}$$

But by Hölder's inequality, $\mathbb{E}\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} \geq \left(\mathbb{E}\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}}$ and clearly, from this and (2.4) we have:

$$\mathbb{E}\left\|\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{1}{2}}\right\|^p \geq \left\|\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^p. \tag{2.5}$$

Now, if $1 \leq p < 2$, then as a direct consequence of Minkowski's integral inequality:

(2.6)

$$\int_X \mathbb{E}\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} d\mu \geq \int_X \left(\sum_{i=1}^n (\mathbb{E}|a_i|^p)^{\frac{2}{p}} |f_i|^2\right)^{\frac{p}{2}} d\mu \geq c\left\|\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^p.$$

On the other hand,

$$\mathbb{E}\left\|\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{1}{2}}\right\|^{2p} = \int_\Omega \left(\int_X \left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{p}{2}} d\mu\right)^2 d\mathbf{P}. \tag{2.7}$$

If we define $g(x,\omega) = \left(\sum_{i=1}^n |a_i(\omega) f_i(x)|^2\right)^{\frac{p}{2}}$, then, by Minkowski's inequality, we have the following bound on (2.7):

$$\int_\Omega \left(\int_X g(x,\omega) d\mu\right)^2 d\mathbf{P} \leq \left(\int_X \left(\int_\Omega |g(x,\omega)|^2 d\mathbf{P}\right)^{\frac{1}{2}} d\mu\right)^2.$$

Now, $\int_\Omega |g(x,\omega)|^2 d\mathbf{P} = \mathbb{E}\left(\sum_{i=1}^n |a_i f_i(x)|^2\right)^p$. Then by the triangle inequality:

$$\begin{aligned}\mathbb{E}\left(\sum_{i=1}^n |a_i f_i(x)|^2\right)^p &\leq \left(\sum_{i=1}^n \left(\mathbb{E}|a_i f_i(x)|^{2p}\right)^{\frac{1}{p}}\right)^p \\ &= \left(\sum_{i=1}^n (\mathbb{E}|a_i|^{2p})^{\frac{1}{p}} |f_i(x)|^2\right)^p \leq C\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i(x)|^2\right)^p,\end{aligned} \tag{2.8}$$

where the last inequality follows from $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^2)^p$, $\forall\, j$.

Hence:

$$\begin{aligned}&\int_X \left(\int_\Omega |g(x,\omega)|^2 d\mathbf{P}\right)^{\frac{1}{2}} d\mu \\ &\quad \leq C^{\frac{1}{2}} \int_X \left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{p}{2}} d\mu = C^{\frac{1}{2}}\left\|\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|_{L^p(X)}^p,\end{aligned}$$

and from this it is immediate that

$$\mathbb{E}\left\|\left(\sum_{i=1}^n |a_i f_i|^2\right)^{\frac{1}{2}}\right\|^{2p} \leq C\left\|\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^{2p}. \tag{2.9}$$

By equations (2.9) and (2.4) or (2.6) we have the following bounds:

$$\mathbb{E}\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^p \geq A_p^p \mathbb{E}\left\|\left(\sum_{i=1}^{n}|a_i f_i|^2\right)^{\frac{1}{2}}\right\|^p \geq k_p A_p^p \left\|\left(\sum_{i=1}^{n}\mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^p \tag{2.10}$$

and

(2.11)

$$\mathbb{E}\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^{2p} \leq B_p^{2p} \mathbb{E}\left\|\left(\sum_{i=1}^{n}|a_i f_i|^2\right)^{\frac{1}{2}}\right\|^{2p} \leq C B_p^{2p} \left\|\left(\sum_{i=1}^{n}\mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^{2p}.$$

Recalling (2.2), also from (2.10) and (2.11), we have:

$$\frac{\left(\mathbb{E}\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^p\right)^2}{\mathbb{E}\left\|\sum_{j=1}^{n} a_j\, f_j\right\|^{2p}} \geq \frac{A_p^{2p}\left\|\left(\sum_{i=1}^{n}\mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^{2p}}{B_p^{2p} C\left\|\left(\sum_{i=1}^{n}\mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^{2p}},$$

and then we get the desired result. □

Now, let us prove a result which is a consequence of Theorem 2.1:

Theorem 2.3. *Let $\{X_i\}_i$ be a sequence of random elements in $L^p(X,\Sigma,\mu)$ such that $\mathbb{E}X_i = 0$ $[\mu]$-a.e. and $\{X_i(x,\cdot)\}_i$ are independent for almost all $x \in X$. Then if $\sum_i X_i$ converges in the norm topology of $L^p(X\times\Omega)$, then it converges $[\mu]$-a.e. a.s.*

Proof. This proof is a standard argument. First we begin by transferring Theorem 2.1 for random variables to this context:

$$\begin{aligned}
&\int_X \mathbf{P}\left((x,\omega)\in X\times\Omega : \bigvee_{j=1}^{n}\left|\sum_{i=m+1}^{m+j} X_i(x,\omega)\right| > \delta\right) d\mu \\
&\leq \frac{1}{\delta^p}\int_X \mathbb{E}\left|\sum_{i=m+1}^{m+n} X_i\right|^p d\mu \qquad \text{(By Theorem 2.1)} \\
&= \frac{1}{\delta^p}\mathbb{E}\left\|\sum_{i=m+1}^{m+n} X_i\right\|^p_{L^p(X,\Sigma,\mu)} \qquad \text{(By Fubini's theorem).}
\end{aligned} \tag{2.12}$$

Now, with this maximal inequality we have: in $X\times\Omega$ write $\nu = \mu\times\mathbf{P}$; taking $\delta > 0$ and $m\in\mathbb{N}$ then:

$$\left\{(x,\omega) : \sup_{j\in\mathbb{N}}\left|\sum_{i=m+1}^{m+j} X_i(x,\omega)\right| > \delta\right\} \subset \bigcup_{n\in\mathbb{N}} D_n\ ,$$

where $D_n = \left\{(x,\omega) : \bigvee_{j=1}^{n} \left|\sum_{i=m+1}^{m+j} X_i(x,\omega)\right| > \delta\right\}$. Clearly $D_n \subset D_{n+1}$. Then

$$\begin{aligned}\nu\left\{(x,\omega) : \sup_{j\in\mathbb{N}}\left|\sum_{i=m+1}^{m+j} X_i(x,\omega)\right| > \delta\right\} &\leq \nu\left(\bigcup_{n\in\mathbb{N}} D_n\right) = \lim_{n\to\infty}\nu(D_n) \\ &\leq \frac{K_p^p}{\delta^p}\lim_{n\to\infty}\mathbb{E}\left\|\sum_{i=m+1}^{n} X_i\right\|^p = C(m,\delta) < \infty \qquad \text{(By equation (2.12))}.\end{aligned}$$

Since $\sum_{i=1}^n X_i$ is Cauchy in $L^p(X\times\Omega)$, this implies:

$$\lim_{m\to\infty}\nu\left\{(x,\omega) : \sup_{j\in\mathbb{N}}\left|\sum_{i=m+1}^{m+j} X_i(x,\omega)\right| > \delta\right\} = 0\,. \tag{2.13}$$

Define $E_{n\,\delta} = \left\{(x,\omega) : \sup_{j,k>n}\left|\sum_{i=k+1}^{j} X_i(x,\omega)\right| > 2\delta\right\}$. Then

$$E_{n\,\delta} \subset \left\{\sup_{j\in\mathbb{N}}\left|\sum_{i=n+1}^{n+j} X_i(x,\omega)\right| > \delta\right\}\,,$$

so that $E_{n+1\,\delta} \subset E_{n\,\delta}$. From this and equation (2.13) we have:

$$\nu\left(\bigcap_{n\in\mathbb{N}} E_{n\,\delta}\right) = \lim_{n\to\infty}\nu(E_{n\,\delta}) = 0 \Longrightarrow \nu\left(\bigcup_{\delta\in\mathbb{Q}_{>0}}\bigcap_{n\in\mathbb{N}} E_{n\,\delta}\right) = 0\,. \qquad \square$$

3. Main results

First, let us note that if $\sup_{n\in\mathbb{N}}\left\|\left(\sum_{i=1}^n \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^p < \infty$ under the conditions of Proposition 2.1 part b), then it is inmediate that $S_n = \sum_{i=1}^n a_i f_i$ is a Cauchy sequence in $L^p(X\times\Omega)$ (equations (3.4) and (3.5)) and since for $\lambda > 0$: $\mathbf{P}(\|S_n - S_m\| > \lambda) \leq \frac{\mathbb{E}\|S_n - S_m\|^p}{\lambda^p}$, then it is a Cauchy sequence in probability (in the sense for random elements [7], [2]), but the convergence in probability of sums of random independent elements implies a.s. convergence ([2], Chap. 2). A similar argument holds for the case of l^p-stable sequences. Now we need a converse of this fact, which is not as trivial.

Proposition 3.1. *a) Let $\{f_j\}_{j\in\mathbb{N}}$ be an l^p-stable sequence and $\{a_j\}_{j\in\mathbb{N}}$ be a sequence of independent random variables such that there exists a constant $C > 0$ such that $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^p)^2\ \forall j$. Then if $\sum_{i=1}^\infty a_i f_i$ converges in the norm topology of E a.s., then*

$$\sum_{i=1}^\infty \mathbb{E}|a_i|^p < \infty\,.$$

b) Let $\{f_j\}_{j\in\mathbb{N}} \subset L^p(X,\Sigma,\mu)$ $(p \geq 2)$ be a basic unconditional sequence and $\{a_j\}_{j\in\mathbb{N}}$ a sequence of independent random variables such that there exists a constant $C > 0$, $\mathbb{E}|a_j|^{2p} \leq C(\mathbb{E}|a_j|^2)^p\ \forall\, j$. If $\sum_{i=1}^\infty a_i f_i$ converges in the norm topology of $L^p(X,\Sigma,\mu)$ a.s., then

$$\left\|\left(\sum_{i=1}^\infty \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\|^p < \infty\,.$$

If $1 \le p < 2$*, the last assertion remains true with the additional condition:* $(\mathbb{E}|a_i|^p)^{\frac{1}{p}} \ge c(\mathbb{E}|a_i|^2)^{\frac{1}{2}}$.

Proof of part a). Take $\lambda \in (0,1)$, define

$$D_n = \left\{ \omega \in \Omega : \left\| \sum_{j=1}^{n} a_j\, f_j \right\|^p > \lambda \mathbb{E} \left\| \sum_{j=1}^{n} a_j\, f_j \right\|^p \right\},$$

and define

$$D = \overline{\lim_{n\to\infty}} D_n = \bigcap_{p=1}^{\infty} \bigcup_{n=p}^{\infty} D_n\ . \tag{3.1}$$

By Proposition 2.1, $\exists\, k > 0$ such that $\mathbf{P}(D_n) \ge k(1-\lambda)^2$ for all n, but

$$\mathbf{P}(\overline{\lim_{n\to\infty}} D_n) \ge \overline{\lim_{n\to\infty}} \mathbf{P}(D_n) \ge\ k(1-\lambda)^2 >\ 0.$$

Then $\mathbf{P}(D) > 0$.

From this last fact, $D \cap \{\omega \in \Omega : \sum_i a_i f_i$ *converges in* $E\} \ne \varnothing$, equivalently, $\exists\, \omega \in D$ such that $\sum_i a_i(\omega) f_i$ converges in $(E, \|.\|)$ and this implies: $\exists\, M > 0$ such that

$$\sup_{n\in\mathbb{N}} \left\| \sum_{j=1}^{n} a_j(\omega)\, f_j \right\|^p \le\ M\ .$$

By equation (3.1) there exist infinitely many n's, such that for this $\omega \in D$:

$$\begin{aligned} \infty > M \ge \left\| \sum_{j=1}^{n} a_j(\omega)\, f_j \right\|^p &> \lambda \mathbb{E} \left\| \sum_{j=1}^{n} a_j\, f_j \right\|^p \ge \lambda c_p^p \sum_{i=1}^{n} \mathbb{E}|a_i|^p \\ \Longrightarrow \sum_{i=1}^{\infty} \mathbb{E}|a_i|^p\ &<\ \infty, \end{aligned} \tag{3.2}$$

and the proof of a) is complete. □

Proof of part b). The proof is almost the same as for Part a). Instead of the bound (3.2), recalling the bound of equation (2.10) we have:

$$\infty > M \ge \left\| \sum_{j=1}^{n} a_j(\omega)\, f_j \right\|^p > \lambda \mathbb{E} \left\| \sum_{j=1}^{n} a_j\, f_j \right\|^p \ge \lambda k_p A_p^p \left\| \left(\sum_{i=1}^{n} \mathbb{E}|a_i|^2 |f_i|^2 \right)^{\frac{1}{2}} \right\|^p \tag{3.3}$$

for infinitely many n's and, then by Beppo Levi's theorem:

$$\infty > \lim_{n\to\infty} \left\| \left(\sum_{i=1}^{n} \mathbb{E}|a_i|^2 |f_i|^2 \right)^{\frac{1}{2}} \right\|^p = \left\| \left(\sum_{i=1}^{\infty} \mathbb{E}|a_i|^2 |f_i|^2 \right)^{\frac{1}{2}} \right\|^p .$$ □

3.1. **Main result.** Now, we can prove the desired result:

Proof of Theorem 1.1. Under the hypotheses of a) or b) , we can see that $\sum_{i=1}^{n} a_i f_i$ is a Cauchy sequence in $L^p(X \times \Omega)$. Then both assertions will follow as a consequence of Theorem 2.3:

Part a). We have

$$\mathbb{E}\left\|\sum_{i=m}^{n} a_i f_i\right\|^p \leq K_p^p \mathbb{E}\sum_{i=m}^{n} |a_i|^p = K_p^p \sum_{i=m}^{n} \mathbb{E}|a_i|^p \longrightarrow 0\,, \tag{3.4}$$

when $n, m \longrightarrow \infty$, as a consequence of Proposition 3.1, Part a).

Part b). Again, from Hölder's inequality and equation (2.11):

(3.5)

$$\mathbb{E}\left\|\sum_{i=m}^{n} a_i f_i\right\|^p \leq \left(\mathbb{E}\left\|\sum_{i=m}^{n} a_i f_i\right\|^{2p}\right)^{\frac{1}{2p}} \leq C^{\frac{1}{2p}} B_p \left\|\left(\sum_{i=m}^{n} \mathbb{E}|a_i|^2|f_i|^2\right)^{\frac{1}{2}}\right\| \longrightarrow 0,$$

when $n, m \longrightarrow \infty$, since $\sum_{i=1}^{n} \mathbb{E}|a_i|^2|f_i|^2$ is a Cauchy sequence in $L^{\frac{p}{2}}(X, \Sigma, \mu)$ as a consequence of Proposition 3.1, Part b). □

4. Conclusions

Unconditional basic sequences are very important in the theory of Banach spaces. Another related concept is l^p-stability, and both are important topics in wavelet analysis [8], shift invariant subspaces and sampling. Here we have shown that if $\{f_j\}_j$ is an unconditional basic sequence or an l_p-stable sequence, then, if the random series (1.1) converges in the norm topology a.s., then (1.1) also converges $[\mu]$-almost everywhere a.s. On the other hand, paraphrasing [2] in many circumstances it is hard to find a mathematical object with some prescribed properties, or to prove that a certain family of objects verifies a given property, but it is pretty easy to exhibit random objects which enjoy these properties almost surely. This is the main idea behind the result obtained in Corollary 1.1.

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